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Lie Bialgebras on $\mathbf{k}^3$ and Lagrange Varieties

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Abstract. Lie bialgebras on $\mathbf{k}^3$ and the corresponding Lagrange varieties are classified by means of a pair of quadratic forms on $\mathbf{k}^4$, where $\mathbf{k}$ is a field whose characteristic is not 2. It turns out that any Lagrange variety is composed of two (possibly degenerate) quadratic surfaces in $\mathbf{k}P_3$ defined by the above quadratic forms respectively.
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1. Introduction

The theory of Lie bialgebras comes from the study of quantum groups and integrable hamiltonian systems, and it has broad applications in mathematics and physics. In particular, Lagrange subalgebras, also called Dirac structures, in the Drinfel'd double of a given Lie bialgebra are closely connected to many important notions and research topics in this theory such as Manin triples, Poisson homogenous spaces ([6], [13], [17]), Manin pairs and moment maps ([1], [2]) as well as pure spinors ([2], [10]). It is known that all the Lagrange subalgebras form a subvariety of the Grassmann manifold and have rich algebraic and geometric properties ([4], [7], [12]). Therefore, it is worthwhile to get to the bottom of Lagrange varieties, at least, in low dimensional cases.

The purpose of this paper is to classify three dimensional Lie bialgebras and the corresponding Lagrange varieties. There are already some studies on classifying Lie bialgebras on the field $\mathbb{R}^3$ ([8], [18], [20]). Actually, we can work on an arbitrary field $\mathbf{k}$ which has the characteristic $\neq 2$ by a pure algebraic way, though the motivation comes from a geometric point of view in [15], where Lie-Poisson structures on $\mathbb{R}^3$ are classified by compatible pairs, which are composed of the modular vector and a quadratic function. The idea of using linear Poisson brackets to understand the structure of Lie algebras can be traced back to the work of Lie. In this spirit it has been suggested that this geometric approach be pursued for Lie algebra structures(e.g., see [3], [9]).

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The paper is organized as follows: In section 2, we characterize and classify Lie algebras on $\mathbf{k}^3$ by use of compatible pairs. In section 3, Lie bialgebras on $\mathbf{k}^3$ are classified by means of a pair of quadratic forms on $\mathbf{k}^4$. In addition, a complete list of their standard forms for $\mathbf{k}$ being $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Z}_3$ is given in the appendix. In section 4, the r-matrices are classified. In section 5, we prove that all Lagrange varieties are composed of two (possibly degenerate) quadratic surfaces in $\mathbf{k}P_3$.

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2. 3-dim. Lie algebras over k

Throughout this paper we will use the following notations: (1). $\mathbf{k}$ is a field with the characteristic $\neq 2$. In particular, we take $\mathbf{k}$ to be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Z}_3$ as examples for the classification. (2). $\mathfrak{g} \cong \mathbf{k}^3$ is a three dimensional vector space or a Lie algebra over $\mathbf{k}$ and $\mathfrak{g}^*$ is the dual space. (3). $V \in \wedge^3\mathfrak{g}$ and $V^* \in \wedge^3\mathfrak{g}^*$ are volume forms respectively such that $\langle V, V^*\rangle = 1$. The following identities will be used below.

Lemma 2.1. *With the notations above, we have*

(1) $\eta \wedge i_x V^* = \langle \eta, x\rangle V^*, \quad \forall \eta \in \mathfrak{g}^*, \forall x \in \mathfrak{g};$

(2) $i_{(i_P V^*)} V = P, \quad \forall P \in \wedge^k \mathfrak{g}, \quad k = 1, 2, 3;$

(3) $(Tx) \wedge y \wedge z + x \wedge (Ty) \wedge z + x \wedge y \wedge (Tz) = tr(T) x \wedge y \wedge z, \quad \forall T \in \mathfrak{gl}(\mathfrak{g});$

In order to classify Lie-Poisson structures on $\mathbb{R}^3$, a decomposition of the Poisson structures is given in [15] by use of a pair (k, f), where $k \in \mathbb{R}^3$ is the modular vector and f is a homogenous quadratic function such that $\hat{k} f = 0$. One will see that the same conclusion is true on general fields.

Lemma 2.2. *Let $\mathfrak{g}$ be a three dimensional vector space over $\mathbf{k}$. Then, for $\kappa \in \mathfrak{g}^*$ and $A = A^* : \mathfrak{g}^* \to \mathfrak{g}$, the following defined bracket,*

$$[x, y] \triangleq i_\kappa(x \wedge y) + A(i_{x\wedge y} V^*), \quad \forall x, y \in \mathfrak{g}, \tag{1}$$

is a Lie bracket if and only if $A\kappa = 0$.

Proof. First, we claim that both the brackets defined by

$$[x, y]_\kappa \triangleq i_\kappa(x \wedge y) = \langle \kappa, x\rangle y - \langle \kappa, y\rangle x, \qquad [x, y]_A \triangleq A(i_{x\wedge y} V^*) \tag{2}$$

are Lie brackets on $\mathfrak{g}$. The first one can be checked directly, and the second one can be checked by means of the following identity:

$$[x, y]_A \wedge z + [y, z]_A \wedge x + [z, x]_A \wedge y = 0, \quad \forall x, y, z \in \mathfrak{g}. \tag{3}$$

Therefore, we have

$$\begin{array}{rcl} [[x,y],z]+c.p. &=& [[x,y]_A,z]+[[x,y]_\kappa,z]+c.p. \\ &=& [[x,y]_\kappa,z]_A+[[x,y]_A,z]_A+[[x,y]_\kappa,z]_\kappa+c.p. \\ &=& 2\langle\kappa,x\rangle A i_{y\wedge z}V^*+c.p. \\ (set\ x\wedge y\wedge z=cV) &=& 2cAi_{(i_\kappa V)}V^* \\ (by\, Lemma\, 2.1\,(2)) &=& 2c\langle x\wedge y\wedge z, V^*\rangle A\kappa. \end{array}$$

Thus, the bracket defined by the pair (κ, A) satisfies the Jacobi identity if and only if $A\kappa = 0$ since $Ch(\mathbf{k}) \neq 2$. ■

We will call (κ, A) a *compatible pair* if $A\kappa = 0$. As what has been done for $\mathbf{k} = \mathbb{R}$ in [15], we get the same conclusion for a general field.

Theorem 2.3. *There is one-one correspondence between a Lie algebra $\mathfrak{g}$ on $\mathbf{k}^3$ and a compatible pair (κ, A) such that $[\cdot,\cdot] = [\cdot,\cdot]_\kappa + [\cdot,\cdot]_A$, where $2\kappa \in \mathfrak{g}^*$ corresponds with the modular character of $\mathfrak{g}$.*

Proof. The one side has just been proved above. For the other side, let $d : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$ be the corresponding exterior derivation of the Lie algebra $\mathfrak{g}$. From a map $P \in \mathfrak{g} \otimes \mathfrak{g} : \mathfrak{g}^* \to \mathfrak{g}$ given by $P(\alpha) = -i_{d\alpha}V$, $\forall \alpha \in \mathfrak{g}^*$, we define (κ, A) as

$$A = \frac{1}{2}(P + P^*), \quad \kappa = \frac{1}{2} i_{(P-P^*)}V^*,\ i.e.,\ i_\kappa V = \frac{1}{2}(P - P^*).$$

Then, we have

$$\begin{array}{rcl} \langle\alpha,[x,y]\rangle &=& -\langle d\alpha, x\wedge y\rangle = -\langle i_{i_{d\alpha}V}V^*, x\wedge y\rangle = \langle P(\alpha), i_{x\wedge y}V^*\rangle \\ &=& \langle \frac{1}{2}(P-P^*)(\alpha), i_{x\wedge y}V^*\rangle + \langle A(\alpha), i_{x\wedge y}V^*\rangle \\ &=& \langle i_{\kappa\wedge\alpha}V, i_{x\wedge y}V^*\rangle + \langle\alpha, Ai_{x\wedge y}V^*\rangle \\ &=& \langle \kappa\wedge\alpha, x\wedge y\rangle + \langle\alpha, Ai_{x\wedge y}V^*\rangle \\ &=& \langle\alpha, i_\kappa(x\wedge y) + Ai_{x\wedge y}V^*\rangle. \end{array}$$

So the Lie bracket has just been given by (1). Finally, the fact that 2κ is the modular character can be checked easily. ■

Lemma 2.4. *Let $\mathfrak{g}_i$ $(i = 1, 2)$ be two Lie algebras on $\mathbf{k}^3$ with the compatible pairs (κ_i, A_i). Then $\mathfrak{g}_1$ is isomorphic to $\mathfrak{g}_2$, if and only if there is a $T \in GL(3, \mathbf{k})$ such that*

$$\kappa_2 = T\kappa_1, \qquad T^*A_2T = det(T)A_1.$$

Particularly, the automorphism group of $\mathfrak{g}$ is

$$Aut(\mathfrak{g}) = \{T|\ \ T \in GL(\mathfrak{g}^*),\ \ T\kappa = \kappa,\ \ T^*AT = det(T)A\}. \tag{4}$$

From the discussion above, we see that the key point of the classification of the Lie algebra structures on $\mathbf{k}^3$ is the classification of the $3 \times 3-$symmetric matrices, i.e., the quadratic forms, on this field. Of course, all of the quadratic forms can be diagonalized, but the classification of their equivalence classes seems complicated. See, e.g., [14] for more discussions of quadratic forms. In this paper, as three examples, we list all standard forms just for $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_3$, where the case of $\mathbb{R}$ has been discussed in [15].

Theorem 2.5. *Any compatible pair over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Z}_3$ is isomorphic to one of the following standard forms:*

- $\mathbf{k} = \mathbb{R}:\quad (a > 0)$

$(I).\ \kappa = 0\,(unimodular)$	$(II).\kappa = (0,0,1)^T$
$(1).\ A = 0$	$(7).\ A = 0$
$(2).\ A = I$	$(8).\ A = diag(a,a,0),$
$(3).\ A = diag(1,1,-1)$	$(9).\ A = diag(a,-a,0),$
$(4).\ A = diag(1,1,0)$	$(10).\ A = diag(1,0,0)$
$(5).\ A = diag(1,-1,0)$	
$(6).\ A = diag(1,0,0)$	

- $\mathbf{k} = \mathbb{C}:\quad (0 \leq arg(a) < \pi)$

$(I).\ \kappa = 0\,(unimodular)$	$(II).\kappa = (0,0,1)^T$
$(1).\ A = 0$	$(5).\ A = 0$
$(2).\ A = I$	$(6).\ A = diag(a,a,0),$
$(3).\ A = diag(1,1,0)$	$(7).\ A = diag(1,0,0)$
$(4).\ A = diag(1,0,0)$	

- $\mathbf{k} = \mathbb{Z}_3:$

$(I).\ \kappa = 0\,(unimodular)$	$(II).\kappa = (0,0,1)^T$
$(1).\ A = 0$	$(6).\ A = 0$
$(2).\ A = I$	$(7).\ A = diag(1,1,0)$
$(3).\ A = diag(1,1,0)$	$(8).\ A = diag(1,-1,0)$
$(4).\ A = diag(1,-1,0)$	$(9).\ A = diag(1,0,0)$
$(5).\ A = diag(1,0,0)$	

For a Lie algebra $\mathfrak{g}$ determined by (κ, A), the corresponding exterior derivation $d : \wedge^k\mathfrak{g}^* \to \wedge^{k+1}\mathfrak{g}^*$ splits into $d = d_\kappa + d_A$ according to the split of the Lie bracket in (1).

Lemma 2.6. *With the notations above, we have*

1. $d_\kappa u = -\kappa\wedge u, \quad d_A u = -i_{Au}V^*, \quad$ *i.e.,* $\quad du = -\kappa\wedge u - i_{Au}V^*, \quad \forall u \in \mathfrak{g}^*.$
2. $d_\kappa F = -2\kappa \wedge F, \quad d_A F = 0, \quad$ *i.e.,* $\quad dF = -2\kappa\wedge F, \quad \forall F \in \mathfrak{g}^*\wedge\mathfrak{g}^*.$

Proof. For *1.*, we have

$$\begin{aligned} du(x\wedge y) &= -\langle u,[x,y]\rangle, \quad \forall x,y\in\mathfrak{g} \\ &= \langle u, -\kappa(x)y+\kappa(y)x - A(i_{x\wedge y})V^*\rangle \\ &= -\kappa\wedge u(x\wedge y) - \langle i_{Au}V^*, x\wedge y\rangle. \end{aligned}$$

So $du = -\kappa\wedge u - i_{Au}V^*$. For *2.*, assume $F = u\wedge v$, where $u,v\in\mathfrak{g}^*$. Then

$$\begin{aligned} d_\kappa(u\wedge v) &= (d_\kappa u)\wedge v - u\wedge(d_\kappa v) = -2\kappa\wedge(u\wedge v); \\ d_A(u\wedge v) &= -(i_{Au}V^*)\wedge v + u\wedge(i_{Av}V^*) = \langle -Au,v\rangle V^* + \langle Av,u\rangle V^* = 0. \end{aligned}$$

Thus we have $dF = -2\kappa\wedge F$ ■

Lemma 2.7. *The Schouten bracket on the Gerstenhaber algebra $\wedge^\bullet\mathfrak{g}$ coming from the compatible pair(κ, A) can be represented as follows:*

(1) $[P,x]_A = A(i_PV^*)\wedge x, \quad [P,x]_\kappa = -2\langle\kappa,x\rangle P - (i_\kappa P)\wedge x;$

(2) $[P,Q] = 2\langle Ap,q\rangle V, \quad p \triangleq i_PV^*, \quad q\triangleq i_QV^*;$

(3) $[x,V] = 2\langle\kappa,x\rangle V, \qquad \forall x\in\mathfrak{g}, \quad P,\ Q\in\mathfrak{g}\wedge\mathfrak{g}.$

Proof. For*(1)*, without losing the generality, let $P = a\wedge b$, where $a,b\in\mathfrak{g}$. Then

$$\begin{aligned} [P,x]_A &= [a\wedge b,x]_A = [a,x]_A\wedge b + [x,b]_A\wedge a \\ &= [a,b]_A\wedge x = A(i_PV^*)\wedge x, \\ [P,x]_\kappa &= [a\wedge b,x]_\kappa = [a,x]_\kappa\wedge b + a\wedge[b,x]_\kappa \\ &= -2\kappa(x)a\wedge b + (\kappa(b)a - \kappa(a)b)\wedge x \\ &= -2\kappa(x)P - (i_\kappa P)\wedge x. \end{aligned}$$

For *(2)*, taking $Q = x\wedge y$, then

$$\begin{aligned} [P,Q]_\kappa &= [P,x\wedge y]_\kappa = [P,x]_\kappa\wedge y - [P,y]_\kappa\wedge x \\ &= -2\kappa(x)P\wedge y - (i_\kappa P)\wedge x\wedge y + 2\kappa(y)P\wedge x + (i_\kappa P)\wedge y\wedge x \\ &= -2(i_\kappa P)\wedge x\wedge y - 2P\wedge i_\kappa(x\wedge y) = -2i_\kappa(P\wedge x\wedge y) \\ &= 0, \\ [P,Q]_A &= [P,x\wedge y]_A = [P,x]_A\wedge y - [P,y]_A\wedge x \\ &= A(i_PV^*)\wedge x\wedge y - A(i_PV^*)\wedge y\wedge x = 2\langle A(i_PV^*), i_{x\wedge y}V^*\rangle V \\ &= 2\langle Ap,q\rangle V. \end{aligned}$$

For *(3)*, it is easy to see that $[x,V] = tr(ad_x)V = 2\langle\kappa,x\rangle V$. ■

3. Classification of 3-dim. Lie bialgebras

A Lie bialgebra is a Lie algebra $\mathfrak{g}$ with a cobracket $\delta:\mathfrak{g}\to\mathfrak{g}\wedge\mathfrak{g}$, i.e., its dual, $\delta^*:\mathfrak{g}^*\wedge\mathfrak{g}^*\to\mathfrak{g}^*$, defines a Lie bracket on $\mathfrak{g}^*$, such that

$$\delta[x,y] = [\delta x,y] + [x,\delta y], \quad \forall x,y\in\mathfrak{g}. \tag{5}$$

Suppose that the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$ are defined by (κ, A) and (ξ, B) respectively, where

$$A = A^* : \mathfrak{g}^* \to \mathfrak{g},\ \kappa \in \mathfrak{g}^*,\ A\kappa = 0; \quad B = B^* : \mathfrak{g} \to \mathfrak{g}^*,\ \xi \in \mathfrak{g},\ B\xi = 0.$$

Lemma 3.1. *$(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, if and only if*

$$AB + \xi \otimes \kappa = \langle \kappa, \xi \rangle I_3 \tag{6}$$

where I_3 is the identity map on $\mathfrak{g}$.

Proof. Let δ be the corresponding cobracket on $\mathfrak{g}$. Thus, $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, if and only if

$$(\delta[x, y] - [\delta x, y] - [x, \delta y]) \wedge z = 0, \quad \forall x, y, z \in \mathfrak{g}.$$

From the decompositions $\delta = \delta_\xi + \delta_B$ and $[\cdot, \cdot] = [\cdot, \cdot]_\kappa + [\cdot, \cdot]_A$, we split the computation into four parts:

- $$\begin{aligned} & \delta_\xi[x, y]_\kappa - [\delta_\xi x, y]_\kappa - [x, \delta_\xi y]_\kappa \\ = {} & \delta_\xi(\kappa(x)y - \kappa(y)x) + 2\kappa(y)\delta_\xi x + (i_\kappa \delta_\xi x) \wedge y - 2\kappa(x)\delta_\xi y - (i_\kappa \delta_\xi y) \wedge x \\ = {} & (i_\kappa \delta_\xi x) \wedge y + \kappa(y)\delta_\xi x - (i_\kappa \delta_\xi y) \wedge x - \kappa(x)\delta_\xi y \\ = {} & i_\kappa(\delta_\xi x \wedge y) - i_\kappa(\delta_\xi y \wedge x) \\ = {} & i_\kappa \delta_\xi(x \wedge y), \end{aligned}$$
- $$\begin{aligned} & \delta_\xi[x, y]_A - [\delta_\xi x, y]_A - [x, \delta_\xi y]_A \\ = {} & -\xi \wedge [x, y]_A + [\xi \wedge x, y]_A + [x, \xi \wedge y]_A \\ = {} & \xi \wedge [x, y]_A + x \wedge [y, \xi]_A + y \wedge [\xi, x]_A \\ = {} & 0, \end{aligned}$$
- $$\begin{aligned} & \delta_B[x, y]_\kappa - [\delta_B x, y]_\kappa - [x, \delta_B y]_\kappa \\ = {} & \delta_B(\kappa(x)y - \kappa(y)x) + 2\kappa(y)\delta_B x + (i_\kappa \delta_B x) \wedge y - 2\kappa(x)\delta_B y - (i_\kappa \delta_B y) \wedge x \\ = {} & (i_\kappa \delta_B x) \wedge y + \kappa(y)\delta_B x - (i_\kappa \delta_B y) \wedge x - \kappa(x)\delta_B y \\ = {} & i_\kappa(\delta_B x \wedge y) - i_\kappa(\delta_B y \wedge x) \\ = {} & i_\kappa \delta_B(x \wedge y) \\ = {} & 0, \end{aligned}$$
- $$\begin{aligned} & \delta_B[x, y]_A - [\delta_B x, y]_A - [x, \delta_B y]_A \\ = {} & \delta_B A(i_{x \wedge y} V^*) - A(i_{\delta_B x} V^*) \wedge y + A(i_{\delta_B y} V^*) \wedge x \\ = {} & -i_{BA(i_{x \wedge y} V^*)} V + Ai(_{i_{Bx}} V) V^* \wedge y - Ai(_{i_{By}} V) V^* \wedge x \\ = {} & -i_{BA(i_{x \wedge y} V^*)} V + ABx \wedge y - ABy \wedge x. \end{aligned}$$

Combining the above computations and Lemma 2.1, we have

$$\begin{aligned}
& (\delta[x,y]-[\delta x,y]-[x,\delta y])\wedge z \\
= \; & (i_\kappa\delta_\xi(x\wedge y))\wedge z-(i_{BA(i_{x\wedge y}V^*)}V)\wedge z+ABx\wedge y\wedge z-ABy\wedge x\wedge z \\
= \; & -(i_\kappa 2\xi\wedge x\wedge y)\wedge z-\langle BA(i_{x\wedge y}V^*),z\rangle V-x\wedge y\wedge ABz \\
& +ABx\wedge y\wedge z+x\wedge ABy\wedge z+x\wedge y\wedge ABz \\
= \; & -\langle 2\xi,i_{x\wedge y}V^*\rangle(i_\kappa V)\wedge z-\langle ABz,i_{x\wedge y}V^*\rangle V \\
& +Tr(AB)x\wedge y\wedge z-x\wedge y\wedge ABz \\
= \; & -\langle 2\xi,i_{x\wedge y}V^*\rangle\langle\kappa,z\rangle V+Tr(AB)\langle z,i_{x\wedge y}V^*\rangle V-2\langle ABz,i_{x\wedge y}V^*\rangle V \\
= \; & -\langle[2\xi\otimes\kappa-Tr(AB)I+2AB]z,i_{x\wedge y}V^*\rangle V.
\end{aligned}$$

Since x,y,z are arbitrary, we get that

$$\begin{aligned}
\delta[x,y]-[\delta x,y]-[x,\delta y]=0 \iff & \; 2\xi\otimes\kappa-Tr(AB)I_3+2AB=0 \\
\iff & \; AB+\xi\otimes\kappa=\langle\kappa,\xi\rangle I_3
\end{aligned}$$

by the fact that $tr(\xi\otimes\kappa)=\langle\kappa,\xi\rangle$. ■

We say that (κ,A,ξ,B) is a *compatible quadruple*, if $A\kappa=0$, $B\xi=0$, Equation (6) holds for (κ,A) and (ξ,B). Thus, there is 1-1 correspondence between compatible quadruples and Lie bialgebras on $\mathbf{k}^3$. Obviously, (κ,A,ξ,B) is always compatible if $(\kappa,A)=(0,0)$ or $(\xi,B)=(0,0)$, i.e., one of $\mathfrak{g}$ and $\mathfrak{g}^*$ being abelian. So we only discuss the *nontrivial* compatible quadruples.

An interesting fact is that the three compatibility conditions of a compatible quadruple can be replaced by the one equation of a pair of quadratic forms on $\mathbf{k}^4$, which is also useful to describe the Lagrange varieties in Section 5.

Theorem 3.2. *With the notations above, a quadruple (κ,A,ξ,B) is compatible, if and only if the following equality holds:*

$$\begin{pmatrix} A & \xi \\ \xi^t & 0 \end{pmatrix}\begin{pmatrix} B & \kappa \\ \kappa^t & 0 \end{pmatrix}=\langle\kappa,\xi\rangle I_4 \tag{7}$$

Now we begin to study the classification of Lie bialgebras on $\mathbf{k}^3$. The following lemma can be verified by use of Theorem 2.5 directly.

Lemma 3.3. *Two compatible quadruples $(\kappa_i,A_i;\xi_i,B_i)$, $i=1,2$, are equivalent, i.e., the corresponding Lie bialgebras are isomorphic, if and only if there exists a $T\in GL(\mathfrak{g}^*)$ such that*

$$\kappa_2=T\kappa_1,\quad A_2=|T|(T^*)^{-1}A_1T^{-1};\quad \xi_2=(T^*)^{-1}\xi_1,\quad |T|B_2=TB_1T^*,$$

or, equivalently, for $\hat{T}=diag(T,\,det(T))\in GL(\mathfrak{g}^*\oplus\mathbf{k}V^*)$,

$$\hat{T}^*\begin{pmatrix} A_2 & \xi_2 \\ \xi_2^t & 0 \end{pmatrix}\hat{T}=|T|\begin{pmatrix} A_1 & \xi_1 \\ \xi_1^t & 0 \end{pmatrix};\ |T|\begin{pmatrix} B_2 & \kappa_2 \\ \kappa_2^t & 0 \end{pmatrix}=\hat{T}\begin{pmatrix} B_1 & \kappa_1 \\ \kappa_1^t & 0 \end{pmatrix}\hat{T}^*. \tag{8}$$

Next we discuss the standard form of compatible quadruples. In general, A and B can not be diagonalized at the same time. We always take A as a diagonal matrix and then try to look for a form of B that is as simple as possible.

Lemma 3.4. *Let (κ, A, ξ, B) be compatible a quadruple over $\mathbf{k}$ such that $\kappa \neq 0$ and $\xi \neq 0$. Then it can be represented as the following standard forms: ($a \neq 0$, $\mu \neq 0$, $a_1a_2 \neq 0$)*

$\langle \kappa, \xi\rangle = 0:\ \kappa = (0,0,1)^t,\ A = diag(a, -a, 0)$;

$$\xi = (-a\mu, \pm a\mu, 0)^t,\ B = \begin{pmatrix} 0 & 0 & \mu \\ 0 & 0 & \pm\mu \\ \mu & \pm\mu & 0 \end{pmatrix}; \tag{9}$$

$\langle \kappa, \xi\rangle \neq 0:\ \kappa = (0,0,1)^t,\ A = diag(a_1, a_2, 0)$;

$$\xi = (0,0,\mu)^t,\ B = diag(\mu/a_1, \mu/a_2, 0). \tag{10}$$

Proof. Certainly, we can choose a basis such that $\kappa = (0,0,1)^t$, $\xi = (\xi_1, \xi_2, c)^t$, where $c \triangleq \langle \kappa, \xi\rangle$, and $A = diag(a_1, a_2, 0)$ since A is symmetric and $A\kappa = 0$. Write

$$A = \begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix},\ B = \begin{pmatrix} B_2 & b \\ b^t & b_{33} \end{pmatrix},\ \xi = (\eta, c)^t,\ b^t = (b_1, b_2),\ \eta = (\xi_1, \xi_2) \in \mathbf{k}^2.$$

Then, Lemma 3.1 has the form:

$$\begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_2 & b \\ b^t & b_{33} \end{pmatrix} + \begin{pmatrix} 0 & \eta \\ 0 & c \end{pmatrix} = cI \Longleftrightarrow A_2B_2 = cI, \qquad A_2 b = -\eta, \tag{11}$$

and the condition $B\xi = 0$ is equivalent to

$$B_2\eta + cb = 0, \quad \langle b^t, \eta\rangle + b_{33}c = b_{33}c - \langle b^t, A_2 b\rangle = b_{33}c - a_1b_1^2 - a_2b_2^2 = 0. \tag{12}$$

- For $c = \langle \kappa, \xi\rangle = 0$, first we claim that $b_1b_2 \neq 0$. Say, if $b_2 = 0$, then $a_1b_1 \neq 0$ by $A_2b = -\eta \neq 0$ since we have supposed that $\xi \neq 0$. But we also have $\langle b^t, A_2b\rangle = a_1b_1^2 = 0$ by (12). This contradiction means our claim is true.

In the same way, it is easy to see that A_2 is invertible, i.e., $a_1a_2 \neq 0$, which means that, by Equation (11), $B_2 = 0$ in this case. Therefore, from the fact that

$$a_1b_1^2 + a_2b_2^2 = 0 \Longleftrightarrow \begin{pmatrix} a_1 & 0 \\ 0 & -a_1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & b_2/b_1 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & b_2/b_1 \end{pmatrix},$$

we can choose a new basis such that

$$\kappa = (0,0,1)^t,\ A = diag(a, -a, 0);\ \xi = (-a\mu, \pm a\mu, 0)^t,\ B = \begin{pmatrix} 0 & 0 & \mu \\ 0 & 0 & \pm\mu \\ \mu & \pm\mu & b_{33} \end{pmatrix}.$$

Moreover, since $\mu \neq 0$ we can change b_{33} to zero as in (9) by some $T \in GL(\mathfrak{g}^*)$. This keeps (κ, A) invariant, as shown in Lemma 3.3.

• For $c = \langle \kappa, \xi \rangle \neq 0$, (11) means that $B_2 = A_2^{-1} = diag(c/a_1, c/a_2)$. Note that $b_{33} = 0$ if $b = 0$ by (12). Acutely, since B_2 is invertible B can be always changed to the form given in (10) by some $T \in GL(\mathfrak{g}^*)$, which keeps (κ, A) invariant as follows:

$$T = \begin{pmatrix} I_2 & 0 \\ -b^t B_2^{-1} & 1 \end{pmatrix} \Longrightarrow T \begin{pmatrix} B_2 & b \\ b^t & b_{33} \end{pmatrix} T^* = \begin{pmatrix} B_2 & 0 \\ 0 & 0 \end{pmatrix}, \quad (T^*)^{-1}\xi = (0,0,c)^t.$$

Therefore, Formula (10) holds and the proof of the lemma is finished. ■

If $\kappa = 0$ or $\xi = 0$, i.e., $\mathfrak{g}$ or $\mathfrak{g}^*$ being unimodular, then Equation (6) is reduced to a simple form: $AB = 0$. It is easy to see that A and B can be diagonalized at the same time in this case. Moreover, since the analysis is same, we can just take $\kappa = 0$ and look for $\xi \in \mathfrak{g}$, such that $B\xi = 0$.

Lemma 3.5. *Let (κ, A, ξ, B) be a nontrivial compatible quadruple over* $\mathbf{k}$*, such that $\kappa = 0$ or $\xi = 0$. Then it can be represented as $A = diag(a_1, a_2, a_3)$ and $B = diag(b_1, b_2, b_3)$, satisfying $a_i b_i = 0$ for $i = 1, 2, 3$.*

Of course, the classification of Lie bialgebras on $\mathbf{k}^3$ depends on how the standard form of (κ, A) for a given Lie algebra is defined. The standard forms (κ, A) have been listed in Theorem 2.5 for $\mathbf{k}$ being $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Z}_3$, so that we can classify clearly all compatible quadruples for these cases.

Theorem 3.6. *Any nontrivial compatible quadruple over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Z}_3$ is isomorphic to a standard form listed in Tables **I-III** at the end of this paper.*

Remark 3.7. *In the tables, we use the double index $(\boldsymbol{i}, \boldsymbol{j})$ to denote the corresponding Lie bialgebra. Certainly, $(\boldsymbol{i}, \boldsymbol{j})$ is a Lie bialgebra if and only if $(\boldsymbol{j}, \boldsymbol{i})$ is also a Lie bialgebra. One reason to list both is that, sometimes, one of them is a coboundary bialgebra, but the other is not. Another reason is that they have different standard forms.*

Proof. For $\kappa \neq 0$ and $\xi \neq 0$, the classification is given by combining Lemma 3.4 and Theorem 2.5. These cases are listed as $(\boldsymbol{8}, \boldsymbol{8})$, $(\boldsymbol{9}, \boldsymbol{9})_1$ and $(\boldsymbol{9}, \boldsymbol{9})_2$ in Table **I** for $\mathbf{k} = \mathbb{R}$.

To save space, we only discuss one other in detail by taking $\mathbf{k} = \mathbb{R}$ and $\mathfrak{g} = \mathfrak{o}(2,1)$, which corresponds to **Case (3)** listed in Theorem 2.5. In this case, (κ, A) has the standard form $\kappa = 0$, $A = diag(1, 1, -1)$, and to make a compatible quadruple, we have $B = 0_3$ by Equation (6), $\xi \in \mathfrak{g} \cong \mathbb{R}^3$ can be an arbitrary vector. The pair $(\xi, 0_3)$ corresponds to **Case (7)**, if $\xi \neq 0$. By Lemma 3.3, it is easy to see that

$$(0, A, \xi, 0_3) \sim (0, A, \eta, 0_3) \Longleftrightarrow T\xi = \eta,\ \exists T \in SO(2,1) \Longleftrightarrow \langle A^{-1}\xi, \xi \rangle = \langle A^{-1}\eta, \eta \rangle.$$

Therefore, we can classify all non-abelian dual Lie algebras of $\mathfrak{o}(2,1)$ according to the $SO(2,1)$-invariant quadratic form $\langle A^{-1}\xi, \xi \rangle$ as follows:$(a > 0)$

$(\boldsymbol{3}, \boldsymbol{7})_1$: $\xi = (a, 0, 0)^t$ for $\langle A^{-1}\xi, \xi \rangle = a^2 > 0$;

$(\boldsymbol{3}, \boldsymbol{7})_2$: $\xi = (0, 0, a)^t$ for $\langle A^{-1}\xi, \xi \rangle = -a^2 < 0$;

$(\mathit{3},\mathit{7})_3$: $\xi=(0,1,1)^t$ for $\langle A^{-1}\xi,\xi\rangle=0$.

Similarly, we can obtain the standard forms for all other cases without any difficulty. ■

Remark 3.8. Obviously, we can classify compatible quadruples on a field if we can first classify the 3×3 symmetric matrices on this field, for which we have no general knowledge though we can do more besides $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{Z}_3$.

4. Classification of r-matrices

It is known that $(\mathfrak{g},\mathfrak{g}^*)$ is a coboundary Lie bialgebra, if there is a $R\in\mathfrak{g}\wedge\mathfrak{g}$ satisfying $[[R,R],x]=0,\quad\forall x\in\mathfrak{g}$, such that

$$[u,v]_{\mathfrak{g}^*}=ad^*_{R^\sharp u}v-ad^*_{R^\sharp v}u,\quad\forall u,v\in\mathfrak{g}^*,\tag{13}$$

which is equivalent to the Lie cobracket given by $\delta x=[R,x]$, $\forall x\in\mathfrak{g}$. Such an R is called an r-matrix. In particular, R is said to be triangular if $[R,R]=0$.

In our special case, we also call $r=i_RV^*\in\mathfrak{g}^*$ as an r-matrix when there is no confusion. The following three lemmas will be used to classify all r-matrices for coboundary Lie bialgebras over $\mathbf{k}^3$.

Lemma 4.1. *Let $\mathfrak{g}$ be a Lie algebra with the compatible pair (k,A). Then $R\in\mathfrak{g}\wedge\mathfrak{g}$ is an r-matrix, if and only if $\langle r,Ar\rangle\kappa=0$. In particular, R is triangular, if and only if $\langle r,Ar\rangle=0$.*

This is because, by Lemma 2.7,

$$[[R,R],x]=2\langle r,Ar\rangle[V,x]=-4\langle r,Ar\rangle\langle\kappa,x\rangle,\quad\forall x\in\mathfrak{g}.$$

Lemma 4.2. *For $T\in GL(\mathfrak{g})$, $R'=(T\otimes T)(R)$ and $r'=i_{R'}V^*$, then one has $T^*r'=\det(T)r$.*

Proof. This is because the equality,

$$\langle r',z\rangle=\langle V,R'\wedge z\rangle=|T|\langle V,R\wedge T^{-1}z\rangle=|T|\langle r,T^{-1}z\rangle=|T|\langle (T^*)^{-1}r,z\rangle,$$

holds for any $z\in\mathfrak{g}$. ■

Lemma 4.3. *For a Lie bialgebra $(\mathfrak{g},\mathfrak{g}^*)$ with $(\kappa,A;\xi,B)$ as its compatible quadruple, then the bracket on $\mathfrak{g}^*$ is defined by an r-matrix $r\in\mathfrak{g}^*$ if and only if*

$$\xi=-Ar,\quad B=\kappa\otimes r+r\otimes\kappa\tag{14}$$

Proof. By Lemma 2.1, we have

$$\langle\kappa,x\rangle R+(i_\kappa R)\wedge x=i_\kappa(R\wedge x)=i_\kappa(i_rV\wedge x)=\langle r,x\rangle i_\kappa V.$$

Thus, by Lemma 2.7, we have

$$\begin{aligned}[R,x]&=-2\langle\kappa,x\rangle R-(i_\kappa R)\wedge x+Ar\wedge x\\&=-\langle\kappa,x\rangle R-\langle r,x\rangle i_\kappa V+Ar\wedge x.\end{aligned}$$

Therefore, we see that $\delta x = [R, x]$, i.e., the Lie bracket on $\mathfrak{g}^*$ is defined by the r-matrix R, if and only if

$$-\langle\kappa, x\rangle R - \langle r, x\rangle i_\kappa V + Ar \wedge x = -\xi \wedge x - i_{Bx}V, \quad \forall x \in \mathfrak{g},$$

which is equivalent to that, for $\forall x, y \in \mathfrak{g}$ and $u \triangleq i_{x\wedge y}V^* \in \mathfrak{g}^*$,

$$\begin{aligned} 0 &= (\langle\kappa, x\rangle R + \langle r, x\rangle i_\kappa V - Ar \wedge x - \xi \wedge x - i_{Bx}V) \wedge y \\ &= \langle\kappa, x\rangle R \wedge y + \langle r, x\rangle i_\kappa V \wedge y - Ar \wedge x \wedge y - \xi \wedge x \wedge y - i_{Bx}V \wedge y \\ &= \langle\kappa, x\rangle\langle r, y\rangle V + \langle r, x\rangle\langle\kappa, y\rangle V - \langle Ar, u\rangle V - \langle\xi, u\rangle V - \langle Bx, y\rangle V \\ &= \langle(\kappa \otimes r + r \otimes \kappa - B)x, y\rangle V - \langle Ar + \xi, u\rangle V \\ \Longleftrightarrow &\quad \langle(\kappa \otimes r + r \otimes \kappa - B)x, y\rangle = \langle Ar + \xi, i_{x\wedge y}V^*\rangle = 0, \end{aligned}$$

since, for the above two quadratic forms, the first is symmetric for x and y, but the second is skew-symmetric for x and y. That is

$$\delta = [R, \cdot\,] \quad \Longleftrightarrow \quad \xi = -Ar, \quad B = \kappa \otimes r + r \otimes \kappa.$$

So the conclusion of the Lemma is true. ■

Combining Theorem 3.6 and the three lemmas above, we can classify the r-matrices for Lie bialgebras over $\mathbb{R}^3$ and $\mathbb{C}^3$. We can adjust an r-matrix into a standard form by an element of the automorphism group $Aut(\mathfrak{g}, \mathfrak{g}^*)$, which keeps the standard form of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ invariant by Lemma 4.2.

Theorem 4.4. *For* $\mathbf{k}$ *being* $\mathbb{R}$, $\mathbb{C}$ *or* $\mathbb{Z}_3$, *all the standard forms of r-matrices are listed in Tables* ***I-III*** *at the end of the paper.*

Proof. Here we only give the proof of Case (***5***, ***7***) for $\mathbf{k}$ being $\mathbb{R}$. The discussions for other cases are similar. Since $\kappa = 0$ in this case, $\langle r, Ar\rangle\kappa = 0$ holds naturally. Thus, we need only look for some r satisfying (14) by Lemma 4.1 and Lemma 4.3. As given in the table, we have $A = diag(1, -1, 0)$, $\kappa = 0$, $B = 0$ and

(***5***, ***7***)$_1$**:** $\xi = (0, 0, a)^t$. It is easy to see that Equation (14) has no solution. Thus, the bracket on $\mathfrak{g}^*$ can not be generated by an r-matrix in this case.

(***5***, ***7***)$_2$: $\xi = (1, 0, 0)^t$. The solutions of Equation (14) are as follows: $r = (-1, 0, \lambda)^t$, $\lambda \in \mathbb{R}$. We can take the standard form as, $r = (-1, 0, 0)^t$, by the action of the automorphism group of (***5***, ***7***)$_2$.

(***5***, ***7***)$_3$: $\xi = (1, 1, 0)^t$. The solutions of Equation (14) are as follows: $r = (-1, 1, \lambda)^t$. We can take the standard form as $r = (-1, 1, 0)^t$, which is triangular. ■

It is evident that $r = 0$ corresponds to both $B = 0$ and $\xi = 0$, so that the table of the appendix will only list cases of $r \neq 0$.

5. Lagrange varieties

It is known that, for a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, there is a unique Lie algebra structure on the vector space $D = \mathfrak{g} \oplus \mathfrak{g}^*$, such that both $\mathfrak{g}$ and $\mathfrak{g}^*$ are Lie subalgebras, and the inner product on D

$$(x + \alpha, y + \beta)_+ = \langle x, \beta\rangle + \langle y, \alpha\rangle, \quad \forall x, y \in \mathfrak{g},\ \alpha, \beta \in \mathfrak{g}^*$$

is ad_D-invariant. The Lie algebra D is called the double of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, and $(D, \mathfrak{g}, \mathfrak{g}^*)$ is called a Manin triple.

A Lie subalgebra $L \subset D$ is called a Lagrange subalgebra if L is also a Lagrange subspace, i.e., a maximal isotropic subspace of D with respect to $(\cdot,\cdot)_+$, which is also called a Dirac structure. The set of all Lagrange subalgebras is called the Lagrange variety. The next lemma characters a Lagrange subalgebra.

Lemma 5.1. *([5], [16]) Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a Lie bialgebra. L is defined by a pair $(\mathfrak{h}, \Omega)$, where $\mathfrak{h} = L \cap \mathfrak{g}$, $\Omega \in \mathfrak{g} \wedge \mathfrak{g}$, $\Omega^\sharp : \mathfrak{h}^\perp \to \mathfrak{g}/\mathfrak{h} \cong (\mathfrak{h}^\perp)^*$. Then L is a lagrangian subalgebra if and only if the following three conditions hold:*

(1) $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$.

(2) Ω satisfies the Maurer-Cartan type equation(mod $\mathfrak{h}$), i.e,

$$\delta(\Omega) + \frac{1}{2}[\Omega, \Omega] = 0(mod\mathfrak{h}) \tag{15}$$

(3)$\mathfrak{h}^\perp$ is integrable for the sum bracket $[\cdot,\cdot] + [\cdot,\cdot]_\Omega$, i.e,

$$[u, v] + [u, v]_\Omega \in \mathfrak{h}^\perp, \forall u, v \in \mathfrak{h}^\perp, \tag{16}$$

where $[u, v]_\Omega = ad^*_{\Omega^\sharp u} v - ad^*_{\Omega^\sharp v} u$

Lemma 5.2. *The variety of Lagrange subspaces in $\mathfrak{g} \oplus \mathfrak{g}^*$ has two connected components, both of which are isomorphic to $\mathbf{k}P^3$.*

Proof. We will use the Plucker-Grassmann coordinates to describe a Lagrange subspace. For a Lagrange subspace L defined by a pair $(\mathfrak{h}, \Omega)$, let $w = i_\Omega V^* \in \mathfrak{g}^*$. Next, we work according to the dimension of $\mathfrak{h}$.

- $\dim \mathfrak{h} = 0$. In this case L is the graph of $\Omega^\sharp$: $\mathfrak{g}^* \to \mathfrak{g}$. Taking a basis $\{e^i\}_{i=1}^3$ of $\mathfrak{g}^*$ such that $V^* = e^1 \wedge e^2 \wedge e^3$, then $f_i = e^i + \Omega^\sharp e^i$, $i = 1, 2, 3$ is a basis of L. Let $P_L = f_1 \wedge f_2 \wedge f_3 \in \wedge^3 L$. According to the decomposition

$$\wedge^3(\mathfrak{g} \oplus \mathfrak{g}^*) = \wedge^3\mathfrak{g} \oplus (\wedge^2\mathfrak{g}) \otimes \mathfrak{g}^* \oplus \mathfrak{g} \otimes (\wedge^2\mathfrak{g}^*) \oplus \wedge^3\mathfrak{g}^*,$$

we write $P_L = (P_L^{(3,0)}, P_L^{(2,1)}, P_L^{(1,2)}, P_L^{(0,3)})$. Obviously, $P_L^{(3,0)} = 0$ and $P_L^{(0,3)} = V^*$. It is easy to see that $P_L^{(2,1)} = i_\omega V \wedge \omega$, which is a quadratic map from $\mathfrak{g}^* - \{0\}$ to $(\wedge^2\mathfrak{g}) \otimes \mathfrak{g}^*$. We denote $P_L^{(1,2)}$ by $\Psi(\omega)$, where Ψ is an injective linear map:

$$\Psi : \mathfrak{g}^* \longrightarrow \mathfrak{g} \otimes (\wedge^2\mathfrak{g}^*), \quad \langle \Psi(\omega), x \wedge y \rangle = \Omega^\sharp(i_{x\wedge y}V^*), \quad \forall x, y \in \mathfrak{g},\ w = i_\Omega V^* \in \mathfrak{g}^*.$$

Then, by [19] we get the affine coordinate as well as the Plucker-Grassmanian coordinate of P_L as follows:

$$P_L = (0, i_\omega V \wedge \omega, \Psi(\omega), 1) = [0, i_\omega V \wedge \omega, \lambda\Psi(\omega), \lambda^2], \quad \lambda \neq 0.$$

- $\dim \mathfrak{h} = 2$. In this case $\Omega = 0$ and $L = \mathfrak{h} \oplus \mathfrak{h}^\perp$. Let $\mathfrak{h}^\perp = \mathbf{k}\omega$ for some $\omega \in \mathfrak{g}^*$, since $\dim \mathfrak{h}^\perp = 1$. It is easy to check that $P_L = i_\omega V \wedge \omega \in \wedge^3 L$. Thus,

$$P_L = (0, i_\omega V \wedge \omega, 0, 0) = [0, i_\omega V \wedge \omega, 0, 0].$$

This means that the variety of all Lagrange subspaces $L \subset \mathfrak{g} \oplus \mathfrak{g}^*$ such that $\dim(L \cap \mathfrak{g}) = \{0, 2\}$ is isomorphic to $\mathbf{k}P^3$ with the homogeneous coordinate $[\lambda, \omega]$ for $(\lambda, \omega) \in \mathbf{k} \times \mathfrak{g}^* \cong \mathbf{k}^4$.

It is not difficult to confirm that a Lagrange subspace L satisfying $\dim(L \cap \mathfrak{g}) = \{1, 3\}$ is equivalent to $\dim(L \cap \mathfrak{g}^*) = \{0, 2\}$. Therefore, by similar analysis, we get that the variety of all Lagrange subspaces L such that $\dim(L \cap \mathfrak{g}) = \{1, 3\}$ is also isomorphic to $\mathbf{k}P^3$. ■

Remark 5.3. It is known that Lagrange subspaces can be characterized by pure spinors in $\wedge^\bullet \mathfrak{g}^*$ (see [2], [10]). From the analysis above, it is easy to see that any odd (even) pure spinor is in the form $\omega + \lambda V^*$ $(\lambda + \theta)$ where $\lambda \in \mathbf{k}$, $\omega \in \mathfrak{g}^*$ and $\theta \in \wedge^2 \mathfrak{g}^*$.

After this discussion of the Lagrange subspaces, we are in a position to fix the Lagrange subalgebras.

Theorem 5.4. *Given a Lie bialgebra* $(\mathfrak{g}, \mathfrak{g}^*)$ *with the compatible quadruple* (A, κ, B, ξ), *then the corresponding Lagrange variety is composed of two connected components, both being isomorphic to quadratic surfaces (possibly degenerate) in* $\mathbf{k}P_3$. *More precisely, these two quadratic surfaces are defined by two symmetric matrices on* $\mathbf{k}^4$ *that appeared in Equation (7).*

Proof. For a Lagrange subalgebra L defined by a pair $(\mathfrak{h}, \Omega)$, let $w = i_\Omega V^* \in \mathfrak{g}^*$. Next we work according to the dimension of $\mathfrak{h}$.

- $\dim \mathfrak{h} = 0$. That is, L is the graph of $\Omega^\sharp \colon \mathfrak{g}^* \to \mathfrak{g}$ such that $\delta(\Omega) + \frac{1}{2}[\Omega, \Omega] = 0$, by Lemma 5.1. On one hand, we know that

$$\delta(\Omega) = \delta_\xi(\Omega) + \delta_B(\Omega) = -2\xi \wedge \Omega = -2\xi \wedge i_\omega V = -2\langle \xi, \omega \rangle V,$$

by Lemma 2.1 and Lemma 2.6. On the other hand, we know $\frac{1}{2}[\Omega, \Omega] = \langle A\omega, \omega \rangle V$ by Lemma 2.7. This means that L is a Lagrange subalgebra if and only if

$$\langle A\omega, \omega \rangle - 2\langle \omega, \xi \rangle = 0.$$

The corresponding homogenous equation in $\mathbf{k}P^3$ is

$$\langle A\omega, \omega \rangle - 2\langle \omega, \xi \rangle \lambda = [-\omega, \lambda] \begin{pmatrix} A & \xi \\ \xi^t & 0 \end{pmatrix} [-\omega, \lambda]^t = 0, \quad \lambda \neq 0. \tag{17}$$

- $\dim \mathfrak{h} = 2$. In this case $\Omega = 0$ and $L = \mathfrak{h} \oplus \mathfrak{h}^\perp$ so that Equations (2) and (3) in Lemma 5.1 are naturally satisfied. Therefore, L is a Lagrange subalgebra, if and only if $\mathfrak{h}$ is a two dimensional subalgebra of $\mathfrak{g}$. We see that, for p, $q \in \mathfrak{h}$ such that $\mathfrak{h} = span\{p, q\}$,

$$[p, q] = \kappa(p)q - \kappa(q)p + A\omega, \qquad \omega \triangleq i_{p \wedge q} V^* \in \mathfrak{h}^\perp.$$

Thus $\mathfrak{h}$ is a Lie subalgebra if and only if $[p, q] \wedge p \wedge q = A\omega \wedge p \wedge q = 0$, which is also equivalent to

$$\langle V^*, A\omega \wedge p \wedge q \rangle = \langle i_{p \wedge q} V^*, A\omega \rangle = \langle A\omega, \omega \rangle = 0.$$

So, in this case, L can also be determined by Equation (17) with $\lambda = 0$. Consequently, the Lagrange variety of $\dim(L \cap \mathfrak{g}) = \{0, 2\}$ is isomorphic to a quadratic surface in $\mathbf{k}P^3$ given by the first matrix in Equation (7). Similarly, the Lagrange variety of $\dim(L \cap \mathfrak{g}) = \{1, 3\}$ is isomorphic to a quadratic surface given by the second matrix in Equation (7). ■

We denote these two quadratic surfaces by $S_{A,\xi}$ and $S_{B,\kappa}$, respectively. Note that both of them are nondegenerate if $\langle \kappa, \xi \rangle \neq 0$ by Equation (7) and they are isomorphic to each other since

$$\begin{pmatrix} A & \xi \\ \xi^t & 0 \end{pmatrix}^{-1} = \frac{1}{\langle \kappa, \xi \rangle} \begin{pmatrix} B & \kappa \\ \kappa^t & 0 \end{pmatrix}.$$

On the other hand, the extreme degenerate situation happens when either (A, ξ) or (B, κ) is zero, in this case, the corresponding surface is degenerated to the total space $\mathbf{k}P_3$.

Example 5.5. For $\mathbf{k} = \mathbb{R}$, let $A = I$, $\kappa = 0$ and $B = 0$, $\xi = (a, 0, 0)$, $a > 0$, which is Case (*2*, *7*) in **Table I**. Then $S_{B,\kappa} = \mathbb{R}P^3$, and it is easy to see that $S_{A,\xi}$ is isomorphic to S^2.

Example 5.6. For $\mathbf{k} = \mathbb{R}$, let

$$A = diag(1, 1, -1), \quad \kappa = 0, \quad B = 0, \quad \xi = (\xi_1, \xi_2, \xi_3)$$

where $\xi_i \in \mathbb{R}$, which is Case (*3*, *7*) in **Table I**. Then $S_{B,\kappa} = \mathbb{R}P^3$, and $S_{A,\xi}$ can be written as, by the homogenous coordinates $[x, y, z, w] \in \mathbb{R}P^3$,

$(3, 7)_{\mathbf{1}}$: $x^2 + y^2 - z^2 - w^2 = 0$,
$(3, 7)_{\mathbf{2}}$: $x^2 + y^2 + z^2 - w^2 = 0$,
$(3, 7)_{\mathbf{3}}$: $x^2 + y^2 - z^2 = 0$.

Example 5.7. $\mathbf{k} = \mathbb{R}$, Case (*8*, *8*): In this case, we have

$$A = diag(a, a, 0), \quad \kappa = (0, 0, 1), \quad B = \frac{1}{a} diag(\lambda, \lambda, 0), \quad \xi = (0, 0, \lambda)$$

where $a > 0$, $\lambda \neq 0$. Since the quadratic forms are nondegenerate, which is one of the cases that $\langle \kappa, \xi \rangle \neq 0$ in Lemma 3.4, we get that both $S_{A,\xi}$ and $S_{B,\kappa}$ are isomorphic to the surface $x^2 + y^2 + z^2 - w^2 = 0$.

It is similar for Case $(9, 9)_{\mathbf{1}}$: $A = diag(a, -a, 0)$, $\kappa = (0, 0, 1)$, $B = \frac{1}{a} diag(\lambda, -\lambda, 0)$, $\xi = (0, 0, \lambda)$ where $a > 0$, $\lambda \neq 0$. Both $S_{A,\xi}$ and $S_{B,\kappa}$ are isomorphic to $x^2 + y^2 - z^2 - w^2 = 0$.

Example 5.8. $\mathbf{k} = \mathbb{C}$, Case $(6, 6)_{\mathbf{1}}$: $A = diag(\mu, \mu, 0)$, $\kappa = (0, 0, 1)$, $B = \frac{1}{\mu} diag(b, b, 0)$, $\xi = (0, 0, b)$ where $b \neq 0$, $\mu \neq 0$ and $0 \leq arg(\mu) < \pi$. Since the quadratic forms are nondegenerate in this case, we get that both $S_{A,\xi}$ and $S_{B,\kappa}$ are isomorphic to the surface

$$x^2 + y^2 + z^2 + w^2 = 0, \quad [x, y, z, w] \in \mathbb{C}P^3.$$

Table I. $\mathbf{k}=\mathbb{R}$. $a, \lambda \in \mathbb{R},\ a>0,\ \lambda \neq 0$.

case	A	κ^t	B	ξ^t	r-matrix
$(2,7)$	$diag(1,1,1)$	0	0	$(a,0,0)$	$(-a,0,0)$
$(3,7)_1$	$diag(1,1,-1)$	0	0	$(a,0,0)$	$(-a,0,0)$
$(3,7)_2$	$diag(1,1,-1)$	0	0	$(0,0,a)$	$(0,0,a)$
$(3,7)_3$	$diag(1,1,-1)$	0	0	$(0,1,1)$	$(0,-1,1)$ $(*)$
$(4,6)$	$diag(1,1,0)$	0	$diag(0,0,\pm1)$	0	
$(4,7)_1$	$diag(1,1,0)$	0	0	$(0,0,a)$	
$(4,7)_2$	$diag(1,1,0)$	0	0	$(1,0,0)$	$(-1,0,0)$
$(4,10)$	$diag(1,1,0)$	0	$diag(0,0,\pm1)$	$(a,0,0)$	
$(5,6)$	$diag(1,-1,0)$	0	$diag(0,0,1)$	0	
$(5,7)_1$	$diag(1,-1,0)$	0	0	$(0,0,a)$	
$(5,7)_2$	$diag(1,-1,0)$	0	0	$(1,0,0)$	$(-1,0,0)$
$(5,7)_3$	$diag(1,-1,0)$	0	0	$(1,1,0)$	(-1,1,0) $(*)$
$(5,10)_1$	$diag(1,-1,0)$	0	$\pm diag(0,0,1)$	$(a,0,0)$	
$(5,10)_2$	$diag(1,-1,0)$	0	$diag(0,0,1)$	$(1,1,0)$	
$(6,4)$	$diag(1,0,0)$	0	$\pm diag(0,1,1)$	0	
$(6,5)$	$diag(1,0,0)$	0	$diag(0,1,-1)$	0	
$(6,6)$	$diag(1,0,0)$	0	$diag(0,0,\pm1)$	0	
$(6,7)_1$	$diag(1,0,0)$	0	0	$(0,0,1)$	
$(6,7)_2$	$diag(1,0,0)$	0	0	$(1,0,0)$	
$(6,8)$	$diag(1,0,0)$	0	$diag(0,\lambda,\lambda)$	$(1,0,0)$	
$(6,9)$	$diag(1,0,0)$	0	$diag(0,a,-a)$	$(1,0,0)$	
$(6,10)_1$	$diag(1,0,0)$	0	$diag(0,0,\pm1)$	$(0,a,0)$	
$(6,10)_2$	$diag(1,0,0)$	0	$diag(0,0,\pm1)$	$(1,0,0)$	

Example 5.9. $\mathbf{k}=\mathbb{C}$, Case $(\boldsymbol{2},\boldsymbol{5})$: $A=I,\ \kappa=0,\ B=0,\ \xi=(\xi_1,\xi_2,\xi_3)$ where $\xi_i \in \mathbb{C}$. Then $S_{B,\kappa}=\mathbb{C}P^3$, and $S_{A,\xi}$ is isomorphic to

$$(2,5)_{\mathbf{1}}: x^2+y^2+z^2+w^2=0, \quad (2,5)_{\mathbf{2}}: x^2+y^2+z^2=0,$$

where $[x,y,z,w] \in \mathbb{C}P^3$.

6. Appendix

For $\mathbf{k}$ being $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Z}_3$, all the standard forms of r-matrices are listed in Tables **I-III** below. In these tables, the index $(\boldsymbol{i},\boldsymbol{j})$ denotes the corresponding Lie bialgebra listed in Theorem 2.5, and $(*)$ means that the r-matrix is triangular.

Table I. $\mathbf{k} = \mathbb{R}$. $\kappa^t = (0,0,1)$. $a, \lambda \in \mathbb{R},\ a > 0,\ \lambda \neq 0$.

case	A	κ^t	B	ξ^t	r-matrix
(7, 2)	0		$diag(a,a,a)$	0	
$(7,3)_1$	0		$diag(a,a,-a)$	0	
$(7,3)_2$	0		$diag(a,-a,a)$	0	
$(7,3)_3$	0		$\begin{pmatrix} a & 0 & 0 \\ 0 & 0 & a \\ 0 & a & 0 \end{pmatrix}$	0	
$(7,4)_1$	0		$diag(0,1,1)$	0	
$(7,4)_2$	0		$diag(a,a,0)$	0	
$(7,5)_1$	0		$diag(a,-a,0)$	0	
$(7,5)_2$	0		$diag(0,1,-1)$	0	
$(7,5)_3$	0		$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	0	$(0,1,0)\ (*)$
$(7,6)_1$	0		$diag(1,0,0)$	0	
$(7,6)_2$	0		$diag(0,0,1)$	0	$(0,0,\frac{1}{2})\ (*)$
(8, 6)	$diag(a,a,0)$		$diag(0,0,\pm 1)$	0	$(0,0,\pm\frac{1}{2})\ (*)$
(8, 8)	$diag(a,a,0)$		$\frac{1}{a}\ diag(\lambda,\lambda,0)$	$(0,0,\lambda)$	
(9, 6)	$diag(a,-a,0)$		$diag(0,0,1)$	0	$(0,0,\frac{1}{2})\ (*)$
$(9,9)_1$	$diag(a,-a,0)$		$\frac{1}{a}\ diag(\lambda,-\lambda,0)$	$(0,0,\lambda)$	
$(9,9)_2$	$diag(a,-a,0)$		$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \pm 1 \\ 1 & \pm 1 & 0 \end{pmatrix}$	$(-a,\pm a,0)$	$(1,\pm 1,0)\ (*)$
(10, 4)	$diag(1,0,0)$		$diag(0,\lambda,\lambda)$	0	
$(10,5)_1$	$diag(1,0,0)$		$diag(0,\lambda,-\lambda)$	0	
$(10,5)_2$	diag(1,0,0)		$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	0	$(0,1,0)\ (*)$
$(10,6)_1$	$diag(1,0,0)$		$diag(0,0,\pm 1)$	0	
$(10,6)_2$	$diag(1,0,0)$		$diag(0,\lambda,0)$	0	

Table II. $\mathbf{k} = \mathbb{C}$. $v = (0, 0, 1)$. $b, \mu \in \mathbb{C} - \{0\}$, $0 \leq arg(\mu) < \pi$, $\mathbf{i} = \sqrt{-1}$.

case	A	κ^t	B	ξ^t	r-matrix
$(2,5)_1$	$diag(1,1,1)$	0	0	$(\mu,0,0)$	$(-\mu,0,0)$
$(2,5)_1$	$diag(1,1,1)$	0	0	$(1,\mathbf{i},0)$	$(-1,-\mathbf{i},0)(*)$
$(3,4)$	$diag(1,1,0)$	0	$diag(0,0,1)$	0	
$(3,5)_1$	$diag(1,1,0)$	0	0	$(0,0,\mu)$	
$(3,5)_2$	$diag(1,1,0)$	0	0	$(1,0,0)$	$(-1,0,0)$
$(3,5)_3$	$diag(1,1,0)$	0	0	$(1,\mathbf{i},0)$	$(-1,-\mathbf{i},0)(*)$
$(3,7)_1$	$diag(1,1,0)$	0	$diag(0,0,1)$	$(\mu,0,0)$	
$(3,7)_2$	$diag(1,1,0)$	0	$diag(0,0,1)$	$(1,\mathbf{i},0)$	
$(4,3)$	$diag(1,0,0)$	0	$diag(0,1,1)$	0	
$(4,4)$	$diag(1,0,0)$	0	$diag(0,0,1)$	0	
$(4,5)_1$	$diag(1,0,0)$	0	0	$(0,0,1)$	
$(4,5)_2$	$diag(1,0,0)$	0	0	$(1,0,0)$	
$(4,6)$	$diag(1,0,0)$	0	$diag(0,1,1)$	$(\mu,0,0)$	
$(4,7)_1$	$diag(1,0,0)$	0	$diag(0,1,0)$	$(0,0,\mu)$	
$(4,7)_2$	$diag(1,0,0)$	0	$diag(0,1,0)$	$(1,0,0)$	
$(5,2)_1$	0	v	$diag(\mu,\mu,\mu)$	0	
$(5,2)_2$	0	v	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	0	
$(5,3)_1$	0	v	$diag(\mu,\mu,0)$	0	
$(5,3)_2$	0	v	$diag(0,1,1)$	0	
$(5,3)_3$	0	v	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	0	$(0,1,0)(*)$
$(5,4)_1$	0	v	$diag(1,0,0)$	0	
$(5,4)_2$	0	v	$diag(0,0,1)$	0	$(0,0,\frac{1}{2})\ (*)$
$(6,4)$	$diag(\mu,\mu,0)$	v	$diag(0,0,1)$	0	$(0,0,\frac{1}{2})\ (*)$
$(6,6)_1$	$diag(\mu,\mu,0)$	v	$\frac{1}{\mu}\ diag(b,b,0)$	$(0,0,b)$	
$(6,6)_2$	$diag(\mu,\mu,0)$	v	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \pm\mathbf{i} \\ 1 & \pm\mathbf{i} & 0 \end{pmatrix}$	$(-\mu,\mp\mu\mathbf{i},0)$	$(1,\pm\mathbf{i},0)\ (*)$
$(7,3)_1$	$diag(1,0,0)$	v	$diag(0,b,b)$	0	
$(7,3)_2$	diag(1,0,0)	v	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	0	$(0,1,0)\ (*)$
$(7,4)_1$	$diag(1,0,0)$	v	$diag(0,0,1)$	0	
$(7,4)_2$	$diag(1,0,0)$	v	$diag(0,b,0)$	0	

Table III. $\mathbf{k} = \mathbb{Z}_3$.

case	A	κ^t	B	ξ^t	r-matrix
$(2,6)_1$	$diag(1,1,1)$	0	0	$(1,0,0)$	$(-1,0,0)$
$(2,6)_2$	$diag(1,1,1)$	0	0	$(1,1,0)$	$-(1,1,0)$
$(2,6)_3$	$diag(1,1,1)$	0	0	$(1,1,1)$	$-(1,1,1)$ $(*)$
$(3,5)$	$diag(1,1,0)$	0	$diag(0,0,1)$	0	
$(3,6)_1$	$diag(1,1,0)$	0	0	$(0,0,1)$	
$(3,6)_2$	$diag(1,1,0)$	0	0	$(1,0,0)$	$(-1,0,0)$
$(3,9)_1$	$diag(1,1,0)$	0	$diag(0,0,1)$	$(1,0,0)$	
$(3,9)_2$	$diag(1,1,0)$	0	$diag(0,0,1)$	$(1,1,0)$	
$(4,5)$	$diag(1,-1,0)$	0	$diag(0,0,1)$	0	
$(4,6)_1$	$diag(1,-1,0)$	0	0	$(0,0,1)$	
$(4,6)_2$	$diag(1,-1,0)$	0	0	$(1,0,0)$	$(-1,0,0)$
$(4,6)_3$	$diag(1,-1,0)$	0	0	$(1,1,0)$	$(-1,1,0)$ $(*)$
$(4,9)_1$	$diag(1,-1,0)$	0	$diag(0,0,1)$	$(1,0,0)$	
$(4,9)_2$	$diag(1,-1,0)$	0	$diag(0,0,1)$	$(0,1,0)$	
$(4,9)_3$	$diag(1,-1,0)$	0	$diag(0,0,1)$	$(1,1,0)$	
$(5,3)$	$diag(1,0,0)$	0	$diag(0,1,1)$	0	
$(5,4)$	$diag(1,0,0)$	0	$diag(0,1,-1)$	0	
$(5,5)$	$diag(1,0,0)$	0	$diag(0,0,1)$	0	
$(5,6)_1$	$diag(1,0,0)$	0	0	$(0,0,1)$	
$(5,6)_2$	$diag(1,0,0)$	0	0	$(1,0,0)$	
$(5,7)$	$diag(1,0,0)$	0	$diag(0,1,1)$	$(1,0,0)$	
$(5,8)$	$diag(1,0,0)$	0	$diag(0,1,-1)$	$(1,0,0)$	
$(5,9)_1$	$diag(1,0,0)$	0	$diag(0,0,1)$	$(\pm1,0,0)$	
$(5,9)_2$	$diag(1,0,0)$	0	$diag(0,0,1)$	$(0,\pm1,0)$	

Table III. $\mathbf{k} = \mathbb{Z}_3$.

case	A	κ^t	B	ξ^t	r-matrix
$(6,2)_1$	0	$(0,0,1)$	$diag(1,1,1)$	0	
$(6,2)_2$	0	$(0,0,1)$	$diag(1,-1,1)$	0	
$(6,2)_3$	0	$(0,0,1)$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	0	
$(6,3)_1$	0	$(0,0,1)$	$diag(0,1,1)$	0	
$(6,3)_2$	0	$(0,0,1)$	$diag(1,1,0)$	0	
$(6,4)_1$	0	$(0,0,1)$	$diag(1,-1,0)$	0	
$(6,4)_2$	0	$(0,0,1)$	$diag(0,1,-1)$	0	
$(6,4)_3$	0	$(0,0,1)$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	0	$(0,1,0)$ $(*)$
$(6,5)_1$	0	$(0,0,1)$	$diag(1,0,0)$	0	
$(6,5)_2$	0	$(0,0,1)$	$diag(0,0,1)$	0	$(0,0,-1)$ $(*)$
$(7,5)$	$diag(1,1,0)$	(0,0,1)	$diag(0,0,1)$	0	
$(7,7)$	$diag(1,1,0)$	(0,0,1)	$\pm diag(1,1,0)$	$(0,0,\pm1)$	
$(8,5)$	$diag(1,-1,0)$	(0,0,1)	$diag(0,0,1)$	0	$(0,0,-1)$ $(*)$
$(8,8)_1$	$diag(1,-1,0)$	(0,0,1)	$\pm diag(1,-1,0)$	$(0,0,\pm1)$	
$(8,8)_2$	$diag(1,-1,0)$	(0,0,1)	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \pm1 \\ 1 & \pm1 & 0 \end{pmatrix}$	$(-1,\pm1,0)$	$(1,\pm1,0)$ $(*)$
$(9,3)$	$diag(1,0,0)$	(0,0,1)	$\pm diag(0,1,1)$	0	
$(9,4)_1$	$diag(1,0,0)$	(0,0,1)	$\pm diag(0,1,-1)$	0	
$(9,4)_2$	diag(1,0,0)	(0,0,1)	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$	0	$(0,1,0)$ $(*)$
$(9,5)_1$	$diag(1,0,0)$	(0,0,1)	$diag(0,0,\pm1)$	0	
$(9,5)_2$	$diag(1,0,0)$	(0,0,1)	$diag(0,\pm1,0)$	0	

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