Conical Distributions on the Space of Flat Horocycles

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Abstract. Let $G_0 = K \ltimes p$ be the Cartan motion group associated with a noncompact semisimple Riemannian symmetric pair $(G, K)$. Let $a$ be a maximal abelian subspace of $p$ and let $p = a + q$ be the corresponding orthogonal decomposition. A flat horocycle in $p$ is a $G_0$-translate of $q$. A conical distribution on the space $\Xi_0$ of flat horocycles is an eigendistribution of the algebra $D(\Xi_0)$ of $G_0$-invariant differential operators on $\Xi_0$ which is invariant under the left action of the isotropy subgroup of $G_0$ fixing $q$. We prove that the space of conical distributions belonging to each generic eigenspace of $D(\Xi_0)$ is one-dimensional, and we classify the set of all conical distributions on $\Xi_0$ when $G/K$ has rank one. We also consider the question of the irreducibility of the natural representation of $G_0$ on the eigenspaces of $D(\Xi_0)$.

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1. Introduction and Preliminaries

In this paper we study the flat analogues of conical distributions on the space of horocycles associated with noncompact symmetric spaces. Let $G$ be a connected noncompact real semisimple Lie group with finite center, let $g$ be its Lie algebra, and let $K$ be a maximal compact subgroup of $G$. Let $\theta$ be the corresponding Cartan involution of $G$, and we also let $\theta$ denote its differential on $g$. Let $\mathfrak{k}$ be the Lie algebra of $K$ and $\mathfrak{p}$ its orthogonal complement relative to the Killing form $B$ on $g$, so that $g$ has Cartan decomposition $g = \mathfrak{k} + \mathfrak{p}$. We will generally use the notation in Helgason’s books [10], [11], and [12]. In particular, we let $a$ denote a maximal abelian subspace of $p$, $\Sigma$ the set of restricted roots of $g$ relative to $a$, $W$ the Weyl group of $G$, $g_\alpha$ the restricted root space corresponding to $\alpha \in \Sigma$ and $m_\alpha$ its dimension. In addition, let $a^+$ denote a fixed Weyl chamber in $a$, $\Sigma^+$ the corresponding positive system of restricted roots, and $\rho = 1/2 \sum_{\alpha \in \Sigma^+} m_\alpha \alpha$. We put $n = \sum_{\alpha \in \Sigma^+} g_\alpha$, and let $N$ and $A$ be the analytic subgroups of $G$ with Lie algebras $n$ and $a$, respectively. Then $G$ has Iwasawa decomposition $G = NAK$. Finally, we let $M$ and $M'$ denote the centralizer and normalizer of $A$ in $K$, respectively. Then $W = M'/M$. We let $w$ be the order of $W$. 
We identify \( p \) with \( p^* \) (respectively \( a \) with \( a^* \)) via the restriction of the Killing form \( B \) to \( p \) (respectively \( a \)). In this way, elements of the (complexified) symmetric algebra \( S(p) \) can be viewed as polynomial functions on \( p \), and also as constant coefficient differential operators on \( p \). If \( p \in S(p) \), we let \( \partial(p) \) be the corresponding differential operator on \( p \).

A horocycle in the symmetric space \( G/K \) is an orbit of a conjugate of \( N \) in \( G/K \). The following basic facts about horocycles may be found in Chapter II, §1–3 of [12]. The group \( G \) acts transitively on the space \( \Xi \) of all horocycles, and the isotropy subgroup of \( G \) fixing the identity horocycle \( \xi_0 = N \cdot o \) is \( MN \), so that \( \Xi = G/MN \). The mapping \((kM,a) \mapsto ka \cdot \xi_0 \) is a diffeomorphism of \( K/M \times A \) onto \( \Xi \), and the left \( G \)-invariant measure \( d\xi \) on \( \Xi \) (which is unique up to a constant multiple) is given by

\[
\int_{\Xi} \varphi(\xi) \, d\xi = \int_{K/M} \int_{A} \varphi(kM, a) \, e^{2p(\log a)} \, da \, dk_M, \quad (\varphi \in C_c(K/M))
\]  

(1)

where \( dk_M \) denotes the normalized \( K \)-invariant measure on \( K/M \) and \( da \) denotes the Lebesgue measure on the Euclidean space \( A \).

The algebra \( \mathbb{D}(\Xi) \) of \( G \)-invariant differential operators on \( \Xi \) is isomorphic to \( S(a) \), the symmetric algebra of \( a \), via

\[
\mathcal{D}_p \varphi(k \exp H \cdot \xi_0) = \partial(p)_H \varphi(k \exp H \cdot \xi_0), \quad (p \in S(a))
\]

(2)

Let \( \mathcal{D}'(\Xi) \) denote the space of all distributions on \( \Xi \). If \( D \in \mathbb{D}(\Xi) \) and \( \Psi \in \mathcal{D}'(\Xi) \), the distribution \( D\Psi \) on \( \Xi \) is given by

\[
D\Psi(\varphi) = \Psi(D^* \varphi), \quad (\varphi \in \mathcal{D}(\Xi))
\]

where \( D^* \in \mathbb{D}(\Xi) \) is the adjoint of \( D \) under the invariant measure \( d\xi \). If \( p \in S(a) \), it follows from (1) that

\[
(D_p)^* = D_{e^{-2p^* \rho a^2p}}
\]

(3)

where \( p^* \in S(a) \) is given by \( \partial(p^*) = \partial(p)^* \), the formal adoint of the differential operator \( \partial(p) \) in \( a \).

If \( a^* \) is the complexified dual space of \( a \), then the set of all joint eigendistributions of \( \mathbb{D}(\Xi) \) is parametrized by \( a^* \times \mathcal{D}'(K/M) \). More precisely, if we fix \( \lambda \in a^*_c \), then the relation (3) above implies that the joint eigenspace \( \mathcal{D}'_\lambda(\Xi) = \{ \Psi \in \mathcal{D}'(\Xi) \mid D_p \Psi = p(i\lambda - \rho) \Psi \text{ for all } p \in S(a) \} \) consists precisely of those distributions in \( \Xi \) of the form

\[
\Psi(\varphi) = \int_{K/M} \int_{A} \varphi(kM, a) \, e^{(i\lambda + \rho)(\log a)} \, da \, dS(kM) \quad (\varphi \in \mathcal{D}(\Xi))
\]

(4)

for some \( S \in \mathcal{D}'(K/M) \). (See Proposition 4.4, Chapter II in [12].)

A conical distribution in \( \Xi \) is an \( MN \)-invariant joint eigendistribution of \( \mathbb{D}(\Xi) \). If \( \lambda \) is regular and simple, it turns out that the vector space of conical distributions in \( \mathcal{D}'_\lambda(\Xi) \) is \( w \)-dimensional, and an explicit basis \( \{ \Psi_{\lambda,s} \} \) can be found in [7], each of which is supported in the closure of a Bruhat orbit in \( \Xi \). For exceptional \( \lambda \), the problem of classification of the conical distributions turns
out to be more difficult, although for rank one it is completely solved. (See [7] and Hu’s thesis [13]; the results in these papers are explained in §5–6 of [12].)

In this paper we consider the analogue of conical distributions on the space of flat horocycles in $\mathfrak{p}$. The flat horocycles are the translates, under the Cartan motion group, of the tangent space at the origin $o$ in $G/K$ to the identity horocycle $\xi_0 = N \cdot o$.

To be more precise, let us consider the Cartan motion group $G_0 = K \ltimes \mathfrak{p}$. $G_0$ has group law $(k, X) (k', X') = (kk', X + k \cdot X')$, for $k, k' \in K$ and $X, X' \in \mathfrak{p}$, where we have put $k \cdot X' = Ad k(X')$. The mapping

$$(T, X) \mapsto T + X \quad (T \in \mathfrak{k}, X \in \mathfrak{p})$$

identifies the Lie algebra $\mathfrak{g}_0$ of $G_0$ with $\mathfrak{g}$ as vector spaces. Under this identification, the adjoint representation $Ad_0$ of $G_0$ on $\mathfrak{g}_0$ is given by

$$Ad_0 (k, X)(T' + X') = k \cdot T' + k \cdot X' - [k \cdot T', X]$$

and the Lie bracket $[\ ,\ ]_0$ on $\mathfrak{g}_0$ is given by

$$[T + X, T' + X']_0 = [T, T'] + [T, X'] - [T', X]$$

with $T, T' \in \mathfrak{k}, X, X' \in \mathfrak{p}$, where the Lie brackets on the right are taken in $\mathfrak{g}$. In effect, the Lie bracket on $\mathfrak{g}_0$ is the same as that on $\mathfrak{g}$, except that the subspace $\mathfrak{p}$ has been made abelian.

Now $G_0$ acts transitively on $\mathfrak{p}$ by $(k, X) \cdot Y = X + k \cdot Y$, with $k \in K$ and $X, Y \in \mathfrak{p}$. Let $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$. If we identify $\mathfrak{p}$ with the tangent space $T_oX$, we have $\mathfrak{q} = T_o(N \cdot o)$. A flat horocycle is a translate of $\mathfrak{q}$ by an element of $G_0$. Let $\Xi_0$ be the set of all flat horocycles. Then of course $\Xi_0$ is a homogenous space of $G_0$, and according to Lemma 5.1, Chapter IV of [12], its isotropy subgroup at $\mathfrak{q}$ is $H_\mathfrak{q} = M' \ltimes \mathfrak{q}$.

The flat horocycle Radon transform of $f \in C_c(\mathfrak{p})$ is the function on $\Xi_0$ defined by

$$R f(\xi) = \int_\xi f(X) \ dm(X) \quad (\xi \in \Xi_0)$$

where $dm(X)$ is the Euclidean measure on $\xi$. One may view this as the flat analogue of the horocycle Radon transform on a noncompact symmetric space $G/K$. Properties of this transform, such as an inversion formula, and range and support theorems, have been studied in papers by Helgason and Orloff ([9], [18], [19]). (For a summary, see [12] Chapter IV, §5.) For a brief introduction to harmonic analysis on $\mathfrak{p}$, see [12], Chapter III, §7, which replicates much of the material in [8]. For a sample of the numerous papers dealing with analysis related to Cartan motion groups, see [1], [2], [3], [4], [14].

Let $\mathbb{D}(\Xi_0)$ denote the algebra of all differential operators on $\Xi_0$ invariant under the left action of $G_0$. By analogy with $\Xi = G/MN$, a conical distribution on $\Xi_0$ is a joint eigendistribution of $\mathbb{D}(\Xi_0)$ invariant under the left action of the isotropy subgroup $H_\mathfrak{q}$ of $G_0$ fixing $\mathfrak{q}$. Our aim in this paper is to classify such conical distributions.
In Section 2, we prove that $\mathbb{D}(\Xi_0)$ is isomorphic to the algebra $I(\mathfrak{a})$ of $W$-invariant elements in the symmetric algebra $S(\mathfrak{a})$. This is the analogue for $\Xi_0$ of Theorem 2.2 in [5], which states that $\mathbb{D}(\Xi)$ is isomorphic to $S(\mathfrak{a})$. (See equation (2) above.) It follows that the spaces of joint eigendistributions of $\mathbb{D}(\Xi_0)$ are parametrized by the set $\mathfrak{a}_c^*/W$ of $W$-orbits in $\mathfrak{a}_c^*$. In Section 3, we obtain a general characterization of the joint eigendistributions in $\Xi_0$ similar to that given by the expression (4) above for $\Xi$. In Section 4, we prove our main result (Theorem 4.3 below), which states that in each “generic” joint eigenspace (corresponding to regular $\lambda \in \mathfrak{a}_c^*$), the space of conical distributions is one-dimensional, where we also provide an explicit basis vector.

In Section 5 we show that, by contrast, the space of conical distributions in each joint eigenspace corresponding to non-regular $\lambda$ is infinite-dimensional. The problem of classifying the conical distributions for such $\lambda$ appears to be difficult, although for $G/K$ of rank one (so that $\lambda = 0$), we obtain a complete characterization in Theorem 5.2.

Finally, in Section 6, we consider the natural representation of $G_0$ on the spaces of joint eigendistributions of $\mathbb{D}(\Xi_0)$, relate these to conical distributions, and study the question of irreducibility.

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2. Invariant Differential Operators on $\Xi_0$ and $\tilde{\Xi}_0$

Just as with $\Xi \cong K/M \times A$, it will be convenient to characterize $\Xi_0$ as a vector bundle. For each $s \in W$, we choose a representative $m_s \in M'$. Then the map $\pi : K/M \times W a \to \Xi_0$ given by $\pi(kM,H) = k \cdot (H + q)$ is $w$ to one, with $\pi(kM,H) = \pi(km^{-1}M,sH)$. We can thus identify $\Xi_0$ with the associated bundle $K/M \times_W a$ over $K'/M'$, where $K/M$ can be viewed as a principal bundle over $K'/M'$ with discrete structure group $W = M'/M$. For convenience, we put $[kM,H] = \pi(kM,H)$. It will be clear from the context that this will not be confused with the Lie bracket.

Using the above notation, the action of $G_0$ on $\Xi_0$ is given by

\[ (k,X) \cdot [k_0M,H_0] = X + k \cdot (k_0 \cdot (H_0 + q)) \]
\[ = kk_0 \cdot (H_0 + ((kk_0)^{-1} \cdot X)_a + q) \]
\[ = [kk_0M,H_0 + ((kk_0)^{-1} \cdot X)_a] \] (9)

Here $X_a$ is the orthogonal projection (under the Killing form) of $X \in \mathfrak{p}$ onto $\mathfrak{a}$.

It will also be convenient to note that $G_0$ also acts transitively on the product manifold $\tilde{\Xi}_0 = K/M \times a$ via

\[ (k,X) \cdot (k_0M,H_0) = (kk_0M,H_0 + ((kk_0)^{-1} \cdot X)_a). \] (10)

(That this is a group action is straightforward to verify.) Note that $\tilde{\Xi}_0$, rather than $\Xi_0$, is in a certain sense the limit of the space $G/MN \cong K/M \times A$ of horocycles.
in \( G/K \). The isotropy subgroup of \( G_0 \) at the origin \( \tilde{\xi}_0 = (eM, 0) \in \tilde{\Xi}_0 \) is \( M \rtimes q \).

From (9) and (10), it is immediate that the projection map \( \pi : \tilde{\Xi}_0 \rightarrow \Xi_0 \) commutes with the action of \( G_0 \). It will frequently be useful to do calculations on \( \Xi_0 \) by lifting them up to \( \tilde{\Xi}_0 \). All groups being unimodular, there are unique (up to constant multiple) \( G_0 \)-invariant measures on \( \Xi_0 \) and on \( \tilde{\Xi}_0 \), which we can take in both cases to be \( dk_M dH \).

In this section, our objective is to determine the algebras \( \mathbb{D}(\Xi_0) \) and \( \mathbb{D}(\tilde{\Xi}_0) \) of \( G_0 \)-invariant differential operators on \( \Xi_0 \) and \( \tilde{\Xi}_0 \), respectively.

All algebras here are over \( \mathbb{C} \). Let \( I(p) \) and \( I(a) \) be the subalgebras of \( \text{Ad} K \)-invariant elements of \( S(p) \) and of \( W \)-invariant elements of \( S(a) \), respectively. It is clear that the algebra \( \mathbb{D}(p) \) of \( G_0 \)-invariant differential operators on \( p \) is \( I(p) \). It is also a well-known fact that the restriction mapping \( p \mapsto \bar{p} = p|_a \) is an isomorphism of \( I(p) \) onto \( I(a) \).

Now let \( P \in S(a) \). Then from (10) the differential operator \( D_P \) on \( \tilde{\Xi}_0 \) given by

\[
D_P \Phi(kM, H) = \partial(P)_H \Phi(kM, H) \quad (\Phi \in \mathcal{E}(\tilde{\Xi}_0))
\]

is easily seen to belong to \( \mathbb{D}(\tilde{\Xi}_0) \). If \( P \in I(a) \), we abuse notation and also use \( D_P \) to denote the (well-defined) differential operator on \( \Xi_0 \) given by

\[
D_P \varphi[kM, H] = \partial(P)_H \varphi[kM, H] \quad (\varphi \in \mathcal{E}(\Xi_0))
\]

Then it follows from (9) that \( D_P \in \mathbb{D}(\Xi_0) \). For \( \varphi \in \mathcal{E}(\Xi_0) \), put \( \tilde{\varphi} = \varphi \circ \pi \). Then clearly

\[
(D_P \varphi) = D_P \tilde{\varphi}
\]

For \( P \in S(a) \), we let \( P^* \) be its formal adjoint in \( a \). Then the adjoint of the differential operator \( D_P \) on \( \tilde{\Xi}_0 \) (with respect to the \( G_0 \)-invariant measure \( dk_M dH \)) is \( D_{P^*} \). The same holds for the operator \( D_P \) on \( \Xi_0 \) if \( P \in I(a) \), where the \( G_0 \)-invariant measure \( d\xi \) on \( \Xi_0 \) is fixed so as to satisfy \( \int_{\Xi_0} \varphi(\xi) d\xi = \int_{\Xi_0} \tilde{\varphi}(kM, H) dk_M dH \).

**Theorem 2.1.**

1. The map \( P \mapsto D_P \) is an algebra isomorphism of \( S(a) \) onto \( \mathbb{D}(\tilde{\Xi}_0) \).

2. The map \( P \mapsto D_P \) is an algebra isomorphism of \( I(a) \) onto \( \mathbb{D}(\Xi_0) \).

This theorem is the flat analogue of Theorem 2.2 in [5], which characterizes the algebra \( \mathbb{D}(\Xi) \) of left \( G \)-invariant differential operators on the horocycle space \( \Xi \). (See also Theorem 2.2, Chapter II in [12].) Our proof below is an adaptation of the proof of that theorem.

Let \( H^0_q \equiv M \rtimes q \), so that \( \tilde{\Xi}_0 = G_0/H^0_q \). We let \( m \) denote the Lie algebra of \( M \), and let \( l \) denote the orthogonal complement of \( m \) in \( \mathfrak{l} \) with respect to \( -B \). Then \( \mathfrak{g}_0 \) has the orthogonal decomposition (relative to \( B \) or the inner product \( B_\theta = -B(\cdot, \theta(\cdot)) \) on \( \mathfrak{g} \)) given by

\[
\mathfrak{g}_0 = \mathfrak{g} = (m \oplus q) \oplus l \oplus a.
\]

(14)
Let \( p : g_0 \mapsto g_0H_0^q \) be the coset map from \( G_0 \) onto \( \tilde{\Xi}_0 \), and let \( \tau(g) : g_0H_0^q \mapsto gg_0H_0^q \) be left translation by \( g \in G_0 \) on \( \tilde{\Xi}_0 \). Then \( \tau(k, X) \) is given by (10) and from that we have \( p(k, X) = (kM, (k^{-1} \cdot X)_a) \). Now if \( e_0 = (e, 0) \) is the identity element of \( G_0 \), then (14) shows that \( dp_{e_0} \) is a linear bijection of \( I \oplus a \) onto the tangent space \( T_{e_0}\Xi_0 \). Let \( \sigma \) be the projection of \( g \) onto \( I \oplus a \) according to the decomposition (14). It is straightforward to show that

\[
dp_{e_0} \circ \sigma \circ \text{Ad}_0(h) = d\tau(h) \circ dp_{e_0} \circ \sigma
\]

for all \( h \in H_0 \). Thus the restriction of \( dp_{e_0} \) to \( I \oplus a \) intertwines the representations \( \sigma \circ \text{Ad}_0(h) \) and \( d\tau(h) \) of \( H_0 \) on \( I \oplus a \) and on \( T_{e_0}\Xi_0 \), respectively.

While the pair \( (G_0, H_0^q) \) is not reductive, it is nonetheless possible to determine \( \mathbb{D}(\Xi_0) \) from the elements of the (complexified) symmetric algebra \( S(I \oplus a) \) which are invariant under \( \sigma \circ \text{Ad}(H_0^q) \).

**Lemma 2.2.** \( S(a) \) is precisely the algebra of elements in the symmetric algebra \( S(I \oplus a) \) which are invariant under \( \sigma \circ \text{Ad}(H_0^q) \).

**Proof.** Let \( (m, X) \in H_0^q \). Then according to (6), we have \( \text{Ad}_0(m, X)(H) = H \) for any \( H \in a \). This shows that \( a \), and hence \( S(a) \), is invariant under \( \text{Ad}_0(H_0^q) \) and thus also under \( \sigma \circ \text{Ad}_0(H_0^q) \).

For the converse, let \( \text{ad}_0 \) denote the adjoint representation on the Lie algebra \( g_0 \). Then \( \sigma \circ \text{ad}_0 \) is the representation of the Lie subalgebra \( m \oplus q \) (of \( g_0 \)) on \( I \oplus a \) corresponding to the representation \( \sigma \circ \text{Ad}_0 \) of \( H_0^q \) on the same space. For convenience, for each \( T + X \in m \oplus q \), we let \( d(T + X) \) denote the restriction of \( \sigma \circ \text{ad}_0(T + X) \) to \( I \oplus a \). We then extend \( d(T + X) \) to a derivation of the symmetric algebra \( S(I \oplus a) \).

We will prove that if \( Q \in S(I \oplus a) \) such that

\[
d(Y)Q = 0 \quad \text{for all } Y \in q
\]

then \( Q \in S(a) \). This will then imply that the elements of \( S(I \oplus a) \) invariant under \( \sigma \circ \text{Ad}_0(q) \) belong to \( S(a) \), which will prove the lemma.

For each \( \alpha \in \Sigma^+ \), let \( X_1^\alpha, \ldots, X_m^\alpha \) be an orthonormal basis of the restricted root space \( g_\alpha \) with respect to the inner product \( B_\theta \) on \( g \). Then the vectors \( E_i^\alpha = X_i^\alpha + \theta(X_i^\alpha) \) form an orthogonal basis (with respect to \( -B \)) of the subspace

\[
I_\alpha = \{ T \in \mathfrak{t} \mid \text{ad}(H)^2 T = \alpha(H)^2 T \text{ for all } H \in a \}.
\]

of \( \mathfrak{t} \). Likewise, the vectors \( Y_i^\alpha = X_i^\alpha - \theta(X_i^\alpha) \) form an orthogonal basis (with respect to \( B \)) of

\[
q_\alpha = \{ X \in \mathfrak{p} \mid \text{ad}(H)^2 X = \alpha(H)^2 X \text{ for all } H \in a \}.
\]

Finally, we have \( I = \bigoplus_{\alpha \in \Sigma^+} I_\alpha \) and \( q = \bigoplus_{\alpha \in \Sigma^+} q_\alpha \).

If \( \alpha \neq \beta \), it is easy to check that \( [Y_i^\alpha, E_j^\beta]_0 = [Y_i^\alpha, E_j^\beta] \in q \) and therefore

\[
d(Y_i^\alpha)(E_j^\beta) = 0 \quad (1 \leq i \leq m_\alpha, 1 \leq j \leq m_\beta)
\]
On the other hand, for $1 \leq i, j \leq m_\alpha$,
\[
[Y_\alpha^i, E_j^\alpha]_0 = [Y_\alpha^i, E_j^\alpha] = ([X_\alpha^i, X_j^\alpha] - \theta(X_\alpha^i, X_j^\alpha)) + ([X_\alpha^i, \theta(X_j^\alpha)] - \theta(X_\alpha^i, \theta(X_j^\alpha))]
\]

The first quantity on the right above belongs to $\mathfrak{q}$. If $i \neq j$, then $[X_\alpha^i, \theta(X_j^\alpha)] \in \mathfrak{m}$, so the second expression on the right above vanishes. If $i = j$, then the second quantity on the right equals $2A_\alpha$, where $A_\alpha$ is the vector in $\mathfrak{a}$ such that $B(A_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{a}$.

We conclude that
\[
d(Y_\alpha^i)(E_j^\beta) = \begin{cases} 2A_\alpha & \text{if } \alpha = \beta \text{ and } i = j \\ 0 & \text{otherwise} \end{cases} \quad (17)
\]

Suppose now that $Q \in S(\mathfrak{l} \oplus \mathfrak{a})$ such that $d(Y)Q = 0$ for all $Y \in \mathfrak{q}$. Fix any basis $H_1, \ldots, H_l$ of $\mathfrak{a}$. Then $Q$ can be written uniquely as a polynomial in the $E_j^\beta$ with coefficients in $S(\mathfrak{a})$:
\[
Q = \sum_N \{P_N(H_1, \ldots, H_l) \prod_{\beta \in \Sigma^+} ((E_1^\beta)^{n(\beta,1)} \cdots (E_m^\beta)^{n(\beta,m)}) \}
\]

where the sum ranges over multiindices $N = (n(\beta, j))$ ($1 \leq j \leq m_\beta, \beta \in \Sigma^+$). For convenience, let us put $E(\beta)^{N(\beta)} = (E_1^\beta)^{n(\beta,1)} \cdots (E_m^\beta)^{n(\beta,m)}$ and $P_N = P_N(H_1, \ldots, H_l)$.

Since $d(Y_\alpha^i)H = 0$ for all $H \in \mathfrak{a}$, (17) implies that
\[
d(Y_\alpha^i)Q = 2A_\alpha \sum_{n(\alpha,i) \neq 0} n(\alpha, i) P_N \left( \prod_{\beta \neq \alpha} E(\beta)^{N(\beta)} ((E_1^\alpha)^{n(\alpha,1)} \cdots (E_i^\alpha)^{n(\alpha,i)} \cdots (E_m^\alpha)^{n(\alpha,m)}) \right)
\]

Since the right hand side equals 0, the coefficient of $A_\alpha$ above must equal 0. This coefficient is therefore an empty sum. Since (19) holds for all $Y_\alpha^i$, we conclude that there is only one summand in (18), the one corresponding to $N = 0$. This shows that $Q \in S(\mathfrak{a})$.

Let us recall that, by definition, $H_\mathfrak{q} = M' \ltimes \mathfrak{q}$.

**Corollary 2.3.** The algebra of elements of $S(\mathfrak{l} \oplus \mathfrak{a})$ invariant under $\sigma \circ Ad_0(H_\mathfrak{q})$ is $I(\mathfrak{a})$.

**Proof.** This is clear from Lemma 2.2 and (6).
is a chart on a neighborhood of the identity coset \( eH_0 = \tilde{\xi}_0 \) in \( \tilde{\Xi}_0 \). Suppose that \( D \in \mathbb{D}(\tilde{\Xi}_0) \). Then there is a unique polynomial \( P \) in \( l + r \) variables such that

\[
D \varphi(\tilde{\xi}_0) = P \left( \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_{l+r}} \right) \varphi(\exp(\sum t_i H_i) \cdot \tilde{\xi}_0) \bigg|_{(t) = (0)} \tag{20}
\]

for all \( \varphi \in \mathcal{E}(\tilde{\Xi}_0) \). Now for each \( h \in H_0 \), there is a diffeomorphism \( (t_1, \ldots, t_{l+r}) \mapsto (s_1, \ldots, s_{l+r}) \) on neighborhoods of \( 0 \in \mathbb{R}^{l+r} \) such that

\[
\tau(h) \exp \left( \sum t_i H_i \right) H_0 = \exp \left( \sum s_j H_j \right) H_0.
\]

For convenience, let us put \( \varphi(\exp(\sum t_i H_i) H_0) = \varphi(t_1, \ldots, t_{l+r}) \). Since \( D(\varphi)(\tilde{\xi}_0) = D(\varphi^{\tau(h)})(\tilde{\xi}_0) \), we have

\[
P \left( \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_{l+r}} \right) (\varphi(t_1, \ldots, t_{l+r}) - \varphi(s_1, \ldots, s_{l+r})) \bigg|_{(t) = (0)} \tag{21}
\]

Assume that \( P \) is of order \( N \), let \( P_N \) denote the sum of the highest order terms in \( P \), and write

\[
P_N = \sum_{|J|=N} a_J \left( \frac{\partial}{\partial t_1} \right)^{j_1} \circ \cdots \circ \left( \frac{\partial}{\partial t_{l+r}} \right)^{j_{l+r}}.
\]

If we fix a multiindex \( J \) of order \( N \) and let \( \varphi(t_1, \ldots, t_{l+r}) = t^J = t_1^{j_1} \cdots t_{l+r}^{j_{l+r}} \) near the origin, then (21) shows that

\[
a_J = \sum_{|J|=N} R_{J I} a_I \tag{22}
\]

where \( (R_{J I}) \) is the matrix of the linear operator on the vector space of homogeneous degree \( N \) polynomial functions on \( \mathbb{R}^{l+r} \) extending the operator on \( \mathbb{R}^{l+r} \) whose matrix is the Jacobian matrix \( (\partial s_j / \partial t_i) \) at \( (t) = (0) \). But this Jacobian matrix is also the matrix of \( \sigma \circ \text{Ad}_0(h) \) with respect to the basis \( \{ H_i \} \) of \( \mathfrak{g} \oplus \mathfrak{a} \). Equation (22) thus shows that \( \sum_{|J|=N} a_J H^J \) is invariant under \( \sigma \circ \text{Ad}_0(h) \). Hence by Lemma 2.2, we conclude that \( P_N = P_N(\partial / \partial t_1, \ldots, \partial / \partial t_l) \). Since \( D \) is \( G_0 \)-invariant, we see that

\[
D \varphi(g_0 \cdot \tilde{\xi}_0) = P_N \left( \frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_l} \right) \varphi(g_0 \exp(\sum_{i=1}^l t_i H_i) \cdot \tilde{\xi}_0) \bigg|_{(0)} + \text{lower order terms}
\]

so that \( D - D_{P_N} \) is an element of \( \mathbb{D}(\tilde{\Xi}_0) \) whose order is less than the order of \( D \). A simple induction on the order then completes the proof of the first assertion of Theorem 2.1.

For the second assertion, suppose that \( D \in \mathbb{D}(\Xi_0) \). Then there exists a polynomial \( P \) such that (20) holds for all functions \( \varphi \in \mathcal{E}(\Xi_0) \), with \( \xi_0 \) replacing \( \tilde{\xi}_0 \). With this substitution, the rest of the proof above carries over, with \( h \in H_0^0 \) replaced by \( h \in H_q = M' \ltimes q \), and with \( P_N(H_1, \ldots, H_l) \) \( M' \)-invariant by Corollary 2.3.
3. The Space of Joint Eigendistributions

Suppose that \( \Psi \in \mathcal{D}'(\Xi_0) \) is an eigendistribution of \( \mathcal{D}(\Xi_0) \). Then according to Lemma 3.11, Chapter III of [11], there exists a \( \lambda \in \mathfrak{a}_c^\ast \) (unique up to \( W \)-orbit) such that

\[
D_P \Psi = P(i\lambda) \Psi
\]

for all \( P \in I(\mathfrak{a}) \). We let \( \mathcal{D}'(\xi_0) \) denote the vector space consisting of all \( \Psi \in \mathcal{D}'(\Xi_0) \) satisfying (23).

Any eigendistribution \( \Psi \in \mathcal{D}'(\tilde{\Xi}_0) \) of \( \mathcal{D}(\tilde{\Xi}_0) \) likewise corresponds to a unique \( \lambda \in \mathfrak{a}_c^\ast \) satisfying (23) for all \( P \in S(\mathfrak{a}) \). For a given \( \lambda \), we denote the vector space of all such distributions by \( \mathcal{D}'(\tilde{\Xi}_0) \).

The following can be proved in a manner analogous to the proof of Proposition 4.4, Chapter II in [12].

**Proposition 3.1.** Let \( \Psi \in \mathcal{D}'(\tilde{\Xi}_0) \). Then there is a unique \( S \in \mathcal{D}'(K/M) \) such that

\[
\Psi(\varphi) = \int_{K/M} \int_{\mathfrak{a}} \varphi(kM, H) e^{i\lambda(H)} \, dH \, dS(kM).
\]

Conversely, if \( S \in \mathcal{D}'(K/M) \), then the distribution \( \Psi \) on \( \tilde{\Xi}_0 \) defined above belongs to \( \mathcal{D}'(\tilde{\Xi}_0) \).

If \( F \in \mathcal{E}(\tilde{\Xi}_0) \), we define \( F_\pi \in \mathcal{E}(\Xi_0) \) by

\[
F_\pi[kM, H] = \frac{1}{w} \sum_{s \in W} F(km^{-1}M, sH)
\]

Then the pullback \( \tilde{\Phi} \) of a distribution \( \Phi \in \mathcal{D}'(\Xi_0) \) is defined by

\[
\tilde{\Phi}(F) = \Phi(F_\pi) \quad (F \in \mathcal{D}(\tilde{\Xi}_0))
\]

Note that

\[
\tilde{\Phi}(\varphi) = \Phi(\varphi)
\]

for all \( \Phi \in \mathcal{D}'(\Xi_0) \), \( \varphi \in \mathcal{D}(\Xi_0) \). Let \( P \in I(\mathfrak{a}) \) and \( \Phi \in \mathcal{D}'(\Xi_0) \). Then it is easy to see from (11) and (12) and the fact that \( D_P(F_\pi) = (D_PF)_\pi \), that, in analogy with (13), we have

\[
(D_P\Phi) = D_P\tilde{\Phi}.
\]

Since \( \mathcal{D}(\Xi_0) \) is smaller than \( \mathcal{D}(\tilde{\Xi}_0) \), it is not true that \( \tilde{\Phi} \) belongs to \( \mathcal{D}'(\tilde{\Xi}_0) \) whenever \( \Phi \in \mathcal{D}'(\Xi_0) \). (It is easy to construct smooth counterexamples.) Nonetheless, as we shall see below, we can obtain a result for \( \mathcal{D}'(\tilde{\Xi}_0) \) similar to Proposition 3.1.

Suppose that \( \Phi \in \mathcal{D}'(\Xi_0) \). Then by (26) \( \tilde{\Phi} \) satisfies

\[
D_P(\tilde{\Phi}) = P(i\lambda) \tilde{\Phi} \quad (P \in I(\mathfrak{a}))
\]

Now for functions \( \alpha \in \mathcal{D}(\mathfrak{a}) \) and \( \beta \in \mathcal{E}(K/M) \), let \( \beta \otimes \alpha \) be the function \( \beta(kM)\alpha(H) \) on \( \tilde{\Xi}_0 = K/M \times \mathfrak{a} \). The linear span of such functions is dense in \( \mathcal{D}(\tilde{\Xi}_0) \).
If we fix $\beta \in \mathcal{E}(K/M)$, the map

$$T_\beta : \alpha \in \mathcal{D}(a) \to \widetilde{\Phi}(\beta \otimes \alpha)$$

is a distribution in $a$; in fact, we see from (27) that $T_\beta$ is an eigendistribution of the algebra $I(a)$. Since this algebra contains elliptic elements, it follows that $T_\beta$ is in fact a smooth eigenfunction of $I(a)$, with

$$(\partial(P) T_\beta)(H) = P(i\lambda) T_\beta(H)$$

for all $P \in I(a)$. The space of such eigenfunctions is described in [11], Chapter III, Theorem 3.13. Let $W_\lambda$ denote the subgroup of $W$ consisting of those elements fixing $\lambda$, let $I_\lambda(a)$ be the subalgebra of $W_\lambda$-invariant elements of $S(a)$, and let $H_\lambda$ be the vector space of $W_\lambda$-harmonic polynomial functions on $a$.

Then for each element $s\lambda$ in the orbit $W\cdot \lambda$, there exists a unique polynomial $P_{s\lambda}(\beta)(H)$ in $H_{s\lambda}$, with coefficients depending on $\beta$, such that

$$T_\beta(H) = \sum_{s\lambda \in W\cdot \lambda} P_{s\lambda}(\beta)(H) e^{is\lambda(H)}$$

for all $H \in a$. When $\lambda$ is regular, the $P_{s\lambda}(\beta)$ are just constants (depending, of course, on $\beta$).

For fixed $H \in a$, the map $\beta \in \mathcal{E}(K/M) \to P_{s\lambda}(\beta)(H)$ is continuous, and from this it is not hard to see that the coefficients of the polynomials $P_{s\lambda}(\beta)(H)$ are distributions on $K/M$. More precisely, for each $s\lambda$, fix a basis $P_{s\lambda,j}(H)$ $(1 \leq j \leq r = |W_\lambda|)$ of $H_{s\lambda}$. Then

$$P_{s\lambda}(\beta)(H) = \sum_{j=1}^{r} S_{s\lambda,j}(\beta) P_{s\lambda,j}(H)$$

Each coefficient $S_{s\lambda,j}$ is a distribution on $K/M$ uniquely determined, of course, by the choice of the basis $\{P_{s\lambda,j}\}$. Hence, by (28) (30), and (31), we see that

$$\tilde{\Phi}(F) = \sum_{s\lambda \in W\cdot \lambda} \sum_{j=1}^{r} \int_{K/M} \int_{a} P_{s\lambda,j}(H) F(kM, H) e^{is\lambda(H)} dH dS_{s\lambda,j}(kM)$$

for all $F \in \mathcal{D}(\tilde{\Xi}_0)$ of the form $\beta \otimes \alpha$. Since the $\beta \otimes \alpha$ span a dense subspace of $\mathcal{D}(\tilde{\Xi}_0)$, formula (32) holds for all $F \in \mathcal{D}(\tilde{\Xi}_0)$.

When $\lambda$ is regular, each $H_{s\lambda} = \mathbb{C}$ (so we can take 1 as its basis), and the formula above reduces to

$$\tilde{\Phi}(F) = \sum_{s \in W} \int_{K/M} \int_{a} F(kM, H) e^{is\lambda(H)} dH dS_{s\lambda}(kM)$$

for all $F \in \mathcal{D}(\tilde{\Xi}_0)$.

We now proceed to obtain a more explicit characterization of the eigendistribution $\Phi \in \mathcal{D}'(\Xi_0)$. For this, we note that expression (31) shows that $P_{s\lambda}$ can
be considered as an element of $\mathcal{D}'(K/M) \otimes H_{s\lambda}$, with $P_{s\lambda} = \sum_{j=1}^{m} S_{s\lambda,j} \otimes P_{s\lambda,j}$, so that (32) becomes

$$\tilde{\Phi}(F) = \sum_{s\lambda \in W \cdot \lambda} \int_{a} \int_{K/M} F(kM, H) e^{i\lambda(H)} dP_{s\lambda}(kM)(H) dH$$

(34)

We observe that by (31), each $P_{s\lambda}$ is uniquely determined by $\Phi$.

Now the Weyl group $W$ acts (freely) on both $K/M$ and on $\Xi_0 = K/M \times a$ by $s \cdot kM = km_s^{-1}M$ and $s \cdot (kM, H) = (km_s^{-1}M, sH)$. Thus for each $t \in W$,

$$\tilde{\Phi}(F) = \tilde{\Phi}(F)$$

$$= \sum_{s\lambda \in W \cdot \lambda} \int_{a} \int_{K/M} F(km_t^{-1}M, t \cdot H) e^{i\lambda(H)} dP_{s\lambda}(kM)(H) dH$$

(35)

where we have put $P_{t s\lambda}^t = \sum_{j} S_{t s\lambda,j} \otimes P_{s\lambda,j}$. The right hand side of (35) then equals

$$\sum_{s\lambda \in W \cdot \lambda} \int_{a} \int_{K/M} F(kM, H) e^{i\lambda(H)} t \cdot dP_{s\lambda}(kM)(H) dH$$

(36)

where now $t \cdot P_{s\lambda}^t = \sum_{j} S_{t s\lambda,j} \otimes (t \cdot P_{s\lambda,j})$, an element of $\mathcal{D}'(K/M) \otimes H_{ts\lambda}$. By the uniqueness of the $P_{s\lambda}$, it follows that

$$P_{ts\lambda} = t \cdot P_{s\lambda}^t$$

for all $s, t \in W$. In particular,

$$P_{s\lambda} = s \cdot P_{s\lambda}^s \quad (s \in W)$$

Hence, for any $\varphi \in \mathcal{D}(\Xi_0)$, we have

$$\tilde{\Phi}(\varphi) = \tilde{\Phi}(\tilde{\varphi})$$

$$= \sum_{s\lambda \in W \cdot \lambda} \int_{a} \int_{K/M} \tilde{\varphi}(kM, H) e^{i\lambda(H)} s \cdot dP_{s\lambda}(kM)(H) dH$$

$$= \sum_{s\lambda \in W \cdot \lambda} \int_{a} \int_{K/M} \tilde{\varphi}(km_s^{-1}M, s \cdot H) e^{i\lambda(H)} dP_{s\lambda}(kM)(H) dH$$

$$= |W : \lambda| \int_{a} \int_{K/M} \varphi(kM, H) e^{i\lambda(H)} dP_{\lambda}(kM)(H) dH$$

(37)

If we put $Q_{\lambda} = |W : \lambda| P_{\lambda}$, this leads us to the following result.
Theorem 3.2. Suppose that $\lambda \in \mathfrak{a}_c^*$ and that $\Phi \in \mathcal{D}_\lambda'(\Xi_0)$. Then there exists a unique element $Q_\lambda \in \mathcal{D}'(K/M) \otimes H_\lambda$ such that

$$
\Phi(\varphi) = \int_a \int_{K/M} \varphi[kM, H] e^{i\lambda(H)} dQ_\lambda(kM)(H) dH
$$

for all $\varphi \in \mathcal{D}(\Xi_0)$. Conversely, given any element $Q_\lambda \in \mathcal{D}'(K/M) \otimes H_\lambda$, the expression (38) defines a distribution $\Phi \in \mathcal{D}_\lambda'(\Xi_0)$.

Remarks:

1. Fix a basis $P_1, \ldots, P_r$ of $H_\lambda$. (We may choose this basis to have real coefficients.) If $\Phi \in \mathcal{D}_\lambda'(\Xi_0)$, Theorem 3.2 says that there exist unique distributions $T_j$ on $K/M$ such that

$$
\Phi(\varphi) = \sum_{j=1}^r \int_{K/M} \int_a P_j(H) \varphi[kM, H] e^{i\lambda(H)} dH dT_j(kM)
$$

for all $\varphi \in \mathcal{D}(\Xi_0)$. Conversely, for any distributions $T_j$ on $K/M$, the right hand side of (39) defines a distribution $\Phi \in \mathcal{D}_\lambda'(\Xi_0)$.

2. Equation (39) can also be written as

$$
\Phi(\varphi) = \sum_{j=1}^m \int_{K/M} \partial(P_j^*) \varphi^*[kM, \lambda] dT_j(kM),
$$

where $\varphi^*$ is the (well-defined) Fourier-Laplace transform of $\varphi$:

$$
\varphi^*[kM, \lambda] = \int_a \varphi[kM, H] e^{i\lambda(H)} dH \quad ([kM, \lambda] \in K/M \times_W \mathfrak{a}_c^*)
$$

Proof. Equation (38) follows from (37) by putting $Q_\lambda = |W \cdot \lambda| P_\lambda$. The uniqueness of $Q_\lambda$ is a consequence of the uniqueness of the $P_s$, and in particular, of $P_\lambda$.

Conversely, suppose that $Q_\lambda \in \mathcal{D}'(K/M) \otimes H_\lambda$. If we fix a basis $P_1, \ldots, P_m$ of $H_\lambda$, we can, as in Remark (1) above, write $Q_\lambda = \sum_j S_j \otimes P_j$. The distribution $\Phi$ in (38) is then given by (39), and thus we need to prove that the right hand side of (39) defines a distribution $\Phi \in \mathcal{D}_\lambda'(\Xi_0)$. Now the product $P(H) e^{i\lambda(H)}$ belongs to the joint eigenspace $E_{i\lambda}(a) = \{ \alpha \in E(a) \mid \partial(P) \alpha = P(i\lambda) \alpha \text{ for all } P \in I(a) \}$. Hence for any $Q \in I(a)$, we have

$$
(D_Q(\Phi))(\varphi) = \sum_{j=1}^m \int_{K/M} \int_a \partial(Q^*)\varphi[kM, H] P_j(H) e^{i\lambda(H)} dH dT_j(kM)
$$

$$
= Q(i\lambda) \sum_{j=1}^m \int_{K/M} \int_a \varphi[kM, H] P_j(H) e^{i\lambda(H)} dH dT_j(kM)
$$

$$
= Q(i\lambda) \Phi(\varphi),
$$

for all $\varphi \in \mathcal{D}(\Xi_0)$, proving the theorem.
Corollary 3.3. Suppose that $\lambda$ is regular. Then there is a linear bijection from $\mathcal{D}'(K/M)$ onto $\mathcal{D}'_\lambda(\Xi_0)$ given by
\[
T \mapsto \Phi
\]
\[
\Phi(\varphi) = \int_{K/M} \int_a \varphi[kM,H] e^{i\lambda(H)} dH dT(kM)
\]
\[
= \int_{K/M} \varphi^*[kM,\lambda] dT(kM)
\]
(41)

4. Conical Distributions

By definition, a conical distribution on $\Xi_0$ is an $H_q$-invariant eigendistribution of $\mathcal{D}(\Xi_0)$, where, as we recall, $H_q$ is the isotropy subgroup of $G_0$ fixing $q$: $H_q = M' \ltimes q$.

Suppose that $\Phi$ is a conical distribution on $\Xi_0$ belonging to $\mathcal{D}'_\lambda(\Xi_0)$. ($\lambda$ is of course determined up to $W$-orbit.) First, for simplicity, let us assume that $\lambda$ is regular. Then we see that $\Phi$ satisfies (41), for unique $T \in \mathcal{D}'(K/M)$.

In order to determine this distribution $T$ more explicitly, we first prove that the collection of functions on $K/M$ given by \[\{\varphi^*[kM,\lambda] | \varphi \in \mathcal{D}(\Xi_0)\}\] equals $\mathcal{E}(K/M)$.

For this, we first consider the following easy lemma.

Lemma 4.1. For $f \in \mathcal{D}(a)$ and $\gamma \in \mathcal{E}(a)$, put $(f,\gamma) = \int_a f(H) \gamma(H) dH$. Suppose that $\gamma_1,\ldots,\gamma_m$ are linearly independent elements of $\mathcal{E}(a)$. Then there exist functions $f_1,\ldots,f_m$ in $\mathcal{D}(a)$ such that the $m \times m$ matrix $((f_i,\gamma_j))$ is any prescribed $m \times m$ matrix.

**Proof.** Let $V$ be the linear span of $\gamma_1,\ldots,\gamma_m$. For each $f \in \mathcal{D}(a)$, let $\lambda_f$ be the linear functional on $V$ given by $\lambda_f(\gamma) = (f,\gamma)$. It suffices for us to prove that the linear map $f \mapsto \lambda_f$ maps $\mathcal{D}(a)$ onto $V^*$. But if $f \mapsto \lambda_f$ were not onto, then there would be a nonzero subspace $W$ of $V$ such that $\lambda_f(W) = \{0\}$ for all $f$. But given any nonzero $\gamma \in W$, there is clearly an $f \in \mathcal{D}(a)$ such that $(f,\gamma) \neq 0$, a contradiction.

For every $h \in \mathcal{D}(a)$, let $h^*$ denote its Fourier-Laplace transform
\[
h^*(\lambda) = \int_a h(H) e^{i\lambda(H)} dH \quad (\lambda \in a_\ast)
\]

Lemma 4.2. Let $\lambda \in a_\ast$ be regular. Let $R$ be the linear map from $\mathcal{D}(\Xi_0)$ to $\mathcal{E}(K/M)$ given by $R\varphi(kM) = \varphi^*[kM,\lambda]$. Then $R$ is onto.

**Proof.** The proof requires some care since $\Xi_0$ is not the product manifold $K/M \times a$ but a quotient of it. Note first that Lemma 4.1 implies that for any distinct elements $\lambda_1,\ldots,\lambda_m \in a_\ast$, and any functions $\beta_1,\ldots,\beta_m \in \mathcal{E}(K/M)$, there exists a function $F \in \mathcal{D}(\Xi_0)$ such that $F^*(kM,\lambda_j) = \beta_j(kM)$ for all $k \in K$ and all $j$. In fact since the functions $e^{i\lambda_1},\ldots,e^{i\lambda_m}$ are linearly independent elements
of \( \mathcal{E}(a) \), the lemma implies that there are functions \( h_1, \ldots, h_m \) in \( \mathcal{D}(a) \) such that \( h_i'(\lambda_j) = \delta_{ij} \) for all \( i, j \). Then put \( F(kM, H) = \sum_j \beta_j(kM) h_j(H) \).

Now fix \( \beta \in \mathcal{E}(K/M) \). We will prove that there exists a \( \varphi \in \mathcal{D}(\Xi_0) \) such that \( \varphi^*[kM, \lambda] = \beta(kM) \) for all \( kM \in K/M \).

From the above we know that there exists a function \( F \in \mathcal{D}(\Xi_0) \) such that \( F^*(kM, s\lambda) = \beta(km_sM) \) for all \( kM \in K/M \) and all \( s \in W \). (Here \( m_s \in M' \) is any coset representative of \( K/M \).) Put \( \varphi = F_\pi \), so that \( \varphi \in \mathcal{D}(\Xi_0) \). Then \( \varphi^*[kM, \mu] = (1/w) \cdot \sum_{s \in W} F^*(km^{-1}_sM, s\mu) \) for all \( kM \in K/M \) and all \( \mu \in \mathfrak{a}_c^* \). In particular,

\[
\varphi^*[kM, \lambda] = \frac{1}{w} \sum_{s \in W} F^*(km^{-1}_sM, s\lambda) = \beta(kM)
\]

for all \( kM \in K/M \).

Resuming our investigation of conical distributions, let us assume, as before, that \( \Phi \) is a conical distribution in \( \mathcal{D}'(\Xi_0) \), where \( \lambda \) is a fixed regular element in \( \mathfrak{a}_c^* \). Let \( T \) be the unique element of \( \mathcal{D}'(K/M) \) given by (41).

The \( M' \)-invariance of \( \Phi \) implies that

\[
\int_{K/M} \varphi^*[m'kM, \lambda] dT(kM) = \int_{K/M} \varphi^*[kM, \lambda] dT(kM)
\]  

(42)

for all \( m' \in M' \). By Lemma 4.2, the functions \( \varphi^*[kM, \lambda] \) run through \( \mathcal{E}(K/M) \) as \( \varphi \) runs through \( \mathcal{D}(\Xi_0) \). Thus (42) shows that \( T \) is a left \( M' \)-invariant distribution on \( K/M \).

The \( \mathfrak{q} \)-invariance of \( \Phi \) then shows that

\[
\int_{K/M} \varphi^*[kM, \lambda] dT(kM) = \int_{K/M} \varphi^*[kM, \lambda] e^{-i\lambda(k^{-1}X)_{\mathfrak{a}_c^*}} dT(kM) = \int_{K/M} \varphi^*[kM, \lambda] e^{-iB(k\lambda, X)} dT(kM)
\]  

(43)

By Lemma 4.2, this implies that

\[ T = e^{-iB(k\lambda, X)} T \]  

(44)

for all \( k \in K \) and all \( X \in \mathfrak{q} \).

We will now prove that the property (44) implies that \( T \) has support in the discrete subset \( M'/M \) of \( K/M \). For this, consider any \( k_0 \in K \setminus M' \). Since \( \lambda \) is regular, \( k_0 \cdot A_\lambda \notin \mathfrak{a}_c^* \). It is easy to see that there exists \( X \in \mathfrak{q} \) such that \( B(k_0 \cdot A_\lambda, X) \notin 2\pi \mathbb{Z} \). (This is done by scaling \( X \) if necessary.) Fixing this \( X \), there exists a neighborhood \( U \) of \( k_0M \) in \( K/M \) such that \( B(k \cdot A_\lambda, X) \notin 2\pi \mathbb{Z} \) for all \( kM \in U \). Hence the function \( kM \mapsto e^{iB(k \cdot A_\lambda, X)} - 1 \) is never 0 on \( U \), whereas by (44) the distribution \( (e^{iB(k \cdot A_\lambda, X)} - 1)T \) on \( K/M \) equals 0. This implies that \( T = 0 \) on \( U \). Since \( k_0M \in K/M \) is an arbitrary point in the complement of \( M'/M \), this proves that \( T \) has support in the discrete set \( M'/M \).
In particular, $T$ has the form

$$T = \sum_{s \in W} D_s \delta_{m_s, M}$$  \hspace{1cm} (45)$$

where $D_s$ is a linear differential operator on $K/M$. We will now prove that in fact

$$T = c \sum_{s \in W} \delta_{m_s, M}$$  \hspace{1cm} (46)$$

for some constant $c$. For this, it suffices to prove that near the identity coset $eM$ of $K/M$, $T$ is a multiple of the delta function at $eM$. That is to say, it suffices to prove that for all smooth functions $\beta$ on $K/M$ supported on a small neighborhood of $eM$, then $T(\beta) = c \beta(eM)$. The $M'$-invariance of $T$ then proves (46).

To this end, we introduce local coordinates on $K/M$ near $eM$. Let $T_1, \ldots, T_{m_\alpha}$ be an orthonormal basis (with respect to $-B$) of $l_\alpha$. (We could use $T_{\alpha i} = 2^{-1/2} E_{\alpha i}$ from the proof of Lemma 2.2.) The collection $\{T_j\}_{1 \leq j \leq m_\alpha, \alpha \in \Sigma^+}$ is then an orthonormal basis of $l_\alpha$. We list these basis elements as $T_1, \ldots, T_r$ and assume that $T_j$ belongs to the generalized eigenspace $k_{\alpha_j}$. Then the map

$$\exp(t_1 T_1 + \cdots + t_r T_r) M \mapsto (t_1, \ldots, t_r)$$ \hspace{1cm} (47)$$
defined a chart on a neighborhood $U$ of $eM$ in $K/M$. We assume that $U \cap M'/M = \{ eM \}$.

For each $j$ let us put

$$X_j = -i(B(\alpha_j, \lambda))^{-2} \text{ad}(A_\lambda) T_j.$$ 

Since $\lambda$ is regular, $X_j$ is well defined, and it is easy to see that $X_1, \ldots, X_r$ is a basis of the complexification $q_e$ of $q$, orthogonal with respect to the Killing form on $p_e$. 

Now suppose that $\beta$ is a smooth function on $K/M$ with support in $U$. Then by (45), we have

$$T(\beta) = \sum_J c_J D^J(0)$$ \hspace{1cm} (48)$$

where the sum runs through a finite collection of multiindices $J = (j_1, \ldots, j_r)$, the $c_J$ are constants, and $D^J = \partial^{j_1 + \cdots + j_r}/\partial t_1^{j_1} \cdots \partial t_r^{j_r}$.

In the sum (48), we claim that $c_J = 0$ when $|J| > 0$. Then of course $T(\beta) = c_0(0)$, and this will prove (46). To prove this, let us assume, to the contrary, that $c_J \neq 0$ for some $J \neq 0$. Let $N = \max \{|J| \mid c_J \neq 0\}$. Now by (44) we have

$$\sum_J c_J D^J(0) = \sum_J c_J D^J \left( e^{-iB(\exp(t_1 T_1 + \cdots + t_r T_r) \cdot A_\lambda, X) \beta(0)} \right)$$ \hspace{1cm} (49)$$

for all $X \in q$ and all smooth functions $\beta$ supported in $U$. In (49) choose a $\beta$ which is identically 1 on a small neighborhood of 0. Then the left hand side of (49) is $c_0$. On the other hand, if we write $X = z_1 X_1 + \cdots + z_r X_r$, where $z_j \in \mathbb{C}$, then the right hand side is

$$\sum_J c_J D^J \left( e^{-iB(\exp(t_1 T_1 + \cdots + t_r T_r) \cdot A_\lambda, X)) \beta(0)} \right),$$
a polynomial of degree $N$ in $z_1, \ldots, z_r$. Its homogeneous component of degree $N$ equals

$$\sum_{|J|=N} c_J z^J,$$

where we have put $z^J = z_1^{j_1} \cdots z_r^{j_r}$ when $J = (j_1, \ldots, j_r)$. This yields a contradiction, and we obtain the following result.

**Theorem 4.3.** Suppose that $\lambda \in a^*_c$ is regular. Then the space of conical distributions in $D'_\lambda(\Xi_0)$ is one-dimensional, with basis given by $\Phi_\lambda$, where

$$\Phi_\lambda(\varphi) = \sum_{s \in W} \varphi^*[m_s M, \lambda] \quad (\varphi \in D(\Xi_0)) \quad (50)$$

Proof. If $\Phi$ is a conical distribution in $D'_\lambda(\Xi_0)$ then we have shown that $\Phi$ is a multiple of $\Phi_\lambda$. For the converse, we first observe that Lemma 4.2 implies that there is a $\varphi \in D(\Xi_0)$ such that $\varphi^*[kM, \lambda] = 1$ for all $kM \in K/M$. This shows that the distribution $\Phi_\lambda$ is not zero. Moreover, (41) shows that $\Phi_\lambda$ belongs to $D'_\lambda(\Xi_0)$, with $T = \sum_{s \in W} \delta_{m_s M}$; clearly $T$ satisfies (44) and is $M'$-invariant, so $\Phi_\lambda$ is conical. \(\Box\)

The product manifold $\tilde{\Xi}_0 = K/M \times a$, rather than $\Xi_0$, is in some sense the limiting case of the horocycle space $\Xi \cong K/M \times A$. Thus it makes sense to define a conical distribution on $\tilde{\Xi}_0$ to be a joint eigendistribution of $D(\tilde{\Xi}_0)$ invariant under the left action of the isotropy subgroup $H^0_a = M \times q$ of $\tilde{\xi}_0 = (eM, 0)$. Assume that $\lambda \in a^*_c$ is regular. Then using Proposition 3.1, one can use an argument similar to that used to prove Theorem 4.3 above to conclude that the space of conical distributions in $D'_\lambda(\tilde{\Xi}_0)$ is $w$-dimensional. (We omit the details.)

**Theorem 4.4.** If $\lambda \in a^*_c$ is regular, then the space of conical distributions in $D'_\lambda(\tilde{\Xi}_0)$ has dimension $w$. Any such conical distribution is given by

$$\Psi(\psi) = \int_{K/M} \int_a \psi(kM, H) e^{i\lambda(H)} dH dS(kM) \quad (\psi \in D(\tilde{\Xi}_0))$$

with $S = \sum_{s \in W} c_s \delta_{m_s M}$, for arbitrary scalars $c_s$.

The theorem above is thus a more precise analogue of Theorem 4.9 in [7], which states that, for generic $\lambda \in a^*_c$, the space of conical distributions in $D'_\lambda(\Xi)$ is $w$-dimensional.

5. **Conical Distributions When $\lambda$ is Non-regular**

Just as in the symmetric space case, the problem of characterizing the space of conical distributions in $D'_\lambda(\Xi_0)$ appears to be rather difficult, in general, when $\lambda \in a^*_c$ is not regular. One can, however, show that the space of conical distributions corresponding to any non-regular $\lambda$ is infinite-dimensional. To see this, let $K_\lambda = Z_K(\lambda) = \{k \in K \mid k \cdot \lambda = \lambda\}$ and let $K'_\lambda = \{k \in K \mid k \cdot \lambda \in a^*_c\}$. For any $k \in K'_\lambda$,
there exists an element \( m' \in M' \) such that \( k \cdot \lambda = m' \cdot \lambda \). Thus \( (m')^{-1}k \in K_\lambda \), and so we see that \( K_\lambda' = M'K_\lambda = \cup_{s \in W}m_sK_\lambda = \cup_{s \in W}K_s\lambda m_s \).

Let \( \Sigma_\lambda^+ = \{ \alpha \in \Sigma^+ | B(\alpha, \lambda) = 0 \} \), and as before let \( W_\lambda \) be the subgroup of \( W \) fixing \( \lambda \). Then \( W_\lambda \) is the subroup of \( W \) generated by the reflections along the root hyperplanes in \( \Sigma_\lambda^+ \), and \( M' \cap K_\lambda = \cup_{s \in W_\lambda}m_sM \).

The Lie algebra of \( K_\lambda \) is \( \mathfrak{t}_\lambda = \mathfrak{m} + \sum_\alpha \mathfrak{l}_\alpha \), where the sum is taken over all \( \alpha \) in \( \Sigma_\lambda^+ \). If \( \lambda \) is not regular, then \( \Sigma_\lambda^+ \) is nonempty, and therefore the orbit \( K_\lambda M \) is a submanifold of \( K/M \) of positive dimension. The set \( M'K_\lambda/M \) is a disjoint union of \( |W|/|W_\lambda| \) translates of \( K_\lambda/M \), given by \( m_sK_\lambda/M \), where \( s \) ranges over a set of coset representatives in \( W/W_\lambda \).

Let \( f \) be any continuous function on the orbit \( K_\lambda/M \), invariant under left translation by elements of \( m_sM \), for all \( s \in W_\lambda \). Such \( f \) can be obtained by averaging any continuous function on the orbit by \( M \) and then further averaging by the \( m_s \). The vector space of such \( f \) is infinite-dimensional, since close to the identity coset \( eM \), the space of \( M \)-orbits in \( K_\lambda/M \) is parametrized by the space of \( M \)-orbits on a ball centered at \( 0 \) in \( \sum_{\alpha \in \Sigma_\lambda^+} \mathfrak{l}_\alpha \).

If \( s \in W \), we can extend \( f \) in a well-defined way to the translated orbit \( m_sK_\lambda/M \) by setting \( f(m_skM) = f(kM) \), for all \( k \in K_\lambda \). In this way, \( f \) becomes an \( M' \)-invariant function defined on the union of the translated orbits \( m_sK_\lambda/M \), for all \( s \in W' \).

Now let us define the distribution \( T_f \) on \( K/M \) by

\[
T_f(F) = \sum_s \int_{K_\lambda/M} f(m_s k \lambda M) F(m_s k \lambda M) d(k \lambda)_{M}, \quad (F \in \mathcal{E}(K/M))
\]

where the sum is taken over a set of representatives \( s \) of \( W/W_\lambda \). It is clear from the construction of \( f \) that \( T_f \) is independent of the choice of the \( m_s \) appearing on the right hand side above. \( T_f \) is then an \( M' \)-invariant distribution on \( K/M \).

Now, in accordance with Theorem 3.2, let us define the distribution \( \Phi_f \) in \( \mathcal{D}_\lambda' (\Xi_0) \) by

\[
\Phi_f(\varphi) = \int_{K/M} \int_a \tilde{\varphi}(kM, H) e^{i\lambda(H)} dH dT_f(kM)
\]

(52)

Since \( T_f \) is \( M' \)-invariant, so is \( \Phi_f \). To show that \( \Phi_f \) is \( \mathfrak{q} \)-invariant, we use the expression (51) defining \( T_f \):

\[
\Phi_f(\varphi) = \sum_s \int_{K_\lambda/M} \int_a \tilde{\varphi}(m_s k \lambda M, H) e^{i\lambda(H)} dH f(m_s k \lambda M) d(k \lambda)_{M}
\]

Let \( X \in \mathfrak{q} \). Then by (9) we have

\[
\Phi_f(\varphi^\tau(X)) = \sum_s \int_{K_\lambda/M} \int_a \tilde{\varphi}(m_s k \lambda M, H) e^{i\lambda(H)} e^{iB(m_s k \lambda \cdot \lambda, X)} dH f(m_s k \lambda M) d(k \lambda)_{M}
\]

But \( m_s k \lambda \cdot \lambda \in \mathfrak{a}_\lambda^* \), and thus \( B(m_s k \lambda \cdot \lambda, X) = 0 \), which shows that the right hand side above equals \( \Phi_f(\varphi) \).
Since, as remarked above, the space of all continuous functions $f$ on $K_{\lambda}/M$ invariant under the left action of $m_{s}M$, for all $s \in W_{\lambda}$, is infinite-dimensional, it follows that the space of conical distributions in $D'_{\lambda}(\Xi_{0})$ has infinite dimension.

In the case when the symmetric space $X = G/K$ has rank one; i.e., when $\dim \mathfrak{a} = 1$, it is possible to obtain a complete classification of the space of all conical distributions in $D'_{0}(\Xi_{0})$. In this case, $\Sigma^{+}$ has one or two elements; let $\alpha$ be the indivisible element. Choose $H \in \mathfrak{a}$ such that $\alpha(H) = 1$, and identify $\mathbb{R}$ with $\mathfrak{a}$ by $t \mapsto tH$.

Since we are assuming that $\lambda = 0$, then $W_{\lambda} = W = \{\pm 1\}$, and so the space $H$ of $W_{\lambda}$-harmonic polynomials on $\mathfrak{a}$ has basis $\{1, t\}$. Suppose that $\Phi \in D'_{0}(\Xi_{0})$ is a conical distribution. Then from (39), there exist uniquely determined $M'$-invariant distributions $T_{0}$ and $T_{1}$ on $K/M$ such that

$$
\Phi(\varphi) = \int_{K/M} \int_{-\infty}^{\infty} \bar{\varphi}(kM, tH) \, dt \, dT_{0}(kM) + \int_{K/M} \int_{-\infty}^{\infty} \bar{\varphi}(kM, tH) \, t \, dT_{1}(kM)
$$

(53)

Since $\Phi$ is also invariant under left translation by any $X \in \mathfrak{q}$, we have

$$
\Phi(\varphi) = \int_{K/M} \int_{-\infty}^{\infty} \bar{\varphi}(kM, tH + (k^{-1} \cdot X)_{\mathfrak{a}}) \, dt \, dT_{0}(kM)
$$

$$
+ \int_{K/M} \int_{-\infty}^{\infty} \bar{\varphi}(kM, tH + (k^{-1} \cdot X)_{\mathfrak{a}}) \, t \, dT_{1}(kM)
$$

$$
= \int_{K/M} \int_{-\infty}^{\infty} \bar{\varphi}(kM, tH) \, dt \, dT_{0}(kM) + \int_{K/M} \int_{-\infty}^{\infty} \bar{\varphi}(kM, tH) \, t \, dT_{1}(kM)
$$

$$
- \int_{K/M} \int_{-\infty}^{\infty} \bar{\varphi}(kM, tH) \, B(k \cdot A_{\alpha}, X) \, dt \, dT_{1}(kM)
$$

$$
= \Phi(\varphi) - \int_{K/M} \int_{-\infty}^{\infty} \bar{\varphi}(kM, tH) \, dt \, B(k \cdot A_{\alpha}, X) \, dT_{1}(kM)
$$

Hence

$$
\int_{K/M} \int_{-\infty}^{\infty} \bar{\varphi}(kM, tH) \, dt \, B(k \cdot A_{\alpha}, X) \, dT_{1}(kM) = 0
$$

(54)

for all $\varphi \in D(\Xi_{0})$.

If $T_{0}$ and $T_{1}$ are $M'$-invariant distributions on $K/M$ it is clear that the condition (54) is also sufficient for the distribution $\Phi$ in (53) to be conical in $D'_{0}(\Xi_{0})$. In particular, $T_{0}$ can be arbitrary.

Now it is easy to see that the map $\varphi \mapsto \int_{-\infty}^{\infty} \bar{\varphi}(kM, tH) \, dt$ maps $D(\Xi_{0})$ onto the vector space $E_{M'}(K/M)$ of $C^{\infty}$ functions $F$ on $K/M$ satisfying $F(kM) = F(km^{*}M)$ for all $k \in K$, where $m^{*}$ is any element in $M' \setminus M$. Thus (54) implies that $\Phi$ is conical if and only if the $M'$-invariant distribution $T_{1}$ satisfies the condition

$$
\int_{K/M} F(kM) \, B(k \cdot A_{\alpha}, X) \, dT_{1}(kM) = 0
$$

(55)

for any $X \in \mathfrak{q}$ and all $F \in E_{M'}(K/M)$. As we shall show below, it turns out that all $M'$-invariant distributions on $K/M$ satisfy the condition above.
Assume that we conclude that for any \( l \in \mathbb{R} \), the centralizer and normalizer of \( G \) analytic subgroup of \( \theta \) any nonzero elements \( X, \theta \) of \( g \) this, we use the decomposition in the present case the set \( \Sigma^+ \) consists of \( \alpha \), and possibly \( 2\alpha \), with multiplicities \( m_\alpha \) and \( m_{2\alpha} \), respectively. Let \( H_1 \) be the unit vector in \( \mathfrak{a} \) such that \( \alpha(H_1) > 0 \), and let \( o \) denote the identity coset \( \{ M \} \) in \( K/M \). Then we can endow \( K/M \) with the \( K \)-invariant Riemannian structure induced from the \( Ad \)-invariant inner product on \( I \equiv T_0(K/M) \) given by

\[
(T_\alpha + T_{2\alpha}, T'_\alpha + T'_{2\alpha}) = -\alpha(H_1)^2 B(T_\alpha, T'_\alpha) - 4\alpha(H_1)^2 B(T_{2\alpha}, T'_{2\alpha})
\]

for \( T_\alpha, T'_\alpha \in I_\alpha \) and \( T_{2\alpha}, T'_{2\alpha} \in I_{2\alpha} \). One can easily show that the mapping \( kM \mapsto k \cdot H_1 \) is an isometry from \( K/M \) onto the unit sphere \( S \) (with respect to \( B \)) in \( p \). Whenever it is convenient, we will identify \( K/M \) with \( S \) in this manner.

**Lemma 5.1.** Assume that \( \dim \mathfrak{a} = 1 \). Fix \( m^* \in M' \). Then for every \( kM \in K/M \), there exists an \( m \in M \) such that \( m^*k(m^*)^{-1}M = mkM \).

**Proof.** If \( m^* \in M \), then the result is trivial, so let us assume that \( m^* \) is not in \( M \). It is easy to see that the map \( kM \mapsto m^*k(m^*)^{-1}M \) is a well-defined isometry of \( K/M \). By Theorem 13.2 in [17], the map \( T \mapsto (\exp T)M \) maps \( I \) onto \( K/M \), and clearly \( m^*(\exp T)(m^*)^{-1}M = \exp(Ad(m^*)M) \). Thus it suffices to prove that for each \( T \in I \), there exists \( m \in M \) such that \( Ad(m^*)T = Ad(m)T \).

This assertion can be proved by considering the possible cases for \( m_\alpha \) and \( m_{2\alpha} \). For convenience, let us now provide \( I \) with the inner product given by \(-B\), which we note that \( Ad(m^*) \) leaves invariant. Suppose first that \( m_{2\alpha} > 1 \). Write \( T \in I \) as \( T = T_\alpha + T_{2\alpha} \), with \( T_\alpha \in I_\alpha, T_{2\alpha} \in I_{2\alpha} \). For any \( r, s \geq 0 \), \( AdM \) is transitive on the product of spheres \( \{ T' + T'' \in I_\alpha + I_{2\alpha} \mid \|T'\| = r, \|T''\| = s \} \) ([15]). Since \( Ad(m^*) \) is an isometry on \( I_\alpha \) and on \( I_{2\alpha} \), there exists \( m \in M \) such that \( Ad(m)T_\alpha = Ad(m^*)T_\alpha \) and \( Ad(m)T_{2\alpha} = Ad(m^*)T_{2\alpha} \).

Suppose next that \( m_{2\alpha} = 0 \) and \( m_\alpha > 1 \). Then \( I = I_\alpha \), and since \( AdM \) is transitive on spheres in \( I_\alpha \), our assertion easily holds in this case.

The remaining cases are \( m_{2\alpha} = 1 \) (so \( m_\alpha > 1 \) and \( m_\alpha = 1 \) (so \( m_{2\alpha} = 0 \)). Suppose that \( m_{2\alpha} = 1 \). We claim that \( Adm^* \) is the identity map on \( I_{2\alpha} \). For this, we use the decomposition \( g = g_{-2\alpha} + g_{-\alpha} + m + a + g_\alpha + g_{2\alpha} \). Choose any nonzero elements \( X_\alpha \in g_\alpha \) and \( X_{2\alpha} \in g_{2\alpha} \). Then \( X_\alpha, \theta(X_\alpha), X_{2\alpha}, \) and \( \theta(X_{2\alpha}) \) generate a Lie subalgebra \( g^* \) of \( g \) isomorphic to \( su(2,1) \). Let \( G^* \) be the analytic subgroup of \( G \) with this algebra. Then \( G^* \) has Iwasawa decomposition \( G^* = K^*AN^* \), where \( K^* = G^* \cap K, N^* = G^* \cap N \). If \( M^* \) and \( (M')^* \) denote the centralizer and normalizer of \( a \) in \( K^* \), we also have \( M^* = G^* \cap M \) and \( (M')^* = G^* \cap M' \). Choose any element \( m^*_1 \in (M')^* \setminus M^* \). Then \( m^*_1 \in M' \setminus M \) so there exists an \( m_1 \in M \) such that \( m^*_1 = m^*m_1 \). Now from [10], Chapter IX, §3, \( Ad(m^*_1)X_{2\alpha} = \theta(X_{2\alpha}) \), \( Ad(m^*_1)\theta(X_{2\alpha}) = X_{2\alpha} \), and thus \( Adm^*_1 \) fixes \( X_{2\alpha} + \theta(X_{2\alpha}) \). But this latter vector spans \( I_{2\alpha} \). Since \( AdM \) is the identity map on \( I_{\pm 2\alpha} \) ([12], Chapter III, Lemma 3.8) it follows that \( Adm^* = Ad(m^*_1m_1^{-1}) \) is the identity map on \( I_{\pm 2\alpha} \) as well. Now since \( AdM \) is transitive on spheres in \( I_\alpha \), we conclude that for any \( T \in I_\alpha \) and \( T' \in I_{2\alpha} \), there exists an \( m \in M \) such that \( Ad(m^*)T = Ad(m)T \) and \( Ad(m^*)T' = Ad(m)T' \).

Finally, suppose that \( m_\alpha = 1 \). Choose any nonzero \( X_\alpha \in g_\alpha \). Then \( X_\alpha \) and \( \theta(X_\alpha) \) generate a subalgebra \( g^* \) of \( g \) isomorphic to \( su(1,1) \). Let \( G^* \) be the analytic
Assume that \( (m')^* \in G^* \cap (M' \setminus M) \). There exists an \( m_1 \in M \) such that \( (m')^* = m^* m_1 \). If \( K^* = G^* \cap K \), then \( K^* \) is abelian, so \( \text{Ad}(m'^*) \) is the identity map on \( L_\alpha \). On the other hand, by [12], Chapter III, Lemma 3.8, \( \text{Ad}(M) \) is also the identity map on \( L_\alpha \). Thus \( \text{Ad}(m^*) T = \text{Ad}(m) T = T \) for all \( m \in M \) and \( T \in L_\alpha \).

This covers all the cases and finishes the proof of the lemma. \( \blacksquare \)

We are now in a position to classify the conical distributions in \( D'_0(\Xi_0) \) when \( \dim a = 1 \).

\[ \text{Theorem 5.2.} \quad \text{Assume that } \dim a = 1. \text{ Then the conical distributions in } D'_0(\Xi_0) \text{ are precisely those distributions } \Phi \text{ given by} \]

\[ \Phi(\varphi) = \int_{K/M} \int_{-\infty}^{\infty} \bar{\varphi}(kM, tH) \, dt \, dT_0(kM) + \int_{K/M} \int_{-\infty}^{\infty} \bar{\varphi}(kM, tH) \, dt \, dT_1(kM) \]

(56)

where \( T_0 \) and \( T_1 \) are \( M' \)-invariant distributions on \( K/M \).

\[ \text{Proof.} \quad \text{As remarked in (53), any conical distribution } \Phi \text{ in } D'_0(\Xi_0) \text{ must be of the form (56), with } T_0 \text{ and } T_1 \text{ } M' \text{-invariant.} \]

Conversely, suppose that \( \Phi \in D'(\Xi_0) \) is defined by (56) with \( T_0 \) and \( T_1 \) \( M' \)-invariant. By Theorem 3.2, \( \Phi \) belongs to \( D'_0(\Xi_0) \), and it is clear that \( \Phi \) is \( M' \)-invariant. To prove that \( \Phi \) is conical, it is sufficient to verify that \( T_1 \) satisfies (55) for all \( (k, m') \in E(K/M) \) such that \( F(kM) = F(km^*M) \).

To this end, let us put \( F^#(kM) = \int_{M'} F(m'kM) \, dm' \) for any function \( F \in E(K/M) \), where \( dm' \) is the normalized Haar measure on the compact group \( M' \). Note that since \( T_1 \) is \( M' \)-invariant, \( T_1(F) = T_1(F^#) \) for all \( F \in E(K/M) \).

Lemma 5.1 shows that for any \( kM \in K/M \), there exists \( m_1 \in M \) such that \( m^* k \cdot H_1 = -m^* k (m^*)^{-1} \cdot H_1 = -m_1 k \cdot H_1 \). If \( a : \omega \mapsto -\omega \) denotes the antipodal map on the sphere \( S \), then \( a(k \cdot H_1) = k(m^*)^{-1} \cdot H_1 \), so \( a \) corresponds to the isometry \( kM \mapsto k(m^*)^{-1} \cdot M = K/M \).

Noting that \( F^a(kM) = F(k(m^*)^{-1}M) \), we see from the definition of \( F^# \) that that \( (F^#)^a = (F^#)^a \). On the other hand, for any \( kM \in K/M \), we put \( k \cdot H_1 = \omega \). Applying Lemma 5.1, we have

\[ (F^a)^#(\omega) = \int_{M'} F(m'k(m^*)^{-1}M) \, dm' \]
\[ = \int_{M'} F(m'm^*k(m^*)^{-1}M) \, dm' \]
\[ = \int_{M'} F(m'm_1M) \, dm' \quad \text{(for some } m_1 \in M) \]
\[ = F^#(\omega). \]

In particular, if \( F \in E(K/M) \) corresponds to an odd function on \( S \), we have \( F^# = 0 \).

Now suppose that \( F \in E(K/M) \) satisfies \( F(km^*M) = F(kM) \) for all \( k \in K \). Then \( F \) corresponds to an even function on \( S \), and for each fixed \( X \in q \),
the function \( G(kM) = F(kM)B(k \cdot A_\alpha, X) \) is an odd function on \( S \). It follows that \( G^\# = 0 \) and thus
\[
\int_{K/M} F(kM)B(k \cdot A_\alpha, X) dT_1(kM) = T_1(G)
= T_1(G^\#)
= 0.
\]

Thus (55) holds for all such \( F \), and we conclude that the distribution \( \Phi \) in (56) is conical.

It is curious that the \( M' \)-invariance of any distribution in \( \mathcal{D}'_0(\Xi_0) \) guarantees its \( q \)-invariance.

6. Eigenspace Representations

We conclude this paper by considering the natural representation of \( G_0 \) on the eigenspaces \( \mathcal{D}'_{\lambda}(\Xi_0) \). In particular, we would like to determine the conditions under which this representation is irreducible.

Let \( R \) denote the flat horocycle Radon transform given by (8). Then \( R \) is the Radon transform associated with the double fibration

\[
\begin{array}{c}
p = G_0/K \\
\downarrow \pi \\
\Xi_0 = G_0/H_q \\
\end{array}
\]

\[G_0/(K \cap H_q)\]

(See [6]; for a general introduction to integral transforms associated with group equivariant double fibrations, see [12], Chapter I, §1-3; for details on the flat horocycle Radon transform, see [8] or [12], Chapter IV, §5.) If \( f \in C_c(p) \), then \( Rf \) is given by
\[
Rf[kM, H] = \int_q f(k \cdot (H + Y)) \, dY
\]
where \( dY \) indicates the Euclidean measure on \( q \). Its dual transform is the map \( R^* : C(\Xi_0) \to C(p) \) given by
\[
R^* \varphi(X) = \int_K \varphi(X + k \cdot q) \, dk \quad (\varphi \in C(\Xi_0))
\]
where \( dk \) denotes the normalized Haar measure on the compact group \( K \).

The transforms \( R \) and \( R^* \) are \( G_0 \)-equivariant in the sense that \( R(f \circ l(g)) = (Rf) \circ l(g) \) and \( R^*(\varphi \circ l(g)) = (R^* \varphi) \circ l(g) \), where \( l(g) \) is the natural left action by \( g \in G_0 \) on the homogeneous spaces \( p \) and \( \Xi_0 \). They are also formal adjoints in the sense that
\[
\int_{\Xi_0} Rf(\xi) \varphi(\xi) \, d\xi = \int_p f(X) R^* \varphi(x) \, dx
\]
for all \( f \in C_c(p), \varphi \in C(\Xi_0) \), where, as in Section 2, \( d\xi \) denotes the \( G_0 \)-invariant measure on \( \Xi_0 \) which pulls back to \( dH \, dk_M \) on \( \Xi_0 \).
We equip $\mathcal{D}(\mathfrak{p})$ and $\mathcal{D}(\Xi_0)$ with the usual inductive limit topologies, and their dual spaces $\mathcal{D}'(\mathfrak{p})$ and $\mathcal{D}'(\Xi_0)$ by the corresponding strong topologies. According to Lemma 3.5, Chapter I in [12], $R$ is a continuous linear map from $\mathcal{D}(\mathfrak{p})$ to $\mathcal{D}(\Xi_0)$. Thus we can extend the relation (60) by defining the dual transform $R^* \Phi$ of any $\Phi \in \mathcal{D}'(\Xi_0)$ by

$$R^* \Phi(f) = \Phi(Rf) \quad (f \in \mathcal{D}(\mathfrak{p}))$$

The dual transform $\Phi \mapsto R^* \Phi$ is then a continuous linear map from $\mathcal{D}'(\Xi_0)$ to $\mathcal{D}'(\mathfrak{p})$. This map commutes with the natural left action of $G_0$ on distributions on $\Xi_0$ and $\mathfrak{p}$, respectively.

**Lemma 6.1.** Let $\lambda \in \mathfrak{a}_c^*$, and suppose that $\Phi$ is a distribution in $\mathcal{D}'(\Xi_0)$ given by

$$\Phi(\varphi) = \int_{K/M} \int_\mathfrak{a} \varphi[kM, H] e^{i\lambda(H)} dHdT(kM) \quad (\varphi \in \mathcal{D}(\Xi_0)) \quad (61)$$

where $T \in \mathcal{D}'(K/M)$. Then $R^* \Phi \in \mathcal{E}(\mathfrak{p})$, and is given by

$$R^* \Phi(X) = \int_{K/M} e^{iB(k\cdot A_\lambda, X)} dT(kM) \quad (X \in \mathfrak{p}) \quad (62)$$

**Proof.** If $f \in \mathcal{D}(\mathfrak{p})$, then

$$(R^* \Phi)(f) = \Phi(Rf) = \int_{K/M} \int_\mathfrak{a} Rf[kM, H] e^{i\lambda(H)} dHdT(kM)$$

$$= \int_{K/M} \int_\mathfrak{a} \int_\mathfrak{q} f(k \cdot (H + Y)) dY e^{iB(A_\lambda, H)} dH, dT(kM)$$

$$= \int_{K/M} \int_\mathfrak{p} f(k \cdot X) e^{iB(A_\lambda, X)} dX dT(kM)$$

$$= \int_\mathfrak{p} f(X) \int_{K/M} e^{iB(k\cdot A_\lambda, X)} dT(kM) dX$$

The inner integral on the right is clearly a smooth function of $X \in \mathfrak{p}$. Since it agrees with $R^* \Phi$ on all test functions $f$ on $\mathfrak{p}$, we obtain the lemma.

The right hand side of (62) is the flat analogue of the Poisson transform on the symmetric space $G/K$. We therefore call it the *Poisson transform* of $T$ corresponding to the spectral parameter $\lambda$, and denote it by $P_\lambda(T)$:

$$P_\lambda(T)(X) = \int_{K/M} e^{iB(k\cdot A_\lambda, X)} dT(kM) \quad (X \in \mathfrak{p}) \quad (63)$$

The transform $P_\lambda$ was introduced in [8] in connection with the study of eigenspace representations on $\mathfrak{p}$. Note that the Poisson transform of the function $F_0 \equiv 1$ is the zonal spherical function on $\mathfrak{p}$ corresponding to $\lambda$. 

Lemma 6.1, which relates the dual horocycle and the Poisson transforms, is the analogue of a similar formula ([7], Proposition 4.6) for the dual horocycle transform on $G/K$. If $\lambda$ is regular, then according to Corollary 3.3 above, every $\Phi \in \mathcal{D}_\lambda(\Xi_0)$ has the form (61). However, the analogy to $G/K$ is not completely precise, since when $\lambda$ is not regular, there are $\Phi \in \mathcal{D}_\lambda(\Xi_0)$ which are not of the form (61), so the dual transform $R^*\Phi$ is not necessarily a Poisson transform.

Recall that we have identified the algebra of left $G_0$-invariant differential operators on $p$ with the polynomial algebra $I(p)$, and that the algebra homomorphisms $\alpha : I(p) \to \mathbb{C}$ are given by evaluations $p \mapsto p(\lambda)$ for $\lambda \in a_c^*$, where $\lambda$ is unique up to $W$ orbit. For $\lambda \in a_c^*$, let

$$E_\lambda(p) = \{ f \in E(p) \mid \partial(p) f = p(i\lambda) f \text{ for all } p \in I(p) \}.$$ 

Since the function $X \mapsto e^{iB(k,\lambda,X)}$ belongs to $E_\lambda(p)$ for each $k \in K$, it is easy to see from (63) that the Poisson transform $P_\lambda(T)$ belongs to the joint eigenspace $E_\lambda(X)$ for all $T \in \mathcal{D}'(K/M)$. The joint eigenspace $E_\lambda(p)$ is $G_0$-invariant, and according to [8], Theorem 6.6, the natural representation of $G_0$ on $E_\lambda(p)$ is irreducible if and only if $\lambda$ is regular.

We say that $\lambda \in a_c^*$ is simple if the Poisson transform $P_\lambda$ is injective. An easy convolution argument on $K$ shows that $\lambda$ is simple if and only if the map $F \mapsto P_\lambda(F)$ is injective on $\mathcal{E}(K/M)$. Now according to Theorem 6.2 in [8], $\lambda$ is simple if and only if it is regular. Thus, in view of Corollary 3.3, the dual transform $R^* : \mathcal{D}_\lambda(\Xi_0) \to \mathcal{D}'(p)$ is injective if and only if $\lambda$ is regular.

We now fix some notation. For any $g \in G_0$ and $\varphi \in \mathcal{D}(\Xi_0)$, let $\varphi^{(g)} = \varphi \circ l(g^{-1})$. If $\Phi \in \mathcal{D}'(\Xi_0)$, we let $\Phi^{(g)}$ be the distribution on $\Xi$ given by $\Phi^{(g)}(\varphi) = \Phi(\varphi^{(g^{-1})})$.

The left regular representation of $G_0$ on $\mathcal{D}(\Xi_0)$ is then given by $\tau(g)\varphi = \varphi^{(g)}$ for $g \in G_0$, $\varphi \in \mathcal{D}(\Xi_0)$, and the natural representation of $G_0$ on $\mathcal{D}'(\Xi_0)$ is the contragredient representation, which is given by $\pi(g)\Phi = \Phi^{(g)}$.

If $\lambda \in a_c^*$, then the joint eigenspace $\mathcal{D}_\lambda(\Xi_0)$ is a closed subspace of $\mathcal{D}'(\Xi_0)$ invariant under $\pi$; we let $\pi_\lambda$ denote the restriction of $\pi$ to $\mathcal{D}_\lambda(\Xi_0)$.

**Proposition 6.2.** Suppose that $\lambda \in a_c^*$ is regular. Then the conical distribution $\Phi_\lambda$ in (50) is a cyclic vector for the representation $\pi_\lambda$.

**Proof.** We need to prove that the linear span of the translates $\Phi_\lambda^{(g)}$, for all $g \in G_0$, is dense in $\mathcal{D}_\lambda(\Xi_0)$. Suppose that $L$ belongs to the dual space of $\mathcal{D}_\lambda(\Xi_0)$. Since $\mathcal{D}_\lambda(\Xi_0)$ is a closed subspace of $\mathcal{D}'(\Xi_0)$, we may extend $L$ to a continuous linear functional on $\mathcal{D}'(\Xi_0)$; then because $\mathcal{D}(\Xi_0)$ is reflexive, there exists a $\varphi \in \mathcal{D}(\Xi_0)$ for which $L(\Phi) = \Phi(\varphi)$ for all $\Phi \in \mathcal{D}'(\Xi_0)$.

Now suppose that $L(\Phi_\lambda^{(g)}) = 0$ for all $g \in G_0$. Then by (50),

$$L(\Phi_\lambda^{(g)}) = \Phi^{(g)}(\varphi) = \Phi(\varphi^{(g^{-1})}) = \sum_{s \in W} (\varphi^{(g^{-1})})^* [m_s M, \lambda]$$ (64)
Put \( g = (k, X) \) for some \( k \in K \), \( X \in \mathfrak{p} \). Then by (9), \( l(g)([k_0M, H_0]) = [kk_0M, H_0 + ((kk_0)^{-1} \cdot X)_{a}] \). Hence for each \( s \in W \),

\[
(\varphi^{(g^{-1})})^* [m_sM, \lambda] = \int_\mathfrak{a} \varphi[km_sM, H + ((km_s)^{-1} \cdot X)_{a}] e^{iH} dH
= e^{-iB(km_s, \lambda, X)} \varphi^*[km_sM, \lambda]
\]

(65)

Since \( g \) is arbitrary, we can let \( g = (k, k \cdot H') \) for \( k \in K \), \( H' \in \mathfrak{a} \). Then (64) becomes

\[
\sum_{s \in W} e^{-i s \lambda(H')} \varphi^*[km_sM, \lambda] = 0
\]

(66)

Since \( \lambda \) is regular, the functions \( e^{-i s \lambda} (s \in W) \) are linearly independent, and thus there exist vectors \( H_t \in \mathfrak{a} \), for \( t \in W \), such that the \( \times \times \) matrix \( (e^{-i s \lambda(H_t)}) \) is nonsingular. Replacing \( H' \) in (66) by each \( H_t \), the relations

\[
\sum_{s \in W} e^{-i s \lambda(H_t)} \varphi^*[km_sM, \lambda] = 0 \quad (t \in W)
\]

show in particular that \( \varphi^*[kM, \lambda] = 0 \) for all \( k \in K \). From Corollary 3.3, we conclude that \( \Phi(\varphi) = 0 \) for all \( \Phi \in \mathcal{D}(\Xi_0) \). Hence \( L = 0 \) on \( \mathcal{D}(\Xi_0) \), proving the proposition.

For \( \lambda \in \mathfrak{a}^* \), let \( \mathcal{K}_\lambda(\Xi_0) \) denote the vector space of all distributions \( \Phi \in \mathcal{D}(\Xi_0) \) given by

\[
\Phi(\varphi) = \int_{K/M} \varphi^*[kM, \lambda] F(kM) dkM \quad (\varphi \in \mathcal{D}(K/M))
\]

(67)

where \( F \in L^2(K/M) \). Note that according to Theorem 3.2, the map \( F \mapsto \Phi \) is injective from \( L^2(K/M) \) to \( \mathcal{K}_\lambda(\Xi_0) \). Thus we may endow \( \mathcal{K}_\lambda(\Xi_0) \) with a Hilbert space structure, the norm \( \|\Phi\|_\lambda \) of \( \Phi \) above being the \( L^2 \) norm of \( F \) on \( K/M \). Now a calculation similar to (65) shows that if \( g = (k', X') \), then \( (\varphi^{(g^{-1})})^*[k'M, \lambda] = e^{-iB(k'k^{-1}k', \lambda, X')} \varphi^*[k'M, \lambda] \) for any \( \varphi \in \mathcal{D}(\Xi_0) \). Thus if \( \Phi \) is given by (67), we have

\[
(\pi(\lambda)g)\Phi(\varphi) = \int_{K/M} \varphi^*[k'M, \lambda] e^{-iB(k'k^{-1}k', \lambda, X')} F(kM) dkM
= \int_{K/M} \varphi^*[kM, \lambda] e^{-iB(k, A_\lambda, X')} F((k')^{-1}kM) dkM
\]

(68)

From this one sees that \( \mathcal{K}_\lambda(\Xi_0) \) is invariant under \( \pi_\lambda \). Let \( \pi'_\lambda \) denote the restriction of \( \pi_\lambda \) to \( \mathcal{K}_\lambda(\Xi_0) \). Equation (68) shows that we may identify \( \pi'_\lambda \) with the representation of \( G_0 \) on \( L^2(K/M) \) given by

\[
\pi'_\lambda(k', X') F(kM) = e^{-iB(k, A_\lambda, X')} F((k')^{-1}kM)
\]

(69)

Thus \( \pi'_\lambda \) is the representation of \( G_0 \) induced from the one-dimensional representation \( (m, X) \mapsto e^{-iB(A_\lambda, X)} \) of the subgroup \( M \ltimes \mathfrak{p} \subset G_0 \). The representation \( \pi'_\lambda \) is unitary if and only if \( \lambda \in \mathfrak{a}^* \). If \( \lambda \in \mathfrak{a}^* \) is regular, then Mackey’s imprimitivity
Theorem 6.3. Let $\lambda \in \mathfrak{a}_c^\ast$. Then $\pi_\lambda'$ is irreducible if and only if $\lambda$ is regular.

Proof. Theorem 6.3 is the flat analogue of Proposition 5.3 in [7], and our proof is adapted from the proof of that result.

We first make the following observation. Let $F_0$ be the constant function $F_0(kM) \equiv 1$ on $K/M$. If $\langle \cdot , \cdot \rangle$ denotes the inner product on $L^2(K/M)$, then equation (69) implies that for any $F \in L^2(K/M)$,

$$ \langle \pi'_\lambda(g)F_0, F \rangle = \int_{K/M} e^{-iB(k,\lambda,X)} F(kM) \, dk_M $$

where $g = (k, X) \in G_0$. Thus $F_0$ is a cyclic vector for $\pi'_\lambda$ if and only if $-\lambda$ is simple. But $-\lambda$ is simple if and only if $-\lambda$ is regular, so $F_0$ is cyclic if and only if $\lambda$ is regular.

Now suppose that $\lambda$ is not regular. Let $\mathcal{N}$ denote the kernel of the Poisson transform $P_\lambda$ on $L^2(K/M)$. The Schwartz inequality shows that $\mathcal{N}$ is a closed subspace of $L^2(K/M)$. Since $P_\lambda$ is $G_0$-equivariant, $\mathcal{N}$ is invariant under $\pi'_\lambda$. Finally, since $\lambda$ is not simple, $\mathcal{N} \neq \{0\}$, and since $P_\lambda(F_0)(0) = 1$, we have $\mathcal{N} \neq L^2(K/M)$. This shows that $\pi'_\lambda$ is not irreducible.

Conversely, suppose that $\lambda$ is regular. Let $V$ be a nonzero closed $\pi'_\lambda$-invariant subspace of $L^2(K/M)$. Since $\lambda$ is simple, $P_\lambda(V) \neq \{0\}$, and by the $G_0$-equivariance, there is an $h \in P_\lambda(V)$ such that $h(0) = 1$. Letting $h = P_\lambda(F)$, where $F \in V$, we obtain $\int_{K/M} F(kM) \, dk_M = 1$. Now $F_0(kM) = \int_K F((k')^{-1}kM) \, dk' = \int_K \pi'_\lambda(k') F(k) \, dk'$ for all $kM \in K/M$, so it follows that $F_0 \in V$. But since $\lambda$ is regular, $F_0$ is a cyclic vector for $\pi'_\lambda$, so we conclude that $V = L^2(K/M)$. Hence $\pi'_\lambda$ is irreducible.

Theorem 6.3 now allows to determine the irreducibility of the representation $\pi_\lambda$ of $G_0$ on the eigenspace $\mathcal{D}'_\lambda(\Xi_0)$.

Theorem 6.4. Let $\lambda \in \mathfrak{a}_c^\ast$. Then $\pi_\lambda$ is irreducible if and only if $\lambda$ is regular.

Proof. Suppose first that $\lambda$ is not regular. Then $\lambda$ is not simple, so by Lemma 6.1, the dual transform $R^* : \mathcal{D}'_\lambda(\Xi_0) \rightarrow \mathcal{E}(\mathfrak{p})$ is not injective. By the $G_0$-equivariance and continuity of $R^*$, its kernel $\mathcal{R}$ is thus a nonzero closed invariant subspace of $\mathcal{D}'_\lambda(\Xi_0)$. Moreover by Lemma 6.1, $\mathcal{R} \neq \mathcal{D}'_\lambda(\Xi_0)$, since $P_\lambda(F_0) \neq 0$. This shows that $\pi_\lambda$ is not irreducible.
Next assume that $\lambda$ is regular. Now Corollary 3.3 asserts that we have a bijective linear map $P : T \mapsto \Phi$ from $\mathcal{D}'(K/M)$ onto $\mathcal{D}'(\Xi_0)$ given by

$$\Phi(\varphi) = \int_{K/M} \varphi^*[kM, \lambda] dT(kM) \quad (\varphi \in \mathcal{D}(\Xi_0))$$

This map is continuous since it is the adjoint of the continuous map of $\mathcal{D}(\Xi_0)$ onto $\mathcal{D}(K/M)$ given by

$$\varphi \mapsto \varphi^*[\cdot, \lambda]$$

In particular, this implies that the inclusion map of the Hilbert space $\mathcal{K}_\lambda(\Xi_0)$ into $\mathcal{D}'(\Xi_0)$ is continuous.

Now suppose that $E$ is a closed subspace of $\mathcal{D}'(\Xi_0)$ invariant under $\pi_\lambda$. Then $E \cap \mathcal{K}_\lambda(\Xi_0)$ is a closed $\pi'_\lambda$ invariant subspace of $\mathcal{K}_\lambda(\Xi_0)$. Since $\pi'_\lambda$ is irreducible, we must have $E \cap \mathcal{K}_\lambda(\Xi_0) = \mathcal{K}_\lambda(\Xi_0)$ or $E \cap \mathcal{K}_\lambda(\Xi_0) = \{0\}$.

Let us first consider the case $E \cap \mathcal{K}_\lambda(\Xi_0) = \mathcal{K}_\lambda(\Xi_0)$. Then $E \supset \overline{\mathcal{K}_\lambda(\Xi_0)}$, the closure of $\mathcal{K}_\lambda(\Xi_0)$ in $\mathcal{D}'(\Xi_0)$. But since $P$ is continuous and surjective, and since $L^2(K/M)$ is dense in $\mathcal{D}'(K/M)$, $\mathcal{K}_\lambda(\Xi_0) = P(L^2(K/M))$ is dense in $\mathcal{D}'(\Xi_0) = P(\mathcal{D}'(K/M))$. It follows that $E = \mathcal{D}'(\Xi_0)$.

Next we treat the case $E \cap \mathcal{K}_\lambda(\Xi_0) = \{0\}$. We wish to conclude that $E = \{0\}$. For this, we consider the natural representation $\pi$ of $G_0$ on $\mathcal{D}'(\Xi_0)$. Now $\mathcal{D}'(\Xi_0)$ is a Montel space, hence is barreled and complete. Thus for any $f \in \mathcal{D}(K)$, we obtain a well-defined continuous linear operator $\pi(f)$ on $\mathcal{D}'(\Xi_0)$ given by

$$\pi(f) = \int_K f(k) \pi(k) \, dk$$

Now $\mathcal{D}'(\Xi_0)$ is a closed subspace of $\mathcal{D}'(\Xi_0)$ invariant under $\pi$, so if we approximate $f$ by step functions, we see that $\pi(f) \Phi \in \mathcal{D}'(\Xi_0)$ whenever $\Phi \in \mathcal{D}'(\Xi_0)$. For the same reason, since $E$ is closed in $\mathcal{D}'(\Xi_0)$ (and hence in $\mathcal{D}'(\Xi_0)$), we have $\pi(f) \Phi \in E$ whenever $\Phi \in E$.

Let $\tau$ denote the natural (left) representation of $K$ on $\mathcal{D}'(K/M)$. Suppose that $\Phi \in \mathcal{D}'(\Xi_0)$, so that $\Phi = P(T)$ for some $T \in \mathcal{D}'(K/M)$. Since $P$ is continuous and linear, and commutes with the left action of $K$, we have

$$\pi(f) \Phi = \int_K f(k) \pi(k) P(T) \, dk$$

$$= P\left( \int_K f(k) \pi(k) T \, dk \right)$$

If $f \in \mathcal{D}(K)$, then $\int_K f(k) \tau(k) T \, dk \in \mathcal{D}(K/M)$, so $\pi(f) \Phi \in \mathcal{K}_\lambda(\Xi_0)$. Thus if $\Phi \in E$, we must have $\pi(f) \Phi = 0$. Since $P$ is injective, this implies that $\int_K f(k) \tau(k) T \, dk = 0$. But because $f$ is arbitrary, we obtain $T = 0$, and therefore $\Phi = P(T) = 0$. Hence $E = \{0\}$.

This completes the two cases and shows that $\pi_\lambda$ is irreducible when $\lambda$ is regular.
References


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