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Examples of Self-Iterating Lie Algebras, 2

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Abstract. We study properties of self-iterating Lie algebras in positive characteristic. Let $R = K[t_i | i \in \mathbb{N}]/(t_i^p | i \in \mathbb{N})$ be the truncated polynomial ring. Let $\partial_i = \frac{\partial}{\partial t_i}$, $i \in \mathbb{N}$, denote the respective derivations. Consider the operators

$$\begin{aligned} v_1 &= \partial_1 + t_0(\partial_2 + t_1(\partial_3 + t_2(\partial_4 + t_3(\partial_5 + t_4(\partial_6 + \cdots))))); \\ v_2 &= \qquad\quad \partial_2 + t_1(\partial_3 + t_2(\partial_4 + t_3(\partial_5 + t_4(\partial_6 + \cdots)))). \end{aligned}$$

Let $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2) \subset \mathrm{Der}\, R$ be the restricted Lie algebra generated by these derivations.

We establish the following properties of this algebra in case $p = 2, 3$. a) $\mathbf{L}$ has a polynomial growth with Gelfand-Kirillov dimension $\ln p / \ln((1+\sqrt{5})/2)$. b) the associative envelope $\mathbf{A} = \mathrm{Alg}(v_1, v_2)$ of $\mathbf{L}$ has Gelfand-Kirillov dimension $2 \ln p / \ln((1+\sqrt{5})/2)$. c) $\mathbf{L}$ has a nil-p-mapping. d) $\mathbf{L}$, $\mathbf{A}$ and the augmentation ideal of the restricted enveloping algebra $\mathbf{u} = u_0(\mathbf{L})$ are direct sums of two locally nilpotent subalgebras. The question whether $\mathbf{u}$ is a nil-algebra remains open. e) the restricted enveloping algebra $u(\mathbf{L})$ is of intermediate growth.

These properties resemble those of Grigorchuk and Gupta-Sidki groups.
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1. Introduction: Fibonacci Lie algebras

In this paper we continue the study of self-iterating Lie algebras introduced by the first author in [16], see also further developments in [21], [14]. In this section we give main definitions.

Let K be the ground field of arbitrary characteristic. Let $I = \{0, 1, 2, \dots\}$. Denote also $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. Consider functions $\alpha : I \to \mathbb{N}_0$, which values we denote by α_i, $i \in I$. Denote by $\mathbb{N}_0^I$ the set of functions with finitely many nonzero values α_i. Let $|\alpha| = \sum_{i \in I} \alpha_i$ for $\alpha \in \mathbb{N}_0^I$. Consider the polynomial ring

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$R = K[T_I] = K[t_i \mid i = 0, 1, 2, \dots]$. Let $\alpha \in \mathbb{N}_0^I$, then we denote $\mathbf{t}^\alpha = \prod_{i\in I} t_i^{\alpha_i}$. We have the basis $R = \langle \mathbf{t}^\alpha \mid \alpha \in \mathbb{N}_0^I \rangle_K$. Consider also the ideal of codimension one $R \triangleright R^+ = \langle \mathbf{t}^\alpha \mid \alpha \in \mathbb{N}_0^I,\ |\alpha| > 0 \rangle_K$.

Denote by $\tau : R \to R$ the endomorphism given by $\tau(t_i) = t_{i+1}$ for $i \in I$. Let $\partial_i = \frac{\partial}{\partial t_i}$, $i \in I$, be the partial derivatives of this ring. Denote by $v(t)$ the action of $v \in \operatorname{Der} R$ onto $t \in R$. We define the following two derivations of R:

$$\begin{aligned} v_1 &= \partial_1 + t_0(\partial_2 + t_1(\partial_3 + t_2(\partial_4 + t_3(\partial_5 + t_4(\partial_6 + \cdots))))); \\ v_2 &= \qquad\qquad \partial_2 + t_1(\partial_3 + t_2(\partial_4 + t_3(\partial_5 + t_4(\partial_6 + \cdots)))). \end{aligned}$$

The action on R and products of such operators are well-defined; these operators are so called special derivations, see e.g. [19], [20]. Remark that we can write these derivations recursively:

$$\begin{aligned} v_1 &= \partial_1 + t_0 \tau(v_1); \\ v_2 &= \tau(v_1). \end{aligned}$$

Let $\mathbf{L} = \operatorname{Lie}(v_1, v_2)$ be the Lie subalgebra of $\operatorname{Der} R$ generated by v_1 and v_2. We also consider the associative algebra generated by these derivations $\mathbf{A} = \operatorname{Alg}(v_1, v_2)$. Similarly, define

$$v_i = \tau^{i-1}(v_1) = \partial_i + t_{i-1}(\partial_{i+1} + t_i(\partial_{i+2} + t_{i+1}(\partial_{i+3} + \cdots))), \quad i = 1, 2, \dots. \tag{1}$$

We also can write

$$v_i = \partial_i + t_{i-1} v_{i+1}, \quad i = 1, 2, \dots. \tag{2}$$

Lemma 1.1. *The following commutation relations hold in* $\mathbf{L} = \operatorname{Lie}(v_1, v_2)$

1. $[v_i, v_{i+1}] = v_{i+2}$ *for* $i = 1, 2, \dots$;
2. $[v_i, v_{i+2}] = t_{i-1} v_{i+3}$ *for* $i = 1, 2, \dots$;
3. *in general, for all* $i < j$ *we have*
$$[v_i, v_j] = \Bigg(\prod_{i-1 \le k \le j-3} t_k \Bigg) v_{j+1};$$
4. *for all* $n \ge 1$, $j \ge 0$ *we have the action*
$$v_n(t_j) = \begin{cases} t_{n-1} t_n \cdots t_{j-2}, & n < j; \\ 1, & n = j; \\ 0, & n > j. \end{cases}$$
5. *for all* $k, n \ge 1$
$$[\partial_n, v_k] = \begin{cases} t_{k-1} t_k \cdots t_{n-1} v_{n+2}, & k < n + 1; \\ v_{n+2}, & k = n + 1; \\ 0, & k > n + 1. \end{cases}$$

Proof. We have

$$[v_1, v_2] = [\partial_1 + t_0\tau(v_1), \tau(v_1)] = [\partial_1, \tau(v_1)] = [\partial_1, \partial_2 + t_1\tau^2(v_1)] = \tau^2(v_1) = v_3.$$

Consider the third claim. Let $i < j$, then

$$\begin{aligned}[v_i, v_j] &= [\partial_i + t_{i-1}(\partial_{i+1} + t_i(\cdots + t_{j-3}(\partial_{j-1} + t_{j-2}v_j)\cdots)), v_j]\\ &= [\partial_i + t_{i-1}(\partial_{i+1} + t_i(\cdots + t_{j-3}\partial_{j-1})\ldots), v_j]\\ &= [\partial_i + t_{i-1}(\partial_{i+1} + t_i(\cdots + t_{j-3}\partial_{j-1})\ldots), \partial_j + t_{j-1}v_{j+1}]\\ &= [\partial_i + t_{i-1}(\partial_{i+1} + t_i(\cdots + t_{j-3}\partial_{j-1})\ldots), t_{j-1}v_{j+1}] = t_{i-1}t_i\cdots t_{j-3}v_{j+1}.\end{aligned}$$

The second claim is a partial case of the third. We consider the first relation as a partial case as well.

Consider the forth claim. Remark that $v_n(t_j)$ is nonzero only in the case $n \le j$ and

$$v_n(t_j) = (\partial_n + t_{n-1}(\partial_{n+1} + \cdots + t_{j-2}(\partial_j + \cdots)\ldots))(t_j) = t_{n-1}t_n\cdots t_{j-2}, \quad n < j.$$

The last claim is proved similarly. ■

Let us make some comments on our derivations and possible gradations on them. Recall that $\mathbb{N}_0^I$ is the set of functions $\alpha : I \to \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ with finitely many nonzero values α_i. (we may take the set I to be arbitrary). Consider the formal power series ring that consists of formal sums

$$R = K[[T_I]] = \left\{ \sum_{\alpha\in\mathbb{N}_0^I} \mu_\alpha \mathbf{t}^\alpha \;\middle|\; \mu_\alpha \in K \right\},$$

where the multiplication extends the rule $\mathbf{t}^\alpha\mathbf{t}^\beta = \mathbf{t}^{\alpha+\beta}$, $\alpha, \beta \in \mathbb{N}_0^I$. Let $\epsilon(i) \in \mathbb{N}_0^I$ be such that $\epsilon(i)_j = \delta_{ij}$, Kronecker's delta, where $i, j \in I$. We get elements $t_i = \mathbf{t}^{\epsilon(i)} \in R$ for all $i \in I$ and $\mathbf{t}^\alpha = \prod_{i\in I} t_i^{\alpha_i}$, $\alpha \in \mathbb{N}_0^I$.

Consider so called Lie algebra of *special derivations* [19], [20], [15]. It consists of formal sums

$$\mathbf{W}(T_I, K) = \left\{ \sum_{\alpha\in\mathbb{N}_0^I} \mathbf{t}^\alpha \sum_{j=1}^{m(\alpha)} \lambda_{\alpha,i_j} \frac{\partial}{\partial t_{i_j}} \;\middle|\; \lambda_{\alpha,i_j} \in K,\ i_j \in I \right\}.$$

It is essential that sums are finite at each $\mathbf{t}^\alpha$, $\alpha \in \mathbb{N}_0^I$. One checks that the Lie bracket of such operators is well-defined and that $\mathbf{W}(T_I, K)$ acts by derivations on $K[[T_I]]$. Observe that our Fibonacci Lie algebra consists of special derivations.

Lemma 1.2. *Assign arbitrary weights* $\mathrm{wt}(t_i) = a_i \in \mathbb{C}$, *for all* $i \in I$. *Then this weight function is extended to an additive function on monomials of* R *and* $\mathbf{W}(T_I, K)$.

Proof. The statement is evident for the ring R. We set $\mathrm{wt}(\frac{\partial}{\partial t_i}) = -\mathrm{wt}(t_i) = -a_i$ for all $i \in I$. Consider a monomial $a = \mathbf{t}^\alpha \frac{\partial}{\partial t_j} \in \mathbf{W}(T_I, K)$, $\alpha \in \mathbb{N}_0^I$. Then we set $\mathrm{wt}(a) = \sum_{i\in I} \alpha_i a_i - a_j$. Let also $b = \mathbf{t}^\beta \frac{\partial}{\partial t_k} \in \mathbf{W}(T_I, K)$. A formal check shows that $\mathrm{wt}([a, b]) = \mathrm{wt}(a) + \mathrm{wt}(b)$. ■

Let L be a Lie algebra over a field K of characteristic $p > 0$ and let $\operatorname{ad} x : L \to L$, $\operatorname{ad} x(y) = [x, y]$, for $x, y \in L$, be the adjoint map. Recall that L is called a *restricted Lie algebra* or *Lie p-algebra* [10, 22], if additionally L affords a unary operation $x \mapsto x^{[p]}$, $x \in L$, satisfying

i) $(\lambda x)^{[p]} = \lambda^p x^{[p]}$, for all $\lambda \in K$, $x \in L$;

ii) $\operatorname{ad}(x^{[p]}) = (\operatorname{ad} x)^p$, for all $x \in L$;

iii) for all $x, y \in L$ one has

$$(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{i=1}^{p-1} s_i(x, y), \tag{3}$$

where $is_i(x, y)$ is the coefficient of Z^{i-1} in the following polynomial: $(\operatorname{ad}(Zx + y))^{p-1}(x) \in L[Z]$, where Z is an indeterminate.

Also, $s_i(x, y)$ is a Lie polynomial in x, y of degrees i and $p - i$, respectively.

Suppose that L is a restricted Lie algebra and $X \subset L$. Then by $\operatorname{Lie}_p(X)$ we denote the restricted subalgebra generated by X. Suppose that $H \subset L$ is a Lie subalgebra, i.e. H is a vector subspace that is closed under the Lie bracket. Then by H_p we denote the restricted subalgebra generated by H. Let $u(L)$ denote the restricted enveloping algebra of L and $u_0(L)$ the augmentation ideal of $u(L)$.

We recall the notion of growth. Let A be an associative (or Lie) algebra generated by a finite set X. Denote by $A^{(X,n)}$ the subspace of A spanned by all monomials in X of length not exceeding n. If A is a restricted Lie algebra, then we define [13] $A^{(X,n)} = \langle\, [x_1, \dots, x_s]^{p^k} \mid x_i \in X,\ sp^k \le n \rangle_K$. In either situation, one considers the *growth function* defined by

$$\gamma_A(n) = \gamma_A(X, n) = \dim_K A^{(X,n)}.$$

The growth function clearly depends on the choice of the generating set X. Furthermore, it is easy to see that the exponential growth is the highest possible growth for Lie and associative algebras. The growth function $\gamma_A(n)$ is compared with the polynomial functions n^k, $k \in \mathbb{R}^+$ by computing the *upper and lower Gelfand-Kirillov dimensions* [12], namely

$$\begin{aligned} \overline{\operatorname{GKdim}} A &= \overline{\lim_{n\to\infty}} \frac{\ln \gamma_A(n)}{\ln n}; \\ \underline{\operatorname{GKdim}} A &= \underline{\lim_{n\to\infty}} \frac{\ln \gamma_A(n)}{\ln n}. \end{aligned}$$

This setting assumes that all elements of X have the same weight equal to 1. We shall mainly use a little bit different growth function. Namely, we consider the weight function $\operatorname{wt} v$, $v \in A$ and the growth with respect to it $\tilde{\gamma}_A(r) = \dim_K \langle y \mid y \in A, \operatorname{wt} y \le r \rangle$, $r \in \mathbb{R}$, where the elements of the generating set $X = \{v_1, v_2\}$ have different weights. The standard arguments [12] prove that this growth function yields the same Gelfand-Kirillov dimensions.

2. Main results: Properties of Fibonacci Lie algebras

The goal of the paper is to study properties of Fibonacci restricted Lie algebras in small characteristics $p = 2, 3, 5$.

We modify the example above for the case of positive characteristic. Suppose that $\operatorname{char} K = p > 0$. Denote $I = \{0, 1, 2, \dots\}$ and $\mathbb{N}_p = \{0, 1, \dots, p-1\}$. Let $\mathbb{N}_p^I = \{\alpha : I \to \mathbb{N}_p\}$ be the set of functions with finitely many nonzero values. Now, we consider the truncated polynomial ring

$$R = K[t_i \mid i = 0, 1, 2, \dots]/(t_i^p \mid i = 0, 1, 2, \dots).$$

Let $\alpha \in \mathbb{N}_p^I$, then denote $\mathbf{t}^\alpha = \prod_{i \in I} t_i^{\alpha_i}$. We have the basis $R = \langle \mathbf{t}^\alpha \mid \alpha \in \mathbb{N}_p^I \rangle_K$. Also, consider the ideal $R^+ = \langle \mathbf{t}^\alpha \mid \alpha \in \mathbb{N}_p^I,\ |\alpha| > 0 \rangle_K \triangleleft R$.

Let $v_i \in \operatorname{Der} R$, $i = 1, 2, \dots$ be the derivations as above. Now, let $\mathbf{L} = \operatorname{Lie}_p(v_1, v_2) \subset \operatorname{Der} R$ denote the restricted subalgebra generated by v_1, v_2, it will also be referred to as the *Fibonacci restricted Lie algebra*.

Our goal is to study the restricted Lie algebra $\mathbf{L} = \operatorname{Lie}_p(v_1, v_2)$. We are also interested in properties of the associative envelope $\mathbf{A} = \operatorname{Alg}(v_1, v_2)$ of the operators v_1, v_2. We study this Lie algebra for small characteristics. The main results of the paper are as follows.

Theorem 2.1. *Let* $\operatorname{char} K = p \in \{2, 3\}$. *Consider the Fibonacci restricted Lie algebra* $\mathbf{L} = \operatorname{Lie}_p(v_1, v_2)$ *and its associative envelope* $\mathbf{A} = \operatorname{Alg}(v_1, v_2)$. *Denote* $\lambda = \frac{1+\sqrt{5}}{2}$. *Then*

1. $\operatorname{GKdim} \mathbf{L} = \underline{\operatorname{GKdim}}\, \mathbf{L} = \ln p / \ln \lambda$;
2. $\operatorname{GKdim} \mathbf{A} = \underline{\operatorname{GKdim}}\, \mathbf{A} = 2 \ln p / \ln \lambda$;
3. *L has a nil-p-mapping.*
4. $\mathbf{L}$, $\mathbf{A}$, *and the augmentation ideal of the restricted enveloping algebra* $\mathbf{u} = u_0(\mathbf{L})$ *are direct sums of two locally nilpotent subalgebras*

$$\mathbf{L} = \mathbf{L}_+ \oplus \mathbf{L}_-, \quad \mathbf{A} = \mathbf{A}_+ \oplus \mathbf{A}_-, \quad \mathbf{u} = \mathbf{u}_+ \oplus \mathbf{u}_-.$$

These results will be proved in Sections 4, 5, 6, and 7. In Section 8 we establish also the first claim for $p = 5$. It seems that it is a rather technical problem to prove claims 2,3 for $p = 5$. We start our arguments by introduction of a $\mathbb{Z} \oplus \mathbb{Z}$-gradation by weights on our algebras in Section 3.

The first construction of examples of non-nil rings which are the sum of two locally nilpotent subrings has been carried out by Kelarev [11], thus answering the question of Kegel. Nowadays, there are several other families of such rings. But to the authors knowledge, all previous examples use contracted semigroup algebras or words technique, see [4] and references in it. In our case, $\mathbf{A}$ is of polynomial growth whereas $\mathbf{u}$ is of subexponential growth (see below). The question whether $\mathbf{u}$ is a nil-algebra remains open.

There are analogies between groups and Lie algebras, but these analogies are mainly between properties of the respective Hopf algebras, i.e. (modular) group rings and (restricted) enveloping algebras. Recall that the Grigorchuk group (and its group ring!) has an intermediate growth, more precisely, its growth function can be put between two functions $\exp(n^\alpha)$ and $\exp(n^\beta)$, where $1/2<\alpha<\beta<1$. We show that $u(\mathbf{L})$ has a similar growth function.

To specify the subexponential growth of $u(\mathbf{L})$ let us give some more definitions. Consider two series of functions $\Phi^q_\alpha(n)$, $q=2,3$ of natural argument with the parameter $\alpha\in\mathbb{R}^+$:

$$\Phi^2_\alpha(n)=n^\alpha,$$
$$\Phi^3_\alpha(n)=\exp(n^{\alpha/(\alpha+1)}).$$

We compare functions $f:\mathbb{N}\to\mathbb{R}^+$ by means of the partial order: $f(n)\overset{\mathrm{a}}{\leq} g(n)$ iff there exists $N>0$, such that $f(n)\leq g(n)$, $n\geq N$. Suppose that A is a finitely generated algebra and $\gamma_A(n)$ is its growth function. We define the *dimension of level* q, $q\in\{2,3\}$, and the *lower dimension of level* q by

$$\operatorname{Dim}^q A=\inf\{\alpha\in\mathbb{R}^+\mid \gamma_A(n)\overset{\mathrm{a}}{\leq}\Phi^q_\alpha(n)\},$$
$$\underline{\operatorname{Dim}}^q A=\sup\{\alpha\in\mathbb{R}^+\mid \gamma_A(n)\overset{\mathrm{a}}{\geq}\Phi^q_\alpha(n)\}.$$

The q-dimensions for arbitrary level $q\in\mathbb{N}$ were introduced by the first author in order to specify the subexponential growth of universal enveloping algebras [18]. The condition $\operatorname{Dim}^q A=\underline{\operatorname{Dim}}^q A=\alpha$ means that the growth function $\gamma_A(n)$ behaves like $\Phi^q_\alpha(n)$. Dimensions of level 2 are exactly the upper and lower Gelfand-Kirillov dimensions [5], [12]. Dimensions of level 3 correspond to the superdimensions of [3] up to normalization (see [17]). We prefer to describe the growth of $u(\mathbf{L})$ in terms of $\operatorname{Dim}^3 A$.

Corollary 2.2. *Let* $\operatorname{char} K=p\in\{2,3,5\}$ *and* $\mathbf{L}=\operatorname{Lie}_p(v_1,v_2)$. *Denote* $\lambda=(1+\sqrt{5})/2$ *and* $\theta=\ln p/\ln\lambda$. *Then the growth of the restricted enveloping algebra* $u(\mathbf{L})$ *is intermediate and*

$$\theta-1\leq\operatorname{Dim}^3 u(\mathbf{L})\leq\theta.$$

Proof. By Theorem 2.1 (Theorem 8.1 if $p=5$), we have $\operatorname{Dim}^2\mathbf{L}=\operatorname{GKdim}\mathbf{L}=\theta$. The claim follows from (the proof) of Proposition 1 in [17]. That proposition deals with the growth of the universal enveloping algebra, some minor changes are needed to adopt the proof for the restricted enveloping algebra. ■

Remark 2.3. In order to get the equality $\operatorname{Dim}^3 u(\mathbf{L})=\theta$ it is sufficient to have the asymptotic $\gamma_L(n)-\gamma_L(n-1)\approx Cn^{\theta-1}$, C a constant, $n\to\infty$ [17, Proposition 1]; but we do not have such a statement.

We remark that $\mathbf{L}$ is a *self-similar* Lie algebra. Namely, consider subalgebras $L_i=\operatorname{Lie}(v_i,v_{i+1})$, $i=1,2,\dots$ Then, we have the isomorphisms $\tau^{i-1}:\mathbf{L}\cong L_i$ for all $i=1,2,3,\dots$. On the other hand, we have the embedding

$$\mathbf{L}\hookrightarrow\langle\partial_1\rangle_K\oplus K[t_0]\otimes L_2,\qquad L_2\cong\mathbf{L},$$

where the semidirect product is defined via the action $\partial_1(v_2) = v_3$ and $\partial_1(v_j) = 0$ for $j \geq 3$. These properties resemble those of the Grigorchuk group, Gupta-Sidki group, etc. [8, 7, 6, 9]. Examples of self-similar associative algebras are also introduced in [2].

3. Gradation by weights

In this section we introduce a $\mathbb{Z}\oplus\mathbb{Z}$-gradation by weights on our algebras. First, suppose that K is arbitrary and we consider $\mathbf{L} = \mathrm{Lie}(v_1, v_2)$.

Lemma 3.1. *Let $\mathbf{L} = \mathrm{Lie}(v_1, v_2) \subset \mathrm{Der}\, R$ be a Lie subalgebra. We introduce the weight and superweight functions*

$$\mathrm{wt}\, v_n = -\,\mathrm{wt}\, t_n = \lambda^n, \qquad n = 1, 2, \ldots, \qquad \lambda = \frac{1+\sqrt{5}}{2};$$

$$\mathrm{swt}\, v_n = -\,\mathrm{swt}\, t_n = \bar{\lambda}^{n-2}, \qquad n = 1, 2, \ldots, \qquad \bar{\lambda} = \frac{1-\sqrt{5}}{2}.$$

1. *We have a $\mathbb{Z}\oplus\mathbb{Z}$-gradation $\mathbf{L} = \oplus_{a,b\geq 0}\mathbf{L}_{a,b}$, where $\mathbf{L}_{a,b}$ is spanned by products with a factors v_1 and b factors v_2.*

2. *Both functions are additive on products of homogeneous elements of $\mathbf{L}$.*

3. *Let $v \in \mathbf{L}_{a,b}$, where $a, b \geq 0$. Then*

$$\mathrm{wt}\, v = a\lambda + b\lambda^2, \qquad \mathrm{swt}\, v = -a\lambda + b.$$

Proof. Let us introduce a grading on $\mathbf{L}$ such that v_i are homogeneous. Suppose that we have a weight function $\mathrm{wt}\, v_i = -\,\mathrm{wt}\, t_i = a_i \in \mathbb{R}$, where $i = 1, 2\ldots$ (see Lemma 1.2). Since it is natural to have homogeneous summands in (2), we assume that

$$a_i = \mathrm{wt}\, v_i = \mathrm{wt}\, \partial_i = \mathrm{wt}\, t_{i-1} + \mathrm{wt}\, v_{i+1} = -a_{i-1} + a_{i+1}.$$

Hence, we get the Fibonacci recurrence relation $a_{i+1} = a_i + a_{i-1}$. It is well-known that all solutions are expressed via two functions introduced above: $a_i = \mathrm{wt}\, v_i = \lambda^i$, $i \in \mathbb{N}$, and $a_i = \mathrm{swt}\, v_i = \bar{\lambda}^{i-2}$, $i \in \mathbb{N}$.

Since the weights $\mathrm{wt}\, v_1 = \lambda$ and $\mathrm{wt}\, v_2 = \lambda^2$ are linearly independent over $\mathbb{Z}$, we conclude that $\mathbf{L}$ has the claimed $\mathbb{Z}\oplus\mathbb{Z}$-gradation $\mathbf{L} = \oplus_{a,b\geq 0}\mathbf{L}_{a,b}$, where $\mathbf{L}_{a,b}$ is spanned by products with a factors v_1 and b factors v_2.

Let $v \in \mathbf{L}_{a,b}$, where $a, b \geq 0$. We get $\mathrm{wt}\, v = a\,\mathrm{wt}\, v_1 + b\,\mathrm{wt}\, v_2 = a\lambda + b\lambda^2$ and $\mathrm{swt}\, v = a\,\mathrm{swt}\, v_1 + b\,\mathrm{swt}\, v_2 = a\bar{\lambda}^{-1} + b = -a\lambda + b$. ∎

We introduce a new coordinate system on plane. Let $A = (x, y) \in \mathbb{R}^2$, we define new coordinates (ξ, η) of A that we also refer to as *weight* and *superweight*

$$\begin{aligned} \xi &= \mathrm{wt}(x,y) &&= x\lambda + y\lambda^2, \\ \eta &= \mathrm{swt}(x,y) &&= -x\lambda + y; \end{aligned} \qquad (x, y) \in \mathbb{R}^2.$$

Consider a homogeneous element $v \in \mathbf{L}_{a,b}$, then we denote $\mathrm{Wt}(v) = (a, b) \in \mathbb{R}^2$ and draw v on plane. By Lemma 3.1, the new coordinates (ξ, η) coincide with the weight and superweight functions introduced above. Since the superweight is an additive function, we obtain a "triangular" decomposition as follows.

Corollary 3.2. *For* $\mathbf{L} = \mathrm{Lie}(v_1, v_2)$ *and its universal enveloping algebra* $U = U(\mathbf{L})$ *we have decompositions into direct sums of two subalgebras as follows*

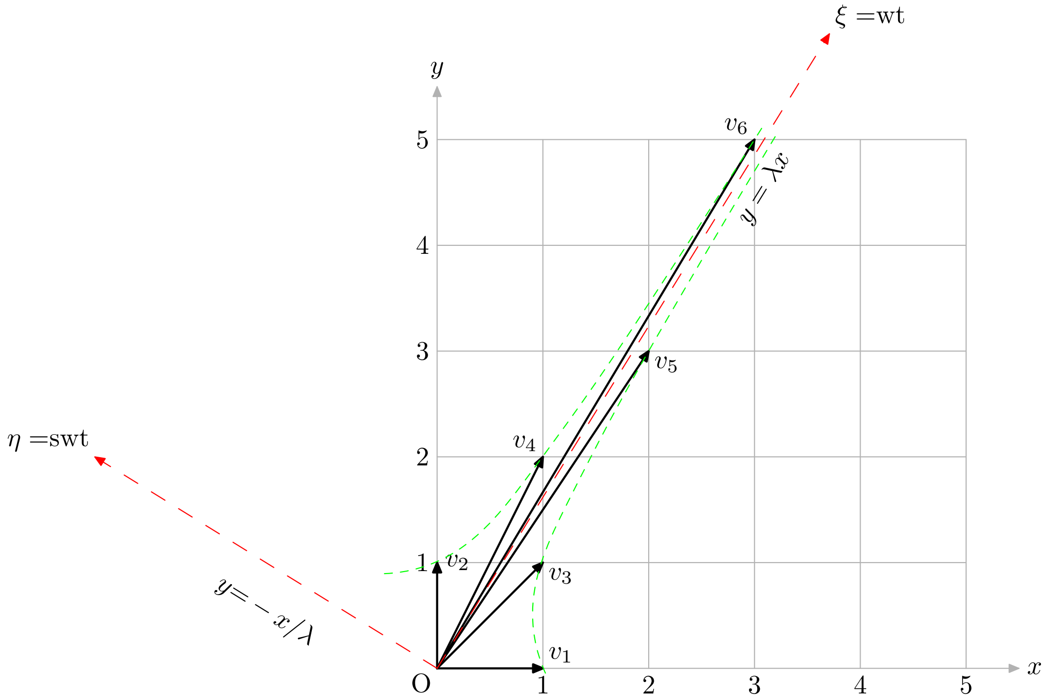
$$\mathbf{L} = \mathbf{L}_+ \oplus \mathbf{L}_-, \quad U = U_+ \oplus U_-,$$

where $\mathbf{L}_+ = \langle v \in \mathbf{L} \mid \mathrm{swt}\, v > 0\rangle_K$, $\mathbf{L}_- = \langle v \in \mathbf{L} \mid \mathrm{swt}\, v < 0\rangle_K$, $U_+ = \langle v \in U \mid \mathrm{swt}\, v > 0\rangle_K$, $U_- = \langle v \in U \mid \mathrm{swt}\, v < 0\rangle_K$.

Recall that $\lambda, \bar\lambda$ are the roots of the equation $x^2 - x - 1 = 0$. The next relations will be used below without special mention:

$$\bar\lambda = \frac{1-\sqrt5}{2} = -\frac{1}{\lambda} = 1 - \lambda;$$
$$\lambda^k + \lambda^{k+1} = \lambda^{k+2}, \ k \ge 0;$$
$$\frac{1}{1-1/\lambda} = \frac{\lambda^2}{\lambda^2-\lambda} = \frac{\lambda^2}{1} = \lambda^2;$$
$$\frac{1}{1-1/\lambda^2} = \frac{\lambda^2}{\lambda^2-1} = \frac{\lambda^2}{\lambda} = \lambda.$$

Let us draw v_is.



Let F_n, $n \ge 0$ be the Fibonacci numbers. We have $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, $n \in \mathbb{Z}$. The following is known as Binet's formula:

$$F_n = \frac{1}{\sqrt5}\left(\left(\frac{1+\sqrt5}{2}\right)^n - \left(\frac{1-\sqrt5}{2}\right)^n\right) = \frac{1}{\sqrt5}\left(\lambda^n - \bar\lambda^n\right), \quad n \in \mathbb{Z}.$$

Lemma 3.3. *Let* $v_n \in \mathbf{L}$ *be as above. Then* $\mathrm{Wt}(v_n) = (F_{n-2}, F_{n-1})$ *for* $n \ge 1$.

Proof. We have $\mathrm{Wt}(v_1)=(1,0)=(F_{-1},F_0)$ and $\mathrm{Wt}(v_2)=(0,1)=(F_0,F_1)$. The statement follows by induction, let $n\ge 0$, then

$$\begin{aligned}\mathrm{Wt}(v_{n+2})&=\mathrm{Wt}([v_n,v_{n+1}])=\mathrm{Wt}(v_n)+\mathrm{Wt}(v_{n+1})\\&=(F_{n-2},F_{n-1})+(F_{n-1},F_n)=(F_n,F_{n+1}).\end{aligned}$$ ∎

The following lemma is a version of Liouville's theorem on approximation of algebraic integers by rational numbers.

Lemma 3.4. *For the points of lattice $(a,b)\in\mathbb{Z}^2\subset\mathbb{R}^2$ we have the inequality*

$$|\,\mathrm{wt}(a,b)\cdot\mathrm{swt}(a,b)|\ge\lambda^2.$$

Proof. Since $t_0=b/a$ is not a root of the polynomial $h(t)=t^2-t-1$ we have $h(b/a)\ne 0$. Moreover,

$$|h(b/a)|=\left|\frac{b^2-ab-a^2}{a^2}\right|\ge\frac{1}{a^2}.$$

We write this inequality in other way

$$\begin{aligned}|h(b/a)|=\left|\frac{b}{a}-\lambda\right|\cdot\left|\frac{b}{a}-\bar\lambda\right|&\ge\frac{1}{a^2};\\|b-a\lambda|\cdot\left|b+\frac{1}{\lambda}a\right|&\ge 1;\\|b-a\lambda|\cdot|b\lambda^2+a\lambda|&\ge\lambda^2;\\|\,\mathrm{swt}(a,b)\cdot\mathrm{wt}(a,b)|&\ge\lambda^2.\end{aligned}$$ ∎

Remark 3.5. This bound is exact. Consider vectors v_n. Recall that $\mathrm{wt}\,v_n=\lambda^n$ and $\mathrm{swt}=\bar\lambda^{n-2}=(-1/\lambda)^{n-2}$. We get $\mathrm{wt}(v_n)\,\mathrm{swt}(v_n)=(-1)^n\lambda^2$ for all $n\in\mathbb{N}$.

It is convenient to embed $\mathbf{L}$ into a bigger Lie subalgebra of $\mathrm{Der}\,R$.

Lemma 3.6. *Let*

$$H=\langle v_1,v_2,v_3,\,t_0^{\alpha_0}t_1^{\alpha_1}\cdots t_{n-4}^{\alpha_{n-4}}\,v_n\mid n\ge 4,\alpha_i\ge 0\rangle_K. \tag{4}$$

1. *H is a Lie subalgebra of $\mathrm{Der}\,R$ and $\mathbf{L}\subset H$;*
2. *denote by $\tilde H$ the span of elements (4) such that $\alpha_{n-4}\le 1$ and $\alpha_{n-5}\le 2$, (which we impose provided that the respective indices are nonnegative). Then $\tilde H$ is also a Lie subalgebra of $\mathrm{Der}\,R$ and $\mathbf{L}\subset\tilde H$.*

Proof. Let us prove that H is a Lie subalgebra. We apply Lemma 1.1 to check that the product of two monomials of type (4) is again expressed via these monomials. Let $n < m$, then

$$[t_0^{\alpha_0}\cdots t_{n-4}^{\alpha_{n-4}} v_n, t_0^{\beta_0}\cdots t_{m-4}^{\beta_{m-4}} v_m] = t_0^{\alpha_0}\cdots t_{n-4}^{\alpha_{n-4}} t_0^{\beta_0}\cdots t_{m-4}^{\beta_{m-4}}\Bigg(\prod_{n-1\le i\le m-3} t_i\Bigg) v_{m+1} \tag{5}$$

$$+ t_0^{\alpha_0}\cdots t_{n-4}^{\alpha_{n-4}}\sum_{\beta_j\neq 0}\Bigg(\prod_{i=0,i\neq j}^{m-4} t_i^{\beta_i}\Bigg)\beta_j t_j^{\beta_j-1} v_n(t_j) v_m, \tag{6}$$

where we used that, by claim 4 of Lemma 1.1, v_m acts on all $t_i^{\alpha_i}$s trivially because $m > n > n-4 \ge i$. Also, $v_n(t_j)$ is nonzero only for $n \le j$, namely

$$v_n(t_j) = \begin{cases} t_{n-1}t_n\cdots t_{j-2}, & n < j; \\ 1, & n = j. \end{cases}$$

In this case, $n \le j \le m-4$ and $n-1 < \cdots < j-2 \le m-6$. We again obtain monomials of type (4). Hence, $H \subset \operatorname{Der} R$ is a Lie subalgebra.

Consider the second claim. We assume that the monomials satisfy the conditions on the last two indices and check that the resulting monomials satisfy these conditions as well. Let $t_0^{\gamma_0}\cdots t_{m-3}^{\gamma_{m-3}} v_{m+1}$ be a resulting monomial of the first type (5). Then $\gamma_{m-3} = 1$ for $n < m-1$ and $\gamma_{m-3} = 0$ for $n = m-1$. We consider the row γ as the sum of two rows $(\beta_0,\dots,\beta_{m-4},0,\dots)$ and $(\alpha_0,\dots,\alpha_{n-4},0,0,1,\dots,1,0,\dots)$, where the 1's are on places $n-1 \le i \le m-3$ and they appear in case $n < m-1$. Thus, we obtain $\gamma_s \le \beta_s + \max\{\alpha_s, 1\}$ for all $s = 0,\dots,m-3$. By inductive assumption, the last possible nonzero α_i is α_{n-4}, since $n < m$, we have $\alpha_{m-4} = 0$. We get $\gamma_{m-4} \le \beta_{m-4} + \max\{0,1\} \le 1+1 = 2$.

Consider monomials of the second type (6) $t_0^{\gamma_0}\cdots t_{m-4}^{\gamma_{m-4}} v_m$. By arguments above, they appear only in the case that $n \le m-4$ and the derivation can increase a power only for t_k such that $k \le m-6$. We conclude that $\gamma_{m-4} \le \beta_{m-4} \le 1$ and $\gamma_{m-5} \le \beta_{m-5} \le 2$. ∎

Corollary 3.7. *Let fv_n, gv_m be monomials* (4) *such that $m-3 \le n < m$, and $f,g \in R$. Then $[fv_n, gv_m] = fg[v_n, v_m]$.*

Proof. Indeed, by computations above, the term (6) appears only in case $n \le m-4$. ∎

Now suppose that $\operatorname{char} K = p > 0$. Let $\operatorname{Lie}(v_1, v_2)$ be the Lie subalgebra generated by brackets only. In this case we introduce the subalgebra H similar to that of Lemma 3.6 as follows

$$\operatorname{Lie}(v_1,v_2) \subset H = \langle v_1, v_2, v_3,\ t_0^{\alpha_0}t_1^{\alpha_1}\cdots t_{n-4}^{\alpha_{n-4}} v_n \mid n \ge 4, \alpha_i \in \{0,1,\dots,p-1\}\rangle_K. \tag{7}$$

4. Fibonacci restricted Lie algebra, char $K = 2$

In this section we consider the case $\operatorname{char} K = 2$. Let A be an associative K-algebra, then

$$(a+b)^2 = a^2 + b^2 + [a,b], \qquad a,b \in A. \tag{8}$$

We get $v_1^2 = (\partial_1 + t_0 v_2)^2 = t_0[\partial_1, v_2] = t_0[\partial_1, \partial_2 + t_1 v_3] = t_0 v_3$. We apply τ and obtain

$$v_i^2 = t_{i-1} v_{i+2}, \qquad i = 1, 2, \dots. \tag{9}$$

Let H_p be the restricted subalgebra generated by H. It is sufficient to add pth powers of the basis of H [10], moreover only the powers (9) are nonzero. These are linearly independent with (7) and we obtain

$$\mathbf{L} \subset H_p = H \oplus \langle t_{n-3} v_n \mid n = 3, 4, \dots \rangle_K. \tag{10}$$

Theorem 4.1. *Let* $\operatorname{char} K = 2$ *and* $\mathbf{L} = \operatorname{Lie}_p(v_1, v_2)$, $\mathbf{A} = \operatorname{Alg}(v_1, v_2)$. *Denote* $\lambda = \frac{1+\sqrt{5}}{2}$. *Then*

1. $\operatorname{GKdim} \mathbf{L} = \underline{\operatorname{GKdim}}\, \mathbf{L} = \ln 2 / \ln \lambda \approx 1.44$;
2. $\operatorname{GKdim} \mathbf{A} = \underline{\operatorname{GKdim}}\, \mathbf{A} = 2 \ln 2 / \ln \lambda \approx 2.88$.

Proof. We use the embedding (10). Fix a number m. Consider a homogeneous element $g \in H_p$ of weight not exceeding m. Then it is a sum of monomials (7) and (9), which we write in the form $w = t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_{n-3}^{\alpha_{n-3}} v_n$, where $\alpha_i \in \{0,1\}$. Let $n \geq 4$, then we have

$$\begin{aligned} m \geq \operatorname{wt}(g) = \operatorname{wt}(w) = \operatorname{wt}(v_n) + \sum_{i=0}^{n-3} \alpha_i \operatorname{wt} t_i = \lambda^n - \sum_{i=0}^{n-3} \alpha_i \lambda^i \geq \lambda^n - \sum_{i=0}^{n-3} \lambda^i \\ > \lambda^n - \frac{\lambda^{n-3}}{1 - 1/\lambda} = \lambda^{n-3}(\lambda^3 - \lambda^2) = \lambda^{n-3}\lambda = \lambda^{n-2}. \end{aligned} \tag{11}$$

We obtain $\lambda^{n-2} < m$. Hence, $n \leq n_0 = 2 + [\ln m / \ln \lambda]$. Remark that $\mathbf{L}$ has only 4 monomials with $n \leq 3$, they are $\{v_1, v_2, v_3, t_0 v_3\}$. We consider the number of monomials w of weight not exceeding m and obtain the bound

$$\begin{aligned} \tilde{\gamma}_{\mathbf{L}}(m) &\leq 4 + \sum_{n=4}^{n_0} 2^{n-2} \leq 4 + \frac{2^{n_0-2}}{1 - 1/2} \leq 4 + 2^{n_0-1} \leq 4 + 2^{1 + \ln m / \ln \lambda} \\ &\leq 4 + 2 m^{\ln 2 / \ln \lambda} \approx C_0 m^{\ln 2 / \ln \lambda}. \end{aligned}$$

This estimate yields the upper bound on the growth of $\mathbf{L}$.

Let us prove the lower bound. We define sets $V_n = \{v_n, t_{n-3} v_n\}$ for all $n \geq 3$. From Lemma 1.1 and (9) it follows that $V_n \subset \mathbf{L}$ for all $n \geq 3$. We also consider another series of sets. Let $W_4 = \{v_4\}$ and

$$W_n = \{t_0^{\alpha_0} \cdots t_{n-5}^{\alpha_{n-5}} v_n \mid \alpha_i \in \{0,1\}\}, \quad n \geq 5. \tag{12}$$

Let us prove by induction on n that $W_n \subset \mathbf{L}$. The base of induction $W_4 \subset \mathbf{L}$ is clear. Fix $n \geq 5$ and assume that $W_{n-1} \subset \mathbf{L}$. By inductive assumption,

$W_{n-1} = \{t_0^{\alpha_0} \cdots t_{n-6}^{\alpha_{n-6}} v_{n-1} \mid \alpha_i \in \{0,1\}\} \subset \mathbf{L}$, also $V_{n-2} = \{v_{n-2}, t_{n-5}v_{n-2}\} \subset \mathbf{L}$. We observe that the pairwise products of these sets yield the whole of the set W_n, thus $W_n \subset \mathbf{L}$.

Fix a number m. Consider numbers n such that $5 \le n \le n_1 = [\ln m/\ln\lambda]$. Then $\mathrm{wt}(W_n) \le \mathrm{wt}(v_n) = \lambda^n \le m$. We count the number of elements (12)

$$\tilde{\gamma}_{\mathbf{L}}(m) \ge \sum_{n=5}^{n_1} 2^{n-4} \ge 2^{n_1-4} \ge 2^{\ln m/\ln\lambda - 5} = \frac{1}{2^5} m^{\ln 2/\ln\lambda}.$$

Hence, $\operatorname{GKdim}\mathbf{L} = \underline{\operatorname{GKdim}}\,\mathbf{L} = \ln 2/\ln\lambda$.

Let us evaluate the growth of $\mathbf{A}$. By our arguments and PBW-theorem, $\mathbf{A}$ is contained in the span of all products of the elements $t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_{n-3}^{\alpha_{n-3}} v_n$, where $\alpha_i \in \{0,1\}$. We move all t_i's to the left (see Claim 4 of Lemma 1.1), and reorder v_i's using the commutation relation and (9). We observe that the appearing products keep the following property at each step, namely, that for each t_a there exists v_b such that $b \ge a+3$. We obtain

$$\mathbf{A} \subset \langle t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_{n-3}^{\alpha_{n-3}} v_1^{\beta_1} v_2^{\beta_2} \cdots v_n^{\beta_n} \mid \alpha_i, \beta_j \in \{0,1\}, \beta_n = 1, n \in \mathbb{N}\rangle_K, \tag{13}$$

where n is maximal such that $\beta_n > 0$. Consider such a monomial of weight not exceeding m. Assume that $n \ge 4$, then

$$m \ge -\sum_{i=0}^{n-3} \alpha_i \lambda^i + \sum_{j=1}^{n} \beta_j \lambda^j \ge \lambda^n - \sum_{i=0}^{n-3} \lambda^i \ge \lambda^n - \frac{\lambda^{n-3}}{(1-1/\lambda)} = \lambda^{n-2},$$

see (11). Similarly, $n \le n_0 = 2 + [\ln m/\ln\lambda]$. Let N be the number of monomials (13) with $n \le 3$. We use the following bound on the number of monomials (13) of weight not exceeding m

$$\begin{aligned}\tilde{\gamma}_{\mathbf{A}}(m) &\le N + \sum_{n=4}^{n_0} 2^{2n-3} \le N + \frac{2^{2n_0-3}}{1-1/4} \le N + \frac{2^{1+2\ln m/\ln\lambda}}{1-1/4} \\ &\le N + \frac{8}{3} m^{2\ln 2/\ln\lambda} \approx C_0 m^{2\ln 2/\ln\lambda}.\end{aligned}$$

This estimate yields the upper bound on the growth of $\mathbf{A}$.

Since all elements (12) are contained in $\mathbf{L}$, we get the monomials

$$\mathbf{A} \supset \{t_0^{\alpha_0} \cdots t_{n-5}^{\alpha_{n-5}} v_n v_{n-1}^{\beta_{n-1}} \cdots v_1^{\beta_1} \mid \alpha_i, \beta_i \in \{0,1\},\ \beta_n = 1\}, \quad n \ge 5. \tag{14}$$

Indeed, the initial segment $t_0^{\alpha_0} \cdots t_{n-5}^{\alpha_{n-5}} v_n$ belongs to $\mathbf{L}$ and the subsequent multiplication by v_is (in the reverse order!) yields an element in $\mathbf{A}$. Denote the monomial above by $\mathbf{t}^\alpha \mathbf{v}^\beta$, where $\alpha, \beta \in \mathbb{N}_p^I$. Introduce the degree $\deg(\mathbf{t}^\alpha \mathbf{v}^\beta) = |\alpha| - |\beta|$, where $|\alpha| = \sum_i \alpha_i$. Let us prove that all monomials (14) are linearly independent. Let $\gamma \in \mathbb{N}_p^I$, and $\mathbf{t}^\gamma = \prod_{i\in I} t_i^{\gamma_i} \in R$, then set $\deg(\mathbf{t}^\gamma) = |\gamma|$. Recall that $v_i = \partial_i + t_{i-1}\partial_{i+1} + t_{i-1}t_i\partial_i + \dots$, let us call ∂_i by the *leading derivation.* Consider the action $(v_n^{\beta_n} \cdots v_1^{\beta_1})(\mathbf{t}^\gamma)$. Observe that only the leading derivations can decrease degree of the polynomial, while the other terms yield summands of higher degree.

As a result, we get 1) a monomial of degree $|\gamma|-|\beta|$ (which can be zero), and 2) polynomials of higher degree. Since we get the monomial of the lowest degree by applying only the leading derivations, we conclude that this monomial does not depend on the order of v_is in the action. Suppose that there exists a linear relation of monomials (14). We write it in the form

$$\sum_{|\alpha|-|\beta|>N} \lambda_{\alpha,\beta}\mathbf{t}^{\alpha}\mathbf{v}^{\beta} + \sum_{|\alpha|-|\beta|=N,\ |\beta|>b} \lambda_{\alpha,\beta}\mathbf{t}^{\alpha}\mathbf{v}^{\beta} + \sum_{|\alpha|-|\beta|=N,\, |\beta|=b} \lambda_{\alpha,\beta}\mathbf{t}^{\alpha}\mathbf{v}^{\beta} = 0, \qquad (15)$$

where $\lambda_{\alpha,\beta}\in K$, N is the minimal degree $\deg(\mathbf{t}^{\alpha}\mathbf{v}^{\beta})=|\alpha|-|\beta|$ of all terms (N may be negative), and b the minimal value of $|\beta|$ of the terms of degree N; let $a=N+b$. We consider the action of (15) onto $\mathbf{t}^{\gamma}\in R$ such that $|\gamma|=b$. The lowest possible degree of the result is a. The first sum yields polynomials of higher degree. The second sum cannot yield polynomials of degree a as well, because to get such a degree we must use the leading derivations for all v_is, but the number $|\beta|$ of these derivations is bigger than $b=\deg \mathbf{t}^{\gamma}$. Hence, the degree a can be achieved only by using the leading derivations of the third sum. We obtain

$$\sum_{|\alpha|=a,|\beta|=b} \lambda_{\alpha,\beta}(t_{i_1}\cdots t_{i_a}\partial_{j_1}\cdots\partial_{j_b})(t_{k_1}\cdots t_{k_b})=0.$$

Let $\lambda_{\alpha^0,\beta^0}\neq 0$ for some $\alpha^0,\beta^0\in\mathbb{N}_p^I$ in this sum. Then we take $\{k_1,\dots,k_b\}$ to be the set of nonzero indices of β^0. Then $\sum_\alpha \lambda_{\alpha,\beta^0}\mathbf{t}^{\alpha}=0$, a contradiction. Hence, monomials (14) are linearly independent.

We evaluate weight of a monomial (14) as follows

$$\mathrm{wt}(\mathbf{t}^{\alpha}\mathbf{v}^{\beta}) = -\sum_{i=0}^{n-5}\alpha_i\lambda^i + \sum_{j=1}^{n}\beta_j\lambda^j \le \sum_{j=1}^{n}\lambda^j < \frac{\lambda^n}{1-1/\lambda} = \lambda^{n+2}.$$

Fix a number m, assume that $\lambda^{n+2}<m$; then all monomials (14) have weights less than m. This is the case for all numbers $n\le n_2=[\ln m/\ln\lambda]-2$. Finally, the number of monomials (14) yields the lower bound for the growth of $\mathbf{A}$

$$\tilde{\gamma}_{\mathbf{A}}(m) \ge \sum_{n=5}^{n_2} 2^{2n-5} \ge 2^{2n_2-5} \ge \frac{1}{2^5}2^{2\ln m/\ln\lambda-11} = \frac{1}{2^{16}}m^{2\ln 2/\ln\lambda}. \qquad \blacksquare$$

Corollary 4.2. *There exist positive constants C, C_1, C_2, m_0 such that*

$$C_1 m^{2\ln 2/\ln\lambda} \le \tilde{\gamma}_{\mathbf{A}}(m) - \tilde{\gamma}_{\mathbf{A}}(m/C) \le C_2 m^{2\ln 2/\ln\lambda}, \quad m\ge m_0.$$

Proof. It remains to prove existence of the lower bound. Let $n_2=[\ln m/\ln\lambda]-2$ be as above. We evaluate weight of a monomial (14)

$$\mathrm{wt}(\mathbf{t}^{\alpha}\mathbf{v}^{\beta}) \ge \lambda^n - \sum_{i=0}^{n-5}\lambda^i \ge \lambda^n - \frac{\lambda^{n-5}}{1-1/\lambda} = \lambda^{n-3}(\lambda^3-1) \ge \lambda^{n-2}.$$

Assume that $\lambda^{n-2}\ge m/C$, it is sufficient to take $n\ge n_3=3+[\ln(m/C)/\ln\lambda]$. Let C be sufficiently large that $n_3<n_2$, and consider monomials (14) for $n_3\le n\le n_2$.

Such monomials have weights between m/C and m, we evaluate their number

$$\sum_{n=n_3}^{n_2} 2^{2n-5} \geq \frac{1}{4-1}(2^{2n_2-3} - 2^{2n_3-5}) \geq \frac{1}{3}(2^{2\ln m/\ln\lambda - 11} - 2^{2\ln(m/C)/\ln\lambda+1})$$
$$\geq \frac{1}{3}\left(2^{-11} m^{2\ln 2/\ln\lambda} - 2\left(\frac{m}{C}\right)^{2\ln 2/\ln\lambda}\right) \geq C_1 m^{2\ln 2/\ln\lambda},$$

provided that C is sufficiently large. ∎

Lemma 4.3. *Let* $\operatorname{char} K = 2$ *and* $\mathbf{L} = \operatorname{Lie}_p(v_1, v_2)$. *Then* $\mathbf{L}$ *has a nil-*p*-mapping.*

Proof. We prove that the bigger subalgebra $H_p \supset \mathbf{L}$ has a nil-p-mapping. Consider $v \in H_p$. Let s be the maximal number such that v_s appears in the decomposition of v. From (10) and (7) we have

$$v = \sum_{i=1}^{s-1} g_i(t_0, \dots, t_{i-3}) v_i + h(t_0, \dots, t_{s-3}) v_s, \tag{16}$$

where $g_i = g_i(t_0, \dots, t_{i-3})$, $i = 1, \dots, s-1$, and $h = h(t_0, \dots, t_{s-3})$ are polynomials from R. We assume that $h \in R^+$. (Otherwise we take the number $s+1$ and consider the decomposition $v = \cdots + h v_{s+1}$, where $h = 0$). We apply the p-mapping to (16) and use (8). Consider v_is with the highest value of i that might appear. The commutators yield (similar to (5) and (6))

$$[g_i v_i, h v_s] = h g_i t_{i-1} \cdots t_{s-3} v_{s+1} + g_i\, v_i(h)\, v_s, \qquad 1 \leq i \leq s-1;$$
$$[g_i v_i, g_j v_j] = \sum_{k=1}^{s} f_k v_k, \quad f_k \in R, \qquad 1 \leq i < j \leq s-1.$$

Since $h \in R^+$ the term with v_{s+1} in the first case goes to $\tilde{h}$ of a presentation similar to (16). Consider the squares. We have $(h v_s)^2 = h^2 v_s^2 = 0$ and the squares arising from the sum yield at most $(g_{s-1} v_{s-1})^2 = g_{s-1}^2 t_{s-2} v_{s+1}$, this term also belongs to $\tilde{h}$. Thus, we obtain the same presentation as (16):

$$v^2 = \sum_{i=1}^{s} \tilde{g}_i(t_0, \dots, t_{i-3}) v_i + \tilde{h}(t_0, \dots, t_{s-2}) v_{s+1}.$$

We iterate the process

$$v^{2^m} = \sum_{i=1}^{s+m-1} \tilde{\tilde{g}}_i(t_0, \dots, t_{i-3}) v_i + \tilde{\tilde{h}}(t_0, \dots, t_{s+m-3}) v_{s+m}. \tag{17}$$

The weight of any homogeneous monomial of v is at least λ. Hence, weights of monomials of v^{2^m} are at least $\lambda 2^m$. Since polynomials only reduce the weight, weights of monomials in (17) are at most $\operatorname{wt}(v_{s+m}) = \lambda^{s+m}$. If $\lambda 2^m > \lambda^{s+m}$, then $v^{2^m} = 0$. Therefore, it is sufficient to take $m > (s-1)\frac{\ln\lambda}{\ln(2/\lambda)}$. (where $\frac{\ln\lambda}{\ln(2/\lambda)} \approx 2.27$.). ∎

Remark that the nil-index of the p-mapping is unbounded. It is sufficient to consider the powers $(v_1 + v_2 + \cdots + v_s)^{2^m}$.

By virtue of Lemma 3.1, we can consider the Hilbert series in two variables

$$\mathcal{H}(\mathbf{L}, x, y) = \sum_{a,b \geq 0} \dim \mathbf{L}_{a,b}\, x^a y^b.$$

Lemma 4.4. *Let* $\operatorname{char} K = 2$ *and* $\mathbf{L} = \operatorname{Lie}_p(v_1, v_2)$. *Then the Hilbert series satisfies the functional relation*

$$\mathcal{H}(\mathbf{L}, x, y) = \mathcal{H}(\mathbf{L}, y, xy)\left(1 + \frac{x}{y}\right) - xy.$$

Proof. Denote $h_{ab} = \dim \mathbf{L}_{a,b}$ for all $a, b \geq 0$. Recall that $\mathbf{L}$ is generated by $X = \{v_1, v_2\}$. Let $B \subset \mathbf{L}$ be the restricted ideal generated by v_2 and v_1^2. Then $\mathbf{L} = \langle v_1 \rangle_K \oplus B$ and it is well-known [1] that the algebra B is generated by the set

$$Y = \{v_2, [v_1, v_2], v_1^2\} = \{v_2, v_3, t_0 v_3\}.$$

Consider homogeneous elements in Y. Remark that we can use $t_0 v_3$ at most once. We have $\operatorname{wt}(v_1) = \lambda$, $\operatorname{wt}(v_2) = \lambda^2 = 1 + \lambda$, $\operatorname{wt}(v_3) = 1 + 2\lambda$, and $\operatorname{wt}(t_0 v_3) = 2\lambda$. Fix numbers $a, b \geq 0$. Let $P_{a,b,0}$ denote the space of homogeneous elements in $Y \setminus \{t_0 v_3\}$ of degrees a, b with respect to v_2 and v_3, respectively. Then $\operatorname{wt} P_{a,b,0} = a \operatorname{wt}(v_2) + b \operatorname{wt}(v_3) = a(1+\lambda) + b(1+2\lambda) = b\lambda + (a+b)(1+\lambda) = b \operatorname{wt}(v_1) + (a+b) \operatorname{wt}(v_2)$. Hence,

$$P_{a,b,0} \subset \mathbf{L}_{b,a+b}. \tag{18}$$

Similarly, let $P_{a,b,1}$ denote the space of homogeneous elements in $Y = \{v_2, v_3, t_0 v_3\}$ of degrees a, b, and 1, respectively. We get $\operatorname{wt} P_{a,b,1} = a(1+\lambda) + b(1+2\lambda) + 2\lambda = (b+2)\lambda + (a+b)(1+\lambda) = (b+2)\operatorname{wt}(v_1) + (a+b)\operatorname{wt}(v_2)$. Thus,

$$P_{a,b,1} \subset \mathbf{L}_{b+2,a+b}. \tag{19}$$

Recall that we have the embedding $\tau : \mathbf{L} \hookrightarrow \mathbf{L}$ given by $\tau(v_i) = v_{i+1}$ for $i \geq 1$. We have

$$P_{a,b,0} = \tau(\mathbf{L}_{a,b}), \qquad P_{a,b,1} = t_0 \tau(\mathbf{L}_{a,b+1}). \tag{20}$$

Consider monomials in X that depend on v_1 only, these are $\{v_1, v_1^2\}$ that yield $\mathcal{H}(\mathbf{L}, x, 0) = \sum_{a \geq 0,\, b=0} h_{a,0} x^a = x + x^2$. From (18), (19), and (20) we get

$$\begin{aligned}
\mathcal{H}(\mathbf{L}, x, y) &= x + \mathcal{H}(B, x, y) = x + \sum_{a,b \geq 0} \dim P_{a,b,0}\, x^b y^{a+b} + \sum_{a,b \geq 0} \dim P_{a,b,1}\, x^{b+2} y^{a+b} \\
&= x + \sum_{a,b \geq 0} h_{a,b}\, y^a (xy)^b + \frac{x}{y} \sum_{a,b \geq 0} h_{a,b+1}\, y^a (xy)^{b+1} \\
&= x + \sum_{a,b \geq 0} h_{a,b}\, y^a (xy)^b + \frac{x}{y} \sum_{a \geq 0, b \geq 1} h_{a,b}\, y^a (xy)^b \\
&= x + \mathcal{H}(\mathbf{L}, y, xy) + \frac{x}{y}\left((\mathcal{H}(\mathbf{L}, x, y) - x - x^2)\Big|_{\substack{x=y,\\ y=xy}} \right) \\
&= \mathcal{H}(\mathbf{L}, y, xy)\left(1 + \frac{x}{y}\right) - xy.
\end{aligned}$$

■

Corollary 4.5. *Set* $\mathcal{H}_1(x,y)=x+y$ *and define recursively*

$$\mathcal{H}_{i+1}(x,y)=\mathcal{H}_i(y,xy)(1+x/y)-xy,\qquad i\geq 2.$$

Then the sequence $\mathcal{H}_i(x,y)$, $i=1,2,\dots$ *converges to* $\mathcal{H}(\mathbf{L},x,y)$ *componentwise.*

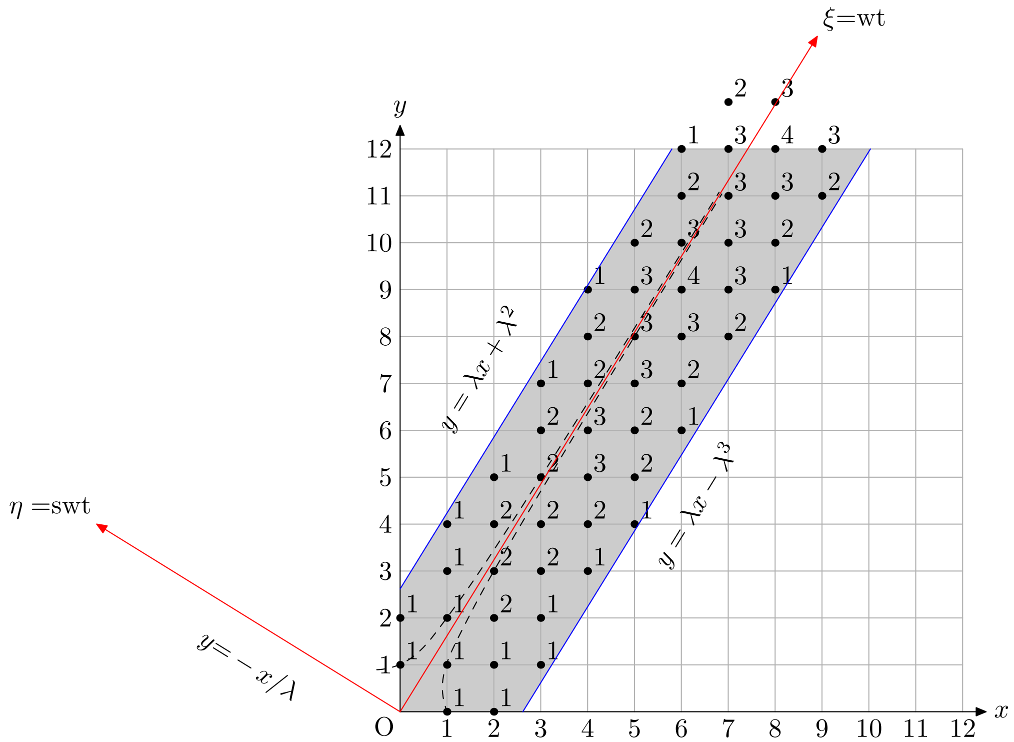
Proof. Let $\mathcal{H}(\mathbf{L},x,y)=\sum_{a+b\geq 1}h_{a,b}x^ay^b$ and $\mathcal{H}_i(x,y)=\sum_{a+b\geq 1}h^{(i)}_{a,b}x^ay^b$. Let us prove by indiction on $j=1,2,\dots$ that $\mathcal{H}(\mathbf{L},x,y)$ and $\mathcal{H}_{2j-1}(x,y)$ have the same coefficients $h_{n,m}=h^{(2j-1)}_{n,m}$ for $n+m\leq j$. Let $j=1$ then by Lemma 3.1 we have $\mathcal{H}(\mathbf{L},x,y)=x+y+\dots$ and the base of induction is true.

Let $j>1$. By Lemma and the recursive relation we have

$$\begin{aligned} h_{n,m}&=h_{m-n,n}+h_{m-n+2,n-1}-\delta_{n,1}\delta_{m,1},\\ h^{(i+1)}_{n,m}&=h^{(i)}_{m-n,n}+h^{(i)}_{m-n+2,n-1}-\delta_{n,1}\delta_{m,1},\quad i\geq 1, \end{aligned}$$

for all $n,m\geq 0$. Here $h_{a,b}$, $h^{(i)}_{a,b}$ are zero if either of indices a,b is negative. We compare sums of indices in the relations above. We have $n+m>(m-n)+n$ if $n>0$ and $n+m>(m-n+2)+(n-1)$ if $n>1$. In case $n=0$ or $n=1$ we apply this relations again and see that sums of indices definitely decrease. Fix numbers n,m such that $n+m\leq j$ and consider coefficients $h_{n,m}$ and $h^{(2j-1)}_{n,m}$. We apply two iterations and conclude that they are expressed in the same way via $h_{a,b}$ and $h^{(2j-3)}_{a,b}$, respectively, the latter coincide by inductive assumption. Hence $h_{n,m}=h^{(2j-1)}_{n,m}$. The induction step is proved. ∎

We apply the corollary and obtain dimensions of homogeneous components of $\mathbf{L}$, a more detailed explanation will be given in the next section.



Also, the first terms of the Hilbert series are as follows

$$\begin{aligned}\mathcal{H}(\mathbf{L},x,y) &= x+y+x^2+xy+y^2+x^2y+xy^2+x^3y+2x^2y^2+xy^3\\ &+x^3y^2+2x^2y^3+xy^4+2x^3y^3+2x^2y^4+x^4y^3+2x^3y^4+x^2y^5\\ &+2x^4y^4+2x^3y^5+x^5y^4+3x^4y^5+2x^3y^6+2x^5y^5+3x^4y^6+x^3y^7\\ &+2x^5y^6+2x^4y^7+x^6y^6+3x^5y^7+2x^4y^8+2x^6y^7+3x^5y^8+x^4y^9\\ &+3x^6y^8+3x^5y^9+2x^7y^8+4x^6y^9+2x^5y^{10}+3x^7y^9+3x^6y^{10}\\ &+x^8y^9+3x^7y^{10}+2x^6y^{11}+2x^8y^{10}+3x^7y^{11}+x^6y^{12}\\ &+3x^8y^{11}+3x^7y^{12}+2x^9y^{11}+4x^8y^{12}+2x^7y^{13}+3x^9y^{12}+3x^8y^{13}+\dots\end{aligned}$$

5. Weight structure in characteristic 2

The goal of this section is to show that weights of $\mathbf{L}$, $\mathbf{A}$ and the restricted enveloping algebra $\mathbf{u}=u_0(\mathbf{L})$ are situated in specific regions on plane. This observation implies that all these three algebras are direct sums of two locally nilpotent subalgebras.

Theorem 5.1. *Let* $\operatorname{char} K=2$ *and* $\mathbf{L}=\operatorname{Lie}_p(v_1,v_2)$, $\mathbf{A}=\operatorname{Alg}(v_1,v_2)$, *and* $\mathbf{u}=u_0(\mathbf{L})$. *Then we have the following regions for weights.*

1. *weights of* $\mathbf{L}$ *lie in the strip* $\lambda x-\lambda^3<y<\lambda x+\lambda^2$;
2. *weights of* $\mathbf{A}$ *lie in the strip* $\lambda x-\lambda^4<y<\lambda x+\lambda^3$;
3. *set* $\theta=\ln 2/\ln\lambda$, $\kappa=\theta/(1+\theta)\approx 0.59$. *There exists* $C>0$ *such that weights of* $\mathbf{u}$ *lie in the region* $|\eta|<C\xi^{\kappa}$;
4. *weight of all three algebras satisfy* $|\eta|\ge\lambda^2/\xi$.

Proof. Let l be the line on plane given by $y=\lambda x$. Recall that we have the linear function $\operatorname{swt}(x,y)=y-\lambda x$ on vectors $(x,y)\in\mathbb{R}^2$. Geometrically, $\operatorname{swt}(x,y)$ is equal to (oriented) length of the vertical segment joining (x,y) with l. By Lemma 3.1 we have $\operatorname{swt}(t_n)=-\operatorname{swt}(v_n)=-\bar\lambda^{n-2}$ for all $n\ge 0$.

By the previous section, all elements of $\mathbf{L}$ are expressed via monomials of type $w=t_0^{\alpha_0}t_1^{\alpha_1}\cdots t_{n-3}^{\alpha_{n-3}}v_n$, where $\alpha_i\in\{0,1\}$. Then

$$\operatorname{swt}(w)=\operatorname{swt}(v_n)+\sum_{i=0}^{n-3}\alpha_i\operatorname{swt}(t_i)=\bar\lambda^{n-2}-\sum_{i=0}^{n-3}\alpha_i\bar\lambda^{i-2}.$$

Since $\bar\lambda<0$, we get bounds as follows

$$\operatorname{swt}(w)\le|\bar\lambda|^{n-2}+\sum_{\substack{i\text{ odd}\\1\le i\le n-3}}|\bar\lambda|^{i-2}<\sum_{j=0}^{\infty}\lambda^{1-2j}=\frac{\lambda}{1-1/\lambda^2}=\frac{\lambda^3}{\lambda^2-1}=\lambda^2;$$

$$\operatorname{swt}(w)\ge-|\bar\lambda|^{n-2}-\sum_{\substack{i\text{ even}\\0\le i\le n-3}}|\bar\lambda|^{i-2}>-\sum_{j=0}^{\infty}\lambda^{2-2j}=-\frac{\lambda^2}{1-1/\lambda^2}=-\lambda^3.$$

We obtain $-\lambda^3 < \operatorname{swt}(w) < \lambda^2$, which is equivalent to the first claim.

Recall that by proof of Theorem 4.1, $\mathbf{A}$ is contained in span of monomials (13) of type $w = t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_{n-3}^{\alpha_{n-3}} v_1^{\beta_1} v_2^{\beta_2} \cdots v_n^{\beta_n}$, where $\alpha_i, \beta_j \in \{0,1\}$, and $\beta_n = 1$. We proceed as above

$$\operatorname{swt}(w) = \sum_{j=1}^{n} \beta_j \operatorname{swt}(v_j) + \sum_{i=0}^{n-3} \alpha_i \operatorname{swt}(t_i) = \sum_{j=1}^{n} \beta_j \bar\lambda^{j-2} - \sum_{i=0}^{n-3} \alpha_i \bar\lambda^{i-2};$$
$$\operatorname{swt}(w) \le \sum_{k=1}^{n} |\bar\lambda|^{k-2} < \frac{\lambda}{1-1/\lambda} = \lambda^3;$$
$$\operatorname{swt}(w) \ge -\sum_{k=0}^{n} |\bar\lambda|^{k-2} > -\frac{\lambda^2}{1-1/\lambda} = -\lambda^4.$$

We obtain that weights of $\mathbf{A}$ lie in the claimed strip $-\lambda^4 < \operatorname{swt}(w) < \lambda^3$.

Let us study weights of $\mathbf{u} = u_0(\mathbf{L})$. Let $\mathbf{L} = \langle w_1, w_2, w_3, \dots \rangle_K$ be a basis the order of which obeys to the weight function. Consider a basis monomial $u = w_{i_1} w_{i_2} \cdots w_{i_k} \in \mathbf{u}$, where $i_1 < i_2 < \cdots < i_k$, and k is a fixed large number. Let us find relation between new coordinates (ξ, η), where $\xi = \operatorname{wt} u$, $\eta = \operatorname{swt} u$. From the first claim we have the estimate

$$|\eta| = \left| \sum_{j=1}^{k} \operatorname{swt} w_{i_j} \right| < k\lambda^3. \tag{21}$$

Now, let us evaluate the weight of $\mathbf{u}$. Denote $\theta = \ln 2 / \ln \lambda$. By the proof of Theorem 4.1 we have the upper bound $\tilde\gamma_{\mathbf{L}}(m) < C_0 m^\theta$, $m \in \mathbb{N}$, where C_0 is some constant. Set

$$m = m(k) = [(k/C_0)^{1/\theta}]. \tag{22}$$

By this setting, $\tilde\gamma_{\mathbf{L}}(m) < C_0 m^\theta < k$. We observe that the total weight of k distinct vectors w_{i_j} is bigger than weight of k first vectors, which, by our construction, contain all basis vectors of weight at most m, by proof of Theorem 4.1, the latter contain the set $W_n = \{t_0^{\alpha_0} \cdots t_{n-5}^{\alpha_{n-5}} v_n \mid \alpha_i \in \{0,1\}\}$ (see (12)):

$$\xi = \operatorname{wt} u = \sum_{j=1}^{k} \operatorname{wt} w_{i_j} \ge \sum_{j=1}^{k} \operatorname{wt} w_j \ge \sum_{\operatorname{wt} w_j \le m} \operatorname{wt} w_j \ge \sum_{w \in W_n} \operatorname{wt} w. \tag{23}$$

Recall that $\operatorname{wt} W_n \le \lambda^n$ and to get $\operatorname{wt} W_n < m$ we just set $n = n(m) = [\ln m / \ln \lambda]$. Let $v = t_0^{\alpha_0} \cdots t_{n-5}^{\alpha_{n-5}} v_n \in W_n$, then

$$\operatorname{wt} v \ge \lambda^n - \sum_{i=0}^{n-5} \lambda^i \ge \lambda^n - \frac{\lambda^{n-5}}{1-1/\lambda} = \lambda^n - \lambda^{n-3} > \lambda^n - \lambda^{n-2} = \lambda^{n-1}.$$

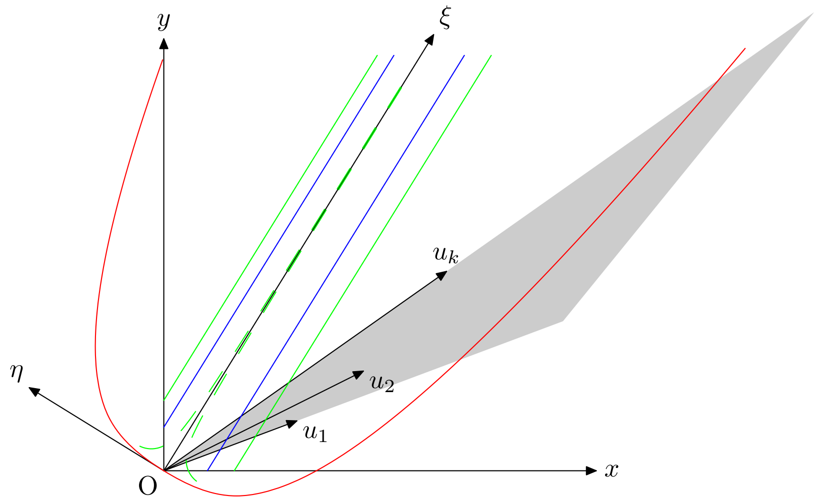
We continue the estimate (23)

$$\xi \ge \sum_{w \in W_n} \operatorname{wt} w \ge \lambda^{n-1} |W_n| = \lambda^{n-1} 2^{n-4} = \frac{1}{8} (2\lambda)^{n-1} \ge \frac{1}{8} (2\lambda)^{\ln m / \ln \lambda - 2}$$
$$= \frac{1}{32\lambda^2} m^{\ln(2\lambda)/\ln\lambda} = \frac{1}{32\lambda^2} m^{1+\theta} \ge \frac{1}{100} m^{1+\theta}. \tag{24}$$

Finally, (21), (22), and (24) yield the desired region

$$|\eta| < k\lambda^3 \le C_0\lambda^3(m+1)^\theta \le C_0\lambda^3\big((100\xi)^{1/(1+\theta)}+1\big)^\theta \le C\xi^{\theta/(1+\theta)}.$$

The last claim follows from Lemma 3.4. ■



Corollary 5.2. *Let* char $K=2$ *and* $\mathbf{L}$, $\mathbf{A}$, *and* $\mathbf{u}=u_0(\mathbf{L})$ *be as above. Then these three algebras are direct sums of two locally nilpotent subalgebras*

$$\mathbf{L}=\mathbf{L}_+\oplus\mathbf{L}_-,\qquad \mathbf{A}=\mathbf{A}_+\oplus\mathbf{A}_-,\qquad \mathbf{u}=\mathbf{u}_+\oplus\mathbf{u}_-,$$

where the decomposition is given by the function of superweight.

Proof. Consider, for example $\mathbf{u}$. Let $\mathbf{u}_+$, $\mathbf{u}_-$ consist of homogeneous monomials with positive, respectively negative superweights, i.e. those that lie above or below the line $y=\lambda x$. Let $u_1,\dots,u_k\in\mathbf{u}_-$ be homogeneous monomials and $A=\mathrm{Alg}(u_1,\dots,u_k)$ be the subalgebra generated by these elements. Let $N\in\mathbb{N}$ and consider $u=\sum_{j;\ n\ge N}\alpha_j u_{j_1}\cdots u_{j_n}$, $\alpha_j\in K$. Then it is geometrically clear that the sums of at least N respective vectors will go out of the shaded region $|\eta|<C\xi^\kappa$ provided that N is sufficiently large (see the picture above). Hence, $A^N=0$. ■

We can give some bounds on degrees of nilpotence of subalgebras of $\mathbf{A}$.

Corollary 5.3. *Let* char $K=2$, $\mathbf{A}$ *be as above. Consider a subalgebra* $A=\mathrm{Alg}(u_1,\dots,u_k)\subset\mathbf{A}$, *where* $u_1,\dots,u_k\in\mathbf{A}_+$ *(or* $u_1,\dots,u_k\in\mathbf{A}_-$*). Let* $C=\max\{\mathrm{wt}\,u_i\mid 1\le i\le k\}$. *Then* $A^N=0$, *where* $N=[C\lambda]+1$ *(or, respectively,* $N=[C\lambda^2]+1$*).*

Proof. Consider the case $u_1,\dots,u_k\in\mathbf{A}_+$. We apply Lemma 3.4

$$|\,\mathrm{swt}(u_i)|\ge\frac{\lambda^2}{\mathrm{wt}(u_i)}\ge\frac{\lambda^2}{C},\qquad 1\le i\le k.$$

Consider a homogeneous element $w\in A^N$, then $\mathrm{swt}\,w\ge N\lambda^2/C=([C\lambda]+1)\lambda^2/C\ge\lambda^3$. By the second statement of the Theorem, $0<\mathrm{swt}\,w<\lambda^3$. Hence $w=0$. We proved that $A^N=0$. ■

6. Fibonacci restricted Lie algebra, char $K = 3$

In this Section, we show that the Fibonacci restricted Lie algebra has the same properties in case $\operatorname{char} K = 3$. Let A be an associative algebra over a field K with $\operatorname{char} K = 3$, then one has the identity

$$(a+b)^3 = a^3 + b^3 + [a,[a,b]] + [b,[b,a]], \qquad a,b \in A. \tag{25}$$

We get $v_1^3 = (\partial_1 + t_0 v_2)^3 = t_0[\partial_1,[\partial_1,v_2]] + t_0^2[v_2,[v_2,\partial_1]] = t_0[\partial_1,v_3] - t_0^2[v_2,v_3] = -t_0^2 v_4$. We apply τ and obtain

$$v_i^3 = -t_{i-1}^2 v_{i+3}, \quad i = 1,2,\dots. \tag{26}$$

Let $\operatorname{Lie}(v_1,v_2) \subset \operatorname{Der} R$ be the Lie subalgebra generated by v_1 and v_2. By Lemma 3.6 we get the Lie subalgebra H that contains $\operatorname{Lie}(v_1,v_2)$. Moreover,

$$\mathbf{L} = \operatorname{Lie}(v_1,v_2)_p \subset H = \langle v_1, v_2, v_3,\, t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_{n-4}^{\alpha_{n-4}}\, v_n \mid n \ge 4, \alpha_i \in \{0,1,2\} \rangle_K. \tag{27}$$

Indeed, from (26) it follows that $H \subset \operatorname{Der} R$ is a restricted subalgebra. In notations of Lemma 3.6, we also have the restricted subalgebra $\tilde{H}_p = \tilde{H} \oplus \langle t_{i-4}^2 v_i \mid i \ge 4 \rangle_K$, but we are not using it. We shall use only embedding (27).

Theorem 6.1. *Let* $\operatorname{char} K = 3$ *and* $\mathbf{L} = \operatorname{Lie}_p(v_1,v_2)$, $\mathbf{A} = \operatorname{Alg}(v_1,v_2)$. *Denote* $\lambda = \frac{1+\sqrt{5}}{2}$. *Then*

1. $\operatorname{GKdim} \mathbf{L} = \underline{\operatorname{GKdim}}\, \mathbf{L} = \ln 3/\ln\lambda \approx 2.28$;
2. $\operatorname{GKdim} \mathbf{A} = \underline{\operatorname{GKdim}}\, \mathbf{A} = 2\ln 3/\ln\lambda \approx 4.56$.

Proof. Consider a monomial $w = t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_{n-4}^{\alpha_{n-4}}\, v_n \in H$ of weight not exceeding m. Assume that $n \ge 4$, then

$$\begin{aligned} m \ge \operatorname{wt}(w) &= \operatorname{wt}(v_n) + \sum_{i=0}^{n-4} \alpha_i \operatorname{wt} t_i = \lambda^n - \sum_{i=0}^{n-4} \alpha_i \lambda^i \ge \lambda^n - 2\sum_{i=0}^{n-4} \lambda^i \\ &> \lambda^n - \frac{2\lambda^{n-4}}{1-1/\lambda} = \lambda^{n-4}(\lambda^4 - 2\lambda^2) = \lambda^{n-4}((\lambda^3+\lambda^2) - 2\lambda^2) = \lambda^{n-3}. \end{aligned}$$

We obtain $\lambda^{n-3} < m$. Hence, $n \le n_0 = 3 + [\ln m/\ln\lambda]$. The number of such monomials w of weight not exceeding m yields the bound

$$\tilde{\gamma}_{\mathbf{L}}(m) \le 3 + \sum_{n=4}^{n_0} 3^{n-3} \le 3 + \frac{3^{n_0-3}}{1-1/3} \le 3 + \frac{3}{2} 3^{\ln m/\ln\lambda} \le 3 + \frac{3}{2} m^{\ln 3/\ln\lambda}.$$

The upper bound on the growth of $\mathbf{L}$ is proved.

Let us prove the lower bound. Consider the sets $V_n = \{v_n, t_{n-4}v_n, t_{n-4}^2 v_n\}$ for $n \ge 4$. The relations $v_{n-3}^3 = -t_{n-4}^2 v_n$ and $[v_{n-3}, v_{n-1}] = t_{n-4} v_n$ prove that $V_n \subset \mathbf{L}$ for all $n \ge 4$. Set also $W_5 = \{v_5\}$ and

$$W_n = \{t_0^{\alpha_0} \cdots t_{n-6}^{\alpha_{n-6}}\, v_n \mid \alpha_i \in \{0,1,2\}\}, \quad n \ge 6.$$

We prove by induction on n that $W_n \subset \mathbf{L}$. The base of induction is $W_5 \subset \mathbf{L}$. Now consider $n \geq 6$. By inductive assumption, $W_{n-1} = \{t_0^{\alpha_0} \cdots t_{n-7}^{\alpha_{n-7}} v_{n-1} \mid \alpha_i \in \{0,1,2\}\} \subset \mathbf{L}$. Also, $V_{n-2} = \{v_{n-2}, t_{n-6}v_{n-2}, t_{n-6}^2 v_{n-2}\} \subset \mathbf{L}$. The pairwise products yield W_n and we get $W_n \subset \mathbf{L}$.

Fix a number m. Consider all numbers n such that $6 \leq n \leq n_1 = [\ln m/\ln\lambda]$. Then $\operatorname{wt}(W_n) \leq \operatorname{wt}(v_n) = \lambda^n \leq m$. We count the number of elements in such W_n

$$\tilde{\gamma}_{\mathbf{L}}(m) \geq \sum_{n=6}^{n_1} 3^{n-5} \geq 3^{n_1-5} \geq 3^{\ln m/\ln\lambda - 6} = \frac{1}{3^6} m^{\ln 3/\ln\lambda}.$$

Hence, $\operatorname{GKdim} \mathbf{L} = \underline{\operatorname{GKdim}}\, \mathbf{L} = \ln 3/\ln\lambda$.

Let us evaluate the growth of $\mathbf{A}$. Similar to Theorem 4.1, we rearrange products of elements (27) and get

$$\mathbf{A} \subset \langle t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_{n-4}^{\alpha_{n-4}} v_1^{\beta_1} \cdots v_n^{\beta_n} \mid \alpha_i, b_j \in \{0,1,2\},\ \beta_n > 0\rangle_K. \tag{28}$$

Consider such a monomial of weight not exceeding m. Assume that $n \geq 4$, then

$$\begin{aligned} m &\geq -\sum_{i=0}^{n-4} \alpha_i \lambda^i + \sum_{j=1}^{n} \beta_j \lambda^j \geq \lambda^n - \sum_{i=0}^{n-4} 2\lambda^i > \lambda^n - \frac{2\lambda^{n-4}}{(1-1/\lambda)} \\ &= \lambda^{n-4}(\lambda^4 - 2\lambda^2) = \lambda^{n-4}((\lambda^3+\lambda^2) - 2\lambda^2) = \lambda^{n-3}. \end{aligned}$$

We obtain $\lambda^{n-3} < m$. Hence, $n \leq n_0 = 3 + [\ln m/\ln\lambda]$. Let N be the number of monomials (28) with $n \leq 3$. We compute the number of monomials (28) of weight not exceeding m and obtain

$$\tilde{\gamma}_{\mathbf{A}}(m) \leq N + \sum_{n=4}^{n_0} 3^{2n-3} \leq N + \frac{3^{2n_0-3}}{1-1/9} \leq N + \frac{3^{5+2\ln m/\ln\lambda}}{8} \leq N + \frac{3^5}{8} m^{2\ln 3/\ln\lambda}.$$

The claimed upper bound on the growth of $\mathbf{A}$ is proved. To get the lower bound, we consider the analogue of the set (14)

$$\mathbf{A} \supset \{t_0^{\alpha_0} \cdots t_{n-6}^{\alpha_{n-6}} v_n v_{n-1}^{\beta_{n-1}} \cdots v_1^{\beta_1} \mid \alpha_i, \beta_i \in \{0,1,2\},\ \beta_n = 1\}, \quad n \geq 6. \tag{29}$$

Indeed, since $W_n \subset \mathbf{L}$ for $n \geq 6$ we see that these monomials belong to $\mathbf{A}$. The proof of Theorem 4.1 shows that this set is linearly independent. Weight of a monomial (29) is bounded from above as

$$-\sum_{i=0}^{n-6} \alpha_i \lambda^i + \sum_{j=1}^{n} \beta_j \lambda^j \leq \sum_{j=1}^{n-1} 2\lambda^j + \lambda^n < \frac{2\lambda^{n-1}}{1-1/\lambda} + \lambda^n = 2\lambda^{n+1} + \lambda^n = \lambda^{n+3}.$$

Fix a number m, assume that $\lambda^{n+3} < m$; then all monomials (29) have weights less than m. This is the case for all numbers $n \leq n_2 = [\ln m/\ln\lambda] - 3$. Finally, the number of monomials (29) yields the lower bound on the growth of $\mathbf{A}$

$$\tilde{\gamma}_{\mathbf{A}}(m) \geq \sum_{n=6}^{n_2} 3^{2n-6} \geq 3^{2n_2-6} \geq 3^{2\ln m/\ln\lambda - 14} = \frac{1}{3^{14}} m^{2\ln 3/\ln\lambda}.$$

This estimate proves the upper bound on the growth of $\mathbf{A}$. ∎

Lemma 6.2. *Let* $\operatorname{char} K = 3$ *and* $\mathbf{L} = \operatorname{Lie}_p(v_1, v_2)$. *Then* $\mathbf{L}$ *has a nil-p-mapping.*

Proof. Consider $v \in \mathbf{L}$. Let s be the maximal number such that v_s appears in the decomposition of v. We have

$$v = \sum_{i=1}^{s-1} g_i(t_0, \dots, t_{i-4})v_i + h(t_0, \dots, t_{s-4})v_s, \tag{30}$$

where $g_i = g_i(t_0, \dots, t_{i-4})$ and $h = h(t_0, \dots, t_{s-4})$ are polynomials from R. As above, we assume that $h \in R^+$. (Otherwise we take the number $s+1$ and consider the decomposition $v = \cdots + hv_{s+1}$, where $h = 0$). We apply the p-mapping to (30) and use (25) and (26). We observe v_is with the highest value of i that might appear. We have three cases.

a) We consider the commutators $[g_{i_1}v_{i_1}, [g_{i_2}v_{i_2}, g_{i_3}v_{i_3}]]$ without the factor hv_s. For example, the extreme case is as follows

$$[g_{s-1}v_{s-1}, [g_{s-2}v_{s-2}, g_{s-1}v_{s-1}]] = g_{s-1}^2 g_{s-2} v_{s+1},$$

where extra summands of type $f_j v_j$, with $f_j \in R$ for $j < s+1$ do not appear by Corollary 3.7. In general case, let $j = \max\{i_1, i_2, i_3\}$, then $j \le s-1$. The multiplication rule (see Lemma 1.1) implies that we can at most twice increase j by one and obtain v_{j+2}, where $j+2 \le s+1$. Thus, we obtain at most $\tilde g v_{s+1}$, where $\tilde g \in R$.

b) Next, consider 3-fold commutators that contain hv_s. We obtain for example

$$[hv_s, [g_{s-1}v_{s-1}, hv_s]] = h^2 g_{s-1} v_{s+2}.$$

In general, we obtain at most v_{s+2} with the factor $h \in R^+$. Indeed, if we try to kill h by some v_j then v_s remains and we obtain at most v_{s+1}. Thus, all the terms with v_{s+2} go to $\tilde h v_{s+2}$, $\tilde h \in R^+$, of a presentation similar to (30).

c) Consider the cubes. We have $(hv_s)^3 = h^3 v_s^3 = 0$ and the cubes arising from the sum (30) yield at most $(g_{s-1}v_{s-1})^3 = -g_{s-1}^3 t_{s-2}^2 v_{s+2}$, this term belongs to $\tilde h v_{s+2}$.

Thus, we obtain a presentation of type (30):

$$v^3 = \sum_{i=1}^{s+1} \tilde g_i(t_0, \dots, t_{i-4})v_i + \tilde h(t_0, \dots, t_{s-2})v_{s+2}.$$

We iterate the process

$$v^{3^m} = \sum_{i=1}^{s+2m-1} \tilde{\tilde g}_i(t_0, \dots, t_{i-4})v_i + \tilde{\tilde h}(t_0, \dots, t_{s+2m-4})v_{s+2m}. \tag{31}$$

The weight of any homogeneous monomial of v is at least λ. Hence, weights of monomials of v^{3^m} are at least $\lambda 3^m$. Since polynomials only reduce the weight, weights of monomials in (31) are at most $\operatorname{wt}(v_{s+2m}) = \lambda^{s+2m}$. If $\lambda 3^m > \lambda^{s+2m}$, then $v^{3^m} = 0$. Therefore, it is sufficient to take $m > (s-1)\frac{\ln\lambda}{\ln(3/\lambda^2)}$, where $\frac{\ln\lambda}{\ln(3/\lambda^2)} \approx 3.53$. ■

7. Weight structure in characteristic 3

In this section we extend the results on weights (Theorem 5.1) to characteristic 3.

Theorem 7.1. *Let* $\operatorname{char} K = 3$ *and* $\mathbf{L} = \operatorname{Lie}_p(v_1, v_2)$, $\mathbf{A} = \operatorname{Alg}(v_1, v_2)$, *and* $\mathbf{u} = u_0(\mathbf{L})$. *Then we have the following regions for weights.*

1. *weights of* $\mathbf{L}$ *lie in the strip* $\lambda x - 2\lambda^3 < y < \lambda x + 2\lambda^2$;
2. *weights of* $\mathbf{A}$ *lie in the strip* $\lambda x - 2\lambda^4 < y < \lambda x + 2\lambda^3$;
3. *set* $\theta = \ln 3/\ln\lambda$, $\kappa = \theta/(1+\theta) \approx 0.695$. *There exists* $C > 0$ *such that weights of* $\mathbf{u}$ *lie in the region* $|\eta| < C\xi^\kappa$;
4. *weight of all three algebras satisfy* $|\eta| \ge \lambda^2/\xi$.

Proof. All elements of $\mathbf{L}$ are expressed via monomials $w = t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_{n-3}^{\alpha_{n-3}} v_n$, where $\alpha_i \in \{0,1,2\}$. We proceed as above

$$\operatorname{swt}(w) = \operatorname{swt}(v_n) + \sum_{i=0}^{n-3} \alpha_i \operatorname{swt}(t_i) = \bar\lambda^{n-2} - \sum_{i=0}^{n-3} \alpha_i \bar\lambda^{i-2},$$

$$\operatorname{swt}(w) \le |\bar\lambda|^{n-2} + 2 \sum_{\substack{i \text{ odd} \\ 1\le i\le n-3}} |\bar\lambda|^{i-2} < 2\sum_{j=0}^{\infty} \lambda^{1-2j} = \frac{2\lambda}{1-1/\lambda^2} = \frac{2\lambda^3}{\lambda^2-1} = 2\lambda^2;$$

$$\operatorname{swt}(w) \ge -|\bar\lambda|^{n-2} - 2 \sum_{\substack{i \text{ even} \\ 0\le i\le n-3}} |\bar\lambda|^{i-2} > -2\sum_{j=0}^{\infty} \lambda^{2-2j} = -\frac{2\lambda^2}{1-1/\lambda^2} = -2\lambda^3.$$

We obtain the strip $-2\lambda^3 < \operatorname{swt}(w) < 2\lambda^2$.

By proof of Theorem 6.1, $\mathbf{A}$ is contained in the span of monomials (13) of type $w = t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_{n-3}^{\alpha_{n-3}} v_1^{\beta_1} v_2^{\beta_2} \cdots v_n^{\beta_n}$, where $\alpha_i, \beta_j \in \{0,1,2\}$, and $\beta_n = 1$. We proceed as above

$$\operatorname{swt}(w) = \sum_{j=1}^{n} \beta_j \operatorname{swt}(v_j) + \sum_{i=0}^{n-3} \alpha_i \operatorname{swt}(t_i) = \sum_{j=1}^{n} \beta_j \bar\lambda^{j-2} - \sum_{i=0}^{n-3} \alpha_i \bar\lambda^{i-2};$$

$$\operatorname{swt}(w) \le 2\sum_{k=1}^{n} |\bar\lambda|^{k-2} < \frac{2\lambda}{1-1/\lambda} = 2\lambda^3;$$

$$\operatorname{swt}(w) \ge -2\sum_{k=0}^{n} |\bar\lambda|^{k-2} > -\frac{2\lambda^2}{1-1/\lambda} = -2\lambda^4.$$

Hence, the weights of $\mathbf{A}$ lie in the claimed strip $-2\lambda^4 < \operatorname{swt}(w) < 2\lambda^3$.

Let us study the weights of $\mathbf{u} = u_0(\mathbf{L})$. Let $\mathbf{L} = \langle w_1, w_2, w_3, \dots \rangle_K$ be a basis the order of which obeys to the weight function. Consider a basis monomial $u = w_{i_1}^{\gamma_1} w_{i_2}^{\gamma_2} \cdots w_{i_k}^{\gamma_k} \in \mathbf{u}$, where $i_1 < i_2 < \cdots < i_k$, $\gamma_i \in \{1,2\}$, and k is a fixed large number. Let us find a relation between new coordinates (ξ,η), where $\xi = \operatorname{wt} u$, $\eta = \operatorname{swt} u$. From the first claim we have the estimate

$$|\eta| = \left| \sum_{j=1}^{k} \gamma_k \operatorname{swt} w_{i_j} \right| < 4k\lambda^3. \tag{32}$$

Now, let us evaluate the weight of $\mathbf{u}$. Denote $\theta = \ln 3/\ln\lambda$. By the proof of Theorem 6.1 we have the upper bound $\tilde{\gamma}_{\mathbf{L}}(m) < C_0 m^\theta$, $m \in \mathbb{N}$, where C_0 is some constant. Set

$$m = m(k) = [(k/C_0)^{1/\theta}]. \tag{33}$$

By this setting, $\tilde{\gamma}_{\mathbf{L}}(m) < C_0 m^\theta < k$. We observe that the total weight of k distinct vectors w_{i_j} is bigger than weight of k first vectors, which, by our construction, contain all basis vectors of weight at most m, by proof of Theorem 6.1, the latter contain the set $W_n = \{t_0^{\alpha_0} \cdots t_{n-6}^{\alpha_{n-6}} v_n \mid \alpha_i \in \{0,1,2\}\}$:

$$\xi = \operatorname{wt} u = \sum_{j=1}^{k} \gamma_j \operatorname{wt} w_{i_j} \geq \sum_{j=1}^{k} \operatorname{wt} w_j \geq \sum_{\operatorname{wt} w_j \leq m} \operatorname{wt} w_j \geq \sum_{w \in W_n} \operatorname{wt} w. \tag{34}$$

Recall that $\operatorname{wt} W_n \leq \lambda^n$ and to get $\operatorname{wt} W_n < m$ we just set $n = n(m) = [\ln m/\ln\lambda]$. Let $v = t_0^{\alpha_0} \cdots t_{n-6}^{\alpha_{n-6}} v_n \in W_n$, then

$$\begin{aligned} \operatorname{wt} v &\geq \lambda^n - 2\sum_{i=0}^{n-6} \lambda^i \geq \lambda^n - \frac{2\lambda^{n-6}}{1-1/\lambda} = \lambda^n - 2\lambda^{n-4} \\ &> \lambda^n - \lambda^{n-3} - \lambda^{n-4} = \lambda^n - \lambda^{n-2} = \lambda^{n-1}. \end{aligned}$$

We continue the estimate (34)

$$\begin{aligned} \xi \geq \sum_{w \in W_n} \operatorname{wt} w \geq \lambda^{n-1}|W_n| = \lambda^{n-1}3^{n-5} = \frac{1}{81}(3\lambda)^{n-1} \geq \frac{1}{81}(3\lambda)^{\ln m/\ln\lambda - 2} \\ = \frac{1}{3^6\lambda^2} m^{\ln(3\lambda)/\ln\lambda} = \frac{1}{3^6\lambda^2} m^{1+\theta} \geq \frac{1}{2000} m^{1+\theta}. \end{aligned} \tag{35}$$

Finally, (32), (33), and (35) yield the desired region

$$|\eta| < 4k\lambda^3 \leq 4C_0\lambda^3(m+1)^\theta \leq 4C_0\lambda^3\big((2000\xi)^{1/(1+\theta)} + 1\big)^\theta \leq C\xi^{\theta/(1+\theta)}. \qquad \blacksquare$$

Corollary 7.2. *Let* $\operatorname{char} K = 3$ *and* $\mathbf{L}$, $\mathbf{A}$, *and* $\mathbf{u} = u_0(\mathbf{L})$ *be as above. Then these three algebras are direct sums of two locally nilpotent subalgebras*

$$\mathbf{L} = \mathbf{L}_+ \oplus \mathbf{L}_-, \quad \mathbf{A} = \mathbf{A}_+ \oplus \mathbf{A}_-, \quad \mathbf{u} = \mathbf{u}_+ \oplus \mathbf{u}_-,$$

where the decomposition is given by the function of superweight.

8. Fibonacci restricted Lie algebra, char $K = 5$

In this Section $\operatorname{char} K = 5$. We compute the pth power of $v_1 = \partial_1 + t_0 v_2$ using the third property of the p-mapping (3). Recall that $\partial_1(v_2) = v_3$ and $\partial_1(v_i) = 0$ for $i > 2$.

$$\begin{aligned}\operatorname{ad}(Z\partial_1 + t_0v_2)^4(\partial_1) &= -\operatorname{ad}(Z\partial_1 + t_0v_2)^3(t_0\partial_1(v_2)) = -\operatorname{ad}(Z\partial_1 + t_0v_2)^3(t_0v_3)\\ &= -t_0^2 \operatorname{ad}(Z\partial_1 + t_0v_2)^2(v_4) = -t_0^3 \operatorname{ad}(Z\partial_1 + t_0v_2)(t_1v_5)\\ &= -t_0^3(Zv_5 + t_0t_1^2t_2v_6).\end{aligned}$$

We get two nonzero terms, namely $s_1(\partial_1, t_0v_2)$ and $s_2(\partial_1, t_0v_2)$. Hence,

$$v_1^5 = (\partial_1 + t_0v_2)^5 = -t_0^3\left(\frac{1}{2}v_5 + t_0t_1^2t_2v_6\right).$$

Thus, in case $\operatorname{char} K = 5$ we have

$$v_i^5 = 2t_{i-1}^3\, v_{i+4} - t_{i-1}^4t_i^2t_{i+1}\, v_{i+5}, \quad i = 1, 2, \dots. \tag{36}$$

By Lemma 3.6 we have the Lie subalgebra $\tilde H \subset \operatorname{Der} R$ as follows

$$\tilde H = \langle v_1, v_2, v_3,\ t_0^{\alpha_0}t_1^{\alpha_1}\cdots t_{n-4}^{\alpha_{n-4}}\, v_n \mid n \ge 4, \alpha_i \in \{0,1,2,3,4\};\ \alpha_{n-4} \le 1, \alpha_{n-5} \le 2\rangle_K.$$

Also $\operatorname{Lie}(v_1, v_2) \subset \tilde H$. Since the first term in (36) does not satisfy the condition $\alpha_{n-5} \le 2$ we have

$$\tilde H_p = \tilde H \oplus \langle v_i^5 \mid i = 0, 1, 2, \dots\rangle_K.$$

Now we use the embedding $\mathbf{L} = \operatorname{Lie}_p(v_1, v_2) \subset \tilde H_p$.

Theorem 8.1. *Let* $\operatorname{char} K = 5$ *and* $\mathbf{L} = \operatorname{Lie}_p(v_1, v_2)$. *Denote* $\lambda = \frac{1+\sqrt5}{2}$. *Then* $\operatorname{GKdim} \mathbf{L} = \underline{\operatorname{GKdim}}\, \mathbf{L} = \ln 5/\ln\lambda \approx 3.34.$

Proof. Denote by N the number of basis monomials of $\tilde H$ such that $n < 6$. Let $w = t_0^{\alpha_0}t_1^{\alpha_1}\cdots t_{n-4}^{\alpha_{n-4}}\, v_n \in \tilde H$ be a monomial of weight not exceeding m and $n \ge 6$. Then

$$\begin{aligned} m \ge \operatorname{wt} w = \operatorname{wt} v_n + \sum_{i=0}^{n-4}\alpha_i \operatorname{wt} t_i = \lambda^n - \sum_{i=0}^{n-4}\alpha_i\lambda^i \ge \lambda^n - \lambda^{n-4} - 2\lambda^{n-5} - 4\sum_{i=0}^{n-6}\lambda^i\\ > \lambda^{n-6}\left(\lambda^6 - \lambda^2 - 2\lambda - \frac{4}{1-1/\lambda}\right) = \lambda^{n-6}(\lambda^6 - 5\lambda^2 - 2\lambda) = \lambda^{n-5}.\end{aligned}$$

(the reader can check that the expression in brackets above is indeed equal to λ). We obtain $\lambda^{n-5} < m$. Hence, $n \le n_0 = 5 + [\ln m/\ln\lambda]$. Then the number of such monomials w of weight not exceeding m is bounded by

$$N + 2\cdot 3\sum_{n=6}^{n_0} 5^{n-5} \le N + \frac{6\cdot 5^{n_0-5}}{1-1/5} \le N + \frac{15}{2}5^{\ln m/\ln\lambda} = N + \frac{15}{2}m^{\ln 5/\ln\lambda}.$$

Consider also elements $v_n^5 \in \tilde{H}_p$ of weight not exceeding m. We have $5\,\mathrm{wt}\,v_n = 5\lambda^n \le m$. The number of such elements is bounded by $C_1 \ln m$ and we obtain the claimed upper estimate on the growth of $\mathbf{L}$.

Let us prove the lower bound. We have $[v_1, v_3] = t_0 v_4 \in \mathbf{L}$. We subsequently multiply $t_0 v_4$ by $v_3, v_4, v_5, \dots$ and conclude that $\{t_0 v_n \mid n \ge 4\} \subset \mathbf{L}$. Next, $[t_0 v_4, t_0 v_5] = t_0^2 v_6 \in \mathbf{L}$, we multiply the last element by $v_5, v_6, \dots$ and obtain that $\{t_0^2 v_n \mid n \ge 6\} \subset \mathbf{L}$. Also $[t_0 v_5, t_0^2 v_6] = t_0^3 v_7 \in \mathbf{L}$ and $[t_0^2 v_6, t_0^2 v_7] = t_0^4 v_8 \in \mathbf{L}$. Similarly, we multiply by v_is and conclude that

$$\{t_0^\alpha v_n \mid 0 \le \alpha \le 4, n \ge 8\} \subset \mathbf{L}. \tag{37}$$

Consider the sets

$$V_n = \{t_{n-8}^\alpha v_n \mid 0 \le \alpha \le 4\}, \quad n \ge 8.$$

We apply τ to (37) and obtain that $V_n \subset \mathbf{L}$ for all $n \ge 8$. Next we set $W_9 = \{v_9\}$ and

$$W_n = \{t_0^{\alpha_0} \cdots t_{n-10}^{\alpha_{n-10}} v_n \mid 0 \le \alpha_i \le 4\}, \quad n \ge 10.$$

We prove by induction on n that $W_n \subset \mathbf{L}$ for all $n \ge 9$. The base of induction is $W_9 \subset \mathbf{L}$. Consider $n \ge 10$. By inductive assumption, $W_{n-1} = \{t_0^{\alpha_0} \cdots t_{n-11}^{\alpha_{n-11}} v_{n-1} \mid 0 \le \alpha_i \le 4\} \subset \mathbf{L}$. Also, $V_{n-2} = \{t_{n-10}^\alpha v_{n-2} \mid 0 \le \alpha \le 4\} \subset \mathbf{L}$. The pairwise products yield that $W_n \subset \mathbf{L}$.

Fix a number m. Consider all numbers n such that $9 \le n \le n_1 = [\ln m / \ln \lambda]$. Then $\mathrm{wt}(W_n) \le \mathrm{wt}(v_n) = \lambda^n \le m$. We count the number of elements in such W_n and obtain

$$\tilde{\gamma}_{\mathbf{L}}(m) \ge \sum_{n=9}^{n_1} 5^{n-9} \ge 5^{n_1 - 9} \ge 5^{\ln m/\ln\lambda - 10} = \frac{1}{5^{10}} m^{\ln 5/\ln\lambda}.$$

Hence, $\mathrm{GKdim}\,\mathbf{L} = \underline{\mathrm{GKdim}}\,\mathbf{L} = \ln 5/\ln\lambda$. ∎

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