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Invariant Semisimple *CR* Structures on the Compact Lie Groups $\mathbf{SU}(n)$ and $\mathbf{SO}(p,\mathbb{R})$, $5 \le p \le 7$

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Abstract. Let G_0 be a compact real Lie group of dimension N and denote by $\mathfrak{g}_0$ its Lie algebra. In an article published in 2004, Charbonnel and the first author studied G_0-invariant *CR* structures on G_0. Such a structure is defined by the fiber of the identity element of G_0 which is a Lie subalgebra $\mathfrak{h}$ of the complexification $\mathfrak{g}$ of $\mathfrak{g}_0$, having trivial intersection with $\mathfrak{g}_0$. If the dimension of the *CR* structure is maximal, that is $\left[\frac{N}{2}\right]$, then Charbonnel and the first author showed that $\mathfrak{h}$ is a solvable Lie algebra. In this note, we are interested in G_0-invariant *CR* structures on G_0 which are defined by a semisimple Lie subalgebra and of maximal dimension. We distinguish two types of these *CR* structures which we shall call *CRSS* structure of type I and of type II. In the case of the group $\mathrm{SU}(n)$, with $n \ge 3$, we show that there exists always a *CRSS* structure of type I, while in the case of $\mathrm{SO}(p,\mathbb{R})$, with $5 \le p \le 7$, we show that a *CRSS* structure of type II exists. We obtain from these structures for each of these groups an almost global *CR* embedding into a finite-dimensional complex vector space.
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1. Introduction

Let G_0 be a compact real Lie group of dimension N and denote by $\mathfrak{g}_0$ its Lie algebra. The notion of a *CR* structure on a $\mathcal{C}^\infty$-manifold is well-known ([2] Baouendi and Trèves). In this paper, we are interested in such structures on the group G_0 which are invariant by the left action of G_0 on the tangent bundle and which are semisimple.

For a real $\mathcal{C}^\infty$-manifold X of dimension N and T a rank r subbundle of the complexification of the tangent bundle, we denote by $\mathcal{L}$ the space of $\mathcal{C}^\infty$-sections of T. We say that T is *formally integrable* if $\mathcal{L}$ is stable with respect to the Lie algebra structure on the space of vector fields on X. The pair (X,T) is a *CR manifold* if T is formally integrable and if given any point $x \in X$, the intersection of the fiber T_x of x in T and its conjugate $\overline{T_x}$ is zero, where by conjugate, we

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mean the conjugation with respect to the complexification of the tangent bundle whose fixed points give the tangent bundle of X. We shall say that T is a *CR structure* on X if (X,T) is a *CR* manifold.

Let (X,T) be a *CR* manifold. A $\mathcal{C}^\infty$ map $f : X \to \mathbb{C}^m$ is called a *CR map* if f is annihilated by the sections of the subbundle T. Furthermore, if f is an embedding, then we shall call f a *CR embedding* ([4] H. Jacobowitz, [1] M.S. Baouendi and L.P. Rothschild).

Let H be a Lie group acting on X. The subbundle T is said to be H-invariant if for all $h \in H$, we have $T_{h.x} = h.T_x$.

Let us consider the action G_0 on itself by left translation. For $\xi \in \mathfrak{g}_0$ and $g \in G_0$, we denote by $g.\xi$ the differential of the map $h \mapsto gh$ of G_0 to G_0 at the identity element e of G_0. The map $(g,\xi) \mapsto g.\xi$ is an isomorphism from $G_0 \times \mathfrak{g}_0$ onto the tangent bundle $\mathrm{T}G_0$ of G_0. This isomorphism allows us to identify $G_0 \times \mathfrak{g}_0$ with $\mathrm{T}G_0$. Denote by $\mathfrak{g}$ the complexification of $\mathfrak{g}_0$. Then the complexification $\mathbb{C} \otimes \mathrm{T}G_0$ of $\mathrm{T}G_0$ can be identified with $G_0 \times \mathfrak{g}$. A G_0-invariant *CR* structure on G_0 is then a *CR* structure on G_0 which is stable under the automorphisms $(g,\xi) \mapsto (hg,\xi)$ of $\mathbb{C} \otimes \mathrm{T}G_0$, where $h \in G_0$.

Observe that a G_0-invariant *CR* structure T on G_0 is determined by its fiber at e, which is a complex Lie subalgebra $\mathfrak{h}_T$ of $\mathfrak{g}$ verifying $\mathfrak{h}_T \cap \mathfrak{g}_0 = \{0\}$. Thus the map $T \mapsto \mathfrak{h}_T$ is a bijection between the set of G_0-invariant *CR* structures on G_0 and the set of complex Lie subalgebras $\mathfrak{h}$ of $\mathfrak{g}$ verifying $\mathfrak{h} \cap \mathfrak{g}_0 = \{0\}$. We call $\mathfrak{h}_T$ the Lie subalgebra corresponding to the *CR* structure T.

In [3], Charbonnel and the first author studied G_0-invariant *CR* structures on G_0 of maximal rank. They showed that such a *CR* structure is of rank $\left[\frac{N}{2}\right]$, and the Lie subalgebra corresponding to it has to be solvable. These *CR* structures of maximal rank do not in general admit any global *CR* embedding into a complex vector space.

In this paper, we study G_0-invariant *CR* structures on G_0 such that the Lie subalgebras corresponding to them are semisimple. We shall call such a *CR* structure a *semisimple* G_0-invariant *CR* structure, or a *CRSS structure*. It follows from [3] that a *CRSS* structure has rank strictly less than $\left[\frac{N}{2}\right]$.

Definition 1.1. A *CRSS* structure T is said to be *maximal* if for any *CRSS* structure T' satisfying $T \subset T'$, we have $T = T'$. Equivalently, T is maximal if the Lie subalgebra corresponding to T is maximal by inclusion among semisimple Lie subalgebras of $\mathfrak{g}$ having trivial intersection with $\mathfrak{g}_0$.

We shall distinguish two types of maximal *CRSS* structures.

Definition 1.2. Let T be a *CRSS* structure on G_0, and $\mathfrak{h}_T$ the Lie subalgebra corresponding to T.

- We say that T is of *type I* if $\mathfrak{h}_T$ is maximal by inclusion in the set of semisimple Lie subalgebras of $\mathfrak{g}$ which are strictly contained in $\mathfrak{g}$.
- We say that T is of *type II* if the rank of T is maximal among *CRSS* structures on G_0.

We show in Section 2 that in the case where $G_0 = \mathrm{SU}(n)$ with $n \geq 3$, *CRSS* structures of type I exist. In this case, $\mathfrak{g}_0 = \mathfrak{su}(n)$ and $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. We construct a family of complex Lie subalgebras of $\mathfrak{sl}(n, \mathbb{C})$, $\{\mathfrak{h}_\varepsilon; \varepsilon \in \mathbb{C}^*,\ |\varepsilon| \neq 1\}$, such that $\mathfrak{h}_\varepsilon \cap \mathfrak{su}(n) = \{0\}$, and $\mathfrak{h}_\varepsilon$ is isomorphic to $\mathfrak{so}(n, \mathbb{C})$. The subbundle T_ε corresponding to $\mathfrak{h}_\varepsilon$ is then a *CRSS* structure of type I.

In Section 3, we treat the case where $G_0 = \mathrm{SO}(p)$ with $5 \leq p \leq 7$. We show that *CRSS* structures of type II exist in these cases, and the Lie subalgebras corresponding to these structures are isomorphic to respectively $\mathfrak{so}(3, \mathbb{C})$, $\mathfrak{so}(3, \mathbb{C}) \times \mathfrak{so}(3, \mathbb{C})$ and $\mathfrak{so}(3, \mathbb{C}) \times \mathfrak{so}(4, \mathbb{C})$. For these cases, we have used the computer program Maple to prove the existence of these structures.

Definition 1.3. Let T be a G_0-invariant *CR* structure on G_0. A map $f : G_0 \to \mathbb{C}^m$ is called an *almost global CR embedding* if f is a *CR* immersion, and if there exists a finite subgroup F of G_0 such that f induces an embedding from G_0/F into $\mathbb{C}^m$.

We show also that all the *CR* structures obtained here have an almost global *CR* embedding into a finite-dimensional complex vector space.

2. *CRSS* structures on SU(n)

In this section, we shall show that there exists a *CRSS* structure of type I on $\mathrm{SU}(n)$ when $n \geq 3$.

The following result is well-known and is a consequence of a more general result on symmetric Lie algebras. For the sake of completeness, we have included a proof in this special case of orthogonal Lie algebras.

Proposition 2.1. *Let $n \geq 3$, then $\mathfrak{so}(n, \mathbb{C})$ is a semisimple Lie subalgebra of $\mathfrak{sl}(n, \mathbb{C})$ which is maximal by inclusion.*

Proof. Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{k} = \mathfrak{so}(n, \mathbb{C})$. We shall consider $\mathfrak{k}$ as the set of fixed points of the involutive automorphism $A \mapsto -{}^tA$. So $\mathfrak{k}$ is the set of antisymmetric matrices in $\mathfrak{g}$. Denote by $\mathfrak{p}$ the set of symmetric matrices in $\mathfrak{g}$. Then $\mathfrak{p}$ is $\mathfrak{k}$-stable, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.

Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$ containing strictly $\mathfrak{k}$, then $\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p}_1$ where $\mathfrak{p}_1 = \mathfrak{h} \cap \mathfrak{p} \neq \{0\}$. Since $\mathfrak{k}$ is reductive, there is a $\mathfrak{k}$-stable complementary subspace $\mathfrak{p}_{-1}$ of $\mathfrak{p}_1$ in $\mathfrak{p}$. Thus

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_1 \oplus \mathfrak{p}_{-1}.$$

One verifies easily for any $i, j \in \{-1, 1\}$ that

$$[\mathfrak{p}_i, \mathfrak{p}_j] \text{ is an ideal of } \mathfrak{k}\ , \text{ and that } [\mathfrak{p}_i, [\mathfrak{p}_j, \mathfrak{p}_j]] \subset \mathfrak{p}_i \cap \mathfrak{p}_j \tag{1}$$

because $[\mathfrak{p}_i, [\mathfrak{p}_j, \mathfrak{p}_j]] \subset [\mathfrak{p}_i, \mathfrak{k}] \subset \mathfrak{p}_i$ and $[\mathfrak{p}_i, [\mathfrak{p}_j, \mathfrak{p}_j]] \subset [[\mathfrak{p}_i, \mathfrak{p}_j], \mathfrak{p}_j] + [\mathfrak{p}_j, [\mathfrak{p}_i, \mathfrak{p}_j]] \subset [\mathfrak{k}, \mathfrak{p}_j] \subset \mathfrak{p}_j$.

Now suppose that $n \geq 3$ and $n \neq 4$, then $\mathfrak{k}$ is simple. So $[\mathfrak{p}_i, \mathfrak{p}_i] = \{0\}$ or $\mathfrak{k}$. If $[\mathfrak{p}_i, \mathfrak{p}_i] = \mathfrak{k}$, then by (1), $[\mathfrak{k}, \mathfrak{p}_{-i}] = \{0\}$, and therefore

$$\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = ([\mathfrak{k}, \mathfrak{k}] + [\mathfrak{p}, \mathfrak{p}]) \oplus [\mathfrak{k}, \mathfrak{p}_1] \oplus [\mathfrak{k}, \mathfrak{p}_{-1}] \subset \mathfrak{k} \oplus \mathfrak{p}_i,$$

hence $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}_i$. Since $\mathfrak{p}_1 \neq \{0\}$, this implies that $[\mathfrak{p}_{-1}, \mathfrak{p}_{-1}] = \{0\}$.

If $[\mathfrak{p}_1, \mathfrak{p}_1] = \{0\}$, then one checks easily that the endomorphism d of $\mathfrak{g}$ verifying

$$d(x) = 0 \text{ if } x \in \mathfrak{k}\ , \text{ and } d(x) = ix \text{ if } x \in \mathfrak{p}_i$$

is a derivation of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, d is interior, and one checks easily that there exists $z \in \mathfrak{k}$ such that $d = \mathrm{ad}_{\mathfrak{g}} z$. But this would mean that z is a non zero element in the centre of $\mathfrak{k}$, which is absurd because $\mathfrak{k}$ is simple. Thus $[\mathfrak{p}_1, \mathfrak{p}_1] = \mathfrak{k}$, and $\mathfrak{h} = \mathfrak{g}$. Hence $\mathfrak{k}$ is maximal.

Finally, when $n = 4$, $\mathfrak{k}$ is semisimple, but not simple. In this special case, one can check directly that $\mathfrak{p}$ is a simple $\mathfrak{k}$-module, and so $\mathfrak{k}$ is maximal. ■

For $\varepsilon \in \mathbb{C}^*$, denote by $D(\varepsilon) = (d_{ij})_{1 \leq i,j \leq n} \in \mathrm{SL}(n, \mathbb{C})$ the matrix where

$$d_{ij} = \varepsilon^{i - \frac{n(n+1)}{2}} \delta_{ij}.$$

Set

$$\mathfrak{h}_\varepsilon = D(\varepsilon) \mathfrak{so}(n, \mathbb{C}) D(\varepsilon)^{-1} \subset \mathfrak{sl}(n, \mathbb{C}).$$

Proposition 2.2. *Let $\varepsilon \in \mathbb{C}^*$ be such that $|\varepsilon| \neq 1$, then*

$$\mathfrak{h}_\varepsilon \cap \mathfrak{su}(n) = \{0\}.$$

So $\mathfrak{h}_\varepsilon$ defines a CRSS structure of type I on $\mathrm{SU}(n)$.

Proof. Let $B = (b_{ij})_{1 \leq i,j \leq n} \in \mathfrak{so}(n, \mathbb{C})$. We have that B is antisymmetric and

$$D(\varepsilon) B D(\varepsilon)^{-1} = (\varepsilon^{i-j} b_{ij})_{1 \leq i,j \leq n}.$$

Now if $D(\varepsilon) B D(\varepsilon)^{-1} \in \mathfrak{su}(n)$, then

$$\varepsilon^{i-j} b_{ij} = -\overline{\varepsilon}^{j-i} \overline{b_{ji}}\ , \text{ and hence } |\varepsilon|^{2(i-j)} b_{ij} = -\overline{b_{ji}} = \overline{b_{ij}}$$

for $1 \leq i, j \leq n$.

So if $|\varepsilon| \neq 1$, then $B = 0$. Hence

$$\mathfrak{h}_\varepsilon \cap \mathfrak{su}(n) = \{0\}.$$

Finally, by Proposition 2.1, the *CRSS* structure on $\mathrm{SU}(n)$ defined by $\mathfrak{h}_\varepsilon$ is of type I. ■

We shall now show that the maximal *CRSS* structure defined by $\mathfrak{h}_\varepsilon$, has an almost global *CR* embedding into the vector space S_n of symmetric n by n complex matrices.

Proposition 2.3. *Let $n \geq 3$, and $\varepsilon \in \mathbb{C}^*$ such that $|\varepsilon| \neq 1$. Denote by F the finite subgroup of $\mathrm{SL}(n, \mathbb{C})$ defined by*

$$F = \{A = (\lambda_i \delta_{ij})_{1 \leq i,j \leq n} \in \mathrm{SL}(n, \mathbb{C}); \lambda_i^2 = 1\}.$$

Then $\mathrm{SU}(n)/F$ can be identified as a real differentiable submanifold of S_n via the maximal CRSS structure defined by $\mathfrak{h}_\varepsilon$.

Thus the group $\mathrm{SU}(n)$, endowed with the maximal CRSS structure defined by $\mathfrak{h}_\varepsilon$, admits an almost global CR embedding into S_n.

Proof. Denote by $H_\varepsilon = D(\varepsilon)\mathrm{SO}(n,\mathbb{C})D(\varepsilon)^{-1}$ the connected closed subgroup of $\mathrm{SL}(n,\mathbb{C})$ whose Lie algebra is $\mathfrak{h}_\varepsilon$. Consider the action of $\mathrm{SL}(n,\mathbb{C})$ on S_n given by

$$\mathrm{SL}(n,\mathbb{C}) \times S_n \to S_n\ ,\ (g,Z) \mapsto gZ^tg.$$

Let $Z_0 = D(\varepsilon^2)$, then H_ε is the stabilizer of Z_0 in $\mathrm{SL}(n,\mathbb{C})$.

Since $\mathfrak{h}_\varepsilon \cap \mathfrak{su}(n) = \{0\}$, $H_\varepsilon \cap \mathrm{SU}(n)$ is a finite subgroup of $\mathrm{SU}(n)$. One deduces that $F = H_\varepsilon \cap \mathrm{SU}(n)$ is the stabilizer of Z_0 in $\mathrm{SU}(n)$.

It follows that the map $\varphi : \mathrm{SU}(n) \to S_n$, $g \mapsto gZ_0{}^tg$, is a *CR* immersion which induces an embedding of $\mathrm{SU}(n)/F$ into the $\mathrm{SU}(n)$-orbit Ω_0 of Z_0, which is a real submanifold of codimension $n+1$ in S_n. Thus $\mathrm{SU}(n)/F$, endowed with this *CRSS* structure, can be identified as a real submanifold Ω_0 of S_n. ■

3. *CRSS* structures on $\mathrm{SO}(p,\mathbb{R})$, $5 \le p \le 7$

In this section, we shall show that there exists a *CRSS* structure of type II on $\mathrm{SO}(p,\mathbb{R})$ for $5 \le p \le 7$.

Proposition 3.1. *Let $5 \le p \le 7$. There exists a semisimple Lie subalgebra $\mathfrak{h}_p$ of $\mathfrak{so}(p,\mathbb{C})$ such that*

$$\mathfrak{h}_p \cap \mathfrak{so}(p,\mathbb{R}) = \{0\}$$

and $\mathfrak{h}_p$ induces a CRSS structure of type II on $\mathrm{SO}(p,\mathbb{R})$. The Lie algebras $\mathfrak{h}_5$, $\mathfrak{h}_6$, $\mathfrak{h}_7$ are isomorphic respectively to $\mathfrak{so}(3,\mathbb{C})$, $\mathfrak{so}(3,\mathbb{C}) \times \mathfrak{so}(3,\mathbb{C})$ and $\mathfrak{so}(3,\mathbb{C}) \times \mathfrak{so}(4,\mathbb{C})$.

Proof. Let $N_p = \dim \mathrm{SO}(p,\mathbb{R})$. Recall that by [3], if $\mathfrak{h}_p$ is semisimple and $\mathfrak{h}_p \cap \mathfrak{so}(p,\mathbb{R}) = \{0\}$, then $\dim_{\mathbb{C}} \mathfrak{h}_p < \left[\frac{N_p}{2}\right]$. Since there are no semisimple Lie algebras of dimension 4, if $\mathfrak{h}_p$ defines a *CRSS* structure, then it would be of type II.

We are therefore left to show that $\mathfrak{h}_p \cap \mathfrak{so}(p,\mathbb{R}) = \{0\}$. All the computations were done by using the program Maple.

i) Let

$$P_5 = \begin{pmatrix} 0 & 1+i & -1 & 1-i & 0 \\ -2-4i & -3+6i & -1-3i & 7+i & 0 \\ 7+i & -5-9i & 5+i & -7+8i & 1-i \\ -3+6i & 10-2i & -2+4i & -5-9i & 1+i \\ -1-3i & -2+4i & -2i & 5+i & -1 \end{pmatrix} \in \mathrm{SO}(5,\mathbb{C}).$$

We identify $\mathfrak{so}(3,\mathbb{C})$ as the Lie subalgebra of $\mathfrak{so}(5,\mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_1+ia_2 & b_1+ib_2 & 0 \\ 0 & -a_1-ia_2 & 0 & c_1+ic_2 & 0 \\ 0 & -b_1-ib_2 & -c_1-ic_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where $a_1, a_2, b_1, b_2, c_1, c_2$ are real numbers.

Set $\mathfrak{h}_5 = {}^tP_5\mathfrak{so}(3,\mathbb{C})P_5$. Any matrix $M = (m_{kl})_{1\le k,l\le 5} \in \mathfrak{h}_5$ is an antisymmetric matrix with

$$\begin{array}{rcl}
m_{12} &=& (1-i)a - b + (3-i)c \\
m_{13} &=& -2a - (1+3i)b + (3-i)c \\
m_{14} &=& -2(1+i)a + (1-i)b + (1-2i)c \\
m_{15} &=& -2(3+i)a + 2(1-3i)b + (3-i)c \\
m_{23} &=& (1+3i)a - 2(1-2i)b - 2(3+i)c \\
m_{24} &=& (-1+2i)a + (-3+i)b - 2(1+2i)c \\
m_{25} &=& 3(1+3i)a - 3(3-i)b - 2(2+i)c \\
m_{34} &=& (-3+i)a + (-4-2i)b - (2-2i)c \\
m_{35} &=& (-4-2i)a + (2-4i)b + 2c \\
m_{45} &=& (8-6i)a + (6+8i)b + (-1+5i)c
\end{array}$$

where $a = a_1 + ia_2,\ b = b_1 + ib_2,\ c = c_1 + ic_2$.

It follows that $M \in \mathfrak{h}_5 \cap \mathfrak{so}(5,\mathbb{R})$ if and only if a_1, a_2, b_1, b_2, c_1 and c_2 verify a Cramer system whose matrix is given by

$$\begin{pmatrix}
-1 & 1 & 0 & -1 & -1 & 3 \\
0 & -2 & -3 & -1 & -1 & 3 \\
-2 & -2 & -1 & 1 & -2 & 1 \\
3 & 1 & 4 & -2 & -2 & -6 \\
2 & -1 & 1 & -3 & -4 & -2 \\
-2 & -6 & -6 & 2 & -1 & 3 \\
9 & 3 & 3 & -9 & -2 & -4 \\
1 & -3 & -2 & -4 & -6 & 2 \\
-2 & -4 & -4 & 2 & 0 & 2 \\
-6 & 8 & 8 & 6 & 5 & -1
\end{pmatrix}$$

which is of rank 6. So $M = 0$.

ii) Let P_6 be the following matrix in $\mathrm{SO}(6,\mathbb{C})$

$$P_6 = \begin{pmatrix}
0 & 1-i & -1 & 1+i & 0 & 0 \\
0 & -4+6i & 4 & -4-6i & 5 & 0 \\
-2+4i & -25-60i & -15+25i & 65-5i & -22+36i & 0 \\
-3-6i & 96+20i & -18-40i & -49+85i & -26-58i & 1-i \\
7-i & -51+81i & 43-3i & -61-74i & 62-4i & 1+i \\
-1+3i & -16-46i & -12+18i & 49-i & -18+26i & -1
\end{pmatrix}.$$

We identify $\mathfrak{so}(3,\mathbb{C}) \times \mathfrak{so}(3,\mathbb{C})$ as the Lie subalgebra of $\mathfrak{so}(6,\mathbb{C})$ consisting of matrices of the form

$$\begin{pmatrix}
0 & a_1+ia_2 & b_1+ib_2 & 0 & 0 & 0 \\
-a_1-ia_2 & 0 & c_1+ic_2 & 0 & 0 & 0 \\
-b_1-ib_2 & -c_1-ic_2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & d_1+id_2 & e_1+ie_2 \\
0 & 0 & 0 & -d_1-id_2 & 0 & f_1+if_2 \\
0 & 0 & 0 & -e_1-ie_2 & -f_1-if_2 & 0
\end{pmatrix}$$

where $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2$ are real numbers.

Set $\mathfrak{h}_6 = {}^tP_6(\mathfrak{so}(3,\mathbb{C}) \times \mathfrak{so}(3,\mathbb{C}))P_6$.

Similar to the case where $p = 5$, we obtain that $M \in \mathfrak{h}_6 \cap \mathfrak{so}(6,\mathbb{R})$ if and only if $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2, e_1, e_2, f_1, f_2$ verify a Cramer system whose matrix is given by

$$\begin{pmatrix}
0 & 0 & -2 & -6 & 16 & 28 & -53 & 19 & -72 & -34 & 34 & -72 \\
0 & 0 & -2 & 4 & 8 & -16 & 19 & 13 & 6 & 32 & -32 & 6 \\
0 & 0 & 6 & -2 & -32 & 4 & -3 & -56 & 53 & -59 & 59 & 53 \\
0 & 0 & 0 & 0 & 10 & -20 & 30 & 20 & 10 & 50 & -50 & 10 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & -1 & 1 & 2 & -3 & -1 \\
0 & 2 & -15 & -20 & 10 & 50 & 30 & -10 & 40 & 20 & -20 & 40 \\
0 & -4 & 25 & 15 & 30 & 20 & 10 & -20 & 30 & -10 & 10 & 30 \\
5 & -5 & 14 & 58 & -3 & 24 & 8 & 4 & 4 & 12 & -12 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 46 & -16 & -34 & 10 & 21 & -19 \\
0 & 2 & -25 & -5 & 50 & -10 & -10 & -30 & 20 & -40 & 40 & 20 \\
-5 & 0 & 22 & -36 & -13 & 19 & 16 & 8 & 8 & 24 & -24 & 8 \\
0 & 0 & 0 & 0 & 0 & 0 & -18 & -12 & 12 & 10 & -13 & -3 \\
5 & 5 & -58 & 14 & -21 & 13 & 8 & 4 & 4 & 12 & -12 & 4 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 49 & 1 & -35 & 11 & 26 \\
0 & 0 & 0 & 0 & 0 & 0 & -26 & -18 & 18 & 14 & -18 & -4
\end{pmatrix}$$

which is of rank 12. So $M = 0$.

iii) Let

$$P_7 = \begin{pmatrix}
1+i & 0 & 0 & -1 & 0 & 0 & 1-i \\
0 & 1-i & -1 & 0 & 1+i & 0 & 0 \\
0 & 1+2i & 1-i & 0 & -2 & 1-i & 0 \\
-1 & 0 & 0 & 1-i & 0 & 0 & 1+i \\
0 & -2 & 1+i & 0 & 1-2i & 1+i & 0 \\
0 & 1-i & 0 & 0 & 1+i & -1 & 0 \\
1-i & 0 & 0 & 1+i & 0 & 0 & -1
\end{pmatrix} \in \mathrm{SO}(7,\mathbb{C}).$$

Set $\mathfrak{h}_7 = {}^tP_7(\mathfrak{so}(3,\mathbb{C}) \times \mathfrak{so}(4,\mathbb{C}))P_7$. A similar argument using the program MAPLE shows that $\mathfrak{h}_7 \cap \mathfrak{so}(7,\mathbb{R}) = \{0\}$. ■

We shall end this section by showing that each of these $\mathrm{SO}(p,\mathbb{R})$, endowed with the maximal *CRSS* structure defined by $\mathfrak{h}_p$, admits an almost global *CR* embedding.

Let $\mathbf{r} = (r_1, \ldots, r_k) \in (\mathbb{N}^*)^k$ be such that $p = r_1 + \cdots + r_k$. Denote by Φ the non-degenerate symmetric bilinear form on $\mathbb{C}^p$ whose isotropy group is $\mathrm{SO}(p,\mathbb{C})$. Let $(e_1, \ldots, e_p)$ be an orthonormal basis of $\mathbb{C}^p$ with respect to Φ. Let U_1 denote the subspace spanned by $e_1, \ldots, e_{r_1}$. For $2 \le i \le k$, denote by U_i the subspace spanned by $e_{r_1+\cdots+r_{i-1}+1}, \ldots, e_{r_1+\cdots+r_i}$. So $\mathbb{C}^n = U_1 \oplus \cdots \oplus U_k$ is an orthogonal decomposition of $\mathbb{C}^p$.

The natural action of $\mathrm{SO}(p,\mathbb{C})$ on $\mathbb{C}^p$ extends naturally to an action on $\bigwedge^{r_i} \mathbb{C}^p$, hence also on the vector space

$$E_p(\mathbf{r}) = (\textstyle\bigwedge^{r_1} \mathbb{C}^p) \times \cdots \times (\textstyle\bigwedge^{r_k} \mathbb{C}^p),$$

which has dimension $\binom{p}{r_1} + \cdots + \binom{p}{r_k}$.

Let $v_1 = e_1 \wedge \cdots \wedge e_{r_1}$. For $2 \leq i \leq k$, let

$$v_i = e_{r_1+\cdots+r_{i-1}+1} \wedge \cdots \wedge e_{r_1+\cdots+r_i} \in \bigwedge^{r_i} \mathbb{C}^p.$$

Let $x \in \mathrm{SO}(p,\mathbb{C})$ be an element in the stabilizer $H(\mathbf{r})$ of $(v_1,\ldots,v_k) \in E_p(\mathbf{r})$. Then x leaves each U_i invariant, and so $x \in \mathrm{SO}(r_1,\mathbb{C}) \times \cdots \times \mathrm{SO}(r_k,\mathbb{C})$. Conversely, it is clear that any element of $\mathrm{SO}(r_1,\mathbb{C}) \times \cdots \times \mathrm{SO}(r_k,\mathbb{C})$ stabilizes $(v_1,\ldots,v_k)$. So $H(\mathbf{r}) = \mathrm{SO}(r_1,\mathbb{C}) \times \cdots \times \mathrm{SO}(r_k,\mathbb{C})$.

Applying the above in our three cases with $\mathbf{r}_5 = (1,3,1)$, $\mathbf{r}_6 = (3,3)$ and $\mathbf{r}_7 = (3,4)$, $\mathfrak{h}_p$ is conjugate to the Lie algebra of $H(\mathbf{r}_p)$. Using the same arguments as in Proposition 2.3 and the appropriate conjugation, we obtain the following result:

Proposition 3.2. *Let $5 \leq p \leq 7$. The group $\mathrm{SO}(p,\mathbb{R})$, endowed with the maximal CRSS structure defined by $\mathfrak{h}_p$, admits an almost global CR embedding into $E_p(\mathbf{r}_p)$.*

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