Journal of Lie Theory
Volume **20** (2010) 3–15

A Combinatorial Basis for the Free Lie Algebra of the Labelled Rooted Trees

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Communicated by A. Fialowski

Abstract. The pre-Lie operad is an operad structure on the species $\mathcal{T}$ of labelled rooted trees. A result of F. Chapoton shows that the pre-Lie operad is a free twisted Lie algebra over a field of characteristic zero, that is $\mathcal{T} = \mathcal{L}ie \circ \mathcal{F}$ for some species $\mathcal{F}$. Indeed Chapoton proves that any section of the indecomposables of the pre-Lie operad, viewed as a twisted Lie algebra, gives such a species $\mathcal{F}$. In this paper, we first construct an explicit vector space basis of $\mathcal{F}[S]$ when S is a linearly ordered set. We deduce the associated explicit species $\mathcal{F}$, solution to the equation $\mathcal{T} = \mathcal{L}ie \circ \mathcal{F}$. As a corollary the graded vector space $(\mathcal{F}[\{1,\dots,n\}])_{n\in\mathbb{N}}$ forms a sub non-symmetric operad of the pre-Lie operad $\mathcal{T}$.
Mathematics Subject Classification 2000: 18D, 05E, 17B.
Key Words and Phrases: Free Lie algebra, rooted tree, pre-Lie operad, Lyndon word.

Introduction

One of the most fascinating result in the theory of operads is the Koszul duality between the Lie operad $\mathcal{L}ie$ and the commutative and associative operad $\mathcal{C}om$ [8]. This has inspired many researchers to study this pair of operads and its refinements. One particular instance of this is the study of pre-Lie algebras. A pre-Lie algebra is a **k**-vector space L, where **k** is the ground field, together with a product $*$ satisfying the relation

$$(x*y)*z - x*(y*z) = (x*z)*y - x*(z*y),\ \forall x,y,z \in L.$$

Any pre-Lie algebra gives rise to a Lie algebra with Lie bracket defined by

$$[x,y] = x*y - y*x.$$

In [4], the pre-Lie operad is described in terms of the species $\mathcal{T}$ of labelled rooted trees. We assume in this paper that the ground field **k** is of characteristic

ISSN 0949–5932 / $2.50

zero. This operad sits naturally between $\mathcal{L}ie$ and $\mathcal{A}s$, the operad for associative algebras, for the injective morphisms

$$\mathcal{L}ie \to \mathcal{T} \to \mathcal{A}s$$

factor the usual injective morphism from $\mathcal{L}ie$ to $\mathcal{A}s$.

At the level of algebras, the free pre-Lie algebra generated by a vector space of dimension 1, is the vector space freely generated by unlabelled rooted trees. Indeed the enveloping algebra of its associated Lie algebra $\mathcal{L}^1$ is isomorphic to the Grossman-Larson Hopf algebra [9] and its dual is isomorphic to the Connes-Kreimer Hopf algebra describing renormalisation theory in [6]. Foissy proved that the Lie algebra $\mathcal{L}^1$ is a free Lie algebra in [7]. This result generalizes easily to the following statement: the Lie algebra associated to a free pre-Lie algebra generated by a vector space V is a free Lie algebra generated by a vector space W. However, the proof of Foissy does not give an explicit description of W.

A species [1, 10] is a contravariant functor from the category of finite sets and bijections to the category of vector spaces. The category of species is equivalent to the one of $\mathbb{S}$-modules. An $\mathbb{S}$-module is a sequence $(M(n))_{n\in\mathbb{N}}$, with M_n a right S_n-module. The category of species is endowed with a composition product $\circ$ described in Section 1. An operad is a monoid in the category of species with respect to the composition product.

Foissy's result suggested that, at the level of species, there exists a species $\mathcal{F}$ such that

$$\mathcal{T} = \mathcal{L}ie \circ \mathcal{F}.$$

This was proved by Chapoton in [3]. Furthermore, the morphism of operads $\mathcal{L}ie \to \mathcal{T}$ implies that $\mathcal{T}$ is a twisted Lie algebra (or Lie algebra in the category of species). The species of indecomposable elements is $\mathcal{T}/[\mathcal{T},\mathcal{T}]$. Chapoton proves that any section of the projection $\mathcal{T} \to \mathcal{T}/[\mathcal{T},\mathcal{T}]$ yields a species $\mathcal{F}$, which is a solution to the equation $\mathcal{T} = \mathcal{L}ie \circ \mathcal{F}$.

The purpose of this paper is to describe a specific section of the latter projection. In fact, we do not use the result of Chapoton.

After some preliminaries, we construct in Section 2 a sub vector space $\mathcal{F}[S]$ of $\mathcal{T}[S]$ for any linearly ordered set S and prove that $\mathcal{T}[S] = \mathcal{L}ie(\mathcal{F})[S]$ using the Lyndon permutations to describe the basis in $\mathcal{L}ie$. In Section 3 we describe the species $\mathcal{F}$ and prove that $\mathcal{T} = \mathcal{L}ie \circ \mathcal{F}$ as species. This gives a new proof of Foissy's result and of Chapoton's result. Section 4 is concerned with the study of the operad $\mathcal{T}$, viewed as a non-symmetric operad. Similarly to species, a graded vector space amounts to a contravariant functor from the category of finite linearly ordered sets and ordered bijections (a discrete category) to the category of vector spaces. There is an analog of the composition product in the category of graded vector spaces. A non-symmetric operad is a monoid in this category with respect to this composition product. Any species gives rise to a graded vector space by forgetting the action of the symmetric group and any operad gives rise to a non-symmetric operad. We prove that $\mathcal{F}$ is a sub non-symmetric operad of $\mathcal{T}$.

1. The pre-Lie operad and rooted trees

We first recall the definition of the pre-Lie operad based on labelled rooted trees as in [4]. For $n \in \mathbb{N}^*$, the set $\{1, \ldots, n\}$ is denoted by $[n]$ and $[0]$ denotes the empty set. The symmetric group on k letters is denoted by S_k.

Recall that a species is a contravariant functor from the category of finite sets $\mathbf{Set}^\times$ and bijections to the category of $\mathbf{k}$-vector spaces $\mathbf{Vect}$. Following Joyal in [10], a species is equivalent to an $\mathbb{S}$-module $V = (V_n)_{n \in N}$, that is, a collection of right S_n-modules V_n.

Given two species $\mathcal{A}, \mathcal{B} \colon \mathbf{Set}^\times \to \mathbf{Vect}$ we have the product

$$\mathcal{A} \bullet \mathcal{B}[S] = \bigoplus_{I+J=S} \mathcal{A}[I] \otimes \mathcal{B}[J] \,, \tag{1}$$

where $I + J$ denotes the disjoint union of the sets I and J. The composition of species is defined by

$$\mathcal{A} \circ \mathcal{B}[S] = \bigoplus_{k \geq 0} \mathcal{A}[k] \otimes_{S_k} (\mathcal{B}^{\bullet k}[S]) \,.$$

If $\mathcal{B}[\emptyset] = 0$ the composition of species has the form

$$\mathcal{A} \circ \mathcal{B}[S] = \bigoplus_{\Phi \vdash S} \mathcal{A}[\Phi] \otimes \left(\bigotimes_{\phi \in \Phi} \mathcal{B}[\phi] \right) \,, \tag{2}$$

where $\Phi \vdash S$ denotes that Φ is a set partition of S.

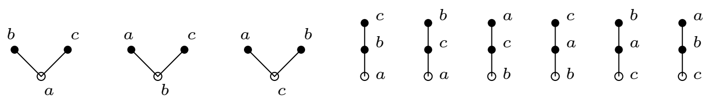
An operad is a monoid in the category of species with respect to the composition product. A (twisted) algebra A over an operad $\mathcal{P}$ is a species together with an evaluation product

$$\mu_A : \mathcal{P} \circ A \to A$$

satisfying the usual condition (see [1] for more details). An algebra over an operad is usually a vector space considered as a species which is always zero except on the emptyset. The terminology "twisted" emphasizes the fact that we generalize the usual definition to any species. For instance $\mathcal{P}$ is the free twisted $\mathcal{P}$-algebra generated by the unit I for the composition product, whereas $\oplus_{n \geq 0} \mathcal{P}[n]/S_n$ is the free $\mathcal{P}$-algebra generated by a 1-dimensional vector space.

Given a finite set S of cardinality n let $\mathcal{T}[S]$ be the vector space freely generated by the labelled rooted trees on n vertices with distinct label chosen in S. For $n = 0$ we set $T[\emptyset] = 0$. This gives us a species.

Example 1.1. The space $\mathcal{T}[\{a, b, c\}]$ is the linear span of the following trees:



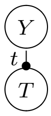
In general there are n^{n-1} such trees on a set of cardinality n (see [2] for more details).

Theorem 1.2. *[4, theorem 1.9] The species $\mathcal{T}$ forms an operad. Algebras over this operad are pre-Lie algebras, that is, vector spaces L together with a product $*$ satisfying the relation*

$$(x*y)*z - x*(y*z) = (x*z)*y - x*(z*y),\ \forall x,y,z \in L.$$

As a consequence $\mathcal{T}$ is the free twisted pre-Lie algebra generated by I. The twisted pre-Lie product is described as follows.

Definition 1.3. Given two disjoint sets I, J and two trees $T \in \mathcal{T}[I]$ and $Y \in \mathcal{T}[J]$ we define

$$T * Y = \sum_{t \in Vert(T)}$$ 

where the sum is over all possible ways of *grafting* the root of the tree Y on a vertex t of T. The root of $T*Y$ is the one of T.

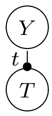
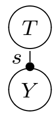
Since any pre-Lie algebra L gives rise to a Lie algebra whose bracket is defined by $[x,y] = x*y - y*x$ there is a morphism of operads

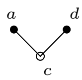
$$\mathcal{L}ie \to \mathcal{T}.$$

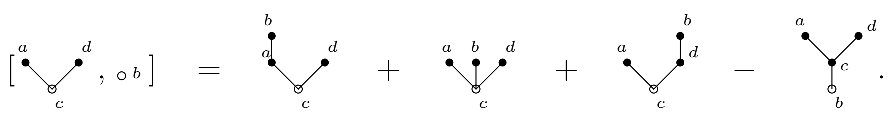
Note that this morphism is injective: an associative algebra is obviously a pre-Lie algebra and the composition of morphisms of operads

$$\mathcal{L}ie \to \mathcal{T} \to \mathcal{A}s$$

is the usual injective morphism from $\mathcal{L}ie$ to $\mathcal{A}s$. As a consequence the species $\mathcal{T}$ is a twisted Lie algebra, that is a Lie monoid in the category of species. It is endowed with the following Lie bracket $[\ ,\ ]\colon \mathcal{T} \bullet \mathcal{T} \to \mathcal{T}$: given two disjoint sets I, J and two trees $T \in \mathcal{T}[I]$ and $Y \in \mathcal{T}[J]$ we define

$$[T,Y] = T*Y - Y*T = \sum_{t \in Vert(T)}$$  $$- \sum_{s \in Vert(Y)}$$  . (3)

Example 1.4. For $T =$  $\in \mathcal{T}[\{a,c,d\}]$ and $Y = \circ\, b \in \mathcal{T}[\{b\}]$ we have that



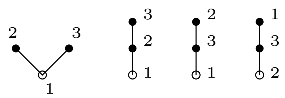
As we mentioned in the Introduction, we shall now describe explicitly a species $\mathcal{F}$ such that $\mathcal{T} = \mathcal{L}ie \circ \mathcal{F}$.

2. $\mathcal{T}[S] = \mathcal{L}ie \circ \mathcal{F}[S]$ as vector spaces

In this section, we show an auxiliary result relating $\mathcal{T}$ to a free Lie algebra over rooted trees that are increasing in the first level. We give an explicit isomorphism using basis. This has the advantage to be explicit but it is not natural. In the next section we will induce an action of the symmetric groups on both side hence giving an identity of species.

Given a finite set S and a linear order on S, let $\mathcal{F}[S]$ be the vector space spanned by the set of S-labelled rooted trees that are increasing at the first level. That is the trees such that the labels increase from the root to the adjacent vertices and no other condition on the other labels. Also, we let $\mathcal{F}[\emptyset] = 0$. At this point, $\mathcal{F}$ is not a species as it depends on an order on S. We will turn this into a species in the next section.

Example 2.1. The space $\mathcal{F}[\{1,2,3\}]$ with the natural order on $\{1,2,3\}$ has basis given by the following trees:



In general there are $(n-1)^{n-1}$ such trees (see e.g. [5] for more details).

For our next result, we also need to consider $\mathcal{L}ie[S]$ as the vector space of multilinear brackets of degree $|S|$. That is the vector space spanned by all brackets of the elements of S (without repetition) modulo the antisymmetry relation and Jacobi identity. It is easy to check that this construction is functorial. We then have that $\mathcal{L}ie$ is a species. In fact, $\mathcal{L}ie$ is an operad and algebras over this operad are the classical Lie algebras. (see e.g. [8]).

Example 2.2. The space $\mathcal{L}ie[\{1,2,3\}]$ is the linear span of the following brackets:

$$[[1,2],3],\ [[1,3],2],\ [[2,1],3],\ [[2,3],1],\ [[3,1],2],\ [[3,2],1],$$
$$[1,[2,3]],\ [1,[3,2]],\ [2,[1,3]],\ [2,[3,1]],\ [3,[1,2]],\ [3,[2,1]].$$

As we will see below, it is well known that this space has dimension equal to two and that a basis is given by $\{[[3,1],2],[3,[2,1]]\}$. In general there are $(n-1)!$ linearly independent brackets [11].

If we are given a linear order on S we can construct an explicit basis of $\mathcal{L}ie[S]$. This is the classical Lyndon basis of $\mathcal{L}ie$ (see [11]). More precisely, $\mathcal{L}ie[S]$ has basis given by the Lyndon permutations with Lyndon bracketing. For our purpose we use the reverse lexicographic order to produce the following basis of $\mathcal{L}ie[S]$. Let $S = \{a < b < \cdots < y < z\}$. A *Lyndon* permutation $\sigma\colon S \to S$ is a permutation such that $\sigma(a) = z$. The Lyndon bracketing $\mathrm{sb}[\sigma]$ of σ is defined recursively. We write $\sigma = \big(\sigma(a), \sigma(b), \ldots, \sigma(z)\big)$ as the list of its values. If $S = \{a\}$ then we define $\mathrm{sb}[\sigma(a)] = a$. If $|S| > 1$, let $k \in S$ be such that $\sigma(k) = y$ the second largest value of S, then define

$$\mathrm{sb}[\sigma(a), \ldots, \sigma(j), \sigma(k), \ldots, \sigma(z)] = \big[\mathrm{sb}[\sigma(a), \ldots, \sigma(j)],\ \mathrm{sb}[\sigma(k), \ldots, \sigma(z)]\big].$$

A basis of $\mathcal{L}ie[S]$ is given by the set $\{\mathrm{sb}[\sigma] : \sigma \text{ is Lyndon}\}$. In the Example 2.2 we have that $(3,1,2)$ and $(3,2,1)$ are the only two Lyndon permutations and $\mathrm{sb}[3,1,2] = [\mathrm{sb}[3,1], \mathrm{sb}[2]] = [[\mathrm{sb}[3], \mathrm{sb}[1]], 2] = [[3,1],2]$. Similarly $\mathrm{sb}[3,2,1] = [3,[2,1]]$.

Even though $\mathcal{F}$ is not a species we can still define $\mathcal{L}ie \circ \mathcal{F}$. Let S be a finite set with a linear order. We define

$$\mathcal{L}ie \circ \mathcal{F}[S] = \bigoplus_{\Phi \vdash S} \mathcal{L}ie[\Phi] \otimes \left(\bigotimes_{\phi \in \Phi} \mathcal{F}[\phi] \right),$$

where for $\Phi \vdash S$ we induce a linear order on each part $\phi \in \Phi$ from the linear order on S.

Theorem 2.3. *$\mathcal{T}[S] = \mathcal{L}ie \circ \mathcal{F}[S]$ as vector spaces.*

Proof. Given a linear order on a finite set S, we construct a linear isomorphism between $\mathcal{T}[S]$ and $\mathcal{L}ie \circ \mathcal{F}[S]$. By definition, $\mathcal{L}ie \circ \mathcal{F}[S]$ is any bracketing of trees of type $\mathcal{F}$ such that the disjoint union of all the labels is S. Since $\mathcal{T}$ is a Lie monoid there is a natural map $\Xi\colon \mathcal{L}ie \circ \mathcal{F}[S] \to \mathcal{T}[S]$. We need to show that this map is injective and surjective.

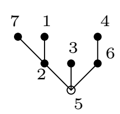
Assume that we have a finite set S and a linear order on S. For $\Phi = \{\phi_1, \phi_2, \ldots, \phi_\ell\} \vdash S$ we have that each part ϕ_i is also ordered. We can then order any set of trees $\{T_i : T_i \in \mathcal{F}[\phi_i],\ 1 \le i \le \ell\}$ using the roots of the trees. It follows that a basis for $\mathcal{L}ie \circ \mathcal{F}[S]$ is given by

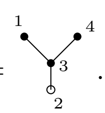
$$\left\{\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}] : {\scriptstyle \Phi=\{\phi_1,\ldots,\phi_\ell\}\vdash S,\ \sigma\colon [\ell]\to[\ell],\ T_i\in\mathcal{F}[\phi_i],\ T_{\sigma(1)}} \text{ has the largest root}\right\}.$$

To complete the proof, we need to show that

$$\left\{\Xi(\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}]) : {\scriptstyle \Phi=\{\phi_1,\ldots,\phi_\ell\}\vdash S,\ \sigma\colon [\ell]\to[\ell],\ T_i\in\mathcal{F}[\phi_i],\ T_{\sigma(1)}} \text{ has the largest root}\right\} \qquad (4)$$

is a basis of $\mathcal{T}[S]$. Using the order on S, we introduce a grading on the basis of labelled rooted trees of $\mathcal{T}[S]$ and show that there exists a triangularity relation between the basis in (4) and the basis of labelled rooted trees. We say that a tree $T \in \mathcal{T}[S]$ is of degree d if the maximal decreasing connected subtree of T from the root has d vertices. For any tree $T \in \mathcal{T}[S]$ we denote by $MD(T)$ its maximal decreasing connected subtree from the root. For example consider

$T_1 =$ 

and $T_2 =$ 

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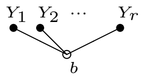
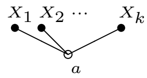
$MD(T_1)$ is built with the vertices labelled by $\{5,3,2,1\}$ and $MD(T_2)$ with the vertex labelled by $\{2\}$. Hence T_1 is of degree 4 and T_2 is of degree 1. Remark that $T \in \mathcal{F}[S]$ if and only if the degree of T is 1.

Given a set partition $\Phi = \{\phi_1, \ldots, \phi_\ell\} \vdash S$, a permutation $\sigma\colon [\ell] \to [\ell]$, a family of trees $\{T_i : T_i \in \mathcal{F}[\phi_i],\ T_{\sigma(1)} \text{ has the largest root}\}$, we claim that in the expansion of $\Xi(\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}])$ there is a unique tree of maximal degree ℓ (with coefficient 1). Furthermore, the correspondence from $\Xi(\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}])$ to its maximal degree term T is such that $MD(T)$ is formed from the vertices labelled by the labels of the roots of $T_1, T_2, \ldots, T_\ell$. In fact the maximal decreasing subtree of any tree in the expansion of $\Xi(\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}])$ is formed from the vertices labelled by a subset of the labels of the roots of $T_1, T_2, \ldots, T_\ell$.

We proceed by induction on ℓ. For $\ell = 1$ we have $\Xi(\mathrm{sb}[T_1]) = T_1$ a unique tree of degree 1. For $\ell = 2$ we are given two trees of degree 1:

$T_1 =$ 

and $T_2 =$ 

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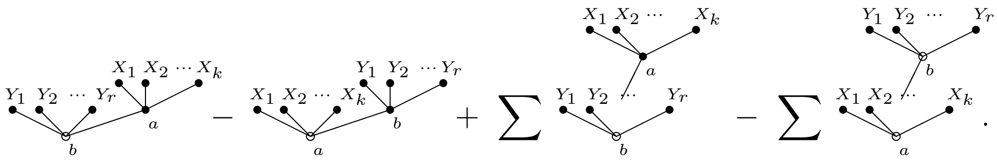
where $T_i \in \mathcal{F}[\phi_i]$. This implies that the root of each Y_j is strictly greater than b and the root of each X_j is strictly greater than a. We assume without loss of generality that $b > a$. When we expand $\Xi(\mathrm{sb}[T_1 T_2]) = [T_1, T_2]$ we obtain



The first term is of degree 2 and its maximal decreasing subtree is built from $\{b, a\}$ the roots of T_1 and T_2. All the other trees in this expansion are of degree 1 and their maximal decreasing subtrees are labelled either by a or by b.

We now assume that $\ell > 2$. To compute $\Xi(\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}])$, let $b_1, b_2, \ldots, b_\ell$ be the roots of $T_{\sigma(1)}, T_{\sigma(2)}, \ldots T_{\sigma(\ell)}$ respectively. By construction we have that $b_1 = \max(b_1, b_2, \ldots, b_\ell)$. Let $b_k = \max(b_2, \ldots, b_\ell)$. That is b_k is the second largest root and $k > 1$. The Lyndon factorization writes

$$\Xi(\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}]) = \big[\Xi(\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(k-1)}]), \Xi(\mathrm{sb}[T_{\sigma(k)} \cdots T_{\sigma(\ell)}])\big].$$

By induction hypothesis we have that

$$\Xi(\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(k-1)}]) = Y_0 + \sum_i c_i Y_i$$

where Y_0 is of degree $k-1$ and $MD(Y_0)$ is formed with vertices labelled by $\{b_1, \ldots, b_{k-1}\}$ and the trees Y_i $(i \neq 0)$ are of degree $< k-1$ where $MD(Y_i)$ are formed with vertices labelled by a subset of $\{b_1, \ldots, b_{k-1}\}$. Similarly,

$$\Xi(\mathrm{sb}[T_{\sigma(k)} \cdots T_{\sigma(\ell)}]) = X_0 + \sum_j d_j X_j$$

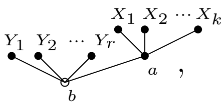
where X_0 is of degree $\ell - k + 1$ and $MD(X_0)$ is formed with vertices labelled by $\{b_k, \ldots, b_\ell\}$ and the trees X_j $(j \neq 0)$ are of degree $< \ell - k + 1$ where $MD(X_j)$ are formed with vertices labelled by a subset of $\{b_k, \ldots, b_\ell\}$. The largest degree term in $[Y_i, X_j]$ must be obtained by either grafting $MD(Y_i)$ in $MD(X_j)$, or by grafting $MD(X_j)$ in $MD(Y_i)$. Hence the largest degree term in $[Y_i, X_j]$ is of degree at most $\deg(Y_i) + \deg(X_j)$. Hence it is sufficient to concentrate our attention on $[Y_0, X_0]$. In this case, recall that b_1 is the largest value, so it must be the root of $MD(Y_0)$. Similarly b_k is the root of $MD(X_0)$. We can get a tree of degree ℓ by grafting X_0

at the root of Y_0. If we graft X_0 anywhere else in Y_0 we get a tree of degree strictly smaller. In fact, since $b_k > \max(b_2, ..., b_{k-1})$, if we graft X_0 on $MD(Y_0)$ (not at the root) or anywhere else, we get a tree of degree equal to $\deg(Y_0) = k-1 < \ell$. On the other hand, since b_1 is maximal, if we graft Y_0 in X_0 we always get a tree of degree equal to $\deg(X_0) = \ell - k + 1 < \ell$.

We now remark that $MD(Z)$ of any term Z in the expansion of $[Y_i, X_j]$, is either $MD(Y_i)$,$MD(X_j)$, the grafting of $MD(Y_i)$ in $MD(X_j)$ or the grafting of $MD(X_j)$ in $MD(Y_i)$. In all cases, the vertices of $MD(Z)$ are labelled by a subset of $\{b_1, \ldots, b_\ell\}$ and this conclude the induction.

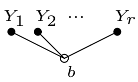
To conclude the triangularity relation we need to show that for any tree $T \in \mathcal{T}[S]$ there is a basis element in the basis (4) with T as its leading degree term. For this we proceed by induction on the degree of T. Our hypothesis is that for any tree $T \in \mathcal{T}[S]$ we can find a set partition $\Phi = \{\phi_1, \ldots, \phi_\ell\} \vdash S$, a permutation $\sigma\colon [\ell] \to [\ell]$ and a family of trees $\{T_i : T_i \in \mathcal{F}[\phi_i],\ T_{\sigma(1)} \text{ has the largest root}\}$, such that T is the leading term of $\Xi(\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}])$. Furthermore, $MD(T)$ is the subtree formed with the vertices labelled by labels of the roots of $T_1, \ldots, T_\ell$.

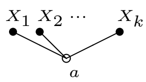
If T is of degree 1, then $T = \Xi(\mathrm{sb}[T])$ and $MD(T)$ is a single vertex. If $deg(T) > 1$, then T is of the form

$T =$ 

, (5)

where a is the largest label adjacent to the root such that $a < b$. Such an a exists since $MD(T)$ is of size $deg(T) > 1$. It is clear that b is the largest value of the labels of $MD(T)$ (it is a decreasing tree, the root has the largest value). By choice, a is the second largest value of the labels of $MD(T)$. We now consider the two subtrees

$Z_1 =$ 

and $Z_2 =$ 

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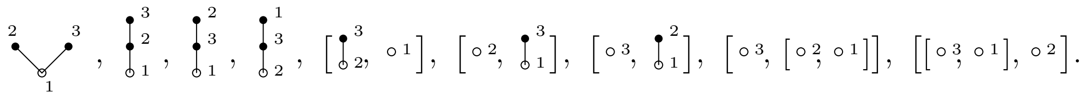
Clearly $\deg(Z_1) < \deg(T)$ and $\deg(Z_2) < \deg(T)$. Hence by induction hypothesis we can find a set partition $\Phi = \{\phi_1, \ldots, \phi_\ell\} \vdash S$, a permutation $\sigma\colon [\ell] \to [\ell]$ and a family of trees $\{T_i : T_i \in \mathcal{F}[\phi_i]\}$ such that $T_{\sigma(1)}$ has root labelled by b and $T_{\sigma(k)}$ has root labelled by a for some $k > 1$. Furthermore $MD(Z_1)$ is the subtree of Z_1 labelled by the labels of the roots of $T_{\sigma(1)}, \ldots, T_{\sigma(k-1)}$ and $MD(Z_2)$ is the subtree of Z_2 labelled by the labels of the roots of $T_{\sigma(k)}, \ldots, T_{\sigma(\ell)}$. We can find this data in such a way that Z_1 is the leading term of $\Xi(\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(k-1)}])$ and Z_2 is the leading term of $\Xi(\mathrm{sb}[T_{\sigma(k)} \cdots T_{\sigma(\ell)}])$.

Using the same argument as before, it is clear that T is the leading term of $[Z_1, Z_2]$. Thus, T is the leading term of

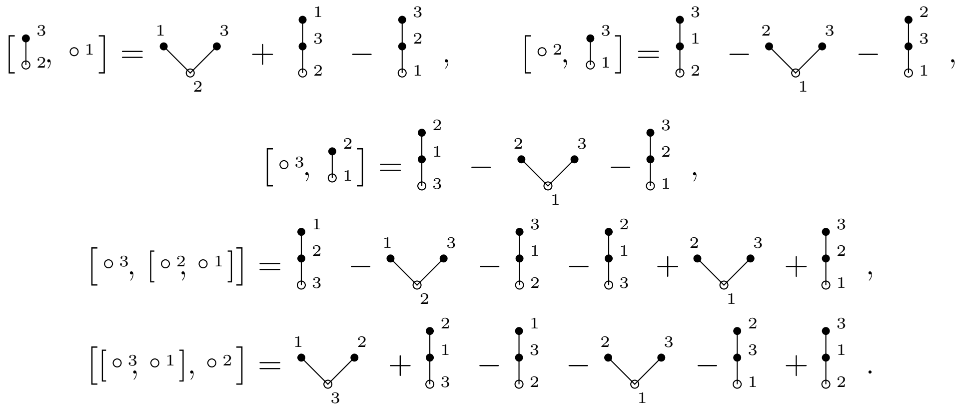
$$\big[\Xi(\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(k-1)}])\,,\, \Xi(\mathrm{sb}[T_{\sigma(k)} \cdots T_{\sigma(\ell)}])\big]. \tag{6}$$

We now need to show that the element in (6) is one of the element in the basis (4). This follows from the fact that a is the second largest element among the labels of the roots of $T_1, \ldots, T_\ell$. In particular it implies that the first step in the Lyndon bracketing of $\Xi(\mathrm{sb}[T_{\sigma(1)}\cdots T_{\sigma(\ell)}])$ is precisely the element in (6). As a conclusion, $MD(T)$ is the subtree of T labelled by the roots of $T_{\sigma(1)},\dots,T_{\sigma(\ell)}$. ■

Example 2.4. Let us compare the basis of $\mathcal{T}[\{1,2,3\}]$ as given in Example 1.1 with the following basis of $(\mathcal{L}ie\circ\mathcal{F})[\{1,2,3\}]$ as given by Eq. (4):



The first four elements are already trees and they correspond to the basis of $\mathcal{F}$ as given in Example 2.1. As we expand the remaining elements in the basis of trees (via Ξ) we get



Each tree of the basis in Example 1.1 appears once as the leading term (the first term) of an expression above.

Remark 2.5. The Lie bracket on $\mathcal{T}$ is filtrated with respect to the degree. More precisely, if we start with two disjoint sets I,J (with a linear order on the disjoint union $I+J$) and two elements in $T\in\mathcal{T}[I]$ and $Y\in\mathcal{T}[J]$ with $\deg(T)=d_1$ and $\deg(Y)=d_2$, then the maximal degree part of $[T,Y]$ is of degree d_1+d_2. This follows from the fact that $MD(Z)$ of the term Z appearing in the maximal degree part of $[T,Y]$ must be obtained from a subtree of either the grafting of $MD(T)$ in $MD(Y)$ or the other way around. Grafting at the root will produce a decreasing tree in one case (hence achieving the degree d_1+d_2). In general other graftings of $MD(T)$ in $MD(Y)$ (or the other way around) may have maximal degree. This did not happen in the proof of the Theorem 2.3 because the label of the roots of $MD(T)$ and $MD(Y)$ where the largest two labels.

3. $\mathcal{T}=\mathcal{L}ie\circ\mathcal{F}$ as species

In the previous section we have shown that $\mathcal{T}[S]$ is isomorphic to $\mathcal{L}ie\circ\mathcal{F}[S]$ as vector spaces for any finite set S with a linear order. Using the basis (4) in Theorem 2.3 we can now define the notion of Lie-degree. We say that an element $\Xi(\mathrm{sb}[T_{\sigma(1)}\cdots T_{\sigma(\ell)}])$ has Lie-degree $\ell=\text{L-deg}(\Xi(\mathrm{sb}[T_{\sigma(1)}\cdots T_{\sigma(\ell)}]))$. For an arbitrary element $\varphi\in\mathcal{T}[S]$ we let $\text{L-deg}(\varphi)=\ell$ if ℓ is the smallest Lie-degree of the basis elements with non-zero coefficient in the expansion of φ in the basis (4). It is clear that given two disjoint sets I,J with linear order on $I+J$,

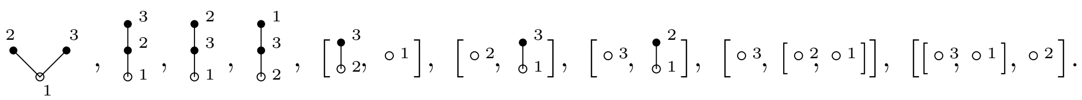
$\varphi \in \mathcal{T}[I]$ and $\psi \in \mathcal{T}[J]$, we have that $\text{L-deg}([\varphi, \psi]) = \text{L-deg}(\varphi) + \text{L-deg}(\psi)$. The notion of Lie-degree is not related to the notion of degree we used in the proof of Theorem 2.3.

We consider the species $[\mathcal{T}, \mathcal{T}]$ which is the image of $[,]\colon \mathcal{T} \bullet \mathcal{T} \to \mathcal{T}$. It follows from our discussion above that $\mathcal{T}[S] \big/ [\mathcal{T}, \mathcal{T}][S]$ is isomorphic to the set of homogeneous elements of Lie-degree equal to one. These elements are precisely $\mathcal{F}[S]$. We thus have the following corollary:

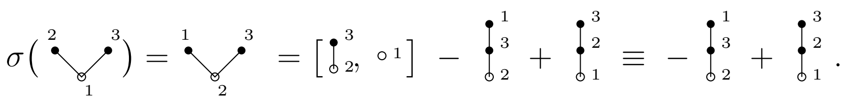
Corollary 3.1. *There is a linear isomorphism* $\mathcal{F}[S] \to \mathcal{T}[S] \big/ [\mathcal{T}, \mathcal{T}][S]$.

Since $\mathcal{T} \big/ [\mathcal{T}, \mathcal{T}]$ is clearly a species, Corollary 3.1 allows us to view $\mathcal{F}$ as a species.

Example 3.2. Using the basis (4) of $\mathcal{T}[3]$, the space $\mathcal{F}[3] = \mathcal{T}[3] \big/ [\mathcal{T}, \mathcal{T}][3]$ is the linear span of the elements:



The first four elements form a basis of $\mathcal{F}[3]$ and the remaining ones are zero modulo $[\mathcal{T}, \mathcal{T}]$. The action of the symmetric group S_3 is given by the action on the quotient. For example, if we consider the transposition $\sigma = (1\,2)$, then



Once we have the identification $\mathcal{F} \cong \mathcal{T} \big/ [\mathcal{T}, \mathcal{T}]$, we have a natural action of the symmetric group on $\mathcal{F}[n]$ and on $\mathcal{L}ie \circ \mathcal{F}[n]$. That is, we have the

Corollary 3.3. $\mathcal{T}[S] = \mathcal{L}ie \circ \mathcal{F}[S]$ *as species.*

Proof. To describe the action of S_n on $\mathcal{L}ie \circ \mathcal{F}[n]$, we consider the natural order on $[n]$ and the basis

$$\left\{ \text{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}] \,:\, {\scriptstyle \Phi = \{\phi_1, \ldots, \phi_\ell\} \vdash S,\ \sigma\colon [\ell] \to [\ell],\ T_i \in \mathcal{F}[\phi_i],\ T_{\sigma(1)}} \text{ has the largest root} \right\}. \qquad (7)$$

A permutation $\pi \in S_n$ acts on a basis element $\text{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}]$ as follows. The permutation π acts on a tree $T_{\sigma(i)}$ by acting on its labels ϕ_i. We write the element $\pi(T_{\sigma(i)})$ as a linear combination of trees in $\mathcal{F}[\pi(\phi_i)]$. We then substitute the results in $\text{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}]$. We use the Jacobi relation and antisymmetry to rewrite the result as a linear combination of elements in the basis (7). The important fact to notice is that the element $\pi\big(\text{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}]\big)$ is a linear combination of elements of the basis (7) with exactly the same Lie-degree (the same number of $\mathcal{F}$-trees bracketed). Hence the matrix representation corresponding to $\pi \in S_n$ acting on the basis (7) is block diagonal, each block corresponds to the Lie-degrees.

On the other hand, we have shown in the Theorem 2.3, that under the map $\Xi\colon \mathcal{L}ie \circ \mathcal{F}[S] \to \mathcal{T}[S]$ the basis (7) is sent to the basis

$$\left\{\Xi(\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}]) : {\scriptstyle \Phi=\{\phi_1,\dots,\phi_\ell\}\vdash S,\ \sigma\colon [\ell]\to[\ell],\ T_i\in\mathcal{F}[\phi_i],\ T_{\sigma(1)}} \text{ has the largest root}\right\}. \tag{8}$$

Now a permutation $\pi \in S_n$ acts on a basis element $\Xi(\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}])$ as follows. We rewrite $\pi(T_{\sigma(i)})$ as a linear combination of basis elements in $\mathcal{T}[\pi(\phi_i)]$. We remark that the Lie-degree of the result is also one but contains higher Lie-degree terms. We then substitute the results in $\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}]$. We use the Jacobi relation and antisymmetry to rewrite the result as a linear combination of elements in the basis (8). The important fact to notice in this case is that the element $\pi\big(\Xi(\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}])\big)$ is a linear combination of elements of the basis (7) with Lie-degree equal or higher than $\Xi(\mathrm{sb}[T_{\sigma(1)} \cdots T_{\sigma(\ell)}])$. Hence the matrix representation corresponding to $\pi \in S_n$ acting on the basis (8) is block triangular, each block corresponding to a Lie-degree. Moreover the block diagonal part of this matrix is exactly the same as the action of π on $\mathcal{L}ie \circ \mathcal{F}$. In characteristic zero, S_n-modules are fully decomposable (by semi-simplicity). It is clear that the S_n-modules $\mathcal{T}[n]$ and $\mathcal{L}ie \circ \mathcal{F}[n]$ have the same block decomposition. Hence they are equivalent as S_n-modules. ■

4. Operations on $\mathcal{F}$

A *non-symmetric operad* is a graded vector space $(P_n)_{n\in\mathbb{N}}$ together with composition products

$$P_n \otimes P_{i_1} \otimes \ldots \otimes P_{i_n} \to P_{i_1+\ldots+i_n}$$

satisfying associativity conditions. As pointed out in the introduction a graded vector space amounts to a contravariant functor from the category of finite linearly ordered sets and ordered bijections to finite dimensional vector spaces. There is a composition product given by

$$P \circ Q[S] = \oplus_{k,S_1,\ldots,S_k} P([k]) \otimes \otimes_{i=1}^{k} Q[S_i]$$

where $[k] = \{1 < \ldots < k\}$ and $S_1 < \ldots < S_k$ in an ordered partition of S. A non-symmetric operad is a monoid in the category of graded vector spaces with respect to this composition product. Obviously, any species becomes a graded vector space when forgetting the action of the symmetric group– or equivalently when restricting on the subcategory of finite linearly ordered sets in the category of finite sets. Besides, any operad is a non-symmetric operad when forgetting the action of the symmetric group.

We prove in this section that $\mathcal{F}$ is a sub non-symmetric operad of $\mathcal{T}$. It is not a suboperad: one needs to forget the action of the symmetric group to prove the result, since $\mathcal{F}$ is not a monoid in the category of species.

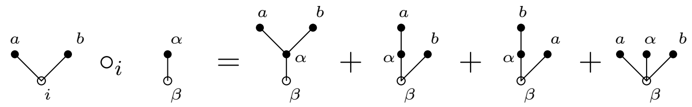
We recall first the operad structure on $\mathcal{T}$ as explained in [4]. To give the operadic structure it is enough to explain the composition on two elements, that is, the compositions

$$\circ_i : \mathcal{T}[I] \otimes \mathcal{T}[J] \to \mathcal{T}[I \setminus \{i\} + J],$$

for two disjoint sets I and J and for $i \in I$. Let $T \in \mathcal{T}[I]$, $In(T,i)$ the set of incoming edges at the vertex labelled by i in T. The composition is defined by

$$T \circ_i S = \sum_{f:In(T,i)\to J} T \circ_i^f S$$

where $T \circ_i^f S$ is the rooted tree obtained by substituting the tree S for the vertex i in T. The outgoing edge of i, if it exists, becomes the outgoing edge of the root of S, whereas the incoming edges of i are grafted on the vertices of S following the map f. The root is the root of T or the root of S if i is the root of T. Here is an example:



If I and J are endowed with a linear order then $I \setminus \{i\} + J$ is endowed with the order on I and J and for all x in I one has $x < J$ if and only if $x < i$ and $x > J$ if and only if $x > i$. Recall that the degree of a tree T is the number of vertices of $MD(T)$, the maximal decreasing connected subtree of T from the root. Given a linearly ordered set I, the vector space $\mathcal{T}[I]$ is filtered by the degree: $F_d\mathcal{T}[I]$ is spanned by the trees T of degree less than d. Therefore $\mathcal{F}[I] = F_1\mathcal{T}[I]$.

Theorem 4.1. *Let I, J be two linearly ordered sets and let $i \in I$. The composition*

$$\circ_i : \mathcal{T}[I] \otimes \mathcal{T}[J] \to \mathcal{T}[I \setminus \{i\} + J]$$

maps $F_d\mathcal{T}[I] \otimes F_e\mathcal{T}[J]$ to $F_{d+e-1}\mathcal{T}[I \setminus \{i\} + J]$.

Proof. Let $T \in \mathcal{T}[I]$, $S \in \mathcal{T}[J]$ and $f : In(T,i) \to J$.

If i is a vertex of $MD(T)$ then any vertex x in T lying in the path from the root of T to i satisfies $x > i$. Then $x > J$ and x is a vertex of $MD(T \circ_i^f S)$. Also any vertex of $MD(S)$ is a vertex of $MD(T \circ_i^f S)$. Let y be a vertex of T attached to i by an edge E in $In(T,i)$. If y is a vertex of $MD(T)$ then $y < i$ and $y < J$. If f sends E to a vertex of $MD(S)$ then y is a vertex of $MD(T \circ_i^f S)$. If it doesn't then the degree of $T \circ_i^f S$ is strictly less than $d + e - 1$. If y is not a vertex of $MD(T)$ then $y > i$ and $y > J$, hence y is not a vertex of $MD(T \circ_i^f S)$.

It is also clear that any other vertex of T which is not in $MD(T)$ or any vertex which is not in $MD(S)$ is not in $MD(T \circ_i^f S)$. As a consequence the degree of $T \circ_i^f S$ is less than $d + e - 1$. It is exactly $d + e - 1$ if f sends any vertex of $MD(T)$ attached to i by an edge in $In(T,i)$ to a vertex of $MD(S)$. In this case $MD(T \circ_i^f S) = MD(T) \circ_i^{\tilde{f}} MD(S)$ where $\tilde{f}$ is the restriction of f to the set of edges of $MD(T)$.

If i is not a vertex of $MD(T)$ then one sees easily that $MD(T \circ_i^f S) = MD(T)$. ∎

Corollary 4.2. *The collection $(\mathcal{F}[n])_{n\geq 1}$ forms a sub non-symmetric operad of the pre-Lie operad $\mathcal{T}$.*

Proof. We apply the previous theorem to $d = e = 1$. ∎

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Received March 11, 2009
and in final form December 17, 2009