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Cartan–Helgason Theorem, Poisson Transform, and Furstenberg–Satake Compactifications

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Abstract. The connections between the objects mentioned in the title are used to give a short proof of the Cartan–Helgason theorem and a natural construction of the compactifications.
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For a real semisimple Lie group G we write, as usual, K for a maximal compact subgroup and MAN for a minimal parabolic. The Cartan–Helgason theorem consists of two parts, one of these (Prop. 2 here) states that a finite-dimensional irreducible representation (ρ, V) of G has a K-fixed vector e if and only if it has an MN-fixed vector (which is then a highest weight vector v^+ of ρ). Of course, when ρ is a faithful representation, the orbit $\rho(G)e$ gives an imbedding of the symmetric space $X \simeq G/K$ into the space PV of lines in V while $\rho(G)v^+$ gives an imbedding of a space G/B where $B \supset MAN$ (i.e. B is a parabolic subgroup, and G/B is one of the Poisson boundaries of X in the sense of [2], [12]). In PV now G/B appears as part of the topological boundary of the image of X.

Two things follow from these observations. First, they give a natural approach to the Cartan–Helgason theorem [3], [5, p. 535], [6, p. 139] which we are splitting into Propositions 1 and 2: the proof of Proposition 2 is based on the Poisson transform. Second, we can consider the full closure of X in PV. It turns out that this is always one of the compactifications constructed originally by Satake [13] and reconstructed later in other ways in [2], [12], [1], [9]. In fact, the construction we sketch here may be conceptually the simplest of all. As J. A. Wolf tells me, the idea of using spherical representations to construct compactifications had also been suggested by R. Hermann some time ago.

1. In the following $\mathfrak{g}$ will be a real semisimple Lie algebra, $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ its decomposition under a Cartan involution $\theta, \mathfrak{g}^{\mathbb{C}}$ its complexification, $\mathfrak{u} = \mathfrak{k} + i\mathfrak{p}$. We choose a maximal subalgebra $\mathfrak{a} \subset \mathfrak{p}$ and complete it to a Cartan subalgebra $\mathfrak{h} = \mathfrak{a} + \mathfrak{t}$ of $\mathfrak{g}$ (so $\mathfrak{t} \subset \mathfrak{k}$). We identify $\mathfrak{h}^{\mathbb{C}}$ with its dual under the Killing form.

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The roots with respect to $\mathfrak{h}^{\mathbb{C}}$ span the real form $\mathfrak{h}_0 = \mathfrak{a} + \mathrm{i}\mathfrak{t}$ of $\mathfrak{h}^{\mathbb{C}}$. The restriction of the Killing form to $\mathfrak{h}_0$ is positive definite, we denote it by $(.\,|\,.)$. Given our identification, a restricted root ($\mathfrak{a}$-root) is the same as the orthogonal projection of an $\mathfrak{h}^{\mathbb{C}}$-root onto $\mathfrak{a}$. For an $\mathfrak{h}^{\mathbb{C}}$-root α we denote the corresponding root space in $\mathfrak{g}^{\mathbb{C}}$ by $\mathfrak{g}_\alpha$. For an $\mathfrak{a}$-root γ we denote the corresponding root space by $\mathfrak{g}^\gamma$. We choose an ordering of $\mathfrak{h}_0$ and we set $\mathfrak{n} = \sum_{\gamma>0} \mathfrak{g}^\gamma, \bar{\mathfrak{n}} = \theta\mathfrak{n}$. $G^{\mathbb{C}}$ will be the simply connected group with Lie algebra $\mathfrak{g}^{\mathbb{C}}$. The analytic subgroups of $G^{\mathbb{C}}$ for $\mathfrak{g}, \mathfrak{k}, \mathfrak{a}, \mathfrak{n}, \bar{\mathfrak{n}}, \mathfrak{u}$ will be denoted $G, K, A, N, \bar{N}, U$, while M, M' with Lie algebra $\mathfrak{m}$ will be the centralizer resp. normalizer of $\mathfrak{a}$ in K. $W = M'/M$ is the Weyl group, $\mathfrak{a}^+$ the open positive Weyl chamber.

The weights of a finite dimensional representation (ρ, V) of $\mathfrak{g}^{\mathbb{C}}$ (or, what is the same, of $G^{\mathbb{C}}$) are in $\mathfrak{h}_0$. If Λ is the highest weight we denote by v^+ a highest weight vector. We will always equip V with a Hermitian inner product such that $\rho(U)$ is unitary (hence, $\rho(\mathfrak{a})$ is Hermitian).

The following proposition is the first half of the Cartan–Helgason theorem. Without claiming any originality, for reference in Sec. 2, we give a proof based on some fundamental facts about the structure of $G^{\mathbb{C}}$.

Proposition 1.1. (a) *v^+ is fixed under the connected component M_0 of M iff $\Lambda \in \mathfrak{a}$.*
(b) *v^+ is fixed under M iff, in addition,*

$$\frac{(\Lambda|\gamma)}{(\gamma|\gamma)} \in \mathbb{Z} \tag{1}$$

for all restricted roots γ.
(c) *Any element λ of $\bar{\mathfrak{a}}^+$ satisfying* (1) *is the highest weight of a representation of $\mathfrak{g}^{\mathbb{C}}$.*

Proof. (a) v^+ is M_0-fixed iff $\rho(H)v^+ = 0$ ($\forall H \in \mathrm{i}\mathfrak{t}$) and $\rho(X_\alpha)v^+ = 0$ for all $X_\alpha \in \mathfrak{g}_\alpha$ such that $\mathfrak{g}_\alpha \subset \mathfrak{m}$. Now $\rho(H)v^+ = (\Lambda|H)v^+ = 0$ ($H \in \mathrm{i}\mathfrak{t}$) by itself amounts to $\Lambda \in \mathfrak{a}$. But $\Lambda \in \mathfrak{a}$ automatically also implies $\rho(X_\alpha)v^+ = 0$ for $\mathfrak{g}_\alpha \subset \mathfrak{m}$, i.e. for $\alpha \perp \Lambda$: In fact by the weight-string property (e.g. [7, p. 114]) $\Lambda \pm \alpha$ are either both weights or neither one is, and the first possibility is excluded by the maximality of Λ.

(b) By a result of Satake [13] (cf. also [4, p. 435]) $M = Z_1 M_0$, where $Z_1 = \exp(\mathrm{i}\mathfrak{a}) \cap K$. So we have to show only that v^+ is Z_1-fixed iff (1) holds. It is well known (e.g. [4, p. 322]) that, since $G^{\mathbb{C}}$ and therefore U are simply connected, $\exp \mathrm{i}H \in K$ for H in $\mathfrak{a}$ is equivalent to H being in the lattice generated by the vectors $\frac{\pi}{(\gamma|\gamma)}\gamma$ with the simple restricted roots γ. Since $\rho(\exp \mathrm{i}H)v^+ = e^{\mathrm{i}(\Lambda|\mathrm{H})}v^+$, Z_1-invariance of v^+ amounts exactly to (1).

(c) We only have to check the standard integrality condition with respect to $\mathfrak{h}^{\mathbb{C}}$-roots. For such a root α we denote by $\bar{\alpha}$ its restriction (projection) to $\mathfrak{a}$. It is well known [4, p. 322] that $\frac{(\alpha|\alpha)}{(\bar{\alpha}|\bar{\alpha})} = 1, 2$ or 4, with 4 only when $2\bar{\alpha}$ is also a restricted root. In the first two cases $2\frac{(\Lambda|\alpha)}{(\alpha|\alpha)} \in \mathbb{Z}$ trivially. In the third case $2\frac{(\Lambda|\alpha)}{(\alpha|\alpha)} = \frac{(\Lambda|2\bar{\alpha})}{(2\bar{\alpha}|2\bar{\alpha})}$ is again in $\mathbb{Z}$, by (1). ∎

Now we come to the second half of the Cartan–Helgason theorem. The proof given here is the main point of this article.

Proposition 1.2. *An irreducible representation (ρ, V) of G has a K-fixed vector iff it has an MN-fixed vector.*

Proof. Again, the existence of an MN-fixed vector amounts to v^+ being M-fixed.

Suppose v^+ is M-fixed. Then $f(g) = \rho(g)v^+$ is a V-valued function transforming as $f(gm(\exp H)n) = e^{(\Lambda|H)}f(g)$ for $g \in G, m \in M, H \in \mathfrak{a}, n \in N$. (It is a section lifted to G of a homogeneous line bundle tensored with V.) Its Poisson transform is

$$(P_{\Lambda+\rho}f)(g) = \int_K \rho(gk)v^+ \mathrm{d}k$$

(in standard notation, with ρ the half-sum of positive $\mathfrak{a}$-roots; cf. [14, p. 81]). This can also be written $\rho(g)e$, where $e = \int_K \rho(k)v^+ \mathrm{d}k$ is K-fixed, we have to show only that $e \neq 0$. Now the Fatou-type theorem of Michelson [11] (see also [14, p. 83], [6, p. 120]) says that, for $H \in \mathfrak{a}^+$.

$$\lim_{t\to\infty} e^{-t(\Lambda|H)}\rho(\exp tH)e = cf(e) = cv^+$$

with $c > 0$. It follows that $e \neq 0$.

(We also note that, since $\Lambda \in \mathfrak{a}^+$, the standard proof of the Fatou-type theorem can be considerably simplified: The convergence of the integral defining c is obvious without even using the explicit expression of the Jacobian of the map $\bar{n} \to k(\bar{n})$.)

Conversely, suppose there exists a K-invariant $e \neq 0$. The weight spaces V_λ for different weights are mutually orthogonal. We write $e = \sum e_\lambda$ with $e_\lambda \in V_\lambda$. Now $e_\Lambda \neq 0$, because otherwise $(e \mid v^+) = 0$, hence $(e \mid \rho(k)\rho(a)\rho(n)v^+) = (\rho(k^{-1})e \mid \rho(a)\rho(n)v^+) = 0$ for all $k \in K, a \in A, n \in N$, which is impossible.

We have $\rho(\exp tH)e = \sum_\lambda e^{t\lambda(H)}e_\lambda$. For $H \in \mathfrak{a}^+$, we have $\Lambda(H) > \lambda(H)$ for all weights $\lambda \neq \Lambda$. Hence

$$\lim_{t\to\infty} e^{-t\Lambda(H)}\rho(\exp tH)e = e_\Lambda.$$

Since e_Λ is a limit of M-fixed vectors, it is M-fixed. ∎

2. We continue with the setup of the preceding section, we consider a faithful representation (ρ, V) of $\mathfrak{g}^{\mathbb{C}}$ with highest weight $\Lambda \in \mathfrak{a}$. When $e \neq 0$ is a K-fixed vector, the map $g \cdot o \longmapsto \rho(g)e$ (we denote by $o \in X$ the point corresponding to K) is an equivariant imbedding of X into V. This is clear since in each simple factor the K-part is a maximal subgroup. We write $\tilde{v}$ for the image of v in the projective space PV. Then we also have that $g \cdot o \longmapsto \rho(g)\tilde{e}$ is an imbedding $X \to PV$. This is so because $\rho(g)$ is scalar only when $g \in Z$, the center of G, and Z is contained in K.

Since we also have $Z \subset M$, we see that $g \longmapsto (\rho(g)v^+)^{\sim}$ is an equivariant map $G \to PV$. The stabilizer B of $(v^+)^{\sim}$ contains MAN by Prop. 2 (so is a parabolic group). So we have X and G/B imbedded in PV and the proof of Prop. 2 shows that G/B is a part of the topological boundary of X.

We may also note that, denoting by V' the orthocomplement of e in V and imbedding it into PV by $v' \longmapsto (e+v')\tilde{}$ the images of $X, G/B$ are actually contained in a bounded part of the vector space V'. This is clear for $\rho(A)\tilde{e}$, since $\rho(a)e$ is a positive combination of the orthogonal system formed by the e_λ (cf. the proof of Prop. 2). Then it is also true for $\rho(G)\tilde{e} = \rho(K)\rho(A)\tilde{e}$ since $\rho(K)$ acts on V' by rotations.

The closure of the image of X in PV (or V') is a compactification to which the action of G extends naturally. As we will now indicate, what we get in this way are exactly the Furstenberg–Satake compactifications [13], [2], [12], [1], [10]. Satake's construction is a special instance of ours, he works with a special class of representations which he obtains from an arbitrary representation σ as the Cartan product of σ and the contragradient $\sigma^\wedge$ composed with θ, and which are realized on vector spaces of Hermitian matrices. So our construction is close in spirit to Satake's.

We denote by Π the set of positive restricted roots. Λ determines a subset $E_0 \subset \Pi$ defined as those $\gamma \in \Pi$ which are orthogonal to Λ. We denote by $\mathfrak{a}(E_0)$ the common zero-space of the elements of E_0 and by $\mathfrak{a}(E_0)^+$ the subset where all γ in $\Pi - E_0$ take positve values. So $\mathfrak{a}(E_0)^+$ is the largest (open) face of $\overline{\mathfrak{a}^+}$ which is perpendicular to Λ.

Now we can make Proposition 1 a little more precise. The θ-image of the subalgebra $\mathfrak{n}^{E_0} = \sum_{\gamma \perp \mathfrak{a}(E_0)} \mathfrak{g}^\gamma$, and $\mathfrak{a}^{E_0}$, the orthocomplement of $\mathfrak{a}(E_0)$ in $\mathfrak{a}$, annihilate v^+. Together with $\mathfrak{m}+\mathfrak{n}$ they form the subalgebra $\mathfrak{m}(E_0)+\mathfrak{n}(E_0)$ annihilating v^+; here $\mathfrak{m}(E_0) = \mathfrak{m}_K(E_0) + \mathfrak{a}^{E_0} + \mathfrak{n}^{E_0} + \theta\mathfrak{n}^{E_0}$ with $\mathfrak{m}_K(E_0)$ the centralizer of $\mathfrak{a}(E_0)$ in $\mathfrak{k}$ and $\mathfrak{n}(E_0) = \sum_{\gamma \notin \mathfrak{a}^{E_0}} \mathfrak{g}^\gamma$.

We write $B(E_0)$ for the group generated by the analytic subgroup corresponding to $\mathfrak{b}(E_0) = \mathfrak{m}(E_0) + \mathfrak{a}(E_0) + \mathfrak{n}(E_0)$ and by M. This is the stabilizer of $(v^+)\tilde{}$ in PV. (As it is well known and easy to prove, with different choices of E_0 these are the only closed subgroups of G containing MAN. The parabolic subgroups are, by definition, their conjugates.)

To describe the closure of $\rho(G)\tilde{e}$ in PV, let now E be any subset of Π. The orbit $X^E = M(E)\cdot o$ is a symmetric subspace of X. The imbedding of X in PV induces an imbedding of X^E as $\rho(M(E))\tilde{e}$. Any point in it can be written as $\rho(k^E)\rho(a^E)\tilde{e}$ with $k^E \in M_K(E)$, $a^E \in A^E$.

We choose an $H \in \mathfrak{a}(E)^+$. We write $e = \sum_\lambda e_\lambda$ where λ runs through the restricted weights of ρ (unlike in Sec. 1 where we worked with the $\mathfrak{h}$-weights). Since Λ is its own restriction, still $e_\Lambda \neq 0$. We have, for all $m_E \in M(E)$,

$$\rho(\exp tH)\rho(m_E)e = \rho(m_E)\sum_\lambda e^{t(\lambda|H)}e_\lambda \tag{2}$$

As $t \to \infty$, the dominating terms in the sum are those with $\lambda \equiv \Lambda(\mathrm{mod}\ \mathfrak{a}^E)$. The limit, on the boundary of the image in PV, will therefore be $\rho(m_E)(e^E)\tilde{}$, where $e^E = \sum_{\lambda \equiv \Lambda(\mathrm{mod}\ \mathfrak{a}^E)} e_\lambda$. The limit of the family of sets $\rho(exptH)\rho(M(E))\tilde{e}$ will be $\rho(m(E))(e^E)\tilde{}$.

The restricted weights such that $\lambda \equiv \Lambda(\mathrm{mod}\ \mathfrak{a}^E)$ are those that arise in the form $\Lambda - \sum m_j\delta_j$ with $\delta_j \in E$. It is not hard to see (cf. [13 , Lemma 8]) that on V^E, the direct sum of the corresponding spaces V_λ, the restriction of ρ to $M(E)$ is an irreducible representation, to be denoted ρ^E. Its highest restricted weight (with respect to $\mathfrak{a}^E$) is the projection Λ^E of Λ onto $\mathfrak{a}^E$, and e^E is an $M_K(E)$-fixed vector of it.

In general $\rho(M(E))(e^E)\tilde{}$ is not a one-to-one image of X^E; there are in general several subsets E for which X^E has the same image. For any $E \subset \Pi$ we have that X^E is the direct product of irreducible symmetric spaces X^{E_i}. The E_i are called the components of E, and E is said to be E_o-connected if none of its components is entirely contained in E_o. Clearly, it is exactly when E is E_o-connected that that the stabilizer of $(e^E)\tilde{}$ is not larger than $M_K(E)$. In this case $\rho(M(E))(e^E)\tilde{}$ is an imbedded image of X^E. Thus, for every E_o-connected E we have an imbedding of X^E into the boundary which we will denote by ι_E. It is interesting to note that, in terms of the vector space V' (identified with its image in PV), the set $\iota_E(X^E)$ is just the parallel translate by $(e^E)\tilde{}$ of $X^E = M(E)\tilde{e}$. (Observe that $\tilde{e}$ is now identified with $0 \in V'$.)

It is easy to show that, for any E_0-connected E, there is a unique maximal set E' such that E' is the union of E and of some components contained in E_0. Writing $E' = E \cup E''$, we have $X^{E'} = X^E \times X^{E''}$. For all $H \in \mathfrak{a}(E')^+$, when the image of $X^{E'}$ in PV is translated by $\rho(\exp tH)$ and we let $t \to \infty$, the limit will be $\iota_E(X^E)$, while $X^{E''}$ will contract and disappear.

We also note that applying $\rho(\exp tH)$ and letting $t \to \infty$ actually moves all points of X into $\iota_E(X^E)$. This follows easily from $X = M(E')A(E')X^{E'}$ which, in turn, is a consequence of the Iwasawa decomposition.

It is easy to see that the boundary consists of the K-images of the sets $\iota_E(X^E)$. In fact if $\{k_\nu a_\nu \cdot o\}$ is a sequence of points in X tending to infinity, by compactness it has a subsequence tending to a point in $k \cdot \iota_E(X^E)$ for some E and some $k \in K$.

To determine the stabilizers of boundary points, let E be E_0-connected and let $H \in \mathfrak{a}(E')^+$. Then the stabilizer of $\exp tH \cdot o$ is $K^{\exp tH}$. As $t \to \infty$, this group gets deformed into $M_K(E')N(E')$. (Indeed, any element of $M_K(E')N(E')$ can be written $mn = m \exp \sum_\gamma X^\gamma$, $(X^\gamma \in \mathfrak{g}, (\gamma|H) > 0)$ and is the limit of $k_t^{\exp tH}$ with $k_t = m \exp \sum e^{-t(\gamma|H)}(X^\gamma + \theta X^\gamma)$ in K. As explained in [10, Sec. 3] this is the basic phenomenon behind Bolyai's and Lobachevsky's definition of horicycles.) It follows easily that the stabilizer of $\iota_E(o)$ is $M_K(E')A(E')N(E')$ and the stabilizer of $\iota_E(X^E)$ is $B(E')$.

In this way all the Satake axioms ([13, p. 100], [10, Sec. 4]) are verified, so our construction gives exactly the same compactifications as the original one of Satake.

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