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The Construction of Hom-Lie Bialgebras

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Abstract. Motivated by recent literature on Hom-Lie algebras and the construction of Lie bialgebras, we investigate the construction of Hom-Lie bialgebras as a generalization of Lie bialgebras. In this paper, we mainly show how to construct (triangular coboundary) Hom-Lie bialgebras both through Hom-Lie algebras and Hom-Lie coalgebras. As examples, we consider the construction of (triangular coboundary) Hom-Lie bialgebras on the three-dimensional Heisenberg algebra, on the split simple Lie algebra $sl(2)$ and on E^3, the three-dimensional Euclidean space.
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1. Introduction and preliminaries

Generalizations of Hopf algebras over fields have quite a long history. The weakening of the (co)associativity leads to Hom-Hopf algebra which is twisted by a linear endomorphism. That is, the associativity of the algebra structure is replaced by the Hom-associativity, namely

$$\alpha(a)(bc) = (ab)\alpha(c),$$

where α is an endomorphism of the algebra. The Hom-associativity was introduced in [13]. In recent years, Hom-structures have been investigated by some scholars in [2, 3, 4, 5, 11, 12, 13, 14, 17, 18]. Hom-algebras were first defined for Lie algebras. Earlier precursors of Hom-Lie algebras can be found in [7]. In [6, 9, 10], Hom-Lie algebras were introduced to describe the structure on certain q-deformations of the Witt and the Virasoro algebras by Silvestrov and his collaborators. In [9], the class of quasi-Lie algebras and subclasses of quasi-Hom-Lie algebras and Hom-Lie algebras have been introduced. These classes of algebras are tailored in the way suitable for simultaneous treatment of the Lie algebras, Lie superalgebras and color Lie algebras and their q-deformations. The idea in the Hom-Lie algebras is that

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the Jacobi identity is replaced by the so called Hom-Jacobi identity through a Lie algebra endomorphism α, namely

$$[[x,y],\alpha(z)]+[[y,z],\alpha(x)]+[[z,x],\alpha(y)]=0.$$

A Lie algebra is a Hom-Lie algebra with $\alpha = id$. More generally, if L is a Lie algebra and α is a Lie algebra endomorphism, then L becomes a Hom-Lie algebra with the bracket $[x,y]_\alpha = \alpha([x,y])$ which was introduced by Yau in [18].

The classical Yang-Baxter equation (CYBE), also known as the classical triangle equation, was investigated by Sklyanin in the context of quantum inverse scattering method. The CYBE in a Lie algebra L states $[r^{12},r^{13}]+[r^{12},r^{23}]+[r^{13},r^{23}]=0$ for $r\in L^{\otimes 2}$. The CYBE and QYBE (quantum Yang-Baxter equation) are collectively known as the YBE, which were first introduced by Yang, Baxter and McGuire. The various forms of the YBE are used in physics. A twisted Hom type generalization of the YBE called Hom-Yang-Baxter equation (HYBE) was introduced in [19, 20] by Yau. The HYBE states

$$(\alpha\otimes B)\circ(B\otimes\alpha)\circ(\alpha\otimes B)=(B\otimes\alpha)\circ(\alpha\otimes B)\circ(B\otimes\alpha).$$

where α is an endomorphism of the vector space V, and $B: V^{\otimes 2}\to V^{\otimes 2}$ is a bilinear map that commutes with $\alpha^{\otimes 2}$. Meanwhile, Yau defined the classical Hom-Yang-Baxter equation (CHYBE) in the same manner and studied Hom-Lie bialgebras in [21].

In [15], Michaelis obtained the structure of a triangular coboundary Lie bialgebra on a Lie algebra containing two linear independent elements a and b satisfying $[a,b]=pb$ for some non-zero $p\in k$, where k denotes a fixed but arbitrary field.

In [22], according to the Lie comodule theory, Zhang constructed some (triangular) Lie bialgebras through Lie coalgebras.

In view of the above works, the motivation of the construction of Hom-Lie bialgebra on a Hom-Lie algebra is natural. The purpose of the present paper is to investigate how to construct Hom-Lie bialgebras both through Hom-Lie algebras and Hom-Lie coalgebras, respectively.

This paper is organized as follows. In Section 1, we recall some basic definitions and give a summary of the fundamental properties concerning Hom-Lie bialgebras. In Section 2, we mainly investigate the construction of (triangular coboundary) Hom-Lie bialgebras from Hom-Lie algebras. In the last section we study how to construct Hom-Lie bialgebras through Hom-Lie coalgebras.

As immediate consequences, by examples, we investigate the construction of (triangular coboundary) Hom-Lie bialgebras on the three-dimensional Heisenberg algebra, on the split three-dimensional simple Lie algebra $sl(2)$ and on E^3, the three-dimensional Euclidean space.

We always work over a fixed field k. For a Lie coalgebra Γ, we write its Lie-cobracket $\Delta:\Gamma\to\Gamma^{\otimes 2}, x\mapsto x_1\otimes x_2$, for any $x\in\Gamma$; for a left $\Gamma-$comodule M, we denote its coaction by $\rho(m)=m_{(-1)}\otimes m_{(0)}$ for any $m\in M$. Let ξ be the cyclic permutation (1 2 3), we denote the symbol $\circlearrowleft$ by the sum over id,ξ,ξ^2. Namely, we denote the Hom-Jacobi identity by $\circlearrowleft [[x,y],\alpha(z)]=0$ in place of

$[[x,y],\alpha(z)]+[[y,z],\alpha(x)]+[[z,x],\alpha(y)]=0$, for any $x,y,z\in\Gamma$. Any unexplained definitions and notations may be found in [16].

In what follows, we recall some concepts and results used in this paper.

Definition 1.1. A Hom-Lie algebra in [12] is a triple $(L,[-,-],\alpha)$ consisting of vector space L, bilinear map $[-,-]:L^{\otimes 2}\to L$ (called the "bracket") and a linear space endomorphism $\alpha:L\to L$ satisfying

$$[x,y]+[y,x]=0, \qquad (anti-symmetric)$$

$$\circlearrowleft [[x,y],\alpha(z)]=0, \qquad (Hom-Jacobi\ \ identity)$$

for any $x,y,z\in L$.

Hom Lie algebras with additional property that α is an algebra homomorphism

$$\alpha[x,y]=[\alpha(x),\alpha(y)]$$

are called multiplicative Hom-Lie algebras. In what follows, unless stated otherwise, whenever we write Hom-Lie algebra we mean multiplicative Hom-Lie algebra.

Let $(L,[-,-])$ be a Lie algebra, and $\alpha:L\to L$ be a Lie algebra endomorphism. Define a new bracket $[-,-]_\alpha$ on L by setting $[x,y]_\alpha=\alpha[x,y]$, then a direct calculation shows that $L_\alpha=(L,[-,-]_\alpha,\alpha)$ is a Hom-Lie algebra.

Definition 1.2. Let $(L,[-,-],\alpha)$ be a Hom-Lie algebra. An L-Hom-Lie module (V,β) consists of a vector space V and a linear endomorphism $\beta:V\to V$ together with a bilinear function $\psi:L\otimes V\to V, x\otimes v\mapsto x\cdot v$ satisfying

$$[x,y]\cdot\beta(v)=\alpha(x)\cdot(y\cdot v)-\alpha(y)\cdot(x\cdot v),$$

$$\beta(x\cdot v)=\alpha(x)\cdot\beta(v),$$

for all $x,y\in L, v\in V$.

It is straightforward that Hom-Lie algebra $(L,[-,-],\alpha)$ is itself an L-Hom-Lie module via its Lie-bracket $[-,-]$. Explicitly, L,α and $[-,-]$ are equivalent to the vector space V, the linear endomorphism β and the bilinear function ψ, respectively.

Let $(L,[-,-],\alpha)$ be a Hom-Lie algebra. For any $x\in L$ and integer number $n\geq 2$, we define the adjoint diagonal action $ad_x:L^{\otimes n}\to L^{\otimes n}$ by

$$ad_x(y_1\otimes\cdots\otimes y_n)=\sum_{i=1}^{n}\alpha(y_1)\otimes\cdots\otimes\alpha(y_{i-1})\otimes[x,y_i]\otimes\alpha(y_{i+1})\cdots\otimes\alpha(y_n).$$

In particular, for $n=2$, we have

$$ad_x(y_1\otimes y_2)=[x,y_1]\otimes\alpha(y_2)+\alpha(y_1)\otimes[x,y_2].$$

Definition 1.3. A Hom-Lie coalgebra introduced in [21], is a triple (Γ, Δ, α) with a vector space Γ, linear map $\Delta : \Gamma \to \Gamma^{\otimes 2}$ (called the "cobracket") and a linear endomorphism $\alpha : \Gamma \to \Gamma$, such that

$$\Delta \circ \alpha = \alpha^{\otimes 2} \circ \Delta, \qquad (co-multiplicativity)$$

$$\Delta + \tau \circ \Delta = 0, \qquad (anti-symmetric)$$

$$\circlearrowleft (\alpha \otimes \Delta) \circ \Delta = 0, \qquad (Hom-co-Jacobi\ \ identity)$$

Definition 1.4. A Hom-Lie bialgebra introduced in [21], is a quadruple $(L, [-,-], \Delta, \alpha)$ in which $(L, [-,-], \alpha)$ is a Hom-Lie algebra and (L, Δ, α) is a Hom-Lie coalgebra such that the following compatibility condition holds, for all $x, y \in L$,

$$\Delta([x, y]) = ad_{\alpha(x)}(\Delta(y)) - ad_{\alpha(y)}(\Delta(x)).$$

Explicitly, the compatibility condition can be written as

$$\Delta([x, y]) = [\alpha(x), y_1] \otimes \alpha(y_2) + \alpha(y_1) \otimes [\alpha(x), y_2] - [\alpha(y), x_1] \otimes \alpha(x_2) - \alpha(x_1) \otimes [\alpha(y), x_2].$$

Definition 1.5. A coboundary Hom-Lie bialgebra $(L, [-,-], \Delta, \alpha, r)$ in [21] consists of a Hom-Lie bialgebra $(L, [-,-], \Delta, \alpha)$ and $r \in Im(id - \tau) \subseteq L \otimes L$, such that for any $x \in L$,

$$\alpha^{\otimes 2}(r) = r, \ \Delta(x) = ad_x(r).$$

Furthermore, if r satisfies the classical Hom-Yang-Baxter equation

$$CH(r) = [r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] = 0,$$

then we call it a triangular coboundary Hom-Lie bialgebra. Here,

$$[r^{12}, r^{13}] = [a_i, a_j] \otimes \alpha(b_i) \otimes \alpha(b_j),$$

$$[r^{12}, r^{23}] = \alpha(a_i) \otimes [b_i, a_k] \otimes \alpha(b_k),$$

$$[r^{13}, r^{23}] = \alpha(a_j) \otimes \alpha(a_k) \otimes [b_j, b_k],$$

where $r^{12} = r \otimes 1 = a_i \otimes b_i \otimes 1, r^{13} = (\tau \otimes id)(1 \otimes r) = a_j \otimes 1 \otimes b_j$ and $r^{23} = 1 \otimes r = 1 \otimes a_k \otimes b_k$.

2. The Construction of Hom-Lie Bialgebras Through Hom-Lie Algebras

In this section we will study Hom-Lie bialgebras further and mainly investigate the construction of Hom-Lie bialgebras from Hom-Lie algebras.

Lemma 2.1. *Let $(L, [-,-], \alpha,)$ be a Hom-Lie algebra. Then $(L^{\otimes 2}, \alpha^{\otimes 2})$ is an L-Hom-Lie module under the adjoint diagonal action $ad_x : L^{\otimes 2} \to L^{\otimes 2}$ by $x \cdot (a \otimes b) = ad_x(a \otimes b)$.*

Proof. For all $x, y \in L, a \otimes b \in L^{\otimes 2}$,

$$\begin{aligned}
&\alpha(x) \cdot (y \cdot (a \otimes b)) - \alpha(y) \cdot (x \cdot (a \otimes b)) \\
&= ad_{\alpha(x)}(ad_y(a \otimes b)) - ad_{\alpha(y)}(ad_x(a \otimes b)) \\
&ad_{\alpha(x)}([y, a] \otimes \alpha(b) + \alpha(a) \otimes [y, b]) - ad_{\alpha(y)}([x, a] \otimes \alpha(b) + \alpha(a) \otimes [x, b]) \\
&= [\alpha(x), [y, a]] \otimes \alpha^2(b) + \alpha[y, a] \otimes [\alpha(x), \alpha(b)] + [\alpha(x), \alpha(a)] \otimes \alpha[y, b] \\
&+\alpha^2(a) \otimes [\alpha(x), [y, b]] - [\alpha(y), [x, a]] \otimes \alpha^2(b) - \alpha[x, a] \otimes [\alpha(y), \alpha(b)] \\
&-[\alpha(y), \alpha(a)] \otimes \alpha[x, b] - \alpha^2(a) \otimes [\alpha(y), [x, b]] \\
&= ([\alpha(x), [y, a]] + [\alpha(y), [a, x]]) \otimes \alpha^2(b) \\
&+\alpha^2(a) \otimes ([\alpha(x), [y, b]] + [\alpha(y), [b, x]]) \\
&= [[x, y], \alpha(a)] \otimes \alpha^2(b) + \alpha^2(a) \otimes [[x, y], \alpha(b)]
\end{aligned}$$

$$\begin{aligned}
[x, y] \cdot \alpha^{\otimes 2}(a \otimes b) &= ad_{[x,y]}(\alpha(a) \otimes \alpha(b)) \\
&= [[x, y], \alpha(a)] \otimes \alpha^2(b) + \alpha^2(a) \otimes [[x, y], \alpha(b)].
\end{aligned}$$

It is easy to show that $\alpha^{\otimes 2}(x \cdot (a \otimes b)) = \alpha[x, a] \otimes \alpha^2(b) + \alpha^2(a) \otimes \alpha[x, b] = \alpha(x) \cdot \alpha^{\otimes 2}(a \otimes b)$. Thus $(L^{\otimes 2}, \alpha^{\otimes 2})$ is an L-Hom-Lie module. ■

Proposition 2.2. *Let $(L, [-, -], \alpha)$ be a Hom-Lie algebra, and $r = a_i \otimes b_i \in Im(id - \tau) \subseteq L^{\otimes 2}$ together with $\alpha^{\otimes 2}(r) = r$. Define $\Delta : L \to L^{\otimes 2}$, $\Delta(x) = ad_x(r)$ for any $x \in L$. Then*

(1) $\Delta + \tau \circ \Delta = 0$.

(2) $\alpha^{\otimes 2} \circ \Delta = \Delta \circ \alpha$.

(3) $\Delta([x, y]) = ad_{\alpha(x)}(\Delta(y)) - ad_{\alpha(y)}(\Delta(x))$ *for any* $x, y \in L$.

Proof. (1) It is straightforward by $r \in Im(id - \tau)$.

(2) For any $x \in L$, since $\alpha^{\otimes 2}(r) = r$, we have

$$\begin{aligned}
\alpha^{\otimes 2} \circ \Delta(x) &= \alpha^{\otimes 2} \circ ad_x(r) \\
&= \alpha[x, a_i] \otimes \alpha^2(b_i) + \alpha^2(a_i) \otimes \alpha[x, b_i] \\
&= [\alpha(x), a_i] \otimes \alpha(b_i) + \alpha(a_i) \otimes [\alpha(x), b_i] \\
&= ad_{\alpha(x)}(r) = \Delta(\alpha(x)).
\end{aligned}$$

(3) By Lemma 2.1, we know that $(L^{\otimes 2}, \alpha^{\otimes 2})$ is an L-Hom-Lie module under the adjoint diagonal action. Hence, for any $x, y \in L$,

$$[x, y] \cdot (\alpha \otimes \alpha)(a_i \otimes b_i) = \alpha(x) \cdot (y \cdot (a_i \otimes b_i)) - \alpha(y) \cdot (x \cdot (a_i \otimes b_i)),$$

that is,

$$ad_{[x,y]}(a_i \otimes b_i) = ad_{\alpha(x)}(ad_y(a_i \otimes b_i)) - ad_{\alpha(y)}(ad_x(a_i \otimes b_i)).$$

So $\Delta([x, y]) = ad_{\alpha(x)}(\Delta(y)) - ad_{\alpha(y)}(\Delta(x))$. ■

Example 2.3. Let $(L, [-, -])$ denote the Lie algebra on E^3 with basis elements $\{e_1, e_2, e_3\}$, whose bracket is given by

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2.$$

Define an endomorphism $\alpha : L \to L$ such that $\alpha(e_1) = e_2, \alpha(e_2) = e_3, \alpha(e_3) = e_1$. We can check easily that α is a morphism of Lie algebra, that is,

$$\alpha[e_1, e_2] = e_1 = [\alpha(e_1), \alpha(e_2)],$$
$$\alpha[e_2, e_3] = e_2 = [\alpha(e_2), \alpha(e_3)],$$
$$\alpha[e_3, e_1] = e_3 = [\alpha(e_3), \alpha(e_1)].$$

Then a Hom-Lie algebra $L' = (L, [-,-]_\alpha = \alpha \circ [-,-], \alpha)$ is obtained.

Set

$$r = e_1 \otimes e_2 - e_2 \otimes e_1 + e_2 \otimes e_3 - e_3 \otimes e_2 + e_3 \otimes e_1 - e_1 \otimes e_3 \in L'^{\otimes 2}.$$

It's not difficult to show that $r \in Im(id - \tau)$ and $\alpha^{\otimes 2}(r) = r$.

For any $x \in L'$, define $\Delta(x) = ad_x(r)$. Then

$$\Delta(e_1) = e_2 \otimes e_1 - e_1 \otimes e_2 + e_2 \otimes e_3 - e_3 \otimes e_2,$$
$$\Delta(e_2) = e_3 \otimes e_2 - e_2 \otimes e_3 + e_3 \otimes e_1 - e_1 \otimes e_3,$$
$$\Delta(e_3) = e_1 \otimes e_3 - e_3 \otimes e_1 + e_1 \otimes e_2 - e_2 \otimes e_1.$$

We can find that $\Delta + \tau \circ \Delta = 0$. For simplicity and clarity, we write (i, j, k) in place of $(e_i \otimes e_j \otimes e_k)$. Then $\circlearrowleft (\alpha \otimes \Delta) \circ \Delta(e_1) = 0$. In fact, since

$$\begin{aligned}
(\alpha \otimes \Delta) \circ \Delta(e_1) &= \alpha(e_2) \otimes \Delta(e_1) - \alpha(e_1) \otimes \Delta(e_2) + \alpha(e_2) \otimes \Delta(e_3) - \alpha(e_3) \otimes \Delta(e_2) \\
&= (3,2,1) - (3,1,2) + (3,2,3) - (3,3,2) \\
&\quad -(2,3,2) + (2,2,3) - (2,3,1) + (2,1,3) \\
&\quad +(3,1,3) - (3,3,1) + (3,1,2) - (3,2,1) \\
&\quad -(1,3,2) + (1,2,3) - (1,3,1) + (1,1,3) \\
&= (3,2,3) - (3,3,2) - (2,3,2) + (2,2,3) \\
&\quad -(2,3,1) + (2,1,3) + (3,1,3) - (3,3,1) \\
&\quad -(1,3,2) + (1,2,3) - (1,3,1) + (1,1,3),
\end{aligned}$$

we have

$$\begin{aligned}
\circlearrowleft (\alpha \otimes \Delta) \circ \Delta(e_1) &= \circlearrowleft ((3,2,3) - (3,3,2) - (2,3,2) + (2,2,3) - (2,3,1) + (2,1,3) \\
&\quad +(3,1,3) - (3,3,1) - (1,3,2) + (1,2,3) - (1,3,1) + (1,1,3)).
\end{aligned}$$

Now, we can check that $\circlearrowleft ((3,2,3) - (3,3,2)) = (3,2,3) - (3,3,2) + (2,3,3) - (3,2,3) + (3,3,2) - (2,3,3)) = 0$.

By analogy,

$$\circlearrowleft (-(2,3,2) + (2,2,3)) = 0,$$
$$\circlearrowleft ((3,1,3) - (3,3,1)) = 0,$$
$$\circlearrowleft (-(1,3,1) + (1,1,3)) = 0.$$

Meanwhile,

$$\begin{aligned}
&\circlearrowleft (-(2,3,1) + (2,1,3) - (1,3,2) + (1,2,3)) \\
&\quad = (2,1,3) - (2,3,1) + (1,2,3) - (1,3,2) \\
&\quad\quad +(1,3,2) - (3,1,2) + (2,3,1) - (3,2,1) \\
&\quad\quad +(3,2,1) - (1,2,3) + (3,1,2) - (2,1,3) \\
&\quad = 0.
\end{aligned}$$

Therefore the summation $\circlearrowleft (\alpha\otimes\Delta)\circ\Delta(e_1)=0$. In the same way,

$$\circlearrowleft (\alpha\otimes\Delta)\circ\Delta(e_2)=0,$$
$$\circlearrowleft (\alpha\otimes\Delta)\circ\Delta(e_3)=0.$$

Thus (L,Δ,α) is a Hom-Lie coalgebra. By Proposition 2.2, the compatibility between $[-,-]_\alpha$ and Δ holds immediately though it also follows by computing directly. Thus $(L,[-,-]_\alpha,\Delta,\alpha,r)$ is a coboundary Hom-Lie bialgebra.

Furthermore, is it triangular? We will solve the problem by checking if the classical Hom-Yang-Baxter equation holds or not, that is, $CH(r)$ equals zero or not.

In fact,

$$\begin{aligned} r_{12} &= e_1\otimes e_2\otimes 1-e_2\otimes e_1\otimes 1+e_2\otimes e_3\otimes 1-e_3\otimes e_2\otimes 1+e_3\otimes e_1\otimes 1-e_1\otimes e_3\otimes 1,\\ r_{13} &= e_1\otimes 1\otimes e_2-e_2\otimes 1\otimes e_1+e_2\otimes 1\otimes e_3-e_3\otimes 1\otimes e_2+e_3\otimes 1\otimes e_1-e_1\otimes 1\otimes e_3,\\ r_{23} &= 1\otimes e_1\otimes e_2-1\otimes e_2\otimes e_1+1\otimes e_2\otimes e_3-1\otimes e_3\otimes e_2\ +1\otimes e_3\otimes e_1-1\otimes e_1\otimes e_3. \end{aligned}$$

It's a tedious checking procedure needing earnest and patience. At last, we get

$$\begin{aligned} CH(r) &= [r^{12},r^{13}]_\alpha+[r^{12},r^{23}]_\alpha+[r^{13},r^{23}]_\alpha\\ &= 3[(1,2,3)+(2,3,1)+(3,1,2)-(2,1,3)\\ &\quad -(3,2,1)]-(1,3,2)\neq 0. \end{aligned}$$

Thus $(L,[-,-]_\alpha,\Delta,\alpha,r)$ is a coboundary Hom-Lie bialgebra but not triangular.

In the following, we'll give the main result of this section.

Theorem 2.4. *Let $(L,[-,-],\alpha)$ be a Hom-Lie algebra, containing linearly independent elements a and b satisfying $[a,b]=p\alpha(a)$ or $[a,b]=p\alpha(b)$ with $0\neq p\in k$. Set*

$$r=a\otimes b-b\otimes a$$

and assume that $\alpha^{\otimes 2}(r)=r$ and define a linear map $\Delta_r: L\to L^{\otimes 2}$ by

$$\Delta_r(x)=ad_x(r)=[x,a]\otimes\alpha(b)+\alpha(a)\otimes[x,b]-\alpha(b)\otimes[x,a]-[x,b]\otimes\alpha(a)$$

for any $x\in L$. Then Δ_r equips L with the structure of a triangular coboundary Hom-Lie bialgebra.

Proof. We just give a proof for the case where $[a,b]=p\alpha(b)$ with $p=1$. For any non-zero $p\in k$ and $[a,b]=p\alpha(a)$, the proof is exactly analogous.

For simplicity, we set

$$f=[[x,a],\alpha(b)],\quad g=[[x,b],\alpha(a)],\quad h=\alpha[x,b]$$

for any $x\in L$. By straightforward computation, the checking procedure needs more patience and carefulness than technique.

For an arbitrary $x\in L$,

$$\begin{aligned} \circlearrowleft (\alpha\otimes\Delta_r)\circ\Delta_r(x) =&\alpha^2(a)\otimes(g+h-f)\otimes\alpha^2(b)-\alpha^2(b)\otimes(g+h-f)\otimes\alpha^2(a)\\ &+\alpha^2(b)\otimes\alpha^2(a)\otimes(g+h-f)-\alpha^2(a)\otimes\alpha^2(b)\otimes(g+h-f)\\ &+(g+h-f)\otimes\alpha^2(b)\otimes\alpha^2(a)-(g+h-f)\otimes\alpha^2(a)\otimes\alpha^2(b)\\ &=0, \end{aligned}$$

since

$$\begin{aligned} g+h-f &= [[x,b],\alpha(a)]+\alpha[x,b]-[[x,a],\alpha(b)] \\ &= [[x,b],\alpha(a)]+[\alpha(x),\alpha(b)]+[[a,x],\alpha(b)] \\ &= [[x,b],\alpha(a)]+[\alpha(x),[a,b]]+[[a,x],\alpha(b)] \\ &= [[x,b],\alpha(a)]+[[b,a],\alpha(x)]+[[a,x],\alpha(b)] \\ &= 0. \end{aligned}$$

And according to Proposition 2.2, $(L,[-,-],\Delta_r,\alpha)$ is a coboundary Hom-Lie bialgebra.

Now, we have only to show that r satisfies the classical Hom-Yang-Baxter equation $CH(r)=0$. Since $r=a\otimes b-b\otimes a$, by Definition 1.5,

$$\begin{aligned} r^{12} &= a\otimes b\otimes 1-b\otimes a\otimes 1, \\ r^{13} &= a\otimes 1\otimes b-b\otimes 1\otimes a, \\ r^{23} &= 1\otimes a\otimes b-1\otimes b\otimes a, \end{aligned}$$

so,

$$\begin{aligned} [r^{12},r^{13}] &= -[a,b]\otimes\alpha(b)\otimes\alpha(a)-[b,a]\otimes\alpha(a)\otimes\alpha(b), \\ [r^{12},r^{23}] &= \alpha(a)\otimes[b,a]\otimes\alpha(b)+\alpha(b)\otimes[a,b]\otimes\alpha(a), \\ [r^{13},r^{23}] &= -\alpha(a)\otimes\alpha(b)\otimes[b,a]-\alpha(b)\otimes\alpha(a)\otimes[a,b], \end{aligned}$$

that is,

$$\begin{aligned} [r^{12},r^{13}] &= -\alpha(b)\otimes\alpha(b)\otimes\alpha(a)+\alpha(b)\otimes\alpha(a)\otimes\alpha(b), \\ [r^{12},r^{23}] &= -\alpha(a)\otimes\alpha(b)\otimes\alpha(b)+\alpha(b)\otimes\alpha(b)\otimes\alpha(a), \\ [r^{13},r^{23}] &= \alpha(a)\otimes\alpha(b)\otimes\alpha(b)-\alpha(b)\otimes\alpha(a)\otimes\alpha(b). \end{aligned}$$

So, $CH(r)=[r^{12},r^{13}]+[r^{12},r^{23}]+[r^{13},r^{23}]=0$. The theorem is proved. ■

In the following three examples, we work over the field $\mathcal{C}$ of complex numbers.

Example 2.5. Let L be a vector space with basis $\{X,Y\}$. Equipped L with a Lie bracket, we get a Lie algebra $(L,[-,-])$ as follows

$$[X,X]=0,\quad [Y,Y]=0,\quad [X,Y]=X,\quad [Y,X]=-X.$$

Given a Lie algebra morphism α on $(L,[-,-])$, the Hom-Lie bracket in the corresponding Hom-Lie algebra $L_\alpha=(L,[-,-]_\alpha=\alpha\circ[-,-],\alpha)$ is determined by the following relations

$$\alpha(X)=X,\quad \alpha(Y)=aX+Y,\quad a\in\mathcal{C}.$$

Then,

$$[X,Y]_\alpha=\alpha[X,Y]=\alpha(X)=X.$$

Set

$$r=X\otimes Y-Y\otimes X$$

with $\alpha^{\otimes 2}(r)=X\otimes(aX+Y)-(aX+Y)\otimes X=X\otimes Y-Y\otimes X=r$ and define

$$\Delta_r(l)=ad_l(r)=[l,X]_\alpha\otimes\alpha(Y)-\alpha(Y)\otimes[l,X]_\alpha+\alpha(X)\otimes[l,Y]_\alpha-[l,Y]_\alpha\otimes\alpha(X),$$

for all $l \in L_\alpha$.

Therefore,

$$\begin{aligned} \Delta_r(X) &= \alpha(X) \otimes [X,Y]_\alpha - [X,Y]_\alpha \otimes \alpha(X) \\ &= X \otimes X - X \otimes X = 0, \\ \Delta_r(Y) &= [Y,X]_\alpha \otimes \alpha(Y) - \alpha(Y) \otimes [Y,X]_\alpha \\ &= -X \otimes (aX+Y) + (aX+Y) \otimes X \\ &= Y \otimes X - X \otimes Y. \end{aligned}$$

According to Theorem 2.4, $(L, [-,-]_\alpha, \Delta_r, \alpha)$ is a triangular coboundary Hom-Lie bialgebra.

Example 2.6. The split three-dimensional simple Lie algebra in [8,pp.14] is $sl(2) = span_{\mathcal{C}}\{X,Y,Z\}$, whose bracket is determined by the relations

$$[X,Y] = 2Y, \ [X,Z] = -2Z, \ [Y,Z] = X.$$

Consider the linear Lie algebra map $\alpha : sl(2) \to sl(2)$,

$$\alpha(X) = X - 2Z, \ \alpha(Y) = X + Y - Z, \ \alpha(Z) = Z.$$

Then we obtain the Hom-Lie bracket

$$\begin{aligned} &[X,Y]_\alpha = 2(X+Y-Z) = -[Y,X]_\alpha, \\ &[X,Z]_\alpha = -2Z = -[Z,X]_\alpha, \\ &[Y,Z]_\alpha = X - 2Z = -[Z,Y]_\alpha \end{aligned}$$

in the corresponding Hom-Lie algebra $sl(2)_\alpha = (sl(2), [-,-]_\alpha, \alpha)$. By taking $a=0, b=1, c=1$ in the matrix α_1 in Theorem 1.5 of [20], we get the above Hom-Lie algebra $sl(2)_\alpha$.

For linearly independent elements X, Z, we know $[X,Z]_\alpha = -2Z = -2\alpha(Z)$. Set

$$r = X \otimes Z - Z \otimes X,$$

with $\alpha^{\otimes 2}(r) = r$ holding and let

$$\Delta_r(s) = ad_s(r),$$

for any $s \in sl(2)$. According to Theorem 2.4, $(sl(2), [-,-]_\alpha, \Delta_r, \alpha)$ is a coboundary triangular Hom-Lie bialgebra with

$$\begin{aligned} &\Delta_r(X) = 2(Z \otimes X - X \otimes Z), \\ &\Delta_r(Y) = 2(Z \otimes X + Z \otimes Y - X \otimes Z - Y \otimes Z), \\ &\Delta_r(Z) = 0. \end{aligned}$$

Example 2.7. Consider the three-dimensional Heisenberg algebra in [1, Example 2] $H = span_{\mathcal{C}}\{X,Y,Z\}$ together with the Lie bracket (the Heisinberg relation)

$$[X,Y] = [X,Z] = 0, \quad [Y,Z] = X.$$

Given a linear map of Lie algebra $\alpha : H \to H$ such that

$$\alpha(X) = X,\ \alpha(Y) = cY,\ \alpha(Z) = c^{-1}Z$$

for some none-zero complex number c. The Hom-Lie bracket in the corresponding Hom-Lie algebra $H_\alpha = (H, [-,-]_\alpha = \alpha \circ [-,-], \alpha)$ is given by the following relations

$$[X,Y]_\alpha = [X,Z]_\alpha = 0 = -[Y,X]_\alpha = -[Z,X]_\alpha,$$
$$[Y,Z]_\alpha = X = -[Z,Y]_\alpha.$$

We know that, H_α does not satisfy the condition containing linearly independent elements a and b such that $[a,b]_\alpha = k\alpha(a)$ or $k\alpha(b)$ as in Theorem 2.4. But we can still construct a Hom-Lie bialgebra on H_α.

Set

$$r = Y \otimes Z - Z \otimes Y$$

and let

$$\Delta_r(h) = ad_h(r),$$

for all $h \in H_\alpha$. Then $\alpha^{\otimes 2}(r) = r$, and

$$\begin{aligned}
&\Delta_r(X) = 0,\\
&\Delta_r(Y) = cY \otimes X - cX \otimes Y,\\
&\Delta_r(Z) = c^{-1}Z \otimes X - c^{-1}X \otimes Z.
\end{aligned}$$

We can check easily that the Hom-co-Jacobi identity and the compatibility hold, so H_α is a coboundary Hom-Lie bialgebra. That is to say, the condition in Theorem 2.4 is just sufficient to construct a Hom-Lie bialgebra but not necessary.

Furthermore, since

$$[r^{12}, r^{13}] = X \otimes Y \otimes Z - X \otimes Z \otimes Y,$$

$$[r^{12}, r^{23}] = Z \otimes X \otimes Y - Y \otimes X \otimes Z,$$

$$[r^{13}, r^{23}] = Y \otimes Z \otimes X - Z \otimes Y \otimes X,$$

then

$$[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] \neq 0.$$

Therefore, H_α is a coboundary but not triangular Hom-Lie bialgebra.

3. The Construction of Hom-Lie Bialgebras Through Hom-Lie Coalgebras

In this section, we mainly construct Hom-Lie bialgebras through Hom-Lie coalgebras.

Definition 3.1. Let (Γ, Δ, α) be a Hom-Lie coalgebra. A Γ-Hom-Lie comodule (V, β) consists of a vector space V and a linear endomorphism $\beta : V \to V$ together with a linear map $\rho : V \to \Gamma \otimes V, v \mapsto v_{(-1)} \otimes v_{(0)}$ satisfying

$$(\Delta \otimes \beta) \circ \rho = (\alpha \otimes \rho) \circ \rho - (\tau \otimes id) \circ (\alpha \otimes \rho) \circ \rho,$$

$$\rho \circ \beta = (\alpha \otimes \beta) \circ \rho,$$

that is, for any $v \in V$, we have

$$v_{(-1)1} \otimes v_{(-1)2} \otimes \beta(v_{(0)}) = \alpha(v_{(-1)}) \otimes v_{(0)(-1)} \otimes v_{(0)(0)} - v_{(0)(-1)} \otimes \alpha(v_{(-1)}) \otimes v_{(0)(0)},$$

$$\beta(v)_{(-1)} \otimes \beta(v)_{(0)} = \alpha(v_{(-1)}) \otimes \beta(v_{(0)}).$$

Lemma 3.2. *Let (Γ, Δ, α) be a Hom-Lie coalgebra, and (M, μ), (N, ν) two Γ-Hom-Lie comodules with comodule structure ρ_M, ρ_N. Then $(M \otimes N, \mu \otimes \nu)$ is a Γ-Hom-Lie comodule under*

$$\rho = \rho_M \otimes \nu + (\tau \otimes id) \circ (\mu \otimes \rho_N),$$

that is, for any $m \in M, n \in N$,

$$\rho : m \otimes n \mapsto m_{(-1)} \otimes m_{(0)} \otimes \nu(n) + n_{(-1)} \otimes \mu(m) \otimes n_{(0)}.$$

Proof. For any $m \in M, n \in N$,

$$\begin{aligned}
&(\Delta \otimes (\mu \otimes \nu)) \circ \rho(m \otimes n)\\
&= (\Delta \otimes (\mu \otimes \nu))(m_{(-1)} \otimes m_{(0)} \otimes \nu(n) + n_{(-1)} \otimes \mu(m) \otimes n_{(0)})\\
&= m_{(-1)1} \otimes m_{(-1)2} \otimes \mu(m_{(0)}) \otimes \nu^2(n) + n_{(-1)1} \otimes n_{(-1)2} \otimes \mu^2(m) \otimes \nu(n_{(0)})\\
&= \alpha(m_{(-1)}) \otimes m_{(0)(-1)} \otimes m_{(0)(0)} \otimes \nu^2(n) - m_{(0)(-1)} \otimes \alpha(m_{(-1)}) \otimes m_{(0)(0)} \otimes \nu^2(n)\\
&+\alpha(n_{(-1)}) \otimes n_{(0)(-1)} \otimes \mu^2(m) \otimes n_{(0)(0)} - n_{(0)(-1)} \otimes \alpha(n_{(-1)}) \otimes \mu^2(m) \otimes n_{(0)(0)}
\end{aligned}$$

$$\begin{aligned}
&[(\alpha \otimes \rho) \circ \rho - (\tau \otimes id) \circ (\alpha \otimes \rho) \circ \rho](m \otimes n)\\
&= \alpha((m \otimes n)_{(-1)}) \otimes (m \otimes n)_{(0)(-1)} \otimes (m \otimes n)_{(0)(0)}\\
&-(m \otimes n)_{(0)(-1)} \otimes \alpha((m \otimes n)_{(-1)}) \otimes (m \otimes n)_{(0)(0)}\\
&= \alpha(m_{(-1)}) \otimes (m_{(0)} \otimes \nu(n))_{(-1)} \otimes (m_{(0)} \otimes \nu(n))_{(0)}\\
&-(m_{(0)} \otimes \nu(n))_{(-1)} \otimes \alpha(m_{(-1)}) \otimes (m_{(0)} \otimes \nu(n))_{(0)}\\
&+\alpha(n_{(-1)}) \otimes (\mu(m) \otimes n_{(0)})_{(-1)} \otimes (\mu(m) \otimes n_{(0)})_{(0)}\\
&-(\mu(m) \otimes n_{(0)})_{(-1)} \otimes \alpha(n_{(-1)}) \otimes (\mu(m) \otimes n_{(0)})_{(0)}\\
&= \alpha(m_{(-1)}) \otimes m_{(0)(-1)} \otimes m_{(0)(0)} \otimes \nu^2(n) + \overline{\alpha(m_{(-1)}) \otimes \nu(n)_{(-1)} \otimes \mu(m_{(0)}) \otimes \nu(n)_{(0)}}\\
&-m_{(0)(-1)} \otimes \alpha(m_{(-1)}) \otimes m_{(0)(0)} \otimes \nu^2(n) \underbrace{- \nu(n)_{(-1)} \otimes \alpha(m_{(-1)}) \otimes \mu(m_{(0)}) \otimes \nu(n)_{(0)}}\\
&\underbrace{+\alpha(n_{(-1)}) \otimes \mu(m)_{(-1)} \otimes \mu(m)_{(0)} \otimes \nu(n_{(0)})} + \alpha(n_{(-1)}) \otimes n_{(0)(-1)} \otimes \mu^2(m) \otimes n_{(0)(0)}\\
&\overline{-\mu(m)_{(-1)} \otimes \alpha(n_{(-1)}) \otimes \mu(m)_{(0)} \otimes \nu(n_{(0)})} - n_{(0)(-1)} \otimes \alpha(n_{(-1)}) \otimes \mu^2(m) \otimes n_{(0)(0)}\\
&= \alpha(m_{(-1)}) \otimes m_{(0)(-1)} \otimes m_{(0)(0)} \otimes \nu^2(n) - m_{(0)(-1)} \otimes \alpha(m_{(-1)}) \otimes m_{(0)(0)} \otimes \nu^2(n)\\
&+\alpha(n_{(-1)}) \otimes n_{(0)(-1)} \otimes \mu^2(m) \otimes n_{(0)(0)} - n_{(0)(-1)} \otimes \alpha(n_{(-1)}) \otimes \mu^2(m) \otimes n_{(0)(0)},
\end{aligned}$$

so, $(\Delta \otimes (\mu \otimes \nu)) \circ \rho = (\alpha \otimes \rho) \circ \rho - (\tau \otimes Id) \circ (\alpha \otimes \rho) \circ \rho$. Meanwhile,

$$\begin{aligned}
\rho \circ (\mu \otimes \nu)(m \otimes n) &= \mu(m)_{(-1)} \otimes \mu(m)_{(0)} \otimes \nu^2(n) + \nu(n)_{(-1)} \otimes \mu^2(m) \otimes \nu(n)_{(0)}\\
&= \alpha(m_{(-1)}) \otimes \mu(m_{(0)}) \otimes \nu^2(n) + \alpha(n_{(-1)}) \otimes \mu^2(m) \otimes \nu(n_{(0)})\\
&= (\alpha \otimes \mu \otimes \nu) \circ \rho(m \otimes n).
\end{aligned}$$

Thus, $(M \otimes N, \mu \otimes \nu)$ is a Γ-Hom-Lie comodule. ■

It is easy to see that Hom-Lie coalgebra (Γ, Δ, α) is itself a Γ-Hom-Lie comodule with the comodule structure $\rho = \Delta$. Therefore, we obtain the following corollary naturally by Lemma 3.2.

Corollary 3.3. *Let (Γ, Δ, α) be a Hom-Lie coalgebra. Then $(\Gamma^{\otimes 2}, \alpha^{\otimes 2})$ is a Γ-Hom-Lie comodule under the Hom-coadjoint diagonal coaction*

$$\rho : \Gamma^{\otimes 2} \to \Gamma^{\otimes 3}; \quad x \otimes y \mapsto x_1 \otimes x_2 \otimes \alpha(y) + y_1 \otimes \alpha(x) \otimes y_2.$$

Proposition 3.4. *Let (Γ, Δ, α) be a Hom-Lie coalgebra, and $\pi : \Gamma^{\otimes 2} \to k$ such that*

$$\pi = -\pi \circ \tau,$$

$$\pi \circ \alpha^{\otimes 2} = \pi.$$

Define

$$[-,-]: \quad \Gamma^{\otimes 2} \xrightarrow{\rho} \Gamma^{\otimes 3} \xrightarrow{id \otimes \pi} \Gamma\,,$$

where $\rho = \Delta \otimes \alpha + (\tau \otimes id) \circ (\alpha \otimes \Delta)$. That is, for any $x, y \in \Gamma$,

$$[-,-] : x \otimes y \mapsto x_1 \pi(x_2 \otimes \alpha(y)) + y_1 \pi(\alpha(x) \otimes y_2).$$

Then, for any $x, y \in \Gamma$,

(1) $[x, y] + [y, x] = 0$*,*

(2) $\alpha[x, y] = [\alpha(x), \alpha(y)]$*,*

(3) $\Delta[x, y] = ad_{\alpha(x)}(\Delta(y)) - ad_{\alpha(y)}(\Delta(x))$*.*

Proof. (1) For any $x, y \in \Gamma$,

$$\begin{aligned} [x, y]+[y, x] &= x_1\pi(x_2\otimes\alpha(y))+y_1\pi(\alpha(x)\otimes y_2)+y_1\pi(y_2\otimes\alpha(x))+x_1\pi(\alpha(y)\otimes x_2) \\ &= x_1\pi(x_2\otimes\alpha(y))+y_1\pi(\alpha(x)\otimes y_2)-y_1\pi(\alpha(x)\otimes y_2)-x_1\pi(x_2\otimes\alpha(y)) \\ &= 0. \end{aligned}$$

(2) For any $x, y \in \Gamma$,

$$\begin{aligned} \alpha[x, y] &= \alpha(x_1)\pi(x_2 \otimes \alpha(y)) + \alpha(y_1)\pi(\alpha(x) \otimes y_2) \\ &= \alpha(x_1)\pi(\alpha(x_2) \otimes \alpha^2(y)) + \alpha(y_1)\pi(\alpha^2(x) \otimes \alpha(y_2)) \\ &= \alpha(x)_1\pi(\alpha(x)_2 \otimes \alpha^2(y)) + \alpha(y)_1\pi(\alpha^2(x) \otimes \alpha(y)_2) \\ &= [\alpha(x), \alpha(y)]. \end{aligned}$$

(3) We know that $(\Gamma^{\otimes 2}, \alpha^{\otimes 2})$ is a Γ-Hom-Lie comodule by Corollary 3.3, and the comodule structure is given by

$$\rho(x \otimes y) = x_1 \otimes x_2 \otimes \alpha(y) + y_1 \otimes \alpha(x) \otimes y_2.$$

So, by Definition 3.1, $\Delta[x,y]$

$$
\begin{aligned}
&= \Delta(x_1\pi(x_2\otimes\alpha(y)) + y_1\pi(\alpha(x)\otimes y_2))\\
&= \Delta\circ(id\otimes\pi)\circ\rho(x\otimes y)\\
&= (id\otimes id\otimes\pi)\circ(\Delta\otimes\alpha\otimes\alpha)\circ\rho(x\otimes y)\\
&= (id\otimes id\otimes\pi)\circ((\alpha\otimes\rho)\circ\rho-(\tau\otimes id)\circ(\alpha\otimes\rho)\circ\rho)(x\otimes y))\\
&= \alpha(x_1)\otimes x_2\pi(x_3\otimes\alpha^2(y)) + \alpha(y_1)\otimes y_2\pi(\alpha^2(x)\otimes y_3)\\
&-x_2\otimes\alpha(x_1)\pi(x_3\otimes\alpha^2(y)) - y_2\otimes\alpha(y_1)\pi(\alpha^2(x)\otimes y_3)\\
&= -\alpha(x_1)\otimes x_2\pi(\alpha^2(y)\otimes x_3) + \alpha(y_1)\otimes y_2\pi(\alpha^2(x)\otimes y_3)\\
&-x_1\otimes\alpha(x_3)\pi(\alpha^2(y)\otimes x_2) + y_1\otimes\alpha(y_3)\pi(\alpha^2(x)\otimes y_2),\\
&ad_{\alpha(x)}(\Delta(y)) - ad_{\alpha(y)}(\Delta(x))\\
&= [\alpha(x),y_1]\otimes\alpha(y_2) + \alpha(y_1)\otimes[\alpha(x),y_2]\\
&-[\alpha(y),x_1]\otimes\alpha(x_2) - \alpha(x_1)\otimes[\alpha(y),x_2]\\
&= \underline{\alpha(x)_1\pi(\alpha(x)_2\otimes\alpha(y_1))\otimes\alpha(y_2)} + y_1\pi(\alpha^2(x)\otimes y_2)\otimes\alpha(y_3)\\
&\underbrace{+\alpha(y_1)\otimes\alpha(x)_1\pi(\alpha(x)_2\otimes\alpha(y_2))} + \alpha(y_1)\otimes y_2\pi(\alpha^2(x)\otimes y_3)\\
&\underbrace{-\alpha(y)_1\pi(\alpha(y)_2\otimes\alpha(x_1))\otimes\alpha(x_2)} - x_1\pi(\alpha^2(y)\otimes x_2)\otimes\alpha(x_3)\\
&\underline{-\alpha(x_1)\otimes\alpha(y)_1\pi(\alpha(y)_2\otimes\alpha(x_2))} - \alpha(x_1)\otimes x_2\pi(\alpha^2(y)\otimes x_3)\\
&\qquad = y_1\otimes\alpha(y_3)\pi(\alpha^2(x)\otimes y_2) + \alpha(y_1)\otimes y_2\pi(\alpha^2(x)\otimes y_3)\\
&-x_1\otimes\alpha(x_3)\pi(\alpha^2(y)\otimes x_2) - \alpha(x_1)\otimes x_2\pi(\alpha^2(y)\otimes x_3),
\end{aligned}
$$

therefore, $\Delta[x,y] = ad_{\alpha(x)}(\Delta(y)) - ad_{\alpha(y)}(\Delta(x))$. ■

According to the above proposition, we obtain the following theorem.

Theorem 3.5. *Let (Γ,Δ,α) be a Hom-Lie coalgebra. Assume that there exist two linearly independent elements $\mu,\nu\in\Gamma^*$ and element $t\in\Gamma$, such that for any $x\in\Gamma$,*

$$(id\otimes\mu)\circ\Delta(x) = \mu(\alpha(x))\alpha(t),$$
$$(id\otimes\nu)\circ\Delta(x) = \nu(\alpha(x))\alpha(t).$$

Set $\pi=\mu\otimes\nu-\nu\otimes\mu$ and assume that $\pi\circ\alpha^{\otimes 2}=\pi$. Define a Hom-Lie bracket $[-,-]$ as in Proposition 3.4. Then $(\Gamma,[-,-],\Delta,\alpha)$ is a Hom-Lie bialgebra.

Proof. Since for any $x,y\in\Gamma$,

$$
\begin{aligned}
[x,y] &= x_1\pi(x_2\otimes\alpha(y)) + y_1\pi(\alpha(x)\otimes y_2)\\
&= x_1\mu(x_2)\nu(\alpha(y)) - x_1\nu(x_2)\mu(\alpha(y))\\
&+y_1\mu(\alpha(x))\nu(y_2) - y_1\nu(\alpha(x))\mu(y_2)\\
&= 2\mu(\alpha(x))\alpha(t)\nu(\alpha(y)) - 2\nu(\alpha(x))\alpha(t)\mu(\alpha(y))\\
&= 2[\mu(x)\nu(y)-\nu(x)\mu(y)]\alpha(t)\\
&= 2\pi(x\otimes y)\alpha(t),
\end{aligned}
$$

we have for any $x,y,z\in\Gamma$,

$$
\begin{aligned}
[\alpha(x),[y,z]] &= [\alpha(x),2\pi(y\otimes z)\alpha(t)]\\
&= 2\pi(y\otimes z)[\alpha(x),\alpha(t)]\\
&= 4\pi(y\otimes z)\pi(x\otimes t)\alpha^2(t)\\
&= 4[\mu(x)\nu(t)\mu(y)\nu(z) - \nu(x)\mu(t)\mu(y)\nu(z)\\
&-\mu(x)\nu(t)\nu(y)\mu(z) + \nu(x)\mu(t)\nu(y)\mu(z)]\alpha^2(t).
\end{aligned}
$$

In the same way,

$$\begin{aligned}
[\alpha(y),[z,x]] &= 4(\mu(y)\nu(t)\mu(z)\nu(x)-\nu(y)\mu(t)\mu(z)\nu(x)\\
&\quad -\mu(y)\nu(t)\nu(z)\mu(x)+\nu(y)\mu(t)\nu(z)\mu(x))\alpha^2(t),\\
[\alpha(z),[x,y]] &= 4(\mu(z)\nu(t)\mu(x)\nu(y)-\nu(z)\mu(t)\mu(x)\nu(y)\\
&\quad -\mu(z)\nu(t)\nu(x)\mu(y)+\nu(z)\mu(t)\nu(x)\mu(y))\alpha^2(t)
\end{aligned}$$

Thus $[\alpha(x),[y,z]]+[\alpha(y),[z,x]]+[\alpha(z),[x,y]]=0$. According to Proposition 3.4, $(\Gamma,[-,-],\Delta,\alpha)$ is a Hom-Lie bialgebra.

■

Example 3.6. Assume that $\Gamma = span\{H,X,Y\}$ is a Lie coalgebra, and the Lie cobracket is given as follows

$$\Delta(H)=0,\ \Delta(X)=X\otimes H-H\otimes X,\ \Delta(Y)=Y\otimes H-H\otimes Y.$$

We can get a Hom-Lie coalgebra $(\Gamma,\Delta_\alpha=\Delta\circ\alpha,\alpha)$ through a non-zero linear self-map $\alpha:\Gamma\to\Gamma$,

$$\alpha(H)=H,\ \alpha(X)=aX,\ \alpha(Y)=a^{-1}Y,$$

for some none-zero a in the field $\mathcal{C}$ of complex numbers. And the structure map is given by

$$\begin{aligned}
\Delta_\alpha(H) &= \Delta\circ\alpha(H)=0,\\
\Delta_\alpha(X) &= a(X\otimes H-H\otimes X)=\alpha^{\otimes 2}\circ\Delta(X),\\
\Delta_\alpha(Y) &= a^{-1}(Y\otimes H-H\otimes Y)=\alpha^{\otimes 2}\circ\Delta(Y).
\end{aligned}$$

Assume that there are two linearly independent elements μ,ν in Γ^* satisfying

$$\mu(H)=\nu(H)=0.$$

Set $\pi=\mu\otimes\nu-\nu\otimes\mu$, we obtain a Hom-Lie bialgebra $(\Gamma,[-,-],\Delta_\alpha,\alpha)$ where $[-,-]$ is defined as in Proposition 3.4.

In fact, $\pi\circ\alpha^{\otimes 2}=\pi$ since

$$\begin{aligned}
\pi\circ\alpha^{\otimes 2}(H\otimes X) &= \pi(H\otimes aX)=0=\pi(H\otimes X),\\
\pi\circ\alpha^{\otimes 2}(H\otimes Y) &= \pi(H\otimes a^{-1}Y)=0=\pi(H\otimes Y),\\
\pi\circ\alpha^{\otimes 2}(X\otimes Y) &= \pi(aX\otimes a^{-1}Y)=\pi(X\otimes Y).
\end{aligned}$$

Next, let $t=-H$, then for any $m\in\Gamma$, we have

$$(id\otimes\mu)\circ\Delta_\alpha(m)=\mu(\alpha(m))\alpha(t),$$

$$(id\otimes\nu)\circ\Delta_\alpha(m)=\nu(\alpha(m))\alpha(t).$$

As a matter of fact,

$$\begin{aligned}
(id\otimes\mu)\circ\Delta_\alpha(H) &= 0,\\
(id\otimes\mu)\circ\Delta_\alpha(X) &= -aH\mu(X)=\mu(\alpha(X))\alpha(t),\\
(id\otimes\mu)\circ\Delta_\alpha(Y) &= (id\otimes\mu)(a^{-1}Y\otimes H-a^{-1}H\otimes Y)\\
&= -a^{-1}H\mu(Y)=\mu(\alpha(Y))\alpha(t).
\end{aligned}$$

For ν, the computing is exactly analogous. Therefore $(\Gamma, [-,-], \Delta_\alpha, \alpha)$ is a Hom-Lie bialgebra by Theorem 3.5 whose Hom-Lie bracket is given by

$$[H, X] = 0, [H, Y] = 0$$

$$[X, Y] = 2(\mu(X)\nu(Y) - \nu(X)\mu(Y))\alpha(t) = 2pH,$$

where $p = \nu(X)\mu(Y) - \mu(X)\nu(Y)$.

Furthermore, we can construct a coboundary Hom-Lie bialgebra for the above Hom-Lie bialgebra.

Set $r = b(H \otimes X - X \otimes H) + c(H \otimes Y - Y \otimes H) + e(X \otimes Y - Y \otimes X)$. Then the following results can be obtained.

$$\begin{aligned} X \cdot r = ad_X(r) &= c\alpha(H) \otimes [X, Y] - c[X, Y] \otimes \alpha(H) \\ &\quad + e\alpha(X) \otimes [X, Y] - e[X, Y] \otimes \alpha(X) \\ &= cH \otimes 2pH - 2pcH \otimes H + eaX \otimes 2pH - 2peH \otimes aX \\ &= 2pae(X \otimes H - H \otimes X), \\ H \cdot r = ad_H(r) &= 0 = \Delta_\alpha(H). \end{aligned}$$

Meanwhile, $X \cdot r = \Delta_\alpha(X) = a(X \otimes H - H \otimes X)$. So $2pae = a$, that is to say $e = \frac{1}{2}p^{-1}$. Similarly, $Y \cdot r = \Delta_\alpha(Y)$ if and only if $e = \frac{1}{2}p^{-1}$. And the condition $\alpha^{\otimes 2}(r) = r$ is needed. That is $\alpha^{\otimes 2}(r) =$

$$\begin{aligned} &= b(H \otimes aX - aX \otimes H) + c(H \otimes a^{-1}Y - a^{-1}Y \otimes H) + e(X \otimes Y - Y \otimes X) \\ &= ba(H \otimes X - X \otimes H) + ca^{-1}(H \otimes Y - Y \otimes H) + e(X \otimes Y - Y \otimes X) \\ &= r. \end{aligned}$$

So $ba = b$ and $ca^{-1} = c$, thus $a = 1$ or $b = c = 0$. If $a = 1$, the constructed Hom-Lie bialgebra is an ordinary Lie bialgebra. If $a \neq 1$, then $b = c = 0$, i.e.

$$r = e(X \otimes Y - Y \otimes X) = \frac{1}{2}p^{-1}(X \otimes Y - Y \otimes X).$$

For $r = \frac{1}{2}p^{-1}(X \otimes Y - Y \otimes X)$, the classical Hom-Yang-Baxter equation

$$\begin{aligned} &[r^{12}, r^{13}] + [r^{12}, r^{23}] + [r^{13}, r^{23}] \\ &\quad = \tfrac{1}{2}p^{-1}(H \otimes X \otimes Y - H \otimes Y \otimes X + Y \otimes H \otimes X \\ &\quad\quad - X \otimes H \otimes Y + X \otimes Y \otimes H - Y \otimes X \otimes H) \\ &\quad \neq 0. \end{aligned}$$

Thus, we construct a coboundary but not triangular Hom-Lie bialgebra $(\Gamma, [-,-], \Delta_\alpha, \alpha, r = \frac{1}{2}p^{-1}(X \otimes Y - Y \otimes X))$ at last.

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