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Graded Nilpotent Lie Algebras of Infinite Type

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Abstract. The paper gives the complete characterization of all graded nilpotent Lie algebras with infinite-dimensional Tanaka prolongation as extensions of graded nilpotent Lie algebras of lower dimension by means of a commutative ideal. We introduce a notion of weak characteristics of a vector distribution and prove that if a bracket-generating distribution of constant type does not have non-zero complex weak characteristics, then its symmetry algebra is necessarily finite-dimensional. The paper also contains a number of illustrative algebraic and geometric examples including the proof that any metabelian Lie algebra with a 2-dimensional center always has an infinite-dimensional Tanaka prolongation.
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1. Introduction

This paper is devoted to the study of non-integrable vector distributions with infinite-dimensional symmetry algebras. The simplest examples of such distributions are contact distributions on odd-dimensional manifolds and, more generally, contact systems on jet spaces $J^k(\mathbb{R}^n, \mathbb{R}^m)$. The symmetry algebras of such distributions are given by Lie–Backlund theorem and are isomorphic to either the Lie algebra of all vector fields on $J^0(\mathbb{R}^n, \mathbb{R}^m) = \mathbb{R}^{n+m}$ for $m \geq 2$ or to the Lie algebra of all contact vector fields on $J^1(\mathbb{R}^n, \mathbb{R})$ for $m = 1$. In both cases the symmetry algebras are infinite-dimensional.

We are interested in only so-called *bracket-generating distributions*, i.e., we always assume that repetitive brackets of vector fields lying in D generate the whole tangent bundle TM. If this is not the case, then D lies in a certain proper completely integrable vector distribution D', and the geometry of D can be essentially reduced to the restrictions of D to the fibers of D'.

We shall also say that the distribution D is *degenerate*, if it possesses non-zero *Cauchy characteristics*, i.e., vector fields $X \in D$ such that $[X, D] \subset D$. It is also clear that any degenerate distribution, even if it is bracket-generating, has an infinite-dimensional symmetry algebra. For the proof and other basic properties of vector distributions we refer to [2, Chapter 2].

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While it seems to be very difficult to provide the complete local description of all vector distributions with infinite-dimensional symmetry algebra, it appears that it is possible to give the complete description of the their symbol algebras.

Namely, let D be a bracket-generating vector distribution on a smooth manifold M. Taking repetitive brackets of vector fields lying in D, we can define *a weak derived flag* of D:

$$0 \subset D \subset D^2 \subset \cdots \subset D^\mu = TM,$$
$$D^0 = 0, \quad D^1 = D, \quad D^{i+1} = [D, D^i],\ i \geq 2.$$

At each point $p \in M$ we can defined the associated graded vector space

$$\mathfrak{m}(p) = \sum_{i<0} \mathfrak{m}_{-i}(p), \quad \mathfrak{m}_{-i}(p) = D^i_p / D^{i-1}_p. \tag{1}$$

Since $[D^i, D^j] \subset D^{i+j}$ for all $i, j \geq 0$, $\mathfrak{m}(p)$ is naturally equipped with a structure of a graded Lie algebra. Namely, if $x \in \mathfrak{m}_{-i}(p)$ and $y \in \mathfrak{m}_{-j}(p)$ are two homogeneous elements in $\mathfrak{m}(p)$, and $X \in D^i$, $Y \in D^j$ are two vector fields such that $X_p + D^{i-1}_p = x$ and $Y_p + D^{j-1}_p = y$, then the value of $[X, Y] + D^{i+j-1}_p$ depends only on x and y, and, thus, defines a graded Lie algebra structure on $\mathfrak{m}(p)$. It is also clear from the definition, that $\mathfrak{m}(p)$ is a nilpotent Lie algebra generated by $\mathfrak{m}_{-1}(p)$.

This Lie algebra is called a symbol of the distribution D at a point $p \in M$, and it plays essential role in study of D and any geometric structures subordinate to D. For example, if D is a contact structure on a smooth manifold of dimension $2n+1$, i.e., a non-degenerate codimension 1 distribution on M, then its symbol is isomorphic to the $(2n+1)$-dimensional Heisenberg Lie algebra at any point $p \in M$.

The family of graded Lie algebras $\mathfrak{m}(p)$ is a basic invariant of any bracket-generating distribution, which includes not only the dimensions of the weak derived series of D, but also a non-trivial algebraic information. In many cases the structure of these algebras has very important geometric consequences and allows to associate various geometric structures with M. One of the most famous examples is E. Cartan paper [3], where he associates a G_2-geometry with any non-degenerate 2-dimensional vector distribution on a 5-dimensional manifold.

We say that the distribution D has constant symbol $\mathfrak{m}$ or is of type $\mathfrak{m}$ if its symbols $\mathfrak{m}(p)$ are isomorphic to $\mathfrak{m}$ for all points $p \in M$. For example, the contact distribution and all contact systems on jet spaces are of constant type.

We shall use the term *graded nilpotent Lie algebra* or simply GNLA for any negatively graded Lie algebra $\mathfrak{m} = \sum_{i=1}^{\mu} \mathfrak{m}_{-i}$ generated by $\mathfrak{m}_{-1}$. We shall call such Lie algebra *non-degenerate*, if $\mathfrak{m}_{-1}$ does not include any non-zero central elements. This has a clear geometric meaning. It is easy to see that the symbol of a bracket-generating distribution D is non-degenerate if and only if D has no non-zero Cauchy characteristics. The largest μ such that $\mathfrak{m}_{-\mu} \neq 0$ is called *the depth* of $\mathfrak{m}$.

It appears that we can derive the exact bound for the dimension of the symmetry algebra of D purely in terms of its symbol $\mathfrak{m}$. Namely, Tanaka [16] has shown in his pioneer works on the geometry of filtered manifolds, that there is a well-defined graded Lie algebra $\mathfrak{g}(\mathfrak{m})$ called Tanaka (or universal) prolongation of $\mathfrak{m}$. It is characterized by the following conditions:

1. $\mathfrak{g}_i(\mathfrak{m}) = \mathfrak{m}_i$ for all $i < 0$;
2. if $[X, \mathfrak{m}] = 0$ for certain $X \in \mathfrak{g}_i(\mathfrak{m})$, $i \geq 0$, then $X = 0$;
3. $\mathfrak{g}(\mathfrak{m})$ is the largest graded Lie algebra satisfying the above two conditions.

The Lie algebra $\mathfrak{g}(\mathfrak{m})$ has also a clear geometric meaning. Namely, let M be a connected simply connected Lie group with the Lie algebra $\mathfrak{m}$. For example, we can identify M with $\mathfrak{m}$ and define the Lie group multiplication in M by means of Campbell–Haussdorf series. Define the distribution D on M by assuming that it is left invariant and that $D_e = \mathfrak{m}_{-1}$. Then $\mathfrak{g}(\mathfrak{m})$ can be naturally identified with the graded Lie algebra associated with the filtered Lie algebra of all germs of infinitesimal symmetries of D at the identity. In particular, if $\mathfrak{g}(\mathfrak{m})$ is finite or infinite-dimensional, so is the symmetry algebra of the corresponding filtration. Such distributions D are called *standard* distributions of type $\mathfrak{m}$.

One of the main results of Tanaka paper [16] can be reformulated as follows.

Tanaka theorem ([16]). *If $\mathfrak{g}(\mathfrak{m})$ is finite-dimensional, then with each distribution $D \subset TM$ of type $\mathfrak{m}$ we can associate a canonical coframe on a certain bundle P over M of dimension $\dim \mathfrak{g}(\mathfrak{m})$. In particular, the symmetry algebra of the distribution D is finite-dimensional, and its dimension is bounded by $\dim \mathfrak{g}(\mathfrak{m})$.*

Thus, if D is a holonomic distribution with infinite-dimensional symmetry algebra, then the Tanaka prolongation of its symbol $\mathfrak{g}(\mathfrak{m})$ should also be infinite-dimensional. The properties of Tanaka prolongation are also studied in [19, 20, 11, 17, 12].

The main result of this paper is Theorem 3.1, which gives the the complete characterization of all GNLA with infinite-dimensional Tanaka prolongation as extensions of graded nilpotent Lie algebras of lower dimension by means of a commutative ideal. Along with this description we also introduce a notion of *weak characteristics* of a vector distribution and prove that if a bracket-generating distribution of constant type does not have non-zero complex weak characteristics, then its symmetry algebra is necessarily finite-dimensional.

This article is closely related to a series of papers devoted to the geometry of 2 and 3-dimensional distributions [1, 4, 5]. These papers show that under very mild non-degeneracy conditions the symmetry algebra of a non-integrable vector distribution becomes finite-dimensional. As we show in Theorem 3.4, the set of all GNLA, whose Tanaka prolongation is infinite-dimensional, forms a closed subvariety in the variety of all GNLA $\mathfrak{m}$ with fixed dimensions of subspaces $\mathfrak{m}_{-i}$, $i > 0$.

Finally, in the last section of the paper we present a number of illustrative algebraic and geometric examples including the proof that any metabelian Lie algebra with a 2-dimensional center always has an infinite-dimensional Tanaka prolongation.

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2. Tanaka and Spencer criteria

Below we assume that all Lie algebras we consider are defined over an arbitrary field k of characteristic 0. We try to be as generic as possible in our algebraic considerations and we explicitly state, when we require the base field k to be algebraically closed.

Let $\mathfrak{m}$ be an arbitrary GNLA and let $\mathfrak{g}(\mathfrak{m})$ be its Tanaka prolongation. We say that $\mathfrak{m}$ is *of finite (infinite) type*, if $\mathfrak{g}(\mathfrak{m})$ is finite-dimensional (resp., infinite-dimensional). Note that this notion is stable with respect to the base field extensions, since each subspace $\mathfrak{g}_i(\mathfrak{m})$, $i \geq 0$ of the Tanaka prolongation can be computed by solving a system of linear equations. Namely, assuming that all subspaces $\mathfrak{g}_i(\mathfrak{m})$, $i < k$ are already computed, the space $\mathfrak{g}_k(\mathfrak{m})$ can be defined as follows:

$$\mathfrak{g}_k(\mathfrak{m}) = \{\phi\colon \mathfrak{m} \to \sum_{i<k} \mathfrak{g}_i(\mathfrak{m}) \mid \phi(\mathfrak{m}_{-i}) \subset \mathfrak{g}_{k-i}(\mathfrak{m}),$$
$$\phi([x,y]) = [\phi(x), y] + [x, \phi(y)], \forall x, y \in \mathfrak{m}\}.$$

Tanaka criterium reduces the question whether Tanaka prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m}$ is finite-dimensional or not to the same question for the standard prolongation (as described, for example, in [15]) of a certain linear Lie algebra. Namely, let $\mathrm{Der}_0(\mathfrak{m})$ be the Lie algebra of all degree-preserving derivations of $\mathfrak{m}$. Define the subalgebra $\mathfrak{h}_0 \subset \mathrm{Der}_0(\mathfrak{m})$ as follows:

$$\mathfrak{h}_0 = \{d \in \mathrm{Der}_0(\mathfrak{m}) \mid d(x) = 0 \text{ for all } x \in \mathfrak{m}_{-i}, i \geq 2\}.$$

We can naturally identify $\mathfrak{h}_0$ with a subspace in $\mathrm{End}(\mathfrak{m}_{-1})$. Then Tanaka criterium can be formulated as follows:

Tanaka criterium ([16]). *Tanaka prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m}$ is finite-dimensional if and only if so is the standard prolongation of $\mathfrak{h}_0 \subset \mathrm{End}(\mathfrak{m}_{-1})$.*

In its turn, Spencer criterium provides a computationally efficient method of detecting whether the standard prolongation of a linear subspace $A \subset \mathrm{End}(V)$ is finite-dimensional or not.

Spencer criterium ([7, 14]). *The standard prolongation of a subspace $A \subset \mathrm{End}(V)$ is finite-dimensional if and only if $A^{\bar{k}} \subset \mathrm{End}(V^{\bar{k}})$ does not contain endomorphisms of rank 1.*

Here by $\bar{k}$ we mean the algebraic closure of our base field k, and by $A^{\bar{k}}$ the subspace in $\mathrm{End}(V^{\bar{k}})$ obtained from A by field extension. A detailed and self-contained proof of Spencer criterium can be found in [13].

We emphasize that this criterium if computationally efficient, as the set of all rank 1 endomorphisms in A is described by a finite set of quadratic polynomials, and, for example, Gröbner basis technique provides an algorithm to determine whether this set of polynomials has a non-zero common root.

In the next section we use both these criteria to prove that all fundamental Lie algebras $\mathfrak{m}$ of infinite type over an algebraically closed field possess a very special algebraic structure.

3. Graded nilpotent Lie algebras of infinite type

Let $\mathfrak{g}$ be an arbitrary finite-dimensional Lie algebra, and let V be an arbitrary $\mathfrak{g}$-module. We recall that a Lie algebra $\bar{\mathfrak{g}}$ is called an extension of $\mathfrak{g}$ by means of V, if $\bar{\mathfrak{g}}$ can be included into the following exact sequence:

$$0 \to V \to \bar{\mathfrak{g}} \to \mathfrak{g} \to 0.$$

In other words, V is embedded into $\bar{\mathfrak{g}}$ as a commutative ideal, the quotient $\bar{\mathfrak{g}}/V$ is identified with $\mathfrak{g}$ and the natural action of $\mathfrak{g} = \bar{\mathfrak{g}}/V$ on V coincides with the predefined $\mathfrak{g}$-module structure on V. Two extensions $\bar{\mathfrak{g}}_1$ and $\bar{\mathfrak{g}}_2$ are called equivalent, if there exists an isomorphism $\bar{\mathfrak{g}}_1 \to \bar{\mathfrak{g}}_2$ identical on V and $\mathfrak{g} \equiv \bar{\mathfrak{g}}_1/V \equiv \bar{\mathfrak{g}}_2/V$.

It is well-known that equivalence classes of such extensions are described by the cohomology space $H^2(\mathfrak{g}, V)$. Namely, if $[\alpha]$ is an element of $H^2(\mathfrak{g}, V)$, where $\alpha \in Z^2(\mathfrak{g}, V)$, then $\bar{\mathfrak{g}}$ can be identified as a vector space with $\mathfrak{g} \times V$ with the Lie bracket given by:

$$[(x_1, v_1), (x_2, v_2)] = ([x_1, x_2], x_1.v_2 - x_2.v_1 + \alpha(x_1, x_2)), \tag{2}$$

$x_1, x_2 \in \mathfrak{g}$, $v_1, v_2 \in V$.

Assume now that both the Lie algebra $\mathfrak{g}$ and the $\mathfrak{g}$-module V are graded, that is $\mathfrak{g} = \sum \mathfrak{g}_i$, $V = \sum V_j$ and $\mathfrak{g}_i.V_j \subset V_{i+j}$. Then the cohomology space $H^2(\mathfrak{g}, V)$ is naturally turned into the graded vector space as well:

$$H^2(\mathfrak{g}, V) = \sum H^2_i(\mathfrak{g}, V).$$

It is easy to see that the Lie algebra $\bar{\mathfrak{g}}$ defined by (2) is also graded if and only if $[\alpha] \in H^2_0(\mathfrak{g}, V)$. Thus, we see that the graded extensions $\bar{\mathfrak{g}}$ of the graded Lie algebra $\mathfrak{g}$ by means of the graded $\mathfrak{g}$-module V are in one to one correspondence with the elements of the vector space $H^2_0(\mathfrak{g}, V)$.

We can introduce an additional group action on $H^2_0(\mathfrak{g}, V)$ as follows. Define $\mathrm{Aut}_0(\mathfrak{g}, V)$ as a subgroup in $\mathrm{Aut}_0(\mathfrak{g}) \times \mathrm{GL}_0(V)$ of the form:

$$\mathrm{Aut}_0(\mathfrak{g}, V) = \{(f, g) \in \mathrm{Aut}_0(\mathfrak{g}) \times \mathrm{GL}_0(V) \mid \\ g(x.v) = f(x).g(v),\ \forall x \in \mathfrak{g}, v \in V\}.$$

This group naturally acts on $H^2_0(\mathfrak{g}, V)$:

$$(f, g).[\alpha] = [g \circ \alpha \circ f^{-1}].$$

It is easy to see that elements of $H^2_0(\mathfrak{g}, V)$ lying in the same orbit of this action correspond to the isomorphic Lie algebras.

In this paper we shall consider *special extensions of GNLA*. Namely, let $\mathfrak{n}$ be an arbitrary (possibly degenerate) GNLA. Fix any subspace $W \subset \mathfrak{n}_{-1}$ and consider $\mathfrak{n}_{-1}/W$ as a graded commutative Lie algebra concentrated in degree -1. Take any graded $\mathfrak{n}_{-1}/W$-module $V = \sum_{i=1}^{\mu} V_{-i}$ generated by V_{-1} (as a module). Assuming that $W.V = 0$ and $\mathfrak{n}_{-i}.V = 0$ for all $i \geq 2$ we can consider V as an $\mathfrak{n}$-module. We call any extension of the GNLA $\mathfrak{n}$ by means of such $\mathfrak{n}$-module V *a special GNLA* extension. Let $[\alpha]$ be the corresponding element of $H^2_0(\mathfrak{n}, V)$. It is easy to see that such extension is non-degenerate if and only if the following two conditions hold:

1. the condition $\mathfrak{n}.v = 0$ implies $v = 0$ for $v \in V_{-1}$;
2. the cocycle α is non-degenerate on $Z(\mathfrak{n}) \cap \mathfrak{n}_{-1}$.

We note that unlike the extensions of GNLA considered in the papers [11, 1], the special extensions defined above are not central extensions of Lie algebras, as the ideal V above does not in general belong to the center, and the action of $\mathfrak{n}$ on V is non-trivial.

The main result of the paper is:

Theorem 3.1. *Let $\mathfrak{m}$ be a non-degenerate GNLA over an algebraically closed field. Then the following conditions are equivalent:*

1. *there exists such $d \in \operatorname{Der}_0(\mathfrak{m})$ that $d(\mathfrak{m}_{-i}) = 0$ for $i \geq 2$ and $\operatorname{rank} d = 1$;*
2. *there exists such $y \in \mathfrak{m}_{-1}$ that $\operatorname{rank} \operatorname{ad} y = 1$; in particular, in this case $[y, \mathfrak{m}_{-i}] = 0$ for all $i \geq 2$;*
3. *$\mathfrak{m}$ can be represented as a special extension of a certain (possibly degenerate) GNLA $\mathfrak{n}$ with $\dim \mathfrak{m}_{-1} = \dim \mathfrak{n}_{-1} + 1$ and $\dim V_{-i} = 1$ for all $i = 1, \ldots, \mu$ and $\mu \geq 2$;*
4. *$\mathfrak{m}$ is of infinite type.*

Proof. (1) $\Rightarrow$ (2). Let $\mathfrak{h}$ be a subalgebra in $\operatorname{Der}_0(\mathfrak{m})$ defined by condition $d(\mathfrak{m}_{-i}) = 0$ for any $d \in \mathfrak{h}$. Let $d \in \mathfrak{h}$ and $\operatorname{rank} d = 1$. Define $W \subset \mathfrak{m}$ as $W = \ker d$. Choose non-zero $x, y \in \mathfrak{m}_{-1}$ such that $d(x) = y$. Then we have $\mathfrak{m} = \langle x \rangle \oplus W$, and $W \supset [\mathfrak{m}, \mathfrak{m}]$. Next,

$$[y, W] = [d(x), W] = -[x, d(W)] = 0.$$

Since $\mathfrak{m}$ is non-degenerate, this implies that $\operatorname{rank} \operatorname{ad} y = 1$ and $[x, y] \neq 0$. We also have

$$[x, d(y)] = -[d(x), y] = -[y, y] = 0.$$

On the other hand,

$$[d(y), W] = -[y, d(W)] = 0.$$

Hence, $[d(y), \mathfrak{m}] = 0$, and from non-degeneracy of $\mathfrak{m}$ it follows that $d(y) = 0$. In particular, d is nilpotent.

(2) $\Rightarrow$ (1). Let $y \in \mathfrak{m}_{-1}$ such that $\operatorname{rank} \operatorname{ad} y = 1$. Let $W = \ker \operatorname{ad} y$. Choose such $x \in \mathfrak{m}_{-1}$ that $[x, y] \neq 0$. Such x always exists by non-degeneracy of $\mathfrak{m}$. It is clear that $\mathfrak{m} = \langle x \rangle \oplus W$ and $y \in W$. It is also clear that $W \supset [\mathfrak{m}, \mathfrak{m}]$. Define $d \in \mathfrak{h}$ as $d(x) = y$ and $d(W) = 0$. It is easy to see that d is indeed a differentiation of $\mathfrak{m}$.

(2) $\Rightarrow$ (3). Again, let x, y be elements in $\mathfrak{m}_{-1}$ such that $\operatorname{rank} \operatorname{ad} y = 1$, $[x, y] \neq 0$, $\mathfrak{m} = \langle x \oplus \rangle W$, $W \supset [\mathfrak{m}, \mathfrak{m}]$ and $[y, W] = 0$. Set $y_1 = y$ and define $y_{i+1} = [x, y_i]$ for all $i \geq 2$. Note that $y_2 \neq 0$. Let μ be the smallest integer such that $y_\mu \neq 0$ and $y_{\mu+1} = 0$. By induction we get:

$$[y_{i+1}, W] = [[x, y_i], W] = [x, [W, y_i]] + [y_i, [x, W]] = 0.$$

Let $V = \langle y_1, \dots y_\mu \rangle$. It is clear that $V \subset W$, $[V, W] = 0$ and $[x, V] \subset V$. Hence, V is a commutative ideal in $\mathfrak{m}$. Put $\mathfrak{n} = \mathfrak{m}/V$. It is easy to see that V is generated by V_{-1} as an $\mathfrak{n}$-module and the action of $[\mathfrak{n}, \mathfrak{n}]$ on V is trivial. Finally, it is evident that $\mathfrak{n}$ is also generated by $\mathfrak{n}_{-1}$ as a graded Lie algebra. Thus, $\mathfrak{m}$ is represented as a special extension:

$$0 \to V \to \mathfrak{m} \to \mathfrak{n} \to 0. \tag{3}$$

(3) $\Rightarrow$ (2). Let $\mathfrak{m}$ be a special extension (3) of $\mathfrak{n}$ with $\dim V_{-i} = 1$, $i = 1, \cdots, \mu$. Take y as a non-zero element in V_{-1}. Since $[\mathfrak{n}, \mathfrak{n}]$ acts trivially on V, we see that $[y, \mathfrak{m}] \subset V_{-2}$. Since V is generated by V_{-1} as an $\mathfrak{n}$-module, we see that $\operatorname{ad} y \neq 0$ and $\operatorname{rank} \operatorname{ad} y = 1$.

(4) $\Leftrightarrow$ (1). This is exactly the combined Tanaka and Spencer criteria. ■

Remark 3.2. Only the last implication uses the fact that the base field is algebraically closed. In case of arbitrary field of characteristic 0 the items (1), (2) and (3) are still equivalent and imply that $\mathfrak{m}$ has infinite type. A simple counterexample for the implication (4) $\Rightarrow$ (1) over $\mathbb{R}$ is given by the 3-dimensional complex Heisenberg Lie algebra (with an obvious grading of depth 2) viewed as a 6-dimensional real Lie algebra.

Remark 3.3. Condition (2) of the Theorem appears already in [12] as a sufficient condition for $\mathfrak{m}$ to be of infinite type.

Item (3) of Theorem 3.1 gives an inductive algorithm for constructing all GNLA of infinite type. Namely, to construct all GNLA $\mathfrak{m}$ of infinite type and $\dim \mathfrak{m}_{-1} = n + 1$ one needs to:

- take an arbitrary GNLA $\mathfrak{n}$ (of finite or infinite type) such that $\dim \mathfrak{n}_{-1} = n$;
- take an arbitrary subspace $W \subset \mathfrak{n}_{-1}$ of codimension 1;
- fix any integer $s \geq 2$ and construct a graded $\mathfrak{n}_{-1}/W$-module $V = \sum_{i=1}^{s} V_{-i}$, which is uniquely defined by the conditions $\dim V_{-i} = 1$, $i = 1, \dots, s$ and $V_{-i-1} = (\mathfrak{n}_{-1}/W).V_{-i}$;
- compute cohomology space $H^2_0(\mathfrak{n}, V)$, where V is treated as an $\mathfrak{n}$-module with the trivial action of $W \oplus [\mathfrak{n}, \mathfrak{n}]$;

- take any $\omega \in H^2_0(\mathfrak{n}, V)$ and define $\mathfrak{m}$ as an extension of $\mathfrak{n}$ by means of V corresponding to ω.

In coordinates this procedure can be reformulated as follows. Fix any convector $\alpha \in \mathfrak{n}^*_{-1}$ and an element $X \in \mathfrak{n}_{-1}$ such that $\alpha(X) = 1$. Extend X to the basis $\{X, Z_1, \dots, Z_r\}$ of $\mathfrak{n}$ consisting of homogeneous elements. Define $\mathfrak{m}$ as a Lie algebra with a basis $\{X, Y_1, \dots, Y_s, Z_1, \dots, Z_r\}$, $s \geq 2$, where

$$
\begin{aligned}
[X, Y_i] &= Y_{i+1}, \quad i = 1, \dots, s-1; \\
[Z_j, Y_i] &= 0, \quad i = 1, \dots, s, \ j = 1, \dots, r; \\
[Y_i, Y_j] &= 0, \quad i, j = 1, \dots, s; \\
[X, Z_i]_{\mathfrak{m}} &= [X, Z_i]_{\mathfrak{n}} + \sum_{j=1}^{s} a_i^j Y_j; \\
[Z_i, Z_j]_{\mathfrak{m}} &= [Z_i, Z_j]_{\mathfrak{n}} + \sum_{k=1}^{s} b_{ij}^k Y_k,
\end{aligned}
$$

where the constants a_i^j and b_{ij}^k define a cocycle $\wedge^2\mathfrak{n} \to V$ of degree 0 (i.e., $\mathfrak{m}$ is a graded Lie algebra) and are viewed modulo the changes of the basis $X \mapsto X + \sum_{i=1}^{s} c_i Y_s$, $Z_i \mapsto Z_i + \sum_{j=1}^{s} d_i^j Y_j$. See the next section for the concrete examples.

Theorem 3.1 provides also an easy way to prove that the set of all graded Lie algebras of infinite type forms a closed algebraic subvariety in the variety of all graded nilpotent Lie algebras. Namely, fix a sequence of integers $k_1, \dots, k_\mu$ and denote by $M(k_1, \dots, k_\mu)$ a variety of all graded nilpotent Lie algebras $\mathfrak{m}$ such that $\dim \mathfrak{m}_{-i} = k_i$ for $i = 1, \dots, \mu$ and $\mathfrak{m}_{-i} = 0$ for $i > \mu$. It is easy to see that it is a closed algebraic variety in the vector space of all skew-symmetric graded algebras (without any conditions on multiplication).

The set of all fundamental graded Lie algebras in $M(k_i)$ is distinguished by conditions $[\mathfrak{m}_{-1}, \mathfrak{m}_{-i}] = \mathfrak{m}_{-i-1}$ for all $i = 1, \dots, \mu - 1$. It is easy to see that these conditions define an open subset (in Zarisski topology) in $M(k_i)$, which we shall denote by $MF(k_i)$.

Define $MF^\infty(k_i)$ as the set of all fundamental Lie algebras of infinite type.

Theorem 3.4. *$MF^\infty(k_i)$ is a closed subvariety in $MF(k_i)$.*

Proof. The existence of an element $y \in \mathfrak{m}_{-1}$ such that $\operatorname{rank} \operatorname{ad} y = 1$ (and, hence, $[y, \mathfrak{m}_{-i}] = 0$ for $i \geq 2$ can be reformulated as an existence of a non-zero solution of a certain system of quadratic equations (2 by 2 minors of the matrix of $\operatorname{ad} y$), whose coefficients are structure constants of the Lie algebra $\mathfrak{m}$. It is well-known [8, Theorem 3.12] that that the conditions when such non-zero solution exists are given in terms of the algebraic equations on the coefficients of this system. ∎

Theorem 3.1 also has a number of immediate geometric applications. Namely, let D be a bracket generating distribution with a constat symbol $\mathfrak{m}$. As usual, we assume that D has no Cauchy characteristics. This implies that $\mathfrak{m}$ is non-degenerate, i.e., does not have non-zero central elements lying in $\mathfrak{m}_{-1}$.

We say that a vector field $Y \in D$ is *a weak characteristic of* D, if the following conditions hold:

1. $Y \in \operatorname{char}(D^i)$ for all $i \geq 2$, or, in other words, $[Y, (D^i)] \subset (D^i)$ for all $i \geq 2$;
2. the subbundle $D' \subset D$ spanned by all $X \in D$ such that $[X, Y] \subset D$ has codimension 1 in D.

Similarly, we can define *complex weak characteristics* of a vector distribution D as sections of the complexified subbundle $D^{\mathbb{C}}$ of the complexified tangent bundle $T^{\mathbb{C}}M$ satisfying the conditions (1) and (2) above, where we replace powers D^i by their complexified versions.

Theorem 3.5. *Assume that D is a bracket generating distribution with a constant symbol $\mathfrak{m}$, that D has no Cauchy characteristics, and that $\dim \operatorname{sym}(D) = \infty$. Then D contains non-trivial complex week characteristics.*

Proof. This immediately follows from item (2) of Theorem 3.1 and the Tanaka theorem [16] that bounds the dimension of $\operatorname{sym}(D)$ by the dimension of the Tanaka prolongation of its symbol $\mathfrak{m}$. ∎

Corollary 3.6. *If $D \cap (\cap_{i\geq 2} \operatorname{char}(D^i)) = 0$, then the symmetry algebra of D is finite-dimensional.*

Proof. Assume that $\operatorname{sym}(D)$ is infinite-dimensional. Then it possesses complex week characteristic Y. Clearly both $\overline{Y}$ and cY, $c \in C$, are also complex week characteristics. Therefore, the complex subbundle spanned by Y and $\overline{Y}$ is stable with respect to the complex conjugation. Hence, it also contains a real one-dimensional subbundle spanned by a certain non-zero vector field X. While X itself might no longer be a week characteristic, it still lies in $D \cap (\cap_{i\geq 2} \operatorname{char}(D^i))$. ∎

Remark 3.7. We formulate Theorem 3.5 and its corollary only for the vector distribution with constant symbol, since the Tanaka theorem on the bound of $\dim \operatorname{sym}(D)$ was stated only for this case. The recent paper of B. Kruglikov [9] generalizes Tanaka theorem to the case of non-constant symbol. In particular, this implies that Theorem 3.5 is also true for bracket-generating distributions with non-constant symbol.

Note that unlike Cauchy characteristics, the set of all weak characteristics does not form a subbundle of D. It defines a certain cone inside D, or more precisely, a so-called cone structure $\pi\colon \mathcal{C} \subset PD$, where for each $p \in M$ the set $\mathcal{C}_p$ is an algebraic variety in the projectivization PD_p of the vector space D_p. It is clear that this cone is naturally associated with the distribution D, and any (local) equivalence map of two distributions maps the corresponding cones of weak characteristics to each other. Let us give several illustrative examples of this cone structure.

Example 3.8. Let, as in Cartan's paper [3], D be a non-degenerate 2-dimensional distribution on a 5-dimensional manifold. Then it has a constant symbol $\mathfrak{m} = \langle X_1, \dots, X_5 \rangle$ with $\mathfrak{m}_{-1} = \langle X_1, X_2 \rangle$ and non-zero Lie brackets given by:

$$\begin{aligned} &[X_1, X_2] = X_3, \\ &[X_1, X_3] = X_4, \quad [X_2, X_3] = X_5. \end{aligned}$$

It is easy to see that there are no non-zero weak characteristics in this case. Thus, the cone structure is trivial in this case, and the symmetry algebra of D is automatically finite-dimensional. Note that the Tanaka prolongation of $\mathfrak{m}$ is exactly the simple exceptional Lie algebra G_2 appearing in [3].

Example 3.9. Let D be a contact distribution on a jet space $J^k(V, W)$, where V and W are two finite-dimensional vector spaces. It is well-known (see, for example, [18]) that D is a standard distribution of type $\mathfrak{m}$, where the GNLA $\mathfrak{m}$ is described as follows:

$$\mathfrak{m} = \sum_{i=0}^{k} \big(S^i(V^*) \otimes W\big) \oplus V.$$

Here both $\sum_{i=0}^{k} S^i(V^*) \otimes W$ and V are abelian subalgebras, and the bracket between them is given by the canonical pairing $S^i(V^*) \times V \to S^{i-1}(V^*)$, $i = 1, \dots, k$. The grading of $\mathfrak{m}$ is defined by $\mathfrak{m}_{-1} = S^k(V^*) \otimes W \oplus V$ and $\mathfrak{m}_{-i} = S^{k+1-i}(V^*) \otimes W$ for $i = 2, \dots, k+1$.

Assume that $k \geq 2$ or $\dim W \geq 2$. Then the set of all elements $x \in \mathfrak{m}_{-1}$ such that $\operatorname{rank} \operatorname{ad} x = 1$ has the form $\alpha^k \otimes w \in S^k(V^*) \otimes W$, where $\alpha \in V^*$, $w \in W$. Thus, we see that it forms a non-trivial cone in $S^k(V^*) \otimes W$, whenever $\dim V \geq 2$. If $k = 1$ and $\dim W = 1$, then the set of all such elements x coincides with all $\mathfrak{m}_{-1}$.

Thus, we see that the cone of weak characteristics for the contact distribution D is always non-trivial and, assuming $k \geq 2$ or $\dim W \geq 2$, spans the vertical subbundle of the canonical projection $J^k(V, W) \to J^{k-1}(V, W)$. This fact is one of the core ideas of the proof of the well-known Lie–Backlund theorem on the symmetries of the contact systems on jet spaces.

4. Examples

4.1. Symbols of 2-dimensional distributions of infinite type. As a first application of Theorem 3.1 we describe the symbols of 2-dimensional bracket generating distributions of infinite type.

Using Theorem 3.1, it is easy to prove that in each dimension $n \geq 2$ there exists a unique (up to isomorphism) GNLA $\mathfrak{m}$ with $\dim \mathfrak{m}_{-1} = 2$. It turns out to coincide with the symbol of the canonical contact system on the jet space $J^{n-2}(\mathbb{R}, \mathbb{R})$.

Indeed, by Theorem 3.1 any such GNLA must be a special extension of a certain GNLA $\mathfrak{n}$ with $\dim \mathfrak{n}_{-1} = 1$. But then $\mathfrak{n} = \mathbb{R}$ is a one-dimensional commutative Lie algebra, and all its special extensions are trivial. In each dimension there is exactly one such extension. It can be described as a Lie algebra $\mathfrak{m}$ with a basis $\langle X, Z_1, \dots, Z_{n-1}\rangle$ and the only non-zero Lie brackets $[X, Z_i] = Z_{i+1}$, $i = 1, \dots, n-2$. Here $\deg X = -1$ and $\deg Z_i = -i$, $i = 1, \dots, n-1$. It is easy to see that the contact system on $J^{n-2}(\mathbb{R}, \mathbb{R})$ is equivalent to the standard distribution of type $\mathfrak{m}$.

4.2. Symbols of 3-dimensional distributions of infinite type. In case of $\dim \mathfrak{m}_{-1} = 3$ we already get a lot of various examples, including the examples of non-trivial special extension of GNLA. As above, any GNLA $\mathfrak{m}$ of infinite type with $\dim \mathfrak{m}_{-1} = 3$ is a special extension of a certain GNLA $\mathfrak{n}$ with $\dim \mathfrak{n}_{-1} = 2$. Note that $\mathfrak{n}$ itself does not need to be of infinite type. As the classification of all such GNLA unknown at the moment, it makes also impossible to classify all infinite type GNLA with $\dim \mathfrak{m}_{-1} = 3$.

As an example, let us describe all special extensions in case, when $\mathfrak{n}$ is a three-dimensional Heisenberg algebra. Then all trivial special extensions of $\mathfrak{n}$ can be described as:

$$\mathfrak{m} = \langle X, Z_1, Z_2, Y_1, \dots, Y_k\rangle,$$

where $\deg X = -1$, $\deg Y_i = -i$, $i = 1, \dots, k$, $\deg Z_j = -j$, $j = 1, 2$. All non-zero Lie brackets are $[X, Z_1] = Z_2$ and $[X, Y_i] = Y_{i+1}$, $i = 1, \dots, k-1$. The Lie algebra $\mathfrak{m}$ is a semidirect product of its subalgebra $\mathfrak{n} = \langle X, Z_1, Z_2\rangle$ and the commutative ideal $V = \langle Y_1, \dots, Y_k\rangle$. Geometrically the standard distribution of type $\mathfrak{m}$ can be described as a canonical contact system on the mixed jet bundle $J^{1,k-1}(\mathbb{R}, \mathbb{R}^2)$.

Any non-trivial special extension of $\mathfrak{n}$ can be obtained by adding non-trivial Lie brackets of the form:

$$\begin{aligned} [X, Z_1] &= Z_2 + aY_2; \\ [X, Z_2] &= bY_3; \\ [Z_1, Z_2] &= cY_3. \end{aligned}$$

By changing Z_1 to $Z_1 - aY_1$ and Z_2 to $Z_2 - bY_2$ we can always obtain $a = b = 0$. If $a \neq 0$, then the Jacobi identity is satisfied only if $k = 3$. In this case we can always scale a to 1. Thus, we see that up to the isomorphism there exists exactly one non-trivial special extension of the three-dimensional Heisenberg Lie algebra. It is given by:

$$\mathfrak{m} = \langle X, Z_1, Z_2, Y_1, Y_2, Y_3\rangle,$$

where non-zero Lie brackets are given by:

$$\begin{aligned} [X, Y_1] &= Y_2, \quad [X, Y_2] = Y_3; \\ [X, Z_1] &= Z_2; \\ [Z_1, Z_2] &= Y_3. \end{aligned}$$

Direct computation shows (see [11], Lie algebra $m6\_3\_4$ in terms of this paper) that an arbitrary symmetry of the standard distribution of type $\mathfrak{m}$ is infinite-dimensional and depends on 4 functions of 1 variable.

4.3. Metabelian Lie algebras. Let $\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}$ be a GNLA of depth 2. The structure of this Lie algebra is completely determined by the skew-symmetric map: $S\colon \wedge^2 \mathfrak{m}_{-1} \to \mathfrak{m}_{-2}$ defined as a restriction of the Lie bracket to $\mathfrak{m}_{-1}$. Below we assume that $\mathfrak{m}$ is non-degenerate.

Consider the dual map $S^*\colon \mathfrak{m}_{-2}^* \to \wedge^2\mathfrak{m}_{-1}^*$. Since $\mathfrak{m}$ is generated by $\mathfrak{m}_{-1}$, the map S^* is injective. Thus, up to the isomorphism the structure of $\mathfrak{m}$ is determined by the subspace $\operatorname{Im} S^* \subset \wedge^2\mathfrak{m}_{-1}^*$. This subspace is considered up to the natural action of the group $GL(\mathfrak{m}_{-1})$ on $\wedge^2\mathfrak{m}_{-1}^*$.

To simplify the notation, let $V = \mathfrak{m}_{-1}$, $n = \dim V$, and let $P = \operatorname{Im} S^* \subset \wedge^2 V^*$, $m = \dim P$. If we fix a basis in V, then $\wedge^2 V^*$ can be identified with a space of all skew-symmetric n by n matrices, P is spanned by m matrices $B_1, \dots, B_m$ and the action of $GL(V)$ on $\wedge^2 V^*$ is given by:

$$X.B = X^t B X, \quad X \in GL(V), B \in \wedge^2 V^*.$$

Let y be an arbitrary element in $\mathfrak{m}_{-1} = V$. It is easy to see that the rank of $\operatorname{ad} y$ is the maximal number of linearly independent skew-symmetric matrices B in P, such that $i_y B \neq 0$. In more detail, let P_y be a subspace in P defined as:

$$P_y = \{B \in P \mid i_y B = 0\}.$$

Then rank of $\operatorname{ad} y$ is equal to to codimension of P_y in P.

Theorem 4.1. *Let $\mathfrak{m} = \mathfrak{m}_{-1} \oplus \mathfrak{m}_{-2}$ be a 2-step complex graded nilpotent Lie algebra with $\dim \mathfrak{m}_{-2} = 2$. Then Tanaka prolongation of $\mathfrak{m}$ is infinite-dimensional.*

Proof. Since the dimension of the Tanaka prolongation is preserved under extension of the base field, we can assume that our base field is algebraically closed. Then according to Theorem 3.1 $\mathfrak{m}$ is of infinite type if and only if there exists a non-zero element $y \in \mathfrak{m}_{-1} = V$ such that P_y is one-dimensional. Let B_1, B_2 be the basis of P. Then the condition $\dim P_y = 1$ is equivalent to the existence of such λ_1, λ_2 that:

$$(\lambda_1 B_1 + \lambda_2 B_2) y = 0.$$

It is clear that this is equivalent to $\det(\lambda_1 B_1 + \lambda_2 B_2) = 0$, which always has non-trivial solutions in case of algebraically closed field. ∎

Note that pencils of skew-symmetric matrices over an algebraically closed field can be effectively classified using Kronecker results on pencils of matrices. This has been done in the work of M. Gauger [6]. Namely, any pair (A, B) of skew-symmetric matrices can be written as one matrix $P = \mu A + \lambda B$, whose entries are linear forms in λ and μ. Such matrices are classified by the following data:

- minimal indices $0 \le m_1 \le m_2 \le m_p$, $p \ge 0$ (in particular, the set of minimal indices can be empty);
- elementary divisors $(\mu + a_1\lambda)^{e_1}$, …, $(\mu + a_q\lambda)^{e_r}$, $(\lambda)^{f_1}$, …, $(\lambda)^{f_s}$ (each of these divisors appears twice).

The canonical form of the pencil P for this data is:

$$P = \begin{pmatrix} \mathcal{M}_{m_1} & & & & & & & & \\ & \ddots & & & & & & & \\ & & \mathcal{M}_{m_p} & & & & & & \\ & & & \mathcal{E}_{e_1}(a_1) & & & & & \\ & & & & \ddots & & & & \\ & & & & & \mathcal{E}_{e_r}(a_r) & & & \\ & & & & & & \mathcal{F}_{f_1} & & \\ & & & & & & & \ddots & \\ & & & & & & & & \mathcal{F}_{f_s} \end{pmatrix} \tag{4}$$

where

$$\mathcal{M}_m = \begin{pmatrix} 0 & M_m \\ -M_m^t & 0 \end{pmatrix}, \quad (2m+1)\times(2m+1), \mathcal{M}_0 = (0);$$
$$\mathcal{E}_n(a) = \begin{pmatrix} 0 & E_n(a) \\ -E_n(a)^t & 0 \end{pmatrix}, \quad (2n)\times(2n);$$
$$\mathcal{F}_n = \begin{pmatrix} 0 & F_n \\ -F_n^t & 0 \end{pmatrix}, \quad (2n)\times(2n);$$

and

$$M_m = \begin{pmatrix} & & & & & \lambda \\ & & & & \lambda & \mu \\ & & & \cdot & \mu & \\ & & \cdot & \cdot & & \\ & \cdot & \cdot & & & \\ \lambda & \cdot & & & & \\ \lambda & \mu & & & & \\ \mu & & & & & \end{pmatrix}, \quad (m+1)\times m;$$

$$E_n(a) = \begin{pmatrix} & & & & \mu + a\lambda \\ & & & \cdot & \lambda \\ & & \cdot & & \\ & \cdot & & \cdot & \\ & & \cdot & & \\ & \mu + a\lambda & \cdot & & \\ \mu + a\lambda & \lambda & & & \end{pmatrix}, \quad n\times n;$$

$$F_n = \begin{pmatrix} & & & & \lambda \\ & & & \cdot & \mu \\ & & \cdot & & \\ & \cdot & & \cdot & \\ & & \cdot & & \\ & \lambda & \cdot & & \\ \lambda & \mu & & & \end{pmatrix}, \quad n\times n;$$

In addition, the elementary divisors are considered up to non-degenerate linear transformations of (λ, μ).

Note also, that the pencil P above corresponds to a non-degenerate 2-step nilpotent Lie algebra if and only if there are no minimal indices 0 among the set of minimal indices $m_1, \dots, m_p$. As we assume that these indices are ordered, this is equivalent to the condition $m_1 > 0$.

Define the following subgroups in $GL(V)$ and the corresponding subalgebras in $\mathfrak{gl}(V)$:

$$\begin{aligned} \mathrm{Aut}(P) &= \{X \in GL(V) \mid X^t B X \subset P \text{ for any } B \in P\}, \\ H_0 &= \{X \in GL(V) \mid X^t B X = B \text{ for any } B \in P\}, \\ \mathrm{Der}(P) &= \{X \in \mathfrak{gl}(V) \mid X^t B + BX \subset P \text{ for any } B \in P\}, \\ \mathfrak{h}_0 &= \{X \in \mathfrak{gl}(V) \mid X^t B + BX = 0 \text{ for any } B \in P\}. \end{aligned}$$

It is clear that:

- $\mathrm{Der}(P)$ and $\mathfrak{h}_0$ are subalgebras in $\mathfrak{gl}(V)$ corresponding to the subgroups $\mathrm{Aut}(P)$ and H_0 respectively;
- H_0 is a subgroup in $\mathrm{Aut}(P)$, $\mathfrak{h}_0$ is a subalgebra in $\mathrm{Der}(P)$;
- $\mathrm{Aut}(P)$ is naturally identified with the the group $\mathrm{Aut}_0(\mathfrak{m})$ of all grading-preserving automorphisms of the Lie algebra $\mathfrak{m}$ and $\mathrm{Der}(P)$ is identified with the Lie algebra $\mathrm{Der}_0(\mathfrak{m})$ of all grading-preserving derivations of $\mathfrak{m}$;
- $\mathfrak{h}_0$ can be identified with all elements in $\mathrm{Der}_0(\mathfrak{m})$ that act trivially on $\mathfrak{m}_{-2}$.

Let us show explicitly that $\mathfrak{h}_0$ always contains a rank 1 element. This will give another proof that the standard prolongation of $\mathfrak{h}_0$ is infinite-dimensional.

Let Q be any of blocks that appear on the diagonal in the canonical form (4) of the pencil P. That is Q is one of the following matrices: $\mathcal{M}_m$ for $m \geq 0$, $\mathcal{E}_e(a)$ for $e \geq 1$, or $\mathcal{F}_f$, $f \geq 1$. We shall call Q an elementary subpencil of the pencil P and denote by $\mathfrak{h}_0(Q)$ the Lie algebra

$$\mathfrak{h}_0(Q) = \{X \in \mathfrak{gl}(k, \mathbb{C}) \mid X^t B + BX = 0 \text{ for all } B \in Q\}.$$

It is easy to see that $\mathfrak{h}_0(Q)$ can be embedded as a subalgebra into $\mathfrak{h}_0$.

So, to prove the theorem it is sufficient to show that $\mathfrak{h}_0(Q)$ contains rank 1 element for each elementary subpencil. Moreover, since elementary divisors are considered up to linear transformations of (λ, μ), we can restrict ourselves only to elementary pencils $\mathcal{M}_m$, $m \geq 1$ and $\mathcal{F}_r$, $r \geq 1$.

In both these cases the subalgebra $\mathfrak{h}_0(Q)$ is easily computed. Namely, denote by $S_{k,l}(\mathbb{C})$ the set of $k \times l$ matrices (a_{ij}) such that $a_{ij} = a_{i+1,j+1}$ for any $1 \leq i < k$, $1 \leq j < l$. Denote also by $T_k(\mathbb{C})$ the set of all $k \times k$ upper triangular matrices that also lie in $S_{k,k}(\mathbb{C})$. Note that both $S_{k,l}(\mathbb{C})$ and $T_k(\mathbb{C})$ contain elements of rank 1 (for example, in the upper right corner).

We have:

$$\mathfrak{h}_0(\mathcal{M}_m) = \left\{ \begin{pmatrix} xE_{m+1} & Y \\ 0 & -xE_m \end{pmatrix} \middle| \; x \in \mathbb{C}, Y \in S_{m+1,m}(\mathbb{C}) \right\};$$

$$\mathfrak{h}_0(\mathcal{F}_r) = \left\{ \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & -Y_{11} \end{pmatrix} \middle| \; Y_{11}, Y_{12}, Y_{21} \in T_r(\mathbb{C}) \right\};$$

As we see, in both cases $\mathfrak{h}_0(Q)$ contains a nilpotent element of rank 1.

4.4. Remarks. 1. It follows form the proof of Theorem 4.1 that $\mathfrak{h}_0$ contains the block diagonal subalgebra, where each block consists of all elements in $\mathfrak{h}_0(Q)$ for the primitive subpencils Q. However, $\mathfrak{h}_0$ can be larger than this direct sum. For example, this happens in the case when there no minimal indices and the set of (λ) and (μ) where each of these divisors appears with multiplicity $2p$ and $2q$ respectively. In this case $\mathfrak{m}$ is isomorphic to the direct sum of two Heisenberg lie algebras of dimension $2p+1$ and $2q+1$. The subalgebra $\mathfrak{h}_0$ is isomorphic to $\mathfrak{sp}(2p,\mathbb{C}) \oplus \mathfrak{sp}(2q,\mathbb{C})$, while $\mathfrak{h}_0(Q)$ for each primitive Q is just $\mathfrak{sl}(2,\mathbb{C})$.

2. It is well-known that there exist 2-step nilpotent Lie algebras with 3-dimensional center, whose Tanaka prolongation is finite-dimensional. The simplest example is a free 2-step nilpotent Lie algebra with 3-dimensional set of generators and the 3-dimensional center:

$$\mathfrak{m}_{-1} = \langle X_1, X_2, X_3 \rangle,$$
$$\mathfrak{m}_{-2} = \langle X_{12}, X_{13}, X_{23} \rangle,$$

where $[X_i, X_j] = X_{ij}$ for $1 \le i < j \le 3$. Its Tanaka prolongation is 21-dimensional and is isomorphic to $\mathfrak{so}(7,\mathbb{C})$. In this case the subalgebra $\mathfrak{h}_0$ is trivial.

We can also generalize this example for arbitrary number of generators. Namely, let:

$$\mathfrak{m}_{-1} = \langle X_1, X_2, \dots, X_k \rangle,$$
$$\mathfrak{m}_{-2} = \langle Y_1, Y_2, Y_3 \rangle,$$

where all non-zero Lie brackets have the form:

$$[X_1, X_k] = [X_2, X_{k-1}] = \dots = Y_1;$$
$$[X_1, X_{k-1}] = [X_2, X_{k-2}] = \dots = Y_2;$$
$$[X_2, X_k] = [X_3, X_{k-1}] = \dots = Y_3.$$

Direct computation shows that Tanaka prolongation $\mathfrak{g}(\mathfrak{m})$ of $\mathfrak{m}$ has the form:

- for $k=3$, $\mathfrak{g}(\mathfrak{m}) \equiv \mathfrak{so}(7,\mathbb{C})$;
- for $k=4$, $\mathfrak{g}(\mathfrak{m}) \equiv \mathfrak{sp}(6,\mathbb{C})$;
- for $k=5$, $\mathfrak{g}(\mathfrak{m}) = \mathfrak{m} \oplus \mathfrak{g}_0$, where $\mathfrak{g}_0 \equiv \mathfrak{gl}(2,\mathbb{C})$ and $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is irreducible;
- for $k \ge 6$ and even, $\mathfrak{g}(\mathfrak{m}) = \mathfrak{m} \oplus \mathfrak{g}_0$, where $\mathfrak{g}_0 \equiv \mathfrak{gl}(2,\mathbb{C}) \times \mathbb{C}$ and $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ splits into the sum of $k/2$ $\mathfrak{gl}(2,\mathbb{C})$-modules $\langle X_i, X_{k/2+i} \rangle$, $i = 1, \dots, k/2-1$;

- for $k \geq 7$ and odd, $\mathfrak{g}(\mathfrak{m}) = \mathfrak{m} \oplus \mathfrak{g}_0$, where $\mathfrak{g}_0 = \mathbb{C}^2$ and the action of $\mathfrak{g}_0$ on $\mathfrak{g}_{-1}$ diagonalizes in the basis $\{X_1, X_2, \dots, X_n\}$.

The above example of 2-step nilpotent Lie algebra with 3-dimensional center shows that the subalgebra $\mathfrak{h}_0$ is trivial for generic 2-step nilpotent Lie algebras with $\dim \mathfrak{m}_{-2} \geq 3$. In particular, a generic GNLA of depth 2 with at least 3-dimensional center is of finite type. However, it is still an open problem to classify all 2-step nilpotent Lie algebras with infinite-dimensional Tanaka prolongation.

3. Let D be a vector distribution of codimension 2 on a smooth manifold M, such that $D^2 = TM$. Then Theorem 4.1 implies that the symbol $\mathfrak{m}(x)$ of D is of infinite type for each $x \in M$. However, this does not imply that such distributions always have infinite-dimensional symmetry algebras.

For example, let D be a generic 3-dimensional distribution on a 5-dimensional space. This case was treated in a historical paper of E. Cartan [3]. The symbol algebra of D is equivalent to the symbol algebra of the canonical contact system on $J^1(\mathbb{R}, \mathbb{R}^2)$ and, thus, is of infinite type. Let D' be a subdistribution of D generated by all weak characteristics of D. Then D' has rank 2. In generic case we have $D = [D', D']$ and, therefore, the symmetry algebras of D and D' coincide. As described in Example 3.8, the symbol of a generic rank 2 distribution on a 5-dimensional manifold is of finite type and, in particular, the symmetry algebra of D is finite-dimensional.

More general examples of distributions with the symbol corresponding to the pencil $\mathcal{M}_m$ are treated in [10]. It is proved that if a subbundle D' generated by all weak characteristics of D is bracket generating, then the symmetry algebra of D is finite-dimensional. Note that in case of the standard distribution D of type $\mathfrak{m}$, where $\mathfrak{m}$ corresponds to the pencil $\mathcal{M}_m$, the subdistribution D' is completely integrable.

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