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Smooth and Weak Synthesis of the Anti-Diagonal in Fourier Algebras of Lie Groups

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Abstract. Let G be a Lie group of dimension n, and let $A(G)$ be the Fourier algebra of G. We show that the anti-diagonal $\check{\Delta}_G = \{(g, g^{-1}) \in G \times G \mid g \in G\}$ is both a set of local smooth synthesis and a set of local weak synthesis of degree at most $[\frac{n}{2}] + 1$ for $A(G \times G)$. We achieve this by using the concept of the cone property in [J. Ludwig and L. Turowska, *Growth and smooth spectral synthesis in the Fourier algebras of Lie groups*, Studia Math. **176** (2006), 139–158]. For compact G, we give an alternative approach to demonstrate the preceding results by applying the ideas developed in [B. E. Forrest, E. Samei and N. Spronk, *Convolutions on compact groups and Fourier algebras of coset spaces*, Studia Math. to appear; arXiv:0705.4277]. We also present similar results for sets of the form HK, where both H and K are subgroups of $G \times G \times G \times G$ of diagonal forms. Our results very much depend on both the geometric and the algebraic structure of these sets.
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1. Introduction

The concept of weak (spectral) synthesis was introduced and studied by Warner in [20]. This is a generalization of the notion of sets of synthesis for a regular Banach algebra of continuous functions. Motivation arose in [20] from studying the question of when the union of two sets of synthesis is a set of synthesis. This concept also appeared in the earlier work of others such as Varopoulos [19] who re-proved the well-known result of L. Schwartz: the $(n-1)$-dimensional sphere S^{n-1} is a set of weak synthesis for the Fourier algebra $A(\mathbb{R}^n)$ of degree $[\frac{n}{2} + 1]$ (see also [12] and the reference therein for more examples).

There is another notion of synthesis, known as smooth synthesis, which is considered for regular Banach algebras of continuous functions on C^∞ manifolds [14] (see also [9] and [13]). This property is, roughly speaking, a suitable concept when the set is a smooth curve. We refer the reader to Domar's survey article

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[4] for more information. An important observation by Domar [3] showed that smoothness of the curve alone is not sufficient to imply its smooth synthesis. However, Domar showed that if a desirable curve in $\mathbb{R}^n$ also satisfies certain curvature condition, then both smooth synthesis and weak synthesis hold [2]. This curvature condition was generalized and modified in [14] (see also [9] and [13]). A closed set satisfying this condition is said to have *the cone property* (see Definition 3).

Let G be a locally compact group. In recent years, owing much of it to the applications of operator space theory to harmonic analysis, the structure of the anti-diagonal $\check{\Delta}_G = \{(g, g^{-1}) \in G \times G \mid g \in G\}$ became closely related to the cohomological property of Fourier algebra $A(G)$. For instance, it is shown in [6] that $A(G)$ is amenable exactly when $\check{\Delta}_G$ is an element of the coset ring of $G \times G$, or exactly when G admits an abelian subgroup of finite index. In an analogous result, it is given in [8, Theorem 2.4] for a compact group G, a full characterization of when $A(G)$ is weakly amenable: when the connected component of the identity G_e is abelian, or equivalently, $\check{\Delta}_G$ is a set of synthesis for $A(G \times G)$ (see also [7, Theorem 3.7]). This can be compared to the well-known result that closed subgroups are of synthesis for Fourier algebras [18, Theorem 3] and $\check{\Delta}_G$ is a subgroup of $G \times G$ exactly when G is abelian. Hence the preceding result shows that the synthesis property of $\check{\Delta}_G$ is inherited from the group structure of G_e or the cohomological behavior of $A(G)$.

In this article, for a Lie group G, we investigate both smooth and weak synthesis of certain diagonal-type sets including the anti-diagonal. The paper is organized as follows.

In Section 2, we state the general background of the concepts and notions required. In Section 3 we summarize and slightly modify the main results in [14, Section 4]. This, in particular, is essential for the results in Section 4 with regard to *local* smooth and weak synthesis.

In Subsection 4, we first show that $\check{\Delta}_G$ is a smooth submanifold of $G \times G$ with the cone property. Then by using the tools developed in [14] and their modification in Section 3, we show that $\check{\Delta}_G$ is both a set of local smooth synthesis and a set of local weak synthesis for $A(G \times G)$. Moreover the degree of the nilpotency is at most $[\frac{n}{2}] + 1$ where n is the dimension of G (Theorem 7). The word "local" is redundant if $A(G)$ has an approximate identity. However since it is not known whether this holds in general, we do not know whether different notations of synthesis and their corresponding local synthesis coincide.

It is shown in [7, Corollary 3.2] that if G is a compact, connected, non-abelian Lie group, then $(\Delta_G \times \Delta_G)\Delta_{G\times G}$ is a set of non-synthesis for $A(G \times G \times G \times G)$. This gives us an example of a set of non-synthesis which is the product of two closed subgroups. In Subsection 4, we show that, for a Lie group G, $(\Delta_G \times \Delta_G)\Delta_{G\times G}$ is both a set of local smooth and local weak synthesis for $A(G \times G \times G \times G)$. Moreover the degree of the nilpotency is at most $[\frac{3n}{2}] + 1$ where n is the dimension of G (Theorem 12). This is done in a similar fashion to that of Subsection 4, i.e. by showing that $(\Delta_G \times \Delta_G)\Delta_{G\times G}$ is a smooth $3n$-dimensional submanifold of $G \times G \times G \times G$ with the cone property.

The rest of the paper is devoted to presenting an alternative approach to prove the preceding results in the case of a compact Lie group G. In Section 5, we prove the projection theorem for smooth and weak synthesis for an arbitrary closed subgroup K (Theorems 13 and 15). The main theory required to achieve this is developed in [7, Sections 1.3 and 2]. We apply these results in Section 6 to obtain the smooth and weak synthesis of $\check{\Delta}_G$ and $(\Delta_G \times \Delta_G)\Delta_{G\times G}$ (Theorem 18).

2. Preliminaries and notations

2.1. Notations of synthesis. Let $\mathcal{A}$ be a Banach algebra contained in $C_0(X)$ for some locally compact Hausdorff space X. We define for any closed subset E of X

$$\begin{aligned} I_{\mathcal{A}}(E) &= \{f \in \mathcal{A} \mid f(x) = 0 \text{ for all } x \in E\}, \\ I^0_{\mathcal{A}}(E) &= \{f \in \mathcal{A} \mid \text{supp} f \cap E = \varnothing, \ \text{supp} f \text{ is compact}\}, \\ J_{\mathcal{A}}(E) &= \overline{\{a \in I_{\mathcal{A}}(E) \mid \text{supp}\, a \text{ is compact}\}}, \end{aligned}$$

where $\text{supp} f = \overline{\{x \in X \mid f(x) \neq 0\}}$ and the closure in $J_{\mathcal{A}}(E)$ is taken with respect to the norm $\|\cdot\|_{\mathcal{A}}$. If X is, in addition, a smooth manifold, then we let $\mathcal{D}(X)$ be the space of all compactly supported C^∞ functions on X and denote by $J^{\mathcal{D}}_{\mathcal{A}}(E)$ the closure (in $\mathcal{A}$) of the space of all elements in $\mathcal{D}(X) \cap \mathcal{A}$ which vanishes on E. When there is no fear of ambiguity, we write $I(E)$ instead of $I_{\mathcal{A}}(E)$, $I_0(E)$ instead of $I^0_{\mathcal{A}}(E)$, $J(E)$ instead of $J_{\mathcal{A}}(E)$, and $J_{\mathcal{D}}(E)$ instead of $J^{\mathcal{D}}_{\mathcal{A}}(E)$.

Suppose that $\mathcal{A}$ is regular on X. We say that E is *a set of synthesis* (*local synthesis*) for $\mathcal{A}$ if $I_0(E)$ is dense in $I(E)$ $(J(E))$. More generally, we say that E is a set of *weak synthesis* (*local weak synthesis*) *for* $\mathcal{A}$ *of degree at most* d if $\overline{I(E)^d} = \overline{I_0(E)}$ $(\overline{J(E)^d} = \overline{I_0(E)})$ for some positive integer d, i.e. if $I(E)/\overline{I_0(E)}$ $(J(E)/\overline{I_0(E)})$ is nilpotent of degree at most d. If X is, in addition, a smooth manifold, then E is *a set of smooth synthesis* for $\mathcal{A}$ if $J_{\mathcal{D}}(E) = I(E)$ and it is *a set of local smooth synthesis* for $\mathcal{A}$ if $J_{\mathcal{D}}(E) = J(E)$ [14] (see also [9] and [13]). It is clear that every set of (weak) synthesis is a set of local (weak) synthesis and the converse holds if $\mathcal{A}$ has an approximate identity with compact support.

Suppose further that X is the Gelfand spectrum of $\mathcal{A}$. Then $I(E)$ is the largest and $\overline{I_0(E)}$ is the smallest closed ideal in $\mathcal{A}$ whose hull is E [1, Proposition 4.1.20]. Thus E is a set of synthesis for $\mathcal{A}$ if and only if there is a unique closed ideal in $\mathcal{A}$ whose hull is E. Let $\mathcal{A}_c$ be the set of all elements in $\mathcal{A}$ with compact support. If $\mathcal{A}_c$ is dense in $\mathcal{A}$, i.e. $\mathcal{A}$ is a *Tauberian algebra* [17], then $J(E)$ is the maximal ideal of $\mathcal{A}$ having E as its hull and being essential as a Banach $\mathcal{A}$-bimodule. So if E is a set of local synthesis, then $J(E)$ is the only closed ideal in $\mathcal{A}$ with this property.

2.2. Fourier algebras. Let G be a locally compact group with a fixed left Haar measure. Given a function f on G the left and right translation of f by $x \in G$ is denoted by $(L_x f)(y) = f(xy)$ and $(R_x f)(y) = f(yx)$, respectively. Let $P(G)$ be the set of all continuous positive definite functions on G and let $B(G)$ be its linear span. The space $B(G)$ can be identified with the dual of the group C^*-algebra $C^*(G)$, this latter being the completion of $L^1(G)$ under its largest

C^*-norm. With pointwise multiplication and the dual norm, $B(G)$ is a regular commutative semisimple Banach algebra. The Fourier algebra $A(G)$ is the closure of $B(G) \cap C_c(G)$ in $B(G)$. It was shown in [5] that $A(G)$ is a commutative regular semisimple Banach algebra whose carrier space is G. Also, if λ is the left regular representation of G on $L^2(G)$ then, up to isomorphism, $A(G)$ is the unique predual of $VN(G)$, the von Neumann algebra generated by the representation λ.

3. General results on smooth and weak synthesis

In [14, Section 4], the smooth and weak synthesis properties were studied in the Fourier algebra $A(G)$ of a Lie group G. In this section, we summarize and slightly modify their main results. These modifications will be used to obtain some of the main results of our article.

3.1. Connection between smooth and weak synthesis. We start with the following theorem which is a generalization of Theorem 4.3 and Corollary 4.4 in [14].

Theorem 1. Let G be a Lie group. Let M be a smooth m-dimensional submanifold of G, and let $E \subseteq M$ be closed in G. Then:

(i) $J_{\mathcal{D}}(E)^{[\frac{m}{2}]+1} = \overline{I_0(E)}$.

(ii) If E is a set of smooth synthesis, then it is a set of weak synthesis of degree at most $[\frac{m}{2}]+1$.

(iii) If E is a set of local smooth synthesis, then it is a set of local weak synthesis of degree at most $[\frac{m}{2}]+1$.

Proof. (i) For simplicity, let $d = [\frac{m}{2}] + 1$. It suffices to show that $J_{\mathcal{D}}(E)^d \subseteq \overline{I_0(E)}$. Let $f \in J_{\mathcal{D}}(E)$ with compact support. We will follow a similar argument to that of [12, proposition 1.6] to demonstrate that $f^d \in \overline{I_0(E)}$. Using the partition of unity, it is sufficient to show that f^d is "locally in $\overline{I_0(E)}$" i.e. for every $x \in G$, there is a neighborhood U_x of x in G and an element $f_x \in \overline{I_0(E)}$ such that $f^d = f_x$ on U_x. Since $A(G)$ is regular, f^d belongs locally to $\overline{I_0(E)}$ at every point $x \in G \setminus E$. On the other hand, for every $x \in E$, let V_x be a compact neighborhood of x in G, and let $E_x = E \cap V_x$. It follows from [14, Theorem 4.3] that

$$J_{\mathcal{D}}(E_x)^d = \overline{I_0(E_x)}.$$

In particular,

$$f^d \in \overline{I_0(E_x)}. \tag{1}$$

Take $h_x \in A(G)$ such that $\operatorname{supp} h_x \subseteq V_x$ and $h_x = 1$ on a neighborhood of x in G, say U_x. Let $\epsilon > 0$. By (1), there is $g_x \in I_0(E_x)$ such that

$$\|f^d - g_x\| < \frac{\epsilon}{\|h_x\|}.$$

Thus $\|h_x f^d - h_x g_x\| < \epsilon$. Moreover, $h_x g_x = 0$ on a neighborhood of E. To see this, we observe that $h_x = 0$ on a neighborhood of $G \setminus V_x$ and $g_x = 0$ on a neighborhood of E_x. So $h_x g_x = 0$ on a neighborhood of $E_x \cup (G \setminus V_x)$ which contains E. Hence $h_x g_x \in I_0(E)$. Since $\epsilon > 0$ was arbitrary, we have

$$h_x f^d \in \overline{I_0(E)}.$$

Moreover, $h_x f^d = f^d$ on U_x since $h_x = 1$ on U_x. That is f belongs locally to $\overline{I_0(E)}$ at every point $x \in E$. Since supp$f$ is compact, an argument using partition of unity implies that $f^d \in \overline{I_0(E)}$. The final result follows from the fact that $J_{\mathcal{D}}(E)$ is the closure of $\{f \in J_{\mathcal{D}}(E) \mid \operatorname{supp} f \text{ is compact}\}$.

Parts (ii) and (iii) follow immediately from part (i). ■

We note that the preceding theorem implies that any closed subset E of a Lie group G with (local) smooth synthesis has (local) weak synthesis. Moreover the degree of the nilpotency is dominated by the dimensions of smooth submanifolds containing E.

3.2. Cone property. Motivated by Theorem 1, we would like to find a sufficient condition that implies a closed set to be of (local) smooth synthesis. One such condition is the cone property that was introduced in [2], [9] and [13] for the case of $\mathbb{R}^n$ and in [14] for a general Lie group.

Definition 2. (cf. [14, Definition 4.5]) Given a Lie group H, let Aut(H) denote its (continuous) automorphism group. It is well known that Aut(H) is also a Lie group. Also, since every continuous homomorphism between Lie groups is analytic (see [10, Ch. II, Theorem 2.6]), each element of Aut(H) is smooth.

(i) Any pair $t \in H$ and $a \in \text{Aut}(H)$ give rise to a mapping $\varphi : H \to H$, $\varphi(s) = a(ts)$, which will be called an *affine transformation* of H.

(ii) We say that a Lie group A is a *group of affine transformation* of H if A acts smoothly by affine transformations on H. Smoothly here means that the mapping $A \times H \to H$, $(a, x) \mapsto a(x)$ is smooth.

(iii) Let Aff(H) denote the group of all affine transformations of H. Here we identify Aff(H) with the semidirect product $H \ltimes_\rho \text{Aut}(H)$, where $\rho : \text{Aut}(H) \to \text{Aff}(H)$ is the identity automorphism. In particular, Aff(H) is a Lie group and acts smoothly on H by the action $\text{Aff}(H) \times H \to H$, $((t, a), s) \mapsto a(ts)$. Moreover if A is any group of affine transformations of H, then the identity map from A into Aff(H) is a smooth monomorphism. We note that image of A under this map does not have to be closed in Aff(H).

Definition 3. (cf. [14, Definition 4.6]) Let H be a Lie group and let M be a smooth m-dimensional submanifold of H. We say that a subset E of M has the *cone property* if the following holds:

(i) E is closed in H.

(ii) For every $x \in E$, there exists an open neighborhood U_x of x in H and a C^∞ mapping ψ_x from an open subset $W_x \subset \mathbb{R}^m$ containing 0 into a Lie group of affine transformations A_x on H such that $\psi_x(0) = \text{id}_H$ and there exists an open subset $W_x^0 \subset W_x$ such that:

(a) 0 is in the closure of W_x^0.

(b) For every $y \in U_x \cap E$, $\psi_x(W_x^0)y$ is contained in E and open in M and the mapping $W_x^0 \to \psi_x(W_x^0)y$, $t \mapsto \psi_x(t)y$, is a diffeomorphism.

Theorem 4. Let G be a Lie group. Let M be a smooth m-dimensional submanifold of G, and let $E \subseteq M$ be a subset with the cone property. Then:

(i) E is a set of local smooth synthesis.

(ii) E is a set of local weak synthesis of degree at most $[\frac{m}{2}] + 1$.

Proof. (i) Let T be an element of $J_{\mathcal{D}}(E)^{\perp}$, and let $f \in A(G)$ with compact support. Then $f \cdot T \in VN(G)$ has compact support and annihilates $J_{\mathcal{D}}(E)$. It is shown in the proof of [14, Theorem 4.3] that $f \cdot T$ must annihilate $I(E)$. Therefore $T = 0$ on $I(E)A_c(G)$, and so, T annihilates $J(E)$. This means that $J_{\mathcal{D}}(E) = J(E)$.

(ii) This follows from part (i) and Theorem 1(iii). ∎

One of the main consequences of the preceding theorem is [14, Corollary 4.9] where it is shown that certain orbits are of local weak synthesis. We note that in the statement of [14, Corollary 4.9] the phrase "weak synthesis" is used instead of "local weak synthesis". However a careful examination of the proof of [14, Corollary 4.9] and a comparison with the argument in Theorem 1 (see also [12, Proposition 1.6]) show that their argument actually proves the local weak synthesis property. For that sake, we state the correct version of that corollary below.

Theorem 5. Let G be a Lie group, and let B be a group of affine transformations of G. Let $\omega \subseteq G$ be a closed m-dimensional B-orbit in G. Then:

(i) ω is a set of local smooth synthesis.

(ii) ω is a set of local weak synthesis of degree at most $[\frac{m}{2}] + 1$.

4. Diagonal-type sets of smooth and weak synthesis

4.1. Anti-diagonal. Let G be an n-dimensional Lie group. Let

$$\check{\Delta}_G = \{(g, g^{-1}) \in G \times G \mid g \in G\},$$

which is called the *anti-diagonal* of $G \times G$. Note that $\check{\Delta}_G$ is a closed smooth n-dimensional submanifold of $G \times G$ since it is equal to the inverse image $\mu^{-1}(e)$, where $\mu : G \times G \to G$ is the multiplication map given by $\mu(g_1, g_2) = g_1 g_2$ and the identity element $e \in G$ is a regular value of μ. Note that the map $\alpha : G \to \check{\Delta}_G$ given by

$$\alpha(g) = (g, g^{-1}) \tag{2}$$

is a diffeomorphism whose inverse is the projection map onto the first factor.

It is shown in [8, Theorem 2.4] that if G is compact with non-abelian connected component, then $\check{\Delta}_G$ is not a set of synthesis for $A(G)$. In this section, we show that for a general Lie group G, $\check{\Delta}_G$ is always a set of local weak synthesis. In particular, it is of weak synthesis when $A(G)$ has an approximate identity. We start with the following lemma which shows that $\check{\Delta}_G$ has the cone property.

Lemma 6. Let G be an n-dimensional Lie group. Then the submanifold $\check{\Delta}_G$ has the cone property in $G \times G$.

Proof. We use the notation in Definition 3. We set $H = G\times G$, $E = M = \check{\Delta}_G$, $m = n$, and $U_x = H = G \times G$ for every $x \in \check{\Delta}_G$. Let $\mathfrak{g}$ denote the Lie algebra of G. Let W be an open subset of $\mathfrak{g} \cong \mathbb{R}^n$ containing 0 such that the exponential map restricted to W, $\exp : W \to \exp(W) \subset G$, is a diffeomorphism. We set $W_x^0 = W_x = W$, and set $A_x = \mathrm{Aff}(G \times G)$ for every $x \in \check{\Delta}_G$. We write $\tilde{t} = \exp(t)$ for short. We define $\psi_x : W \to \mathrm{Aff}(G \times G)$, also independent of x, as follows: for $t \in W$ and $(g_1, g_2) \in G \times G$,

$$\psi_x(t)(g_1, g_2) = (\mathrm{Inn}(\tilde{t}) \times \mathrm{id}_G)((\tilde{t}, \tilde{t}^{-1}) \cdot (g_1, g_2)) = (\tilde{t}^{-1}(\tilde{t}g_1)\tilde{t}, \tilde{t}^{-1}g_2) = (g_1\tilde{t}, \tilde{t}^{-1}g_2),$$

where $\mathrm{Inn}(\tilde{t})$ denotes the inner automorphism of G corresponding to $\tilde{t} \in G$. Each $\psi_x(t)$ is the composition of a left multiplication by $(\tilde{t}, \tilde{t}^{-1})$ and an inner automorphism $\mathrm{Inn}(\tilde{t}) \times \mathrm{id}_G \in \mathrm{Aut}(G \times G)$, and hence is an affine transformation of $G \times G$. Clearly, ψ_x is smooth and $\psi_x(0) = \mathrm{id}_{G\times G}$. For every $y = (g, g^{-1}) \in \check{\Delta}_G$ and $t \in W$, we have

$$\psi_x(t)y = (g\tilde{t}, \tilde{t}^{-1}g^{-1}) = (g\tilde{t}, (g\tilde{t})^{-1}) \in \check{\Delta}_G.$$

The mapping $W \to \psi_x(W)y$ is just the composition $\alpha \circ L_g \circ \exp$, where α is the diffeomorphism given by (2). Since exp is a diffeomorphism on W, the mapping $W \to \psi_x(W)y$ is also a diffeomorphism. ■

Theorem 7. Let G be an n-dimensional Lie group. Then:

(i) $\check{\Delta}_G$ is a set of local smooth synthesis for $A(G)$.

(ii) $\check{\Delta}_G$ is a set of local weak synthesis for $A(G)$ of degree at most $[\frac{n}{2}] + 1$.

Proof. Our results follow from Theorem 4 and Lemma 6. ■

We note that Theorem 7 can also be deduced directly from Theorem 5. This is because $\check{\Delta}_G$ is the orbit generated by the action $G \times (G \times G) \to G \times G$, $(x, (g, h)) \mapsto (gx, x^{-1}h)$ and the identity element $(e, e) \in G \times G$. As it is shown in the proof of Lemma 6, G is a group of affine transformations of $G \times G$ by the above action. Nevertheless, we would like to point out that our original proof of Theorem 7 is more straightforward. The proof of Theorem 5 which is [14, Corollary 4.9], although more general, is based on a "local slicing" of an orbit to subsets with the cone property. However we directly proved in Lemma 6 that in the case of the anti-diagonal, the cone property always holds. Moreover, as we will see in the following proposition, any attempt toward making the anti-diagonal having the cone property is "locally" similar to the construction in the proof of Lemma 6.

Remark 8. In Definition 3, let W'_x be the connected component of $\{0\}$ in $W_x \subseteq \mathbb{R}^m$. Then W'_x is an open connected neighborhood of $0 \in \mathbb{R}^m$. So, by replacing W_x with W'_x and W^0_x with $W^0_x \cap W'_x$, we see that the assumption of Definition 3 still holds. Hence, we can assume that, for every $x \in G$, W_x is connected. This, in particular, implies that ψ_x maps W_x into the connected component of the identity (e, id_G) in Aff(G).

Proposition 9. Let G be a connected, semisimple, Lie group, and let $\mathrm{Inn}(G)$ denote the group of inner automorphisms of G. For every $x \in G$, let ψ_x and W^0_x be the corresponding objects to (x, x^{-1}) in Definition 3 and Remark 8 for $\check{\Delta}_G$. Then there exist smooth maps $\rho_x : W^0_x \to G$ and $\varphi_x : W^0_x \to \mathrm{Inn}(G)$ such that

$$\psi_x(r)(w, z) = (\varphi_x(r)[w\rho_x(r)^{-1}], \varphi_x(r)[\rho_x(r)z])$$

for all $r \in W^0_x$ and $w, z \in G$. That is,

$$\psi_x(r) = [R_{\rho_x(r)^{-1}} \otimes L_{\rho_x(r)}] \circ [\varphi_x(r) \otimes \varphi_x(r)] \quad (x \in G,\ r \in W^0_x).$$

Proof. Let $E = M = \check{\Delta}_G$, $x \in G$, and $U_x = U_{(x,x^{-1})}$ be as in Definition 3 for $\check{\Delta}_G$. There is a symmetric compact neighborhood V of the identity $\{e\}$ in G such that

$$(xV \times Vx^{-1}) \subseteq U_x.$$

Let $v \in V$. Then we have

$$(xv, v^{-1}x^{-1}) = (xv, (xv)^{-1}) \in U_x \cap M. \tag{3}$$

Let $\sigma = \psi_x(r) \in \psi_x(W^0_x)$. Since G is semisimple and connected, the connected component of the identity $((e, e), \mathrm{id}_{G\times G})$ in $\mathrm{Aff}(G \times G)$ is $G \times G \times \mathrm{Inn}(G \times G)$. Therefore, by Definition 2 and Remark 8, there are $a = b \otimes c \in \mathrm{Inn}(G \times G)$ and $(t_1, t_2) \in G \times G$ such that

$$\sigma = L_{(t_1,t_2)}a.$$

Thus, from (3) and Definition 3,

$$(b(t_1xv), c(t_2v^{-1}x^{-1})) = a(t_1xv, t_2v^{-1}x^{-1}) = \sigma(xv, v^{-1}x^{-1}) \in M.$$

That is,

$$b(t_1xv) = c(xvt_2^{-1}), \quad \forall v \in V.$$

In particular,

$$b(t_1x) = c(xt_2^{-1}). \tag{4}$$

Thus we have

$$c(x)c(v)c(t_2^{-1}) = c(xvt_2^{-1}) = b(t_1xv) = b(t_1x)b(v) = c(xt_2^{-1})b(v) = c(x)c(t_2^{-1})b(v).$$

Therefore $b = c \circ \mathrm{Inn}(t_2^{-1})$ on V, and so, $b = c \circ \mathrm{Inn}(t_2^{-1})$ on the closed subgroup H generated by V. Since V is open, so is H. Hence $H = G$ since G is connected. That is,

$$b = c \circ \mathrm{Inn}(t_2^{-1}). \tag{5}$$

Now, from (4) and (5), it follows that

$$b(t_1 x) = c(x t_2^{-1}) = (c \circ \mathrm{Inn}(t_2^{-1}))(t_2^{-1} x) = b(t_2^{-1} x).$$

So $t_1 x = t_2^{-1} x$ which implies that $t_1 = t_2^{-1}$. Therefore, for every $s, s' \in G$,

$$\sigma(s, s') = a(t_1 s, t_2 s') = ((c \circ \mathrm{Inn}(t_2^{-1}))(t_2^{-1} s), c(t_2 s')) = (c(s t_2^{-1}), c(t_2 s')).$$

Now define $\rho_x(r) := t_2$ and $\varphi_x(r) := c$. Then $\rho_x : W_x^0 \to G$ and $\varphi_x : W_x^0 \to \mathrm{Inn}(G)$ are well-defined and

$$\psi_x(r) = \sigma = [R_{\rho_x(r)^{-1}} \otimes L_{\rho_x(r)}] \circ [\varphi_x(r) \otimes \varphi_x(r)].$$

Moreover, since ψ_x is smooth, the above equality implies that both ρ_x and φ_x are smooth. ■

4.2. Certain products of diagonals. Let G be a Lie group, and let

$$\Delta_G = \{(g, g) \in G \times G \mid g \in G\},$$

which is the *diagonal* of $G \times G$. It is well-known that closed subgroups are sets of synthesis for Fourier algebras [18, Theorem 3] (see also [11]). It is shown in [7] that this could fail if we consider products of subgroups. More precisely, it is shown in [7, Corollary 3.2] that if G is compact, connected and non-abelian, then $(\Delta_G \times \Delta_G)\Delta_{G\times G}$ is not a set of synthesis for $A(G^4)$, where $G^4 = G \times G \times G \times G$.

In this section, we show that for a general Lie group, $(\Delta_G \times \Delta_G)\Delta_{G\times G}$ is always a set of local weak synthesis. As in the case of the anti-diagonal, this is done by showing first that $(\Delta_G \times \Delta_G)\Delta_{G\times G}$ has the cone property.

Lemma 10. Let G be an n-dimensional Lie group. Then $(\Delta_G \times \Delta_G)\Delta_{G\times G}$ is a closed smooth submanifold of G^4 of dimension $3n$, and is diffeomorphic to G^3.

Proof. An element in $\Delta_G \times \Delta_G$ is of the form (s, s, v, v). An element in $\Delta_{G\times G}$ is of the form (w, z, w, z). Hence an element in $(\Delta_G \times \Delta_G)\Delta_{G\times G}$ is of the form (sw, sz, vw, vz). Define

$$M = \{(g_1, g_2, g_3, g_4) \in G^4 \mid g_1 g_3^{-1} g_4 g_2^{-1} = e\}.$$

Since $sw(vw)^{-1} vz(sz)^{-1} = e$, we have $(\Delta_G \times \Delta_G)\Delta_{G\times G} \subset M$. Conversely, any element in M is of the form $(g_1, g_1 g_3^{-1} g_4, g_3, g_4)$. If we set $s = g_1$, $v = g_3$, $w = e$, and $z = g_3^{-1} g_4$, then we get $(g_1, g_1 g_3^{-1} g_4, g_3, g_4) = (sw, sz, vw, vz)$ and thus $M \subset (\Delta_G \times \Delta_G)\Delta_{G\times G}$. It follows that $M = (\Delta_G \times \Delta_G)\Delta_{G\times G}$.

Now consider the smooth map $f : G^4 \to G$ given by

$$f(g_1, g_2, g_3, g_4) = g_1 g_3^{-1} g_4 g_2^{-1}. \tag{6}$$

Note that $M = f^{-1}(e)$ is closed in G^4. Clearly, f is surjective. For fixed $g = (g_1, g_2, g_3, g_4) \in G^4$, consider the right multiplication map $\rho : G \to G$ given by

$$\rho(h) = hg_3^{-1}g_4g_2^{-1}.$$

Since ρ is a diffeomorphism, its derivative $(D\rho)_{g_1} : T_{g_1}G \to T_{f(g)}G$ is bijective for every $g_1 \in G$. Since the image of $(D\rho)_{g_1}$ is contained in the image of the derivative $(Df)_g : T_gG^4 \to T_{f(g)}G$, we conclude that $(Df)_g$ must be surjective. It follows that every element of G is a regular value of f. In particular, $e \in G$ is a regular value of f, and consequently $M = f^{-1}(e)$ is a smooth submanifold of G^4 of dimension equal to $\dim(G^4) - \dim G = 4n - n = 3n$.

Finally, a diffeomorphism $\varphi : G^3 \to (\Delta_G \times \Delta_G)\Delta_{G\times G}$ is given by

$$\varphi(g_1, g_2, g_3) = (g_1, g_1g_2^{-1}g_3, g_2, g_3). \tag{7}$$

Note that φ^{-1} is just the projection map $(g_1, g_2, g_3, g_4) \mapsto (g_1, g_3, g_4)$ restricted to $(\Delta_G \times \Delta_G)\Delta_{G\times G}$. ∎

Lemma 11. Let G be an n-dimensional Lie group. Then the closed submanifold $(\Delta_G \times \Delta_G)\Delta_{G\times G}$ has the cone property in G^4.

Proof. We proceed as in the proof of Lemma 6. Using the notation in Definition 3, we set $H = G^4$, $E = M = (\Delta_G \times \Delta_G)\Delta_{G\times G}$, $m = 3\dim G = 3n$, and $U_x = G^4$ for every $x \in (\Delta_G \times \Delta_G)\Delta_{G\times G}$. Let $\mathfrak{g}$ denote the Lie algebra of G. Let W be a bounded open subset of $\mathfrak{g} \cong \mathbb{R}^n$ containing 0 such that the exponential map restricted to W, $\exp : W \to \exp(W) \subset G$, is a diffeomorphism. We set

$$W_x^0 = W_x = W^3 = W \times W \times W \subset \mathfrak{g} \times \mathfrak{g} \times \mathfrak{g} \cong \mathbb{R}^{3n},$$

and set $A_x = \mathrm{Aff}(G^4)$ for every $x \in (\Delta_G \times \Delta_G)\Delta_{G\times G}$. Given $p \in W$, we write $\tilde{p} = \exp(p)$ for short. We define $\psi_x : W^3 \to \mathrm{Aff}(G^4)$, also independent of x, as follows: for $t = (p, q, r) \in W^3$ and $(g_1, g_2, g_3, g_4) \in G^4$,

$$\begin{aligned}\psi_x(t)(g_1, g_2, g_3, g_4) &= (\tilde{p}g_1,\ \tilde{p}g_2\tilde{q}\tilde{p}^{-1},\ \tilde{r}g_3,\ \tilde{r}g_4\tilde{q}\tilde{p}^{-1})\\ &= (\tilde{p}g_1,\ (\tilde{q}\tilde{p}^{-1})^{-1}(\tilde{q}g_2)\tilde{q}\tilde{p}^{-1},\ \tilde{r}g_3,\ (\tilde{q}\tilde{p}^{-1})^{-1}(\tilde{q}\tilde{p}^{-1}\tilde{r}g_4)\tilde{q}\tilde{p}^{-1}).\end{aligned}$$

Each $\psi_x(t)$ is the composition of a left multiplication by $(\tilde{p}, \tilde{q}, \tilde{r}, \tilde{q}\tilde{p}^{-1}\tilde{r})$ and an inner automorphism $(\mathrm{id}_G \times \mathrm{Inn}(\tilde{q}\tilde{p}^{-1}) \times \mathrm{id}_G \times \mathrm{Inn}(\tilde{q}\tilde{p}^{-1})) \in \mathrm{Aut}(G^4)$, and hence is an affine transformation of G^4. Clearly, ψ_x is smooth and $\psi_x(0) = \mathrm{id}_{G^4}$. For every $y = (g_1, g_1g_3^{-1}g_4, g_3, g_4) \in (\Delta_G \times \Delta_G)\Delta_{G\times G}$ and $t = (p, q, r) \in W^3$, we have

$$\begin{aligned}f(\psi_x(t)y) &= f(\tilde{p}g_1,\ \tilde{p}g_1g_3^{-1}g_4\tilde{q}\tilde{p}^{-1},\ \tilde{r}g_3,\ \tilde{r}g_4\tilde{q}\tilde{p}^{-1})\\ &= (\tilde{p}g_1)(\tilde{r}g_3)^{-1}(\tilde{r}g_4\tilde{q}\tilde{p}^{-1})(\tilde{p}g_1g_3^{-1}g_4\tilde{q}\tilde{p}^{-1})^{-1} = e,\end{aligned}$$

where $f : G^4 \to G$ is the smooth map defined in (6). It follows that

$$\psi_x(t)y \in f^{-1}(e) = (\Delta_G \times \Delta_G)\Delta_{G\times G}.$$

It only remains to prove that $W^3 \to \psi_x(W^3)y$ is a diffeomorphism for fixed $y = (g_1, g_1 g_3^{-1} g_4, g_3, g_4) \in (\Delta_G \times \Delta_G)\Delta_{G\times G}$. Let $\varphi : G^3 \to (\Delta_G \times \Delta_G)\Delta_{G\times G}$ be the diffeomorphism defined in (7). It is enough to show that the map

$$\xi : (\exp(W))^3 \to \varphi^{-1}(\psi_x(W^3)y)$$

given by

$$\xi(\tilde{p}, \tilde{q}, \tilde{r}) = (\tilde{p}g_1,\ \tilde{r}g_3,\ \tilde{r}g_4\tilde{q}\tilde{p}^{-1})$$

is a diffeomorphism. Clearly, ξ extends to a smooth map $\hat{\xi} : G^3 \to G^3$. It is easy to check that $\hat{\xi}$ is bijective and the inverse map is given by

$$\hat{\xi}^{-1}(z_1, z_2, z_3) = (z_1 g_1^{-1},\ g_4^{-1} g_3 z_2^{-1} z_3 z_1 g_1^{-1},\ z_2 g_3^{-1}),$$

which is smooth as well. Hence $\hat{\xi}$ is a diffeomorphism, and so is its restriction $\xi = \hat{\xi}|_{(\exp(W))^3}$. ■

Theorem 12. Let G be an n-dimensional Lie group. Then:

(i) $(\Delta_G \times \Delta_G)\Delta_{G\times G}$ is a set of local smooth synthesis for $A(G^4)$.

(ii) $(\Delta_G \times \Delta_G)\Delta_{G\times G}$ is a set of local weak synthesis for $A(G^4)$ of degree at most $[\frac{3n}{2}] + 1$.

Proof. These follow immediately from Theorem 4 and Lemma 11. ■

5. Projection theorem for sets of smooth and weak synthesis

In this section, we prove the projection theorem for sets of weak and smooth synthesis. Let G be a locally compact group, and let $\mathcal{A}(G)$ be a regular Banach algebra of continuous functions on G which is closed under right translations and such that for any $f \in \mathcal{A}(G)$ we have

- $\|R_s f\|_{\mathcal{A}} = \|f\|_{\mathcal{A}}$ for any $s \in G$,
- $s \mapsto R_s f$ is continuous.

If K is a compact subgroup of G we let

$$\mathcal{A}(G : K) = \{f \in \mathcal{A}(G) \mid R_k f = f \text{ for each } k \in K\},$$

which is a closed subalgebra of $\mathcal{A}(G)$ whose elements are constant on left cosets of K. We let G/K denote the space of left cosets with the quotient topology. We define two maps

$$P : \mathcal{A}(G) \to \mathcal{A}(G), \quad Pf = \int_K R_k f\, dk,$$
$$M : \mathcal{A}(G : K) \to C_b(G/K), \quad Mf(sK) = f(s),$$

where $C_b(X)$ denotes the space of bounded continuous functions on a topological space X. The map P is to be regarded as a Bochner integral over the normalized

Haar measure on K; its range is $\mathcal{A}(G:K)$ and P is a (completely) contractive projection. The map M is well-defined by comments above, and its range consists of continuous functions since $\mathcal{A}(G:K) \subset C_b(G:K)$. We note that M is an injective homomorphism and denote its range by $\mathcal{A}(G/K)$. We assign a norm to $\mathcal{A}(G/K)$ in such a way that M is an isometry. We finally define two maps

$$N = M^{-1} : \mathcal{A}(G/K) \to \mathcal{A}(G), \quad \Gamma = M \circ P : \mathcal{A}(G) \to \mathcal{A}(G/K)$$

so that N is an isometric homomorphism and Γ is a quotient map.

The following theorem demonstrates the relationship between (local) weak synthesis for $\mathcal{A}(G)$ and $\mathcal{A}(G/K)$. Its proof follows directly from [7, Theorem 1.4] (see also [7, Corollary 1.5]).

Theorem 13. Let G be a compact group, and let $\mathcal{T}(G)$ denote the space of trigonometric polynomials on G. Let $\mathcal{A}(G)$ be as above and additionally satisfy that $\mathcal{T}(G)\mathcal{A}(G) \subseteq \mathcal{A}(G)$. If E is a closed subset of G/K, let

$$E^* = \{s \in G \mid sK \in E\}.$$

Then, for every $d \in \mathbb{N}$, E is a set of (local) weak synthesis for $\mathcal{A}(G/K)$ of degree at most d if and only if E^* is a set of (local) weak synthesis for $\mathcal{A}(G)$ of degree at most d .

Now let H be a subgroup of a Lie group G. Define

$$C^\infty(G:H) = \{f \in C^\infty(G) \mid f(sh) = f(s),\ \forall s \in G,\ \forall h \in H\}.$$

The proof of the following lemma follows from the well-known fact that the integration and the differentiation on compact Lie groups commute.

Lemma 14. Let G be a Lie group, and let H be a compact subgroup of G. If $f \in C^\infty(G)$, then for $s \in G$ the Haar integral

$$(P_H f)(s) = \int_H f(sr)dr$$

defines an element in $C^\infty(G:H)$.

Theorem 15. Let G be a compact Lie group, let $\mathcal{A}(G)$ be as in Theorem 13, and additionally contain $\mathcal{D}(G)$. If E is a closed subset of G/K, let $E^* = \{s \in G \mid sK \in E\}$. Then E is a set of smooth synthesis for $\mathcal{A}(G/K)$ if and only if E^* is a set of smooth synthesis for $\mathcal{A}(G)$.

Proof. This follows from [7, Theorem 1.4] and Lemma 14. ∎

6. Alternative proofs in the case of a compact Lie group

In this section, we apply Theorem 15 to present alternative proofs of the smooth and weak synthesis results of Section 4 for compact Lie groups (Theorems 7 and 12).

Let G be a *compact* Lie group for the remainder of this section. We use $G \times G$ in place of G, and $K = \Delta = \{(s,s) \mid s \in G\}$ in the setup of Section 5. Since the map

$$(G \times G)/\Delta \to G, \quad (s,e)\Delta \mapsto s \tag{8}$$

is a homeomorphism, we identify the coset space with G. We observe that in this case the map $P : A(G \times G) \to A(G \times G)$ satisfies

$$Pw(s,t) = \int_G w(sr,tr)dr = \int_G w(st^{-1}r,r)dr$$

and the map $M : A(G \times G : \Delta) \to C(G)$ satisfies

$$Mw(s) = w(s,e).$$

The map $\Gamma = M \circ P$, from Section 5, can be regarded as a 'twisted' convolution, for if $A(G \times G)$ contains an elementary function $f \times g$, then for $s \in G$

$$\Gamma(f \times g)(s) = \int_G (f \times g)(st,t)dt = \int_G f(st)g(t)dt = f * \check{g}(s).$$

We denote the image of Γ by $A_\Delta(G)$. We endow $A_\Delta(G)$ with the norm which makes Γ a quotient map. We also note that

$$N : A_\Delta(G) \to A(G \times G : \Delta), \quad Nu(s,t) = u(st^{-1})$$

is an isometry. As in [7, Theorem 2.6], if we repeat the procedure above we obtain

$$A_{\Delta^2}(G) = \Gamma(A_\Delta(G \times G)).$$

We can do a similar construction with the anti-diagonal $\check{\Delta} = \{(s,s^{-1}) \mid s \in G\}$. We let $G \times G/\check{\Delta}$ denote the set of equivalence classes modulo the equivalence relation $(s',t') \sim (s,t)$ if and only if $(s^{-1}s', t't^{-1}) \in \check{\Delta}$, so that $G \times G/\check{\Delta}$, with the quotient topology, is homeomorphic to G via $(s,t) \mapsto st$. We let

$$A(G \times G : \check{\Delta}) = \{u \in A(G \times G) \mid r \diamond u = u \text{ for all } r \in G\},$$

where $(r \diamond u)(s,t) = u(sr, r^{-1}t)$. Similarly as above, $A(G \times G : \check{\Delta})$ is a closed subalgebra of $A(G \times G)$. Also the map

$$\check{\Gamma} : A(G \times G) \to C(G), \quad \check{\Gamma}w(s) = \int_G w(st,t^{-1})dt \tag{9}$$

is surjective, and is injective on $A(G \times G : \check{\Delta})$. We denote the image of $\check{\Gamma}$ by $A_\gamma(G)$. We endow $A_\gamma(G)$ with the norm which makes $\check{\Gamma}$ a quotient map. We also note that

$$\check{N} : A_\gamma(G) \to A(G \times G : \check{\Delta}), \quad \check{N}u(s,t) = u(st) \tag{10}$$

is an isometry.

The proof of the following lemma follows from the well-known fact that the integration and the differentiation on compact Lie groups commute.

Lemma 16. Let G be a compact Lie group. If $F \in C^\infty(G\times G)$, then for $s \in G$ the Haar integral

$$\hat{F}(s) = \int_G F(s,r)dr$$

defines an element in $C^\infty(G)$.

Lemma 17. Let G be a compact Lie group. Then:

(i) $\Gamma \circ N = \mathrm{id}_{C(G)}$ and $\check{\Gamma} \circ \check{N} = \mathrm{id}_{C(G)}$.

(ii) If $f \in C^\infty(G)$, then $Nf \in C^\infty(G \times G)$ and $\check{N}f \in C^\infty(G \times G)$.

(iii) If $f \in C^\infty(G \times G)$, then $\Gamma f \in C^\infty(G)$ and $\check{\Gamma} f \in C^\infty(G)$.

Proof. Part (i) is obvious. For part (ii), let $\mu : G\times G \to G$, $(s,t) \mapsto st$, denote the multiplication map, and let $\mathrm{inv}_G : G \to G$, $g \mapsto g^{-1}$, denote the inverse map. Since μ and inv_G are both smooth, so are the compositions $\check{N}f = f \circ \mu$ and $Nf = f \circ \mu \circ (\mathrm{id}_G \times \mathrm{inv}_G)$. For part (iii), we apply Lemma 16 to the smooth map $F(s,r) = (f \circ (\mu \times \mathrm{id}_G))(s,r) = f(sr,r)$ to conclude that $\hat{F} = \Gamma f$ is smooth. Similarly, we apply Lemma 16 to $F(s,r) = (f \circ (\mu \times \mathrm{inv}_G))(s,r) = f(sr, r^{-1})$ to conclude that $\hat{F} = \check{\Gamma} f$ is smooth. ■

We are now ready to state the main result of this section.

Theorem 18. Let G be an n-dimensional compact Lie group. Then:

(i) $\check{\Delta}_G$ is a set of smooth synthesis for $A(G \times G)$.

(ii) $\check{\Delta}_G$ is a set of weak synthesis for $A(G \times G)$ of degree at most $[\frac{n}{2}] + 1$.

(iii) $(\Delta_G \times \Delta_G)\Delta_{G\times G}$ is a set of smooth synthesis for $A(G^4)$.

(iv) $(\Delta_G \times \Delta_G)\Delta_{G\times G}$ is a set of weak synthesis for $A(G^4)$ of degree at most $[\frac{3n}{2}] + 1$.

Proof. (i) It is noted in the proof of Lemma 2.3 in [8], comparing with (9) and (10), that

$$I(\check{\Delta}_G) = \overline{\mathrm{span}}\{u\check{N}f \mid u \in A(G \times G) \text{ and } f \in I_{A_\gamma(G)}(\{e\})\}.$$

Let us first show that $\{e\}$ is a set of smooth synthesis for $A_\gamma(G)$, i.e.

$$J^{\mathcal{D}}_{A_\gamma(G)}(\{e\}) = I_{A_\gamma(G)}(\{e\}). \tag{11}$$

Since $D(G) \cap A(G)$ is dense in $A(G)$, for every $u \in I_{A_\gamma(G)}(\{e\})$ there is a sequence $\{u_n\} \subset D(G) \cap A(G)$ which converges to u. We note that

$$|u_n(e)| = |u_n(e) - u(e)| \le \|u_n - u\|_\infty \le \|u_n - u\|_{A_\gamma(G)} \overset{n\to\infty}{\longrightarrow} 0.$$

Thus if $u'_n = u_n - u_n(e)1$, then $\{u'_n\} \subset J^{\mathcal{D}}_{A_\gamma(G)}(\{e\})$ with $\lim_{n\to\infty} \|u'_n - u\|_{A_\gamma(G)} = 0$. Hence (11) holds. Now it follows from Lemma 17(ii) that $\check{\Delta}_G$ is a set of smooth synthesis for $A(G \times G)$.

(ii) This follows from part (i) and Theorem 1.

(iii) It is shown in [7, Theorems 1.4 and 2.6] that

$$I_{A_\Delta(G\times G)}(\Delta_G) = \overline{\mathrm{span}}\{uNf \mid u \in A_\Delta(G\times G) \text{ and } f \in I_{A_{\Delta^2}(G)}(\{e\})\}.$$

We can see, similar to (11) that $\{e\}$ is a set of smooth synthesis for $A_{\Delta^2}(G)$, i.e.

$$J^{\mathcal{D}}_{A_{\Delta^2}(G)}(\{e\}) = I_{A_{\Delta^2}(G)}(\{e\}). \tag{12}$$

Thus, from (12) and Lemma 17(ii), it follows that Δ_G is a set of smooth synthesis for $A_\Delta(G\times G)$, and so, $(\Delta_G \times \Delta_G)\Delta_{G\times G}$ is a set of smooth synthesis for $A(G\times G\times G\times G)$ from Theorem 15 and [7, Theorems 1.4 and 2.6].

(iv) This follows from part (iii) and Theorem 1. ■

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