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Conical Distributions on the Space of Flat Horocycles

Fulton B. Gonzalez

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Abstract. Let $G_0 = K \ltimes \mathfrak{p}$ be the Cartan motion group associated with a noncompact semisimple Riemannian symmetric pair (G, K). Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and let $\mathfrak{p} = \mathfrak{a} + \mathfrak{q}$ be the corresponding orthogonal decomposition. A flat horocycle in $\mathfrak{p}$ is a G_0-translate of $\mathfrak{q}$. A conical distribution on the space Ξ_0 of flat horocycles is an eigendistribution of the algebra $\mathbb{D}(\Xi_0)$ of G_0-invariant differential operators on Ξ_0 which is invariant under the left action of the isotropy subgroup of G_0 fixing $\mathfrak{q}$. We prove that the space of conical distributions belonging to each generic eigenspace of $\mathbb{D}(\Xi_0)$ is one-dimensional, and we classify the set of all conical distributions on Ξ_0 when G/K has rank one. We also consider the question of the irreducibility of the natural representation of G_0 on the eigenspaces of $\mathbb{D}(\Xi_0)$.
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1. Introduction and Preliminaries

In this paper we study the flat analogues of conical distributions on the space of horocycles associated with noncompact symmetric spaces. Let G be a connected noncompact real semisimple Lie group with finite center, let $\mathfrak{g}$ be its Lie algebra, and let K be a maximal compact subgroup of G. Let θ be the corresponding Cartan involution of G, and we also let θ denote its differential on $\mathfrak{g}$. Let $\mathfrak{k}$ be the Lie algebra of K and $\mathfrak{p}$ its orthogonal complement relative to the Killing form B on $\mathfrak{g}$, so that $\mathfrak{g}$ has Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$. We will generally use the notation in Helgason's books [10], [11], and [12]. In particular, we let $\mathfrak{a}$ denote a maximal abelian subspace of $\mathfrak{p}$, Σ the set of restricted roots of $\mathfrak{g}$ relative to $\mathfrak{a}$, W the Weyl group of Σ, $\mathfrak{g}_\alpha$ the restricted root space corresponding to $\alpha \in \Sigma$ and m_α its dimension. In addition, let $\mathfrak{a}^+$ denote a fixed Weyl chamber in $\mathfrak{a}$, Σ^+ the corresponding positive system of restricted roots, and $\rho = 1/2 \sum_{\alpha\in\Sigma^+} m_\alpha \alpha$. We put $\mathfrak{n} = \sum_{\alpha\in\Sigma^+} \mathfrak{g}_\alpha$, and let N and A be the analytic subgroups of G with Lie algebras $\mathfrak{n}$ and $\mathfrak{a}$, respectively. Then G has Iwasawa decomposition $G = NAK$. Finally, we let M and M' denote the centralizer and normalizer of A in K, respectively. Then $W = M'/M$. We let w be the order of W.

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We identify $\mathfrak{p}$ with $\mathfrak{p}^*$ (respectively $\mathfrak{a}$ with $\mathfrak{a}^*$) via the restriction of the Killing form B to $\mathfrak{p}$ (respectively $\mathfrak{a}$). In this way, elements of the (complexified) symmetric algebra $S(\mathfrak{p})$ can be viewed as polynomial functions on $\mathfrak{p}$, and also as constant coefficient differential operators on $\mathfrak{p}$. If $p \in S(\mathfrak{p})$, we let $\partial(p)$ be the corresponding differential operator on $\mathfrak{p}$.

A *horocycle* in the symmetric space G/K is an orbit of a conjugate of N in G/K. The following basic facts about horocycles may be found in Chapter II, §1–3 of [12]. The group G acts transitively on the space Ξ of all horocycles, and the isotropy subgroup of G fixing the identity horocycle $\xi_0 = N \cdot o$ is MN, so that $\Xi = G/MN$. The mapping $(kM, a) \mapsto ka \cdot \xi_0$ is a diffeomorphism of $K/M \times A$ onto Ξ, and the left G-invariant measure $d\xi$ on Ξ (which is unique up to a constant multiple) is given by

$$\int_\Xi \varphi(\xi)\, d\xi = \int_{K/M} \int_A \varphi(kM, a)\, e^{2\rho(\log a)}\, da\, dk_M, \qquad (\varphi \in C_c(K/M)) \qquad (1)$$

where dk_M denotes the normalized K-invariant measure on K/M and da denotes the Lebesgue measure on the Euclidean space A.

The algebra $\mathbb{D}(\Xi)$ of G-invariant differential operators on Ξ is isomorphic to $S(\mathfrak{a})$, the symmetric algebra of $\mathfrak{a}$, via

$$D_p\, \varphi(k \exp H \cdot \xi_0) = \partial(p)_H\, \varphi(k\, \exp H \cdot \xi_0), \qquad (p \in S(\mathfrak{a})) \qquad (2)$$

Let $\mathcal{D}'(\Xi)$ denote the space of all distributions on Ξ. If $D \in \mathbb{D}(\Xi)$ and $\Psi \in \mathcal{D}'(\Xi)$, the distribution $D\Psi$ on Ξ is given by

$$D\Psi(\varphi) = \Psi(D^*\varphi), \qquad (\varphi \in \mathcal{D}(\Xi))$$

where $D^* \in \mathbb{D}(\Xi)$ is the adjoint of D under the invariant measure $d\xi$. If $p \in S(\mathfrak{a})$, it follows from (1) that

$$(D_p)^* = D_{e^{-2\rho} \circ p^* \circ e^{2\rho}}, \qquad (3)$$

where $p^* \in S(\mathfrak{a})$ is given by $\partial(p^*) = \partial(p)^*$, the formal adoint of the differential operator $\partial(p)$ in $\mathfrak{a}$.

If $\mathfrak{a}_c^*$ is the complexified dual space of $\mathfrak{a}$, then the set of all joint eigendistributions of $\mathbb{D}(\Xi)$ is parametrized by $\mathfrak{a}_c^* \times \mathcal{D}'(K/M)$. More precisely, if we fix $\lambda \in \mathfrak{a}_c^*$, then the relation (3) above implies that the joint eigenspace $\mathcal{D}'_\lambda(\Xi) = \{\Psi \in \mathcal{D}'(\Xi) \,|\, D_p\, \Psi = p(i\lambda - \rho)\, \Psi \text{ for all } p \in S(\mathfrak{a})\}$ consists precisely of those distributions in Ξ of the form

$$\Psi(\varphi) = \int_{K/M} \int_A \varphi(kM, a)\, e^{(i\lambda+\rho)(\log a)}\, da\, dS(kM) \qquad (\varphi \in \mathcal{D}(\Xi)) \qquad (4)$$

for some $S \in \mathcal{D}'(K/M)$. (See Proposition 4.4, Chapter II in [12].)

A *conical distribution* in Ξ is an MN-invariant joint eigendistributon of $\mathbb{D}(\Xi)$. If λ is regular and simple, it turns out that the vector space of conical distributions in $\mathcal{D}'_\lambda(\Xi)$ is w-dimensional, and an explicit basis $\{\Psi_{\lambda,s}\}$ can be found in [7], each of which is supported in the closure of a Bruhat orbit in Ξ. For exceptional λ, the problem of classification of the conical distributions turns out to be more difficult, although for rank one it is completely solved. (See [7] and Hu's thesis [13]; the results in these papers are explained in §5–6 of [12].)

In this paper we consider the analogue of conical distributions on the space of flat horocycles in $\mathfrak{p}$. The flat horocycles are the translates, under the Cartan motion group, of the tangent space at the origin o in G/K to the identity horocycle $\xi_0 = N \cdot o$.

To be more precise, let us consider the Cartan motion group $G_0 = K \ltimes \mathfrak{p}$. G_0 has group law $(k, X)\,(k', X') = (kk', X + k \cdot X')$, for k, $k' \in K$ and X, $X' \in \mathfrak{p}$, where we have put $k \cdot X' = Ad\,k(X')$. The mapping

$$(T, X) \mapsto T + X \qquad\qquad (T \in \mathfrak{k},\ X \in \mathfrak{p}) \tag{5}$$

identifies the Lie algebra $\mathfrak{g}_0$ of G_0 with $\mathfrak{g}$ as vector spaces. Under this identification, the adjoint representation Ad_0 of G_0 on $\mathfrak{g}_0$ is given by

$$\mathrm{Ad}_0\,(k, X)\,(T' + X') = k \cdot T' + k \cdot X' - [k \cdot T', X] \tag{6}$$

and the Lie bracket $[\quad,\quad]_0$ on $\mathfrak{g}_0$ is given by

$$[T + X, T' + X']_0 = [T, T'] + [T, X'] - [T', X] \tag{7}$$

with T, $T' \in \mathfrak{k}$, X, $X' \in \mathfrak{p}$, where the Lie brackets on the right are taken in $\mathfrak{g}$. In effect, the Lie bracket on $\mathfrak{g}_0$ is the same as that on $\mathfrak{g}$, except that the subspace $\mathfrak{p}$ has been made abelian.

Now G_0 acts transitively on $\mathfrak{p}$ by $(k, X) \cdot Y = X + k \cdot Y$, with $k \in K$ and X, $Y \in \mathfrak{p}$. Let $\mathfrak{q}$ be the orthogonal complement of $\mathfrak{a}$ in $\mathfrak{p}$. If we identify $\mathfrak{p}$ with the tangent space T_oX, we have $\mathfrak{q} = T_o(N \cdot o)$. A *flat horocycle* is a translate of $\mathfrak{q}$ by an element of G_0. Let Ξ_0 be the set of all flat horocycles. Then of course Ξ_0 is a homogenous space of G_0, and according to Lemma 5.1, Chapter IV of [12], its isotropy subgroup at $\mathfrak{q}$ is $H_{\mathfrak{q}} = M' \ltimes \mathfrak{q}$.

The flat horocycle Radon transform of $f \in C_c(\mathfrak{p})$ is the function on Ξ_0 defined by

$$Rf(\xi) = \int_{\xi} f(X)\, dm(X) \qquad (\xi \in \Xi_0) \tag{8}$$

where $dm(X)$ is the Euclidean measure on ξ. One may view this as the flat analogue of the horocycle Radon transform on a noncompact symmetric space G/K. Properties of this transform, such as an inversion formula, and range and support theorems, have been studied in papers by Helgason and Orloff ([9], [18], [19]). (For a summary, see [12] Chapter IV, §5.) For a brief introduction to harmonic analysis on $\mathfrak{p}$, see [12], Chapter III, §7, which replicates much of the material in [8]. For a sample of the numerous papers dealing with analysis related to Cartan motion groups, see [1], [2], [3], [4], [14].

Let $\mathbb{D}(\Xi_0)$ denote the algebra of all differential operators on Ξ_0 invariant under the left action of G_0. By analogy with $\Xi = G/MN$, a *conical distribution* on Ξ_0 is a joint eigendistribution of $\mathbb{D}(\Xi_0)$ invariant under the left action of the isotropy subgroup $H_{\mathfrak{q}}$ of G_0 fixing $\mathfrak{q}$. Our aim in this paper is to classify such conical distributions.

In Section 2, we prove that $\mathbb{D}(\Xi_0)$ is isomorphic to the algebra $I(\mathfrak{a})$ of W-invariant elements in the symmetric algebra $\mathcal{S}(\mathfrak{a})$. This is the analogue for Ξ_0 of Theorem 2.2 in [5], which states that $\mathbb{D}(\Xi)$ is isomorphic to $\mathcal{S}(\mathfrak{a})$. (See equation (2) above.) It follows that the spaces of joint eigendistributions of $\mathbb{D}(\Xi_0)$ are parametrized by the set $\mathfrak{a}_c^*/W$ of W orbits in $\mathfrak{a}_c^*$. In Section 3, we obtain a general characterization of the joint eigendistributions in Ξ_0 similar to that given by the expression (4) above for Ξ. In Section 4, we prove our main result (Theorem 4.3 below), which states that in each "generic" joint eigenspace (corresponding to regular $\lambda \in \mathfrak{a}_c^*$), the space of conical distributions is one-dimensional, where we also provide an explicit basis vector.

In Section 5 we show that, by contrast, the space of conical distributions in each joint eigenspace corresponding to non-regular λ is infinite-dimensional. The problem of classifying the conical distributions for such λ appears to be difficult, although for G/K of rank one (so that $\lambda = 0$), we obtain a complete characterization in Theorem 5.2.

Finally, in Section 6, we consider the natural representation of G_0 on the spaces of joint eigendistributions of $\mathbb{D}(\Xi_0)$, relate these to conical distributions, and study the question of irreducibility.

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2. Invariant Differential Operators on Ξ_0 and $\widetilde{\Xi}_0$

Just as with $\Xi \cong K/M \times A$, it will be convenient to characterize Ξ_0 as a vector bundle. For each $s \in W$, we choose a representative $m_s \in M'$. Then the map $\pi : K/M \times \mathfrak{a} \to \Xi_0$ given by $\pi(kM, H) = k \cdot (H + \mathfrak{q})$ is w to one, with $\pi(kM, H) = \pi(km_s^{-1}M, sH)$. We can thus identify Ξ_0 with the associated bundle $K/M \times_W \mathfrak{a}$ over K/M', where K/M can be viewed as a principal bundle over K/M' with discrete structure group $W = M'/M$. For convenience, we put $[kM, H] = \pi(kM, H)$. It will be clear from the context that this will not be confused with the Lie bracket.

Using the above notation, the action of G_0 on Ξ_0 is given by

$$\begin{aligned}(k, X) \cdot [k_0M, H_0] &= X + k \cdot (k_0 \cdot (H_0 + \mathfrak{q})) \\ &= kk_0 \cdot \big(H_0 + ((kk_0)^{-1} \cdot X)_{\mathfrak{a}} + \mathfrak{q}\big) \\ &= [kk_0M, H_0 + ((kk_0)^{-1} \cdot X)_{\mathfrak{a}}] \end{aligned} \tag{9}$$

Here $X_{\mathfrak{a}}$ is the orthogonal projection (under the Killing form) of $X \in \mathfrak{p}$ onto $\mathfrak{a}$.

It will also be convenient to note that G_0 also acts transitively on the product manifold $\widetilde{\Xi}_0 = K/M \times \mathfrak{a}$ via

$$(k, X) \cdot (k_0M, H_0) = (kk_0M, H_0 + ((kk_0)^{-1} \cdot X)_{\mathfrak{a}}). \tag{10}$$

(That this is a group action is straighforward to verify.) Note that $\widetilde{\Xi}_0$, rather than Ξ_0, is in a certain sense the limit of the space $G/MN \cong K/M \times A$ of horocycles in G/K. The isotropy subgroup of G_0 at the origin $\widetilde{\xi}_0 = (eM, 0) \in \widetilde{\Xi}_0$ is $M \ltimes \mathfrak{q}$. From (9) and (10), it is immediate that the projection map $\pi : \widetilde{\Xi}_0 \to \Xi_0$ commutes with the action of G_0. It will frequently be useful to do calculations on Ξ_0 by lifting them up to $\widetilde{\Xi}_0$. All groups being unimodular, there are unique (up to constant multiple) G_0-invariant measures on Ξ_0 and on $\widetilde{\Xi}_0$, which we can take in both cases to be $dk_M\, dH$.

In this section, our objective is to determine the algebras $\mathbb{D}(\Xi_0)$ and $\mathbb{D}(\widetilde{\Xi}_0)$ of G_0-invariant differential operators on Ξ_0 and $\widetilde{\Xi}_0$, respectively.

All algebras here are over $\mathbb{C}$. Let $I(\mathfrak{p})$ and $I(\mathfrak{a})$ be the subalgebras of $\mathrm{Ad}\, K$-invariant elements of $S(\mathfrak{p})$ and of W-invariant elements of $S(\mathfrak{a})$, respectively. It is clear that the algebra $\mathbb{D}(\mathfrak{p})$ of G_0-invariant differential operators on $\mathfrak{p}$ is $I(\mathfrak{p})$. It is also a well-known fact that the restriction mapping $p \mapsto \overline{p} = p|_{\mathfrak{a}}$ is an isomorphism of $I(\mathfrak{p})$ onto $I(\mathfrak{a})$.

Now let $P \in \mathcal{S}(\mathfrak{a})$. Then from (10) the differential operator D_P on $\widetilde{\Xi}_0$ given by

$$D_P\, \Phi(kM, H) = \partial(P)_H\, \Phi(kM, H) \qquad (\Phi \in \mathcal{E}(\widetilde{\Xi}_0)) \tag{11}$$

is easily seen to belong to $\mathbb{D}(\widetilde{\Xi}_0)$. If $P \in I(\mathfrak{a})$, we abuse notation and also use D_P to denote the (well-defined) differential operator on Ξ_0 given by

$$D_P\, \varphi[kM, H] = \partial(P)_H\, \varphi[kM, H] \qquad (\varphi \in \mathcal{E}(\Xi_0)) \tag{12}$$

Then it follows from (9) that $D_P \in \mathbb{D}(\Xi_0)$. For $\varphi \in \mathcal{E}(\Xi_0)$, put $\widetilde{\varphi} = \varphi \circ \pi$. Then clearly

$$(D_P \varphi)\widetilde{} = D_P \widetilde{\varphi} \tag{13}$$

For $P \in \mathcal{S}(\mathfrak{a})$, we let P^* be its formal adjoint in $\mathfrak{a}$. Then the adjoint of the differential operator D_P on $\widetilde{\Xi}_0$ (with respect to the G_0-invariant measure $dk_M\, dH$) is D_{P^*}. The same holds for the operator D_P on Ξ_0 if $P \in I(\mathfrak{a})$, where the G_0-invariant measure $d\xi$ on Ξ_0 is fixed so as to satisfy $\int_{\Xi_0} \varphi(\xi)\, d\xi = \int_{\widetilde{\Xi}_0} \widetilde{\varphi}(kM, H)\, dk_M\, dH$.

Theorem 2.1. *1. The map $P \mapsto D_P$ is an algebra isomorphism of $\mathcal{S}(\mathfrak{a})$ onto $\mathbb{D}(\widetilde{\Xi}_0)$.*

2. The map $P \mapsto D_P$ is an algebra isomorphism of $I(\mathfrak{a})$ onto $\mathbb{D}(\Xi_0)$.

This theorem is the flat analogue of Theorem 2.2 in [5], which characterizes the algebra $\mathbb{D}(\Xi)$ of left G-invariant differential operators on the horocycle space Ξ. (See also Theorem 2.2, Chapter II in [12].) Our proof below is an adaptation of the proof of that theorem.

Let $H_{\mathfrak{q}}^0 = M \ltimes \mathfrak{q}$, so that $\widetilde{\Xi}_0 = G_0/H_{\mathfrak{q}}^0$. We let $\mathfrak{m}$ denote the Lie algebra of M, and let $\mathfrak{l}$ denote the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k}$ with respect to $-B$. Then $\mathfrak{g}_0$ has the orthogonal decomposition (relative to B or the inner product $B_\theta = -B(\cdot, \theta(\cdot))$ on $\mathfrak{g}$) given by

$$\mathfrak{g}_0 = \mathfrak{g} = (\mathfrak{m} \oplus \mathfrak{q}) \oplus \mathfrak{l} \oplus \mathfrak{a}. \tag{14}$$

Let $p : g_0 \mapsto g_0 H^0_{\mathfrak{q}}$ be the coset map from G_0 onto $\widetilde{\Xi}_0$, and let $\tau(g) : g_0 H^0_{\mathfrak{q}} \mapsto g g_0 H^0_{\mathfrak{q}}$ be left translation by $g \in G_0$ on $\widetilde{\Xi}_0$. Then $\tau(k, X)$ is given by (10) and from that we have $p(k, X) = (kM, (k^{-1} \cdot X)_{\mathfrak{a}})$. Now if $e_0 = (e, 0)$ is the identity element of G_0, then (14) shows that dp_{e_0} is a linear bijection of $\mathfrak{l} \oplus \mathfrak{a}$ onto the tangent space $T_{\widetilde{\xi}_0} \widetilde{\Xi}_0$. Let σ be the projection of $\mathfrak{g}$ onto $\mathfrak{l} \oplus \mathfrak{a}$ according to the decomposition (14). It is straightforward to show that

$$dp_{e_0} \circ \sigma \circ \mathrm{Ad}_0\,(h) = d\tau(h) \circ dp_{e_0} \circ \sigma \tag{15}$$

for all $h \in H_0$. Thus the restriction of dp_{e_0} to $\mathfrak{l} \oplus \mathfrak{a}$ intertwines the representations $\sigma \circ \mathrm{Ad}_0\,(h)$ and $d\tau(h)$ of H_0 on $\mathfrak{l} \oplus \mathfrak{a}$ and on $T_{\widetilde{\xi}_0} \widetilde{\Xi}_0$, respectively.

While the pair $(G_0, H^0_{\mathfrak{q}})$ is not reductive, it is nonetheless possible to determine $\mathbb{D}(\widetilde{\Xi}_0)$ from the elements of the (complexified) symmetric algebra $S(\mathfrak{l} \oplus \mathfrak{a})$ which are invariant under $\sigma \circ \mathrm{Ad}\,(H^0_{\mathfrak{q}})$.

Lemma 2.2. *$S(\mathfrak{a})$ is precisely the algebra of elements in the symmetric algebra $S(\mathfrak{l} \oplus \mathfrak{a})$ which are invariant under $\sigma \circ Ad_0\,(H^0_{\mathfrak{q}})$.*

Proof. Let $(m, X) \in H^0_{\mathfrak{q}}$. Then according to (6), we have $\mathrm{Ad}_0\,(m, X)\,(H) = H$ for any $H \in \mathfrak{a}$. This shows that $\mathfrak{a}$, and hence $S(\mathfrak{a})$, is invariant under $\mathrm{Ad}_0\,(H^0_{\mathfrak{q}})$ and thus also under $\sigma \circ \mathrm{Ad}_0\,(H^0_{\mathfrak{q}})$.

For the converse, let ad_0 denote the adjoint representation on the Lie algebra $\mathfrak{g}_0$. Then $\sigma \circ \mathrm{ad}_0$ is the representation of the Lie subalgebra $\mathfrak{m} \oplus \mathfrak{q}$ (of $\mathfrak{g}_0$) on $\mathfrak{l} \oplus \mathfrak{a}$ corresponding to the representation $\sigma \circ \mathrm{Ad}_0$ of $H^0_{\mathfrak{q}}$ on the same space. For convenience, for each $T + X \in \mathfrak{m} \oplus \mathfrak{q}$, we let $d(T + X)$ denote the restriction of $\sigma \circ \mathrm{ad}_0(T + X)$ to $\mathfrak{l} \oplus \mathfrak{a}$. We then extend $d(T + X)$ to a derivation of the symmetric algebra $S(\mathfrak{l} \oplus \mathfrak{a})$.

We will prove that if $Q \in S(\mathfrak{l} \oplus \mathfrak{a})$ such that

$$d(Y)\,Q = 0 \qquad \text{for all } Y \in \mathfrak{q} \tag{16}$$

then $Q \in S(\mathfrak{a})$. This will then imply that the elements of $S(\mathfrak{l} \oplus \mathfrak{a})$ invariant under $\sigma \circ \mathrm{Ad}_0\,(\mathfrak{q})$ belong to $S(\mathfrak{a})$, which will prove the lemma.

For each $\alpha \in \Sigma^+$, let $X^\alpha_1, \cdots, X^\alpha_{m_\alpha}$ be an orthonormal basis of the restricted root space $\mathfrak{g}_\alpha$ with respect to the inner product B_θ on $\mathfrak{g}$. Then the vectors $E^\alpha_i = X^\alpha_i + \theta(X^\alpha_i)$ form an orthogonal basis (with respect to $-B$) of the subspace

$$\mathfrak{l}_\alpha = \{T \in \mathfrak{k} \,|\, \mathrm{ad}\,(H)^2\, T = \alpha(H)^2\, T \text{ for all } H \in \mathfrak{a}\}.$$

of $\mathfrak{k}$. Likewise, the vectors $Y^\alpha_i = X^\alpha_i - \theta(X^\alpha_i)$ form an orthogonal basis (with respect to B) of

$$\mathfrak{q}_\alpha = \{X \in \mathfrak{p} \,|\, \mathrm{ad}\,(H)^2\, X = \alpha(H)^2\, X \text{ for all } H \in \mathfrak{a}\}.$$

Finally, we have $\mathfrak{l} = \oplus_{\alpha \in \Sigma^+} \mathfrak{l}_\alpha$ and $\mathfrak{q} = \oplus_{\alpha \in \Sigma^+} \mathfrak{q}_\alpha$.

If $\alpha \neq \beta$, it is easy to check that $[Y^\alpha_i, E^\beta_j]_0 = [Y^\alpha_i, E^\beta_j] \in \mathfrak{q}$ and therefore

$$d(Y^\alpha_i)(E^\beta_j) = 0 \qquad (1 \leq i \leq m_\alpha,\ 1 \leq j \leq m_\beta)$$

On the other hand, for $1 \le i, j \le m_\alpha$,

$$[Y_i^\alpha, E_j^\alpha]_0 = [Y_i^\alpha, E_j^\alpha] \\ = ([X_i^\alpha, X_j^\alpha] - \theta[X_i^\alpha, X_j^\alpha]) + ([X_i^\alpha, \theta(X_j^\alpha)] - \theta[X_i^\alpha, \theta(X_j^\alpha)])$$

The first quantity on the right above belongs to $\mathfrak{q}$. If $i \neq j$, then $[X_i^\alpha, \theta(X_j^\alpha)] \in \mathfrak{m}$, so the second expression on the right above vanishes. If $i = j$, then the second quantity on the right equals $2A_\alpha$, where A_α is the vector in $\mathfrak{a}$ such that $B(A_\alpha, H) = \alpha(H)$ for all $H \in \mathfrak{a}$.

We conclude that

$$d(Y_i^\alpha)(E_j^\beta) = \begin{cases} 2A_\alpha & \text{if } \alpha = \beta \text{ and } i = j \\ 0 & \text{otherwise} \end{cases} \tag{17}$$

Suppose now that $Q \in S(\mathfrak{l} \oplus \mathfrak{a})$ such that $d(Y)\,Q = 0$ for all $Y \in \mathfrak{q}$. Fix any basis $H_1, \dots, H_l$ of $\mathfrak{a}$. Then Q can be written uniquely as a polynomial in the E_j^β with coefficients in $S(\mathfrak{a})$:

$$Q = \sum_N \big\{ P_N(H_1, \dots, H_l) \prod_{\beta \in \Sigma^+} ((E_1^\beta)^{n(\beta,1)} \cdots (E_{m_\beta}^\beta)^{n(\beta, m_\beta)}) \big\} \tag{18}$$

where the sum ranges over multiindices $N = (n(\beta, j))$ $(1 \le j \le m_\beta,\ \beta \in \Sigma^+)$. For convenience, let us put $E(\beta)^{N(\beta)} = (E_1^\beta)^{n(\beta,1)} \cdots (E_{m_\beta}^\beta)^{n(\beta,m_\beta)}$ and $P_N = P_N(H_1, \dots, H_l)$.

Since $d(Y_i^\alpha)\,H = 0$ for all $H \in \mathfrak{a}$, (17) implies that

$$d(Y_i^\alpha)\,Q = \\ 2A_\alpha \sum_{n(\alpha,i) \neq 0} n(\alpha, i)\, P_N \big(\prod_{\beta \neq \alpha} E(\beta)^{N(\beta)} ((E_1^\alpha)^{n(\alpha,1)} \cdots (E_i^\alpha)^{n(\alpha,i)-1} \cdots (E_{m_\alpha}^\alpha)^{n(\alpha,m_\alpha)}) \big) \tag{19}$$

Since the right hand side equals 0, the coefficient of A_α above must equal 0. This coefficient is therefore an empty sum. Since (19) holds for all Y_i^α, we conclude that there is only one summand in (18), the one corresponding to $N = 0$. This shows that $Q \in S(\mathfrak{a})$. ∎

Let us recall that, by definition, $H_\mathfrak{q} = M' \ltimes \mathfrak{q}$.

Corollary 2.3. *The algebra of elements of $S(\mathfrak{l} \oplus \mathfrak{a})$ invariant under $\sigma \circ Ad_0(H_\mathfrak{q})$ is $I(\mathfrak{a})$.*

Proof. This is clear from Lemma 2.2 and (6). ∎

The rest of the proof of the first assertion in Theorem 2.1 proceeds exactly as in Helgason's book ([12], Theorem 2.2, Chapter II). For completeness, we include it. It will be convenient here to denote the basis $\{E_i^\alpha\}$ of $\mathfrak{l}$ by $H_{l+1}, \dots, H_{l+r}$. Then for some $\delta > 0$, the inverse of the map

$$(t_1, \dots, t_{l+r}) \mapsto \exp(t_1 H_1 + \cdots + t_{l+r} H_{l+r})\, H_0 \qquad (\textstyle\sum t_i^2 < \delta^2)$$

is a chart on a neighborhood of the identity coset $eH_0 = \widetilde{\xi}_0$ in $\widetilde{\Xi}_0$. Suppose that $D \in \mathbb{D}(\widetilde{\Xi}_0)$. Then there is a unique polynomial P in $l+r$ variables such that

$$D\varphi(\widetilde{\xi}_0) = P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_{l+r}}\right) \varphi(\exp(\sum t_i H_i) \cdot \widetilde{\xi}_0)\Big|_{(t)=(0)} \tag{20}$$

for all $\varphi \in \mathcal{E}(\widetilde{\Xi}_0)$. Now for each $h \in H_0$, there is a diffeomeorphism $(t_1, \dots, t_{l+r}) \mapsto (s_1, \dots, s_{l+r})$ on neighborhoods of $0 \in \mathbb{R}^{l+r}$ such that

$$\tau(h) \exp\left(\sum t_i H_i\right) H_0 = \exp\left(\sum s_j H_j\right) H_0.$$

For convenience, let us put $\varphi(\exp(\sum t_i H_i)\, H_0) = \varphi(t_1, \dots, t_{l+r})$. Since $D(\varphi)(\widetilde{\xi}_0) = D(\varphi^{\tau(h)})(\widetilde{\xi}_0)$, we have

$$P\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_{l+r}}\right) (\varphi(t_1, \dots, t_{l+r}) - \varphi(s_1, \dots, s_{l+r}))\Big|_{(t)=(0)} \tag{21}$$

Assume that P is of order N, let P_N denote the sum of the highest order terms in P, and write

$$P_N = \sum_{|J|=N} a_J \left(\frac{\partial}{\partial t_1}\right)^{j_1} \circ \cdots \circ \left(\frac{\partial}{\partial t_{l+r}}\right)^{j_{l+r}}.$$

If we fix a multiindex J of order N and let $\varphi(t_1, \dots, t_{l+r}) = t^J = t_1^{j_1} \dots t_{l+r}^{j_{l+r}}$ near the origin, then (21) shows that

$$a_J = \sum_{|I|=N} R_{JI}\, a_I \tag{22}$$

where (R_{JI}) is the matrix of the linear operator on the vector space of homogeneous degree N polynomial functions on $\mathbb{R}^{l+r}$ extending the operator on $\mathbb{R}^{l+r}$ whose matrix is the Jacobian matrix $(\partial s_j/\partial t_i)$ at $(t) = (0)$. But this Jacobian matrix is also the matrix of $\sigma \circ \mathrm{Ad}_0\,(h)$ with respect to the basis $\{H_i\}$ of $\mathfrak{l} \oplus \mathfrak{a}$. Equation (22) thus shows that $\sum_{|J|=N} a_J H^J$ is invariant under $\sigma \circ \mathrm{Ad}_0\,(h)$. Hence by Lemma 2.2, we conclude that $P_N = P_N(\partial/\partial t_1, \dots, \partial/\partial t_l)$. Since D is G_0-invariant, we see that

$$D\varphi(g_0 \cdot \widetilde{\xi}_0) = P_N\left(\frac{\partial}{\partial t_1}, \dots, \frac{\partial}{\partial t_l}\right) \varphi(g_0\, \exp(\sum_{i=1}^{l} t_i H_i) \cdot \widetilde{\xi}_0)\Big|_{(0)} + \text{lower order terms}$$

so that $D - D_{P_N}$ is an element of $\mathbb{D}(\widetilde{\Xi}_0)$ whose order is less than the order of D. A simple induction on the order then completes the proof of the first assertion of Theorem 2.1.

For the second assertion, suppose that $D \in \mathbb{D}(\Xi_0)$. Then there exists a polynomial P such that (20) holds for all functions $\varphi \in \mathcal{E}(\Xi_0)$, with ξ_0 replacing $\widetilde{\xi}_0$. With this substitution, the rest of the proof above carries over, with $h \in H_{\mathfrak{q}}^0$ replaced by $h \in H_{\mathfrak{q}} = M' \ltimes \mathfrak{q}$, and with $P_N(H_1, \dots, H_l)$ M'-invariant by Corollary 2.3.

3. The Space of Joint Eigendistributions

Suppose that $\Psi \in \mathcal{D}'(\Xi_0)$ is an eigendistribution of $\mathbb{D}(\Xi_0)$. Then according to Lemma 3.11, Chapter III of [11], there exists a $\lambda \in \mathfrak{a}_c^*$ (unique up to W-orbit) such that

$$D_P\Psi = P(i\lambda)\,\Psi \tag{23}$$

for all $P \in I(\mathfrak{a})$. We let $\mathcal{D}'_\lambda(\Xi_0)$ denote the vector space consisting of all $\Psi \in \mathcal{D}'(\Xi_0)$ satisfying (23).

Any eigendistribution $\Psi \in \mathcal{D}'(\widetilde{\Xi}_0)$ of $\mathbb{D}(\widetilde{\Xi}_0)$ likewise corresponds to a unique $\lambda \in \mathfrak{a}_c^*$ satisfying (23) for all $P \in \mathcal{S}(\mathfrak{a})$. For a given λ, we denote the vector space of all such distributions by $\mathcal{D}'_\lambda(\widetilde{\Xi}_0)$.

The following can be proved in a manner analogous to the proof of Proposition 4.4, Chapter II in [12].

Proposition 3.1. *Let $\Psi \in \mathcal{D}'_\lambda(\widetilde{\Xi}_0)$. Then there is a unique $S \in \mathcal{D}'(K/M)$ such that*

$$\Psi(\varphi) = \int_{K/M}\int_{\mathfrak{a}} \varphi(kM,H)\,e^{i\lambda(H)}\,dH\,dS(kM). \tag{24}$$

Conversely, if $S \in \mathcal{D}'(K/M)$, then the distribution Ψ on $\widetilde{\Xi}_0$ defined above belongs to $\mathcal{D}'_\lambda(\widetilde{\Xi}_0)$.

If $F \in \mathcal{E}(\widetilde{\Xi}_0)$, we define $F_\pi \in \mathcal{E}(\Xi_0)$ by

$$F_\pi[kM,H] = \frac{1}{w}\sum_{s\in W} F(km_s^{-1}M, sH)$$

Then the pullback $\widetilde{\Phi}$ of a distribution $\Phi \in \mathcal{D}'(\Xi_0)$ is defined by

$$\widetilde{\Phi}(F) = \Phi(F_\pi) \qquad\qquad (F \in \mathcal{D}(\widetilde{\Xi}_0)) \tag{25}$$

Note that

$$\widetilde{\Phi}(\widetilde{\varphi}) = \Phi(\varphi)$$

for all $\Phi \in \mathcal{D}'(\Xi_0)$, $\varphi \in \mathcal{D}(\Xi_0)$. Let $P \in I(\mathfrak{a})$ and $\Phi \in \mathcal{D}'(\Xi_0)$. Then it is easy to see from (11) and (12) and the fact that $D_P(F_\pi) = (D_PF)_\pi$, that, in analogy with (13), we have

$$(D_P\Phi)\widetilde{} = D_P\widetilde{\Phi}. \tag{26}$$

Since $\mathbb{D}(\Xi_0)$ is smaller than $\mathbb{D}(\widetilde{\Xi}_0)$, it is not true that $\widetilde{\Phi}$ belongs to $\mathcal{D}'_\lambda(\widetilde{\Xi}_0)$ whenever $\Phi \in \mathcal{D}'_\lambda(\Xi_0)$. (It is easy to construct smooth counterexamples.) Nonetheless, as we shall see below, we can obtain a result for $\mathcal{D}'_\lambda(\Xi_0)$ similar to Proposition 3.1.

Suppose that $\Phi \in \mathcal{D}'_\lambda(\Xi_0)$. Then by (26) $\widetilde{\Phi}$ satisfies

$$D_P(\widetilde{\Phi}) = P(i\lambda)\,\widetilde{\Phi} \qquad\qquad (P \in I(\mathfrak{a})) \tag{27}$$

Now for functions $\alpha \in \mathcal{D}(\mathfrak{a})$ and $\beta \in \mathcal{E}(K/M)$, let $\beta \otimes \alpha$ be the function $\beta(kM)\,\alpha(H)$ on $\widetilde{\Xi}_0 = K/M \times \mathfrak{a}$. The linear span of such functions is dense in $\mathcal{D}(\widetilde{\Xi}_0)$.

If we fix $\beta \in \mathcal{E}(K/M)$, the map

$$T_\beta : \alpha \in \mathcal{D}(\mathfrak{a}) \to \widetilde{\Phi}(\beta \otimes \alpha) \tag{28}$$

is a distribution in $\mathfrak{a}$; in fact, we see from (27) that T_β is an eigendistribution of the algebra $I(\mathfrak{a})$. Since this algebra contains elliptic elements, it follows that T_β is in fact a smooth eigenfunction of $I(\mathfrak{a})$, with

$$(\partial(P)\, T_\beta)(H) = P(i\lambda)\, T_\beta(H) \tag{29}$$

for all $P \in I(\mathfrak{a})$. The space of such eigenfunctions is described in [11], Chapter III, Theorem 3.13. Let W_λ denote the subgroup of W consisting of those elements fixing λ, let $I_\lambda(\mathfrak{a})$ be the subalgebra of W_λ-invariant elements of $\mathcal{S}(\mathfrak{a})$, and let H_λ be the vector space of W_λ-harmonic polynomial functions on $\mathfrak{a}$.

Then for each element $s\lambda$ in the orbit $W\cdot\lambda$, there exists a unique polynomial $P_{s\lambda}(\beta)(H)$ in $H_{s\lambda}$, with coefficients depending on β, such that

$$T_\beta(H) = \sum_{s\lambda \in W\cdot\lambda} P_{s\lambda}(\beta)(H)\, e^{is\lambda(H)} \tag{30}$$

for all $H \in \mathfrak{a}$. When λ is regular, the $P_{s\lambda}(\beta)$ are just constants (depending, of course, on β).

For fixed $H \in \mathfrak{a}$, the map $\beta \in \mathcal{E}(K/M) \to P_{s\lambda}(\beta)(H)$ is continuous, and from this it is not hard to see that the coefficients of the polynomials $P_{s\lambda}(\beta)(H)$ are distributions on K/M. More precisely, for each $s\lambda$, fix a basis $P_{s\lambda,j}(H)$ $(1 \le j \le r = |W_\lambda|)$ of $H_{s\lambda}$. Then

$$P_{s\lambda}(\beta)(H) = \sum_{j=1}^{r} S_{s\lambda,j}(\beta)\, P_{s\lambda,j}(H) \tag{31}$$

Each coefficient $S_{s\lambda,j}$ is a distribution on K/M uniquely determined, of course, by the choice of the basis $\{P_{s\lambda,j}\}$. Hence, by (28) (30), and (31), we see that

$$\widetilde{\Phi}(F) = \sum_{s\lambda \in W\cdot\lambda} \sum_{j=1}^{r} \int_{K/M} \int_{\mathfrak{a}} P_{s\lambda,j}(H)\, F(kM, H)\, e^{is\lambda(H)}\, dH\, dS_{s\lambda,j}(kM) \tag{32}$$

for all $F \in \mathcal{D}(\widetilde{\Xi}_0)$ of the form $\beta \otimes \alpha$. Since the $\beta \otimes \alpha$ span a dense subspace of $\mathcal{D}(\widetilde{\Xi}_0)$, formula (32) holds for all $F \in \mathcal{D}(\widetilde{\Xi}_0)$.

When λ is regular, each $H_{s\lambda} = \mathbb{C}$ (so we can take 1 as its basis), and the formula above reduces to

$$\widetilde{\Phi}(F) = \sum_{s \in W} \int_{K/M} \int_{\mathfrak{a}} F(kM, H)\, e^{is\lambda(H)}\, dH\, dS_{s\lambda}(kM) \tag{33}$$

for all $F \in \mathcal{D}(\widetilde{\Xi}_0)$.

We now proceed to obtain a more explicit characterization of the eigendistribution $\Phi \in \mathcal{D}'_\lambda(\Xi_0)$. For this, we note that expression (31) shows that $P_{s\lambda}$ can be considered as an element of $\mathcal{D}'(K/M)\otimes H_{s\lambda}$, with $P_{s\lambda}=\sum_{j=1}^m S_{s\lambda,j}\otimes P_{s\lambda,j}$, so that (32) becomes

$$\widetilde{\Phi}(F)=\sum_{s\lambda\in W\cdot\lambda}\int_{\mathfrak{a}}\int_{K/M}F(kM,H)\,e^{is\lambda(H)}\,dP_{s\lambda}(kM)(H)\,dH \tag{34}$$

We observe that by (31), each $P_{s\lambda}$ is uniquely determined by Φ.

Now the Weyl group W acts (freely) on both K/M and on $\widetilde{\Xi}_0=K/M\times\mathfrak{a}$ by $s\cdot kM=km_s^{-1}M$ and $s\cdot(kM,H)=(km_s^{-1}M,sH)$. Thus for each $t\in W$,

$$\begin{aligned}\widetilde{\Phi}(F)&=\widetilde{\Phi}^t(F)\\&=\sum_{s\lambda\in W\cdot\lambda}\int_{\mathfrak{a}}\int_{K/M}F(km_t^{-1}M,t\cdot H)\,e^{is\lambda(H)}\,dP_{s\lambda}(kM)(H)\,dH\\&=\sum_{s\lambda\in W\cdot\lambda}\int_{\mathfrak{a}}\int_{K/M}F(kM,t\cdot H)\,e^{is\lambda(H)}\,dP^t_{s\lambda}(kM)(H)\,dH\end{aligned} \tag{35}$$

where we have put $P^t_{s\lambda}=\sum_j S^t_{s\lambda,j}\otimes P_{s\lambda,j}$. The right hand side of (35) then equals

$$\sum_{s\lambda\in W\cdot\lambda}\int_{\mathfrak{a}}\int_{K/M}F(kM,H)\,e^{its\lambda(H)}\,t\cdot dP^t_{s\lambda}(kM)(H)\,dH \tag{36}$$

where now $t\cdot P^t_{s\lambda}=\sum_j S^t_{s\lambda,j}\otimes(t\cdot P_{s\lambda,j})$, an element of $\mathcal{D}'(K/M)\otimes H_{ts\lambda}$. By the uniqueness of the $P_{s\lambda}$, it follows that

$$P_{ts\lambda}=t\cdot P^t_{s\lambda}$$

for all $s,t\in W$. In particular,

$$P_{s\lambda}=s\cdot P^s_\lambda \qquad (s\in W)$$

Hence, for any $\varphi\in\mathcal{D}(\Xi_0)$, we have

$$\begin{aligned}\Phi(\varphi)&=\widetilde{\Phi}(\widetilde{\varphi})\\&=\sum_{s\lambda\in W\cdot\lambda}\int_{\mathfrak{a}}\int_{K/M}\widetilde{\varphi}(kM,H)\,e^{is\lambda(H)}\,s\cdot dP^s_\lambda(kM)(H)\,dH\\&=\sum_{s\lambda\in W\cdot\lambda}\int_{\mathfrak{a}}\int_{K/M}\widetilde{\varphi}(km_s^{-1}M,s\cdot H)\,e^{i\lambda(H)}\,dP_\lambda(kM)(H)\,dH\\&=|W\cdot\lambda|\int_{\mathfrak{a}}\int_{K/M}\widetilde{\varphi}(kM,H)\,e^{i\lambda(H)}\,dP_\lambda(kM)(H)\,dH\\&=|W\cdot\lambda|\int_{\mathfrak{a}}\int_{K/M}\varphi[kM,H]\,e^{i\lambda(H)}\,dP_\lambda(kM)(H)\,dH\end{aligned} \tag{37}$$

If we put $Q_\lambda=|W\cdot\lambda|\,P_\lambda$, this leads us to the following result.

Theorem 3.2. *Suppose that* $\lambda \in \mathfrak{a}_c^*$ *and that* $\Phi \in \mathcal{D}'_\lambda(\Xi_0)$. *Then there exists a unique element* $Q_\lambda \in \mathcal{D}'(K/M) \otimes H_\lambda$ *such that*

$$\Phi(\varphi) = \int_{\mathfrak{a}} \int_{K/M} \varphi[kM, H]\, e^{i\lambda(H)}\, dQ_\lambda(kM)(H)\, dH \tag{38}$$

for all $\varphi \in \mathcal{D}(\Xi_0)$. *Conversely, given any element* $Q_\lambda \in \mathcal{D}'(K/M) \otimes H_\lambda$, *the expression* (38) *defines a distribution* $\Phi \in \mathcal{D}'_\lambda(\Xi_0)$.

Remarks:

1. Fix a basis $P_1, \ldots, P_r$ of H_λ. (We may choose this basis to have real coefficients.) If $\Phi \in \mathcal{D}'_\lambda(\Xi_0)$, Theorem 3.2 says that there exist unique distributions T_j on K/M such that

$$\Phi(\varphi) = \sum_{j=1}^{r} \int_{K/M} \int_{\mathfrak{a}} P_j(H)\, \varphi[kM, H]\, e^{i\lambda(H)}\, dH\, dT_j(kM) \tag{39}$$

for all $\varphi \in \mathcal{D}(\Xi_0)$. Conversely, for any distributions T_j on K/M, the right hand side of (39) defines a distribution $\Phi \in \mathcal{D}'_\lambda(\Xi_0)$.

2. Equation (39) can also be written as

$$\Phi(\varphi) = \sum_{j=1}^{m} \int_{K/M} \partial(P_j^*)\, \varphi^*[kM, \lambda]\, dT_j(kM), \tag{40}$$

where φ^* is the (well-defined) Fourier-Laplace transform of φ:

$$\varphi^*[kM, \lambda] = \int_{\mathfrak{a}} \varphi[kM, H]\, e^{i\lambda(H)}\, dH \qquad ([kM, \lambda] \in K/M \times_W \mathfrak{a}_c^*)$$

Proof. Equation (38) follows from (37) by putting $Q_\lambda = |W \cdot \lambda|\, P_\lambda$. The uniqueness of Q_λ is a consequence of the uniqueness of the $P_{s\lambda}$, and in particular, of P_λ.

Conversely, suppose that $Q_\lambda \in \mathcal{D}'(K/M) \otimes H_\lambda$. If we fix a basis $P_1, \ldots, P_m$ of H_λ, we can, as in Remark (1) above, write $Q_\lambda = \sum_j S_j \otimes P_j$. The distribution Φ in (38) is then given by (39), and thus we need to prove that the right hand side of (39) defines a distribution $\Phi \in \mathcal{D}'_\lambda(\Xi_0)$. Now the product $P(H)\, e^{i\lambda(H)}$ belongs to the joint eigenspace $\mathcal{E}_{i\lambda}(\mathfrak{a}) = \{\alpha \in \mathcal{E}(\mathfrak{a}) \,|\, \partial(P)\, \alpha = P(i\lambda)\, \alpha \text{ for all } P \in I(\mathfrak{a})\}$. Hence for any $Q \in I(\mathfrak{a})$, we have

$$\begin{aligned}
(D_Q(\Phi))(\varphi) &= \sum_{j=1}^{m} \int_{K/M} \int_{\mathfrak{a}} \partial(Q^*)\varphi[kM, H]\, P_j(H)\, e^{i\lambda(H)}\, dH\, dT_j(kM) \\
&= Q(i\lambda) \sum_{j=1}^{m} \int_{K/M} \int_{\mathfrak{a}} \varphi[kM, H]\, P_j(H)\, e^{i\lambda(H)}\, dH\, dT_j(kM) \\
&= Q(i\lambda)\, \Phi(\varphi),
\end{aligned}$$

for all $\varphi \in \mathcal{D}(\Xi_0)$, proving the theorem. ∎

Corollary 3.3. *Suppose that λ is regular. Then there is a linear bijection from $\mathcal{D}'(K/M)$ onto $\mathcal{D}'_\lambda(\Xi_0)$ given by*

$$\begin{aligned} T &\mapsto \Phi \\ \Phi(\varphi) &= \int_{K/M} \int_{\mathfrak{a}} \varphi[kM, H]\, e^{i\lambda(H)}\, dH\, dT(kM) \\ &= \int_{K/M} \varphi^*[kM, \lambda]\, dT(kM) \end{aligned} \tag{41}$$

4. Conical Distributions

By definition, a *conical distribution* on Ξ_0 is an $H_{\mathfrak{q}}$-invariant eigendistribution of $\mathbb{D}(\Xi_0)$, where, as we recall, $H_{\mathfrak{q}}$ is the isotropy subgroup of G_0 fixing $\mathfrak{q}$: $H_{\mathfrak{q}} = M' \ltimes \mathfrak{q}$.

Suppose that Φ is a conical distribution on Ξ_0 belonging to $\mathcal{D}'_\lambda(\Xi_0)$. ($\lambda$ is of course determined up to W-orbit.) First, for simplicity, let us assume that λ is regular. Then we see that Φ satisfies (41), for unique $T \in \mathcal{D}'(K/M)$.

In order to determine this distribution T more explicitly, we first prove that the collection of functions on K/M given by $\{\varphi^*[kM, \lambda] \mid \varphi \in \mathcal{D}(\Xi_0)\}$ equals $\mathcal{E}(K/M)$.

For this, we first consider the following easy lemma.

Lemma 4.1. *For $f \in \mathcal{D}(\mathfrak{a})$ and $\gamma \in \mathcal{E}(\mathfrak{a})$, put $(f, \gamma) = \int_{\mathfrak{a}} f(H)\, \gamma(H)\, dH$. Suppose that $\gamma_1, \ldots, \gamma_m$ are linearly independent elements of $\mathcal{E}(\mathfrak{a})$. Then there exist functions $f_1, \ldots, f_m$ in $\mathcal{D}(\mathfrak{a})$ such that the $m \times m$ matrix $((f_i, \gamma_j))$ is any prescribed $m \times m$ matrix.*

Proof. Let V be the linear span of $\gamma_1, \ldots, \gamma_m$. For each $f \in \mathcal{D}(\mathfrak{a})$, let λ_f be the linear functional on V given by $\lambda_f(\gamma) = (f, \gamma)$. It suffices for us to prove that the linear map $f \mapsto \lambda_f$ maps $\mathcal{D}(\mathfrak{a})$ onto V^*. But if $f \mapsto \lambda_f$ were not onto, then there would be a nonzero subspace W of V such that $\lambda_f(W) = \{0\}$ for all f. But given any nonzero $\gamma \in W$, there is clearly an $f \in \mathcal{D}(\mathfrak{a})$ such that $(f, \gamma) \neq 0$, a contradiction. ■

For every $h \in \mathcal{D}(\mathfrak{a})$, let h^* denote its Fourier-Laplace transform

$$h^*(\lambda) = \int_{\mathfrak{a}} h(H)\, e^{i\lambda(H)}\, dH \qquad (\lambda \in \mathfrak{a}_c^*)$$

Lemma 4.2. *Let $\lambda \in \mathfrak{a}_c^*$ be regular. Let R be the linear map from $\mathcal{D}(\Xi_0)$ to $\mathcal{E}(K/M)$ given by $R\varphi(kM) = \varphi^*[kM, \lambda]$. Then R is onto.*

Proof. The proof requires some care since Ξ_0 is not the product manifold $K/M \times \mathfrak{a}$ but a quotient of it. Note first that Lemma 4.1 implies that for any distinct elements $\lambda_1, \ldots, \lambda_m \in \mathfrak{a}_c^*$, and any functions $\beta_1, \ldots, \beta_m \in \mathcal{E}(K/M)$, there exists a function $F \in \mathcal{D}(\widetilde{\Xi}_0)$ such that $F^*(kM, \lambda_j) = \beta_j(kM)$ for all $k \in K$ and all j. In fact since the functions $e^{i\lambda_1}, \ldots, e^{i\lambda_m}$ are linearly independent elements of $\mathcal{E}(\mathfrak{a})$, the lemma implies that there are functions $h_1, \cdots, h_m$ in $\mathcal{D}(\mathfrak{a})$ such that $h_i^*(\lambda_j) = \delta_{ij}$ for all i, j. Then put $F(kM, H) = \sum_j \beta_j(kM)\, h_j(H)$.

Now fix $\beta \in \mathcal{E}(K/M)$. We will prove that there exists a $\varphi \in \mathcal{D}(\Xi_0)$ such that $\varphi^*[kM, \lambda] = \beta(kM)$ for all $kM \in K/M$.

From the above we know that there exists a function $F \in \mathcal{D}(\widetilde{\Xi}_0)$ such that $F^*(kM, s\lambda) = \beta(km_s M)$ for all $kM \in K/M$ and all $s \in W$. (Here $m_s \in M'$ is any coset representative of s.) Put $\varphi = F_\pi$, so that $\varphi \in \mathcal{D}(\Xi_0)$. Then $\varphi^*[kM, \mu] = (1/w) \cdot \sum_{s \in W} F^*(km_s^{-1}M, s\mu)$ for all $kM \in K/M$ and all $\mu \in \mathfrak{a}_c^*$. In particular,

$$\begin{aligned} \varphi^*[kM, \lambda] &= \frac{1}{w} \sum_{s \in W} F^*(km_s^{-1}M, s\lambda) \\ &= \beta(kM) \end{aligned}$$

for all $kM \in K/M$. ∎

Resuming our investigation of conical distributions, let us assume, as before, that Φ is a conical distribution in $\mathcal{D}'_\lambda(\Xi_0)$, where λ is a fixed regular element in $\mathfrak{a}_c^*$. Let T be the unique element of $D'(K/M)$ given by (41).

The M' invariance of Φ implies that

$$\int_{K/M} \varphi^*[m'kM, \lambda]\, dT(kM) = \int_{K/M} \varphi^*[kM, \lambda]\, dT(kM) \tag{42}$$

for all $m' \in M'$. By Lemma 4.2, the functions $\varphi^*[kM, \lambda]$ run through $\mathcal{E}(K/M)$ as φ runs through $\mathcal{D}(\Xi_0)$. Thus (42) shows that T is a left M'-invariant distribution on K/M.

The $\mathfrak{q}$-invariance of Φ then shows that

$$\begin{aligned} \int_{K/M} \varphi^*[kM, \lambda]\, dT(kM) &= \int_{K/M} \varphi^*[kM, \lambda]\, e^{-i\lambda((k^{-1}X)_{\mathfrak{a}})}\, dT(kM) \\ &= \int_{K/M} \varphi^*[kM, \lambda]\, e^{-iB(kA_\lambda, X)}\, dT(kM) \end{aligned} \tag{43}$$

By Lemma 4.2, this implies that

$$T = e^{-iB(k \cdot A_\lambda, X)}\, T \tag{44}$$

for all $k \in K$ and all $X \in \mathfrak{q}$.

We will now prove that the property (44) implies that T has support in the discrete subset M'/M of K/M. For this, consider any $k_0 \in K \setminus M'$. Since λ is regular, $k_0 \cdot A_\lambda \notin \mathfrak{a}_c^*$. It is easy to see that there exists $X \in \mathfrak{q}$ such that $B(k_0 \cdot A_\lambda, X) \notin 2\pi\mathbb{Z}$. (This is done by scaling X if necessary.) Fixing this X, there exists a neighborhood U of k_0M in K/M such that $B(k \cdot A_\lambda, X) \notin 2\pi\mathbb{Z}$ for all $kM \in U$. Hence the function $kM \mapsto e^{iB(k \cdot A_\lambda, X)} - 1$ is never 0 on U, whereas by (44) the distribution $(e^{iB(k \cdot A_\lambda, X)} - 1)T$ on K/M equals 0. This implies that $T = 0$ on U. Since $k_0M \in K/M$ is an arbitrary point in the complement of M'/M, this proves that T has support in the discrete set M'/M.

In particular, T has the form

$$T = \sum_{s \in W} D_s \, \delta_{m_s M} \tag{45}$$

where D_s is a linear differential operator on K/M. We will now prove that in fact

$$T = c \sum_{s \in W} \delta_{m_s M} \tag{46}$$

for some constant c. For this, it suffices to prove that near the identity coset eM of K/M, T is a multiple of the delta function at eM. That is to say, it suffices to prove that for all smooth functions β on K/M supported on a small neighborhood of eM, then $T(\beta) = c\,\beta(eM)$. The M'-invariance of T then proves (46).

To this end, we introduce local coordinates on K/M near eM. Let $T_1, \dots, T_{m_\alpha}$ be an orthonormal basis (with respect to $-B$) of $\mathfrak{l}_\alpha$. (We could use $T_i^\alpha = 2^{-1/2} E_i^\alpha$ from the proof of Lemma 2.2.) The collection $\{T_j\}_{1 \le j \le m_\alpha, \alpha \in \Sigma^+}$ is then an orthonormal basis of $\mathfrak{l}$. We list these basis elements as $T_1, \dots, T_r$ and assume that T_j belongs to the generalized eigenspace $\mathfrak{k}_{\alpha_j}$. Then the map

$$\exp(t_1 T_1 + \cdots + t_r T_r)M \mapsto (t_1, \dots, t_r) \tag{47}$$

defines a chart on a neightborhood U of eM in K/M. We assume that

$$U \cap M'/M = \{eM\}.$$

For each j let us put $X_j = -i(B(\alpha_j, \lambda))^{-2} \operatorname{ad}(A_\lambda) T_j$. Since λ is regular, X_j is well defined, and it is easy to see that $X_1, \dots, X_r$ is a basis of the complexification $\mathfrak{q}_c$ of $\mathfrak{q}$, orthogonal with respect to the Killing form on $\mathfrak{p}_c$.

Now suppose that β is a smooth function on K/M with support in U. Then by (45), we have

$$T(\beta) = \sum_J c_J D^J \beta(0) \tag{48}$$

where the sum runs through a finite collection of multiindices $J = (j_1, \dots, j_r)$, the c_J are constants, and $D^J = \partial^{j_1 + \cdots + j_r} / \partial t_1^{j_1} \cdots \partial t_r^{j_r}$.

In the sum (48), we claim that $c_J = 0$ when $|J| > 0$. Then of course $T(\beta) = c_0\,\beta(0)$, and this will prove (46). To prove this, let us assume, to the contrary, that $c_J \neq 0$ for some $J \neq 0$. Let $N = \max\{|J| \,|\, c_J \neq 0\}$. Now by (44) we have

$$\sum_J c_J D^J \beta(0) = \sum_J c_J D^J \left(e^{-iB(\exp(t_1 T_1 + \cdots + t_r T_r) \cdot A_\lambda, X)} \beta \right)(0) \tag{49}$$

for all $X \in \mathfrak{q}$ and all smooth functions β supported in U. In (49) choose a β which is identically 1 on a small neighborhood of 0. Then the left hand side of (49) is c_0. On the other hand, if we write $X = z_1 X_1 + \cdots + z_r X_r$, where $z_j \in \mathbb{C}$, then the right hand side is

$$\sum_J c_J D^J \left(e^{-iB(\exp(t_1 T_1 + \cdots + t_r T_r) \cdot A_\lambda, X)} \right)(0),$$

a polynomial of degree N in $z_1, \dots, z_r$. Its homogeneous component of degree N equals

$$\sum_{|J|=N} c_J \, z^J,$$

where we have put $z^J = z_1^{j_1} \cdots z_r^{j_r}$ when $J = (j_1, \dots, j_r)$. This yields a contradiction, and we obtain the following result.

Theorem 4.3. *Suppose that $\lambda \in \mathfrak{a}_c^*$ is regular. Then the space of conical distributions in $\mathcal{D}'_\lambda(\Xi_0)$ is one-dimensional, with basis given by Φ_λ, where*

$$\Phi_\lambda(\varphi) = \sum_{s \in W} \varphi^*[m_s M, \lambda] \qquad (\varphi \in \mathcal{D}(\Xi_0)) \tag{50}$$

Proof. If Φ is a conical distribution in $\mathcal{D}'_\lambda(\Xi_0)$ then we have shown that Φ is a multiple of Φ_λ.

For the converse, we first observe that Lemma 4.2 implies that there is a $\varphi \in \mathcal{D}(\Xi_0)$ such that $\varphi^*[kM, \lambda] = 1$ for all $kM \in K/M$. This shows that the distribution Φ_λ is not zero. Moreover, (41) shows that Φ_λ belongs to $\mathcal{D}'_\lambda(\Xi_0)$, with $T = \sum_{s \in W} \delta_{m_s M}$; clearly T satisfies (44) and is M'-invariant, so Φ_λ is conical. ■

The product manifold $\widetilde{\Xi}_0 = K/M \times \mathfrak{a}$, rather than Ξ_0, is in some sense the limiting case of the horocycle space $\Xi \cong K/M \times A$. Thus it makes sense to define a *conical distribution* on $\widetilde{\Xi}_0$ to be a joint eigendistribution of $\mathbb{D}(\widetilde{\Xi}_0)$ invariant under the left action of the isotropy subgroup $H^0_{\mathfrak{q}} = M \times \mathfrak{q}$ of $\widetilde{\xi}_0 = (eM, 0)$. Assume that $\lambda \in \mathfrak{a}_c^*$ is regular. Then using Proposition 3.1, one can use an argument similar to that used to prove Theorem 4.3 above to conclude that the space of conical distributions in $\mathcal{D}'_\lambda(\widetilde{\Xi}_0)$ is w-dimensional. (We omit the details.)

Theorem 4.4. *If $\lambda \in \mathfrak{a}_c^*$ is regular, then the space of conical distributions in $\mathcal{D}'_\lambda(\widetilde{\Xi}_0)$ has dimension w. Any such conical distribution is given by*

$$\Psi(\psi) = \int_{K/M} \int_{\mathfrak{a}} \psi(kM, H) \, e^{i\lambda(H)} \, dH \, dS(kM) \qquad (\psi \in \mathcal{D}(\widetilde{\Xi}_0))$$

with $S = \sum_{s \in W} c_s \, \delta_{m_s M}$, for arbitrary scalars c_s.

The theorem above is thus a more precise analogue of Theorem 4.9 in [7], which states that, for generic $\lambda \in \mathfrak{a}_c^*$, the space of conical distributions in $\mathcal{D}'_\lambda(\Xi)$ is w-dimensional.

5. Conical Distributions When λ is Non-regular

Just as in the symmetric space case, the problem of characterizing the space of conical distributions in $\mathcal{D}'_\lambda(\Xi_0)$ appears to be rather difficult, in general, when $\lambda \in \mathfrak{a}_c^*$ is not regular. One can, however, show that the space of conical distributions corresponding to any non-regular λ is infinite-dimensional. To see this, let $K_\lambda = Z_K(\lambda) = \{k \in K \mid k \cdot \lambda = \lambda\}$ and let $K'_\lambda = \{k \in K \mid k \cdot \lambda \in \mathfrak{a}_c^*\}$. For any $k \in K'_\lambda$, there exists an element $m' \in M'$ such that $k \cdot \lambda = m' \cdot \lambda$. Thus $(m')^{-1}k \in K_\lambda$, and so we see that $K'_\lambda = M'K_\lambda = \cup_{s \in W} m_s K_\lambda = \cup_{s \in W} K_{s \cdot \lambda} m_s$.

Let $\Sigma^+_\lambda = \{\alpha \in \Sigma^+ \mid B(\alpha, \lambda) = 0\}$, and as before let W_λ be the subgroup of W fixing λ. Then W_λ is the subroup of W generated by the reflections along the root hyperplanes in Σ^+_λ, and $M' \cap K_\lambda = \cup_{s \in W_\lambda} m_s M$.

The Lie algebra of K_λ is $\mathfrak{k}_\lambda = \mathfrak{m} + \sum_\alpha \mathfrak{l}_\alpha$, where the sum is taken over all α in Σ^+_λ. If λ is not regular, then Σ^+_λ is nonempty, and therefore the orbit K_λ/M is a submanifold of K/M of positive dimension. The set $M'K_\lambda/M$ is a disjoint union of $|W|/|W_\lambda|$ translates of K_λ/M, given by $m_s\, K_\lambda/M$, where s ranges over a set of coset representatives in W/W_λ.

Let f be any continuous function on the orbit K_λ/M, invariant under left translation by elements of $m_s M$, for all $s \in W_\lambda$. Such f can be obtained by averaging any continuous function on the orbit by M and then further averaging by the m_s. The vector space of such f is infinite-dimensional, since close to the identity coset eM, the space of M-orbits in K_λ/M is parametrized by the space of M-orbits on a ball centered at 0 in $\sum_{\alpha \in \Sigma^+_\lambda} \mathfrak{l}_\alpha$.

If $s \in W$, we can extend f in a well-defined way to the translated orbit $m_s K_\lambda/M$ by setting $f(m_s kM) = f(kM)$, for all $k \in K_\lambda$. In this way, f becomes an M'-invariant function defined on the union of the translated orbits $m_s K_\lambda/M$, for all $s \in W$.

Now let us define the distribution T_f on K/M by

$$T_f(F) = \sum_s \int_{K_\lambda/M} f(m_s k_\lambda M)\, F(m_s k_\lambda M)\, d(k_\lambda)_M, \qquad (F \in \mathcal{E}(K/M)) \tag{51}$$

where the sum is taken over a set of representatives s of W/W_λ. It is clear from the construction of f that T_f is independent of the choice of the m_s appearing on the right hand side above. T_f is then an M'-invariant distribution on K/M.

Now, in accordance with Theorem 3.2, let us define the distribution Φ_f in $\mathcal{D}'_\lambda(\Xi_0)$ by

$$\Phi_f(\varphi) = \int_{K/M} \int_{\mathfrak{a}} \widetilde{\varphi}(kM, H)\, e^{i\lambda(H)}\, dH\, dT_f(kM) \tag{52}$$

Since T_f is M'-invariant, so is Φ_f. To show that Φ_f is $\mathfrak{q}$-invariant, we use the expression (51) defining T_f:

$$\Phi_f(\varphi) = \sum_s \int_{K_\lambda/M} \int_{\mathfrak{a}} \widetilde{\varphi}(m_s k_\lambda M, H)\, e^{i\lambda(H)}\, dH\, f(m_s k_\lambda M)\, d(k_\lambda)_M$$

Let $X \in \mathfrak{q}$. Then by (9) we have

$$\Phi_f(\varphi^{\tau(X)}) = \\ = \sum_s \int_{K_\lambda/M} \int_{\mathfrak{a}} \widetilde{\varphi}(m_s k_\lambda M, H)\, e^{i\lambda(H)}\, e^{iB(m_s k_\lambda \cdot \lambda, X)}\, dH\, f(m_s k_\lambda M)\, d(k_\lambda)_M$$

But $m_s k_\lambda \cdot \lambda \in \mathfrak{a}^*_c$, and thus $B(m_s k_\lambda \cdot \lambda, X) = 0$, which shows that the right hand side above equals $\Phi_f(\varphi)$.

Since, as remarked above, the space of all continuous functions f on K_λ/M invariant under the left action of $m_s M$, for all $s \in W_\lambda$, is infinite-dimensional, it follows that the space of conical distributions in $\mathcal{D}'_\lambda(\Xi_0)$ has infinite dimension.

In the case when the symmetric space $X = G/K$ has rank one; i.e., when $\dim \mathfrak{a} = 1$, it is possible to obtain a complete classification of the space of all conical distributions in $\mathcal{D}'_0(\Xi_0)$. In this case, Σ^+ has one or two elements; let α be the indivisible element. Choose $H \in \mathfrak{a}$ such that $\alpha(H) = 1$, and identify $\mathbb{R}$ with $\mathfrak{a}$ by $t \mapsto tH$.

Since we are assuming that $\lambda = 0$, then $W_\lambda = W = \{\pm 1\}$, and so the space H of W_λ-harmonic polynomials on $\mathfrak{a}$ has basis $\{1, t\}$. Suppose that $\Phi \in \mathcal{D}'_0(\Xi_0)$ is a conical distribution. Then from (39), there exist uniquely determined M'-invariant distributions T_0 and T_1 on K/M such that

$$\Phi(\varphi) = \int_{K/M} \int_{-\infty}^{\infty} \widetilde{\varphi}(kM, tH)\, dt\, dT_0(kM) + \int_{K/M} \int_{-\infty}^{\infty} \widetilde{\varphi}(kM, tH)\, t\, dt\, dT_1(kM) \tag{53}$$

Since Φ is also invariant under left translation by any $X \in \mathfrak{q}$, we have

$$\begin{aligned}
\Phi(\varphi) &= \int_{K/M} \int_{-\infty}^{\infty} \widetilde{\varphi}(kM, tH + (k^{-1}\cdot X)_{\mathfrak{a}})\, dt\, dT_0(kM) \\
&\qquad + \int_{K/M} \int_{-\infty}^{\infty} \widetilde{\varphi}(kM, tH + (k^{-1}\cdot X)_{\mathfrak{a}})\, t\, dt\, dT_1(kM) \\
&= \int_{K/M} \int_{-\infty}^{\infty} \widetilde{\varphi}(kM, tH)\, dt\, dT_0(kM) + \int_{K/M} \int_{-\infty}^{\infty} \widetilde{\varphi}(kM, tH)\, t\, dt\, dT_1(kM) \\
&\qquad - \int_{K/M} \int_{-\infty}^{\infty} \widetilde{\varphi}(kM, tH)\, B(k\cdot A_\alpha, X) dt\, dT_1(kM) \\
&= \Phi(\varphi) - \int_{K/M} \int_{-\infty}^{\infty} \widetilde{\varphi}(kM, tH)\, dt\;\; B(k\cdot A_\alpha, X)\, dT_1(kM)
\end{aligned}$$

Hence

$$\int_{K/M} \int_{-\infty}^{\infty} \widetilde{\varphi}(kM, tH)\, dt\;\; B(k\cdot A_\alpha, X)\, dT_1(kM) = 0 \tag{54}$$

for all $\varphi \in \mathcal{D}(\Xi_0)$.

If T_0 and T_1 are M'-invariant distributions on K/M it is clear that the condition (54) is also sufficient for the distribution Φ in (53) to be conical in $\mathcal{D}'_0(\Xi_0)$. In particular, T_0 can be arbitrary.

Now it is easy to see that the map $\varphi \mapsto \int_{-\infty}^{\infty} \widetilde{\varphi}(kM, tH)\, dt$ maps $\mathcal{D}(\Xi_0)$ onto the vector space $\mathcal{E}_{M'}(K/M)$ of C^∞ functions F on K/M satisfying $F(kM) = F(km^*M)$ for all $k \in K$, where $m*$ is any element in $M' \setminus M$. Thus (54) implies that Φ is conical if and only if the M'-invariant distribution T_1 satisfies the condition

$$\int_{K/M} F(kM)\, B(k\cdot A_\alpha, X)\, dT_1(kM) = 0 \tag{55}$$

for any $X \in \mathfrak{q}$ and all $F \in \mathcal{E}_{M'}(K/M)$. As we shall show below, it turns out that *all* M'-invariant distributions on K/M satisfy the condition above.

In the present case the set Σ^+ consists of α, and possibly 2α, with multiplicities m_α and $m_{2\alpha}$, respectively. Let H_1 be the unit vector in $\mathfrak{a}$ such that $\alpha(H_1) > 0$, and let o denote the identity coset $\{M\}$ in K/M. Then we can endow K/M with the K-invariant Riemannian structure induced from the $\operatorname{Ad} M$-invariant inner product on $\mathfrak{l} \cong T_o(K/M)$ given by

$$\langle T_\alpha + T_{2\alpha}, T'_\alpha + T'_{2\alpha} \rangle = -\alpha(H_1)^2\, B(T_\alpha, T'_\alpha) - 4\alpha(H_1)^2\, B(T_{2\alpha}, T'_{2\alpha})$$

for $T_\alpha, T'_\alpha \in \mathfrak{l}_\alpha$ and $T_{2\alpha}, T'_{2\alpha} \in \mathfrak{l}_{2\alpha}$. One can easily show that the mapping $kM \mapsto k \cdot H_1$ is an isometry from K/M onto the unit sphere S (with respect to B) in $\mathfrak{p}$. Whenever it is convenient, we will identify K/M with S in this manner.

Lemma 5.1. *Assume that* $\dim \mathfrak{a} = 1$. *Fix* $m^* \in M'$. *Then for every* $kM \in K/M$, *there exists an* $m \in M$ *such that* $m^* k (m^*)^{-1} M = mkM$.

Proof. If $m^* \in M$, then the result is trivial, so let us assume that m^* is not in M. It is easy to see that the map $kM \mapsto m^* k (m^*)^{-1} M$ is a well-defined isometry of K/M. By Theorem 13.2 in [17], the map $T \mapsto (\exp T)\, M$ maps $\mathfrak{l}$ onto K/M, and clearly $m^* (\exp T)(m^*)^{-1} M = \exp(\operatorname{Ad}(m^*)\, T) M$. Thus it suffices to prove that for each $T \in \mathfrak{l}$, there exists $m \in M$ such that $\operatorname{Ad}(m^*)\, T = \operatorname{Ad}(m)\, T$.

This assertion can be proved by considering the possible cases for m_α and $m_{2\alpha}$. For convenience, let us now provide $\mathfrak{l}$ with the inner product given by $-B$, which we note that $\operatorname{Ad}(m^*)$ leaves invariant. Suppose first that $m_{2\alpha} > 1$. Write $T \in \mathfrak{l}$ as $T = T_\alpha + T_{2\alpha}$, with $T_\alpha \in \mathfrak{l}_\alpha$, $T_{2\alpha} \in \mathfrak{l}_{2\alpha}$. For any $r, s \geq 0$, $\operatorname{Ad} M$ is transitive on the product of spheres $\{T' + T'' \in \mathfrak{l}_\alpha + \mathfrak{l}_{2\alpha} \mid \|T'\| = r,\ \|T''\| = s\}$ ([15]). Since $\operatorname{Ad}(m^*)$ is an isometry on $\mathfrak{l}_\alpha$ and on $\mathfrak{l}_{2\alpha}$, there exists $m \in M$ such that $\operatorname{Ad}(m) T_\alpha = \operatorname{Ad}(m^*) T_\alpha$ and $\operatorname{Ad}(m) T_\alpha = \operatorname{Ad}(m^*) T_{2\alpha}$.

Suppose next that $m_{2\alpha} = 0$ and $m_\alpha > 1$. Then $\mathfrak{l} = \mathfrak{l}_\alpha$, and since $\operatorname{Ad} M$ is transitive on spheres in $\mathfrak{l}_\alpha$, our assertion easily holds in this case.

The remaining cases are $m_{2\alpha} = 1$ (so $m_\alpha > 1$) and $m_\alpha = 1$ (so $m_{2\alpha} = 0$). Suppose that $m_{2\alpha} = 1$. We claim that $\operatorname{Ad} m^*$ is the identity map on $\mathfrak{l}_{2\alpha}$. For this, we use the decomposition $\mathfrak{g} = \mathfrak{g}_{-2\alpha} + \mathfrak{g}_{-\alpha} + \mathfrak{m} + \mathfrak{a} + \mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$. Choose any nonzero elements $X_\alpha \in \mathfrak{g}_\alpha$ and $X_{2\alpha} \in \mathfrak{g}_{2\alpha}$. Then X_α, $\theta(X_\alpha)$, $X_{2\alpha}$, and $\theta(X_{2\alpha})$ generate a Lie subalgebra $\mathfrak{g}^*$ of $\mathfrak{g}$ isomorphic to $\mathrm{su}(2,1)$. Let G^* be the analytic subgroup of G with this algebra. Then G^* has Iwasawa decomposition $G^* = K^* A N^*$, where $K^* = G^* \cap K$, $N^* = G^* \cap N$. If M^* and $(M')^*$ denote the centralizer and normalizer of $\mathfrak{a}$ in K^*, we also have $M^* = G^* \cap M$ and $(M')^* = G^* \cap M'$. Choose any element $m_1^* \in (M')^* \setminus M^*$. Then $m_1^* \in M' \setminus M$ so there exists an $m_1 \in M$ such that $m_1^* = m^* m_1$. Now from [10], Chapter IX, §3, $\operatorname{Ad}(m_1^*)\, X_{2\alpha} = \theta(X_{2\alpha})$, $\operatorname{Ad}(m_1^*)\, \theta(X_{2\alpha}) = X_{2\alpha}$, and thus $\operatorname{Ad} m_1^*$ fixes $X_{2\alpha} + \theta(X_{2\alpha})$. But this latter vector spans $\mathfrak{l}_{2\alpha}$. Since $\operatorname{Ad} M$ is the identity map on $\mathfrak{l}_{\pm 2\alpha}$ ([12], Chapter III, Lemma 3.8) it follows that $\operatorname{Ad} m^* = \operatorname{Ad}(m_1^*\, m_1^{-1})$ is the identity map on $\mathfrak{l}_{\pm 2\alpha}$ as well. Now since $\operatorname{Ad} M$ is transitive on spheres in $\mathfrak{l}_\alpha$, we conclude that for any $T \in \mathfrak{l}_\alpha$ and $T' \in \mathfrak{l}_{2\alpha}$, there exists an $m \in M$ such that $\operatorname{Ad}(m^*)\, T = \operatorname{Ad}(m)\, T$ and $\operatorname{Ad}(m^*)\, T' = \operatorname{Ad}(m)\, T' = T'$.

Finally, suppose that $m_\alpha = 1$. Choose any nonzero $X_\alpha \in \mathfrak{g}_\alpha$. Then X_α and $\theta(X_\alpha)$ generate a subalgebra $\mathfrak{g}^*$ of $\mathfrak{g}$ isomorphic to $\mathrm{su}(1,1)$. Let G^* be the analytic subgroup of G with Lie algebra $\mathfrak{g}^*$, and let $(m')^* \in G^* \cap (M' \setminus M)$. There exists an $m_1 \in M$ such that $(m')^* = m^* m_1$. If $K^* = G^* \cap K$, then K^* is abelian, so $\mathrm{Ad}(m')^*$ is the identity map on $\mathfrak{l}_\alpha$. On the other hand, by [12], Chapter III, Lemma 3.8, $\mathrm{Ad}\,(M)$ is also the identity map on $\mathfrak{l}_\alpha$. Thus $\mathrm{Ad}\,(m^*)\,T = \mathrm{Ad}\,(m)\,T = T$ for all $m \in M$ and $T \in \mathfrak{l}_\alpha$.

This covers all the cases and finishes the proof of the lemma. ∎

We are now in a position to classify the conical distributions in $\mathcal{D}_0'(\Xi_0)$ when $\dim \mathfrak{a} = 1$.

Theorem 5.2. *Assume that* $\dim \mathfrak{a} = 1$. *Then the conical distributions in* $\mathcal{D}_0'(\Xi_0)$ *are precisely those distributions* Φ *given by*

$$\Phi(\varphi) = \int_{K/M} \int_{-\infty}^{\infty} \widetilde{\varphi}(kM, tH)\, dt\, dT_0(kM) + \int_{K/M} \int_{-\infty}^{\infty} \widetilde{\varphi}(kM, tH)\, t\, dt\, dT_1(kM) \tag{56}$$

where T_0 *and* T_1 *are* M'*-invariant distributions on* K/M.

Proof. As remarked in (53), any conical distribution Φ in $\mathcal{D}_0'(\Xi_0)$ must be of the form (56), with T_0 and T_1 M'-invariant.

Conversely, suppose that $\Phi \in \mathcal{D}'(\Xi_0)$ is defined by (56) with T_0 and T_1 M'-invariant. By Theorem 3.2, Φ belongs to $\mathcal{D}_0'(\Xi_0)$, and it is clear that Φ is M'-invariant. To prove that Φ is conical, it is sufficient to verify that T_1 satisfies (55) for all $F \in \mathcal{E}(K/M)$ such that $F(kM) = F(km^*M)$.

To this end, let us put $F^\#(kM) = \int_{M'} F(m'kM)\, dm'$ for any function $F \in \mathcal{E}(K/M)$, where dm' is the normalized Haar measure on the compact group M'. Note that since T_1 is M'-invariant, $T_1(F) = T_1(F^\#)$ for all $F \in \mathcal{E}(K/M)$.

Lemma 5.1 shows that for any $kM \in K/M$, there exists $m_1 \in M$ such that $m^*k \cdot H_1 = -m^*k(m^*)^{-1} \cdot H_1 = -m_1 k \cdot H_1$. If $a : \omega \mapsto -\omega$ denotes the antipodal map on the sphere S, then $a(k \cdot H_1) = k(m^*)^{-1} \cdot H_1$, so a corresponds to the isometry $kM \mapsto k(m^*)^{-1}M$ of K/M.

Noting that $F^a(kM) = F(k(m^*)^{-1}M)$, we see from the definition of $F^\#$ that that $(F^\#)^a = (F^a)^\#$. On the other hand, for any $kM \in K/M$, we put $k \cdot H_1 = \omega$. Applying Lemma 5.1, we have

$$\begin{aligned}
(F^a)^\#(\omega) &= \int_{M'} F(m'k(m^*)^{-1}M)\, dm' \\
&= \int_{M'} F(m'm^*k(m^*)^{-1}M)\, dm' \\
&= \int_{M'} F(m'm_1kM)\, dm' \qquad \text{(for some } m_1 \in M) \\
&= F^\#(\omega).
\end{aligned}$$

In particular, if $F \in \mathcal{E}(K/M)$ corresponds to an odd function on S, we have $F^\# = 0$.

Now suppose that $F \in \mathcal{E}(K/M)$ satisfies $F(km^*M) = F(kM)$ for all $k \in K$. Then F corresponds to an even function on S, and for each fixed $X \in \mathfrak{q}$, the function $G(kM) = F(kM)\,B(k \cdot A_\alpha, X)$ is an odd function on S. It follows that $G^{\#} = 0$ and thus

$$\begin{aligned}\int_{K/M} F(kM)\,B(k \cdot A_\alpha, X)\,dT_1(kM) &= T_1(G)\\ &= T_1(G^{\#})\\ &= 0.\end{aligned}$$

Thus (55) holds for all such F, and we conclude that the distribution Φ in (56) is conical. ■

It is curious that the M'-invariance of any distribution in $\mathcal{D}_0'(\Xi_0)$ guarantees its $\mathfrak{q}$-invariance.

6. Eigenspace Representations

We conclude this paper by considering the natural representation of G_0 on the eigenspaces $\mathcal{D}_\lambda'(\Xi_0)$. In particular, we would like to determine the conditions under which this representation is irreducible.

Let R denote the flat horocycle Radon transform given by (8). Then R is the Radon transform associated with the double fibration

$$\begin{array}{ccccc} & & G_0/(K \cap H_{\mathfrak{q}}) & & \\ & \swarrow^{p} & & \searrow^{\pi} & \\ \mathfrak{p} = G_0/K & & & & \Xi_0 = G_0/H_{\mathfrak{q}} \end{array} \tag{57}$$

(See [6]; for a general introduction to integral transforms associated with group equivariant double fibrations, see [12], Chapter I, §1-3; for details on the flat horocycle Radon transform, see [8] or [12], Chapter IV, §5.) If $f \in C_c(\mathfrak{p})$, then Rf is given by

$$Rf[kM, H] = \int_{\mathfrak{q}} f(k \cdot (H + Y))\,dY \tag{58}$$

where dY indicates the Euclidean measure on $\mathfrak{q}$. Its *dual transform* is the map $R^* : C(\Xi_0) \to C(\mathfrak{p})$ given by

$$R^*\varphi(X) = \int_K \varphi(X + k \cdot \mathfrak{q})\,dk \qquad (\varphi \in C(\Xi_0)) \tag{59}$$

where dk denotes the normalized Haar measure on the compact group K.

The transforms R and R^* are G_0-equivariant in the sense that $R(f \circ l(g)) = (Rf) \circ l(g)$ and $R^*(\varphi \circ l(g)) = (R^*\varphi) \circ l(g)$, where $l(g)$ is the natural left action by $g \in G_0$ on the homogeneous spaces $\mathfrak{p}$ and Ξ_0. They are also formal adjoints in the sense that

$$\int_{\Xi_0} Rf(\xi)\,\varphi(\xi)\,d\xi = \int_{\mathfrak{p}} f(X)\,R^*\varphi(x)\,dx \tag{60}$$

for all $f \in C_c(\mathfrak{p})$, $\varphi \in C(\Xi_0)$, where, as in Section 2, $d\xi$ denotes the G_0-invariant measure on Ξ_0 which pulls back to $dH\,dk_M$ on $\widetilde{\Xi}_0$.

We equip $\mathcal{D}(\mathfrak{p})$ and $\mathcal{D}(\Xi_0)$ with the usual inductive limit topologies, and their dual spaces $\mathcal{D}'(\mathfrak{p})$ and $\mathcal{D}'(\Xi_0)$ by the corresponding strong topologies. According to Lemma 3.5, Chapter I in [12], R is a continuous linear map from $\mathcal{D}(\mathfrak{p})$ to $\mathcal{D}(\Xi_0)$. Thus we can extend the relation (60) by defining the dual transform $R^*\Phi$ of any $\Phi \in \mathcal{D}'(\Xi_0)$ by

$$R^*\Phi(f) = \Phi(Rf) \qquad (f \in \mathcal{D}(\mathfrak{p}))$$

The dual transform $\Phi \mapsto R^*\Phi$ is then a continuous linear map from $\mathcal{D}'(\Xi_0)$ to $\mathcal{D}'(\mathfrak{p})$. This map commutes with the natural left action of G_0 on distributions on Ξ_0 and $\mathfrak{p}$, respectively.

Lemma 6.1. *Let $\lambda \in \mathfrak{a}_c^*$, and suppose that Φ is a distribution in $\mathcal{D}'_\lambda(\Xi_0)$ given by*

$$\Phi(\varphi) = \int_{K/M} \int_{\mathfrak{a}} \varphi[kM, H]\, e^{i\lambda(H)}\, dH\, dT(kM) \qquad (\varphi \in \mathcal{D}(\Xi_0)) \tag{61}$$

where $T \in \mathcal{D}'(K/M)$. Then $R^*\Phi \in \mathcal{E}(\mathfrak{p})$, and is given by

$$R^*\Phi(X) = \int_{K/M} e^{i\,B(k\cdot A_\lambda, X)}\, dT(kM) \qquad (X \in \mathfrak{p}) \tag{62}$$

Proof. If $f \in \mathcal{D}(\mathfrak{p})$, then

$$\begin{aligned}
(R^*\Phi)\,(f) &= \Phi\,(Rf) \\
&= \int_{K/M} \int_{\mathfrak{a}} Rf[kM, H]\, e^{i\lambda(H)} dH\, dT(kM) \\
&= \int_{K/M} \int_{\mathfrak{a}} \int_{\mathfrak{q}} f(k\cdot(H+Y))\, dY\, e^{i\,B(A_\lambda, H)}\, dH, dT(kM) \\
&= \int_{K/M} \int_{\mathfrak{p}} f(k\cdot X)\, e^{i\,B(A_\lambda . X)}\, dX\, dT(kM) \\
&= \int_{\mathfrak{p}} f(X) \int_{K/M} e^{i\,B(k\cdot A_\lambda, X)}\, dT(kM)\, dX
\end{aligned}$$

The inner integral on the right is clearly a smooth function of $X \in \mathfrak{p}$. Since it agrees with $R^*\Phi$ on all test functions f on $\mathfrak{p}$, we obtain the lemma. ∎

The right hand side of (62) is the flat analogue of the Poisson transform on the symmetric space G/K. We therefore call it the *Poisson transform* of T corresponding to the spectral parameter λ, and denote it by $\mathcal{P}_\lambda(T)$:

$$P_\lambda(T)(X) = \int_{K/M} e^{i\,B(k\cdot A_\lambda, X)}\, dT(kM) \qquad (X \in \mathfrak{p}) \tag{63}$$

The transform P_λ was introduced in [8] in connection with the study of eigenspace representations on $\mathfrak{p}$. Note that the Poisson transform of the function $F_0 \equiv 1$ is the zonal spherical function on $\mathfrak{p}$ corresponding to λ.

Lemma 6.1, which relates the dual horocycle and the Poisson transforms, is the analogue of a similar formula ([7], Proposition 4.6) for the dual horocycle transform on G/K. If λ is regular, then according to Corollary 3.3 above, every $\Phi \in \mathcal{D}'_\lambda(\Xi_0)$ has the form (61). However, the analogy to G/K is not completely precise, since when λ is not regular, there are $\Phi \in \mathcal{D}'_\lambda(\Xi_0)$ which are not of the form (61), so the dual transform $R^*\Phi$ is not necessarily a Poisson transform.

Recall that we have identified the algebra of left G_0-invariant differential operators on $\mathfrak{p}$ with the polynomial algebra $I(\mathfrak{p})$, and that the algebra homomorphisms $\alpha : I(\mathfrak{p}) \to \mathbb{C}$ are given by evaluations $p \mapsto p(\lambda)$ for $\lambda \in \mathfrak{a}_c^*$, where λ is unique up to W orbit. For $\lambda \in \mathfrak{a}_c^*$, let

$$\mathcal{E}_\lambda(\mathfrak{p}) = \{f \in \mathcal{E}(\mathfrak{p}) \,|\, \partial(p)\, f = p(i\lambda)\, f \text{ for all } p \in I(\mathfrak{p})\}.$$

Since the function $X \mapsto e^{i\,B(k\cdot A_\lambda, X)}$ belongs to $\mathcal{E}_\lambda(\mathfrak{p})$ for each $k \in K$, it is easy to see from (63) that the Poisson transform $P_\lambda(T)$ belongs to the joint eigenspace $\mathcal{E}_\lambda(X)$ for all $T \in \mathcal{D}'(K/M)$. The joint eigenspace $\mathcal{E}_\lambda(\mathfrak{p})$ is G_0-invariant, and according to [8], Theorem 6.6, the natural representation of G_0 on $\mathcal{E}_\lambda(\mathfrak{p})$ is irreducible if and only if λ is regular.

We say that $\lambda \in \mathfrak{a}_c^*$ is *simple* if the Poisson transform P_λ is injective. An easy convolution argument on K shows that λ is simple if and only if the map $F \mapsto P_\lambda(F)$ is injective on $\mathcal{E}(K/M)$. Now according to Theorem 6.2 in [8], λ is simple if and only if it is regular. Thus, in view of Corollary 3.3, the dual transform $R^* : \mathcal{D}'_\lambda(\Xi_0) \to \mathcal{D}'(\mathfrak{p})$ is injective if and only if λ is regular.

We now fix some notation. For any $g \in G_0$ and $\varphi \in \mathcal{D}(\Xi_0)$, let $\varphi^{l(g)} = \varphi \circ l(g^{-1})$. If $\Phi \in \mathcal{D}'(\Xi_0)$, we let $\Phi^{l(g)}$ be the distribution on Ξ given by $\Phi^{l(g)}(\varphi) = \Phi(\varphi^{l(g^{-1})})$.

The left regular representation of G_0 on $\mathcal{D}(\Xi_0)$ is then given by $\tau(g)\varphi = \varphi^{l(g)}$ for $g \in G_0$, $\varphi \in \mathcal{D}(\Xi_0)$, and the natural representation of G_0 on $\mathcal{D}'(\Xi_0)$ is the contragredient representation, which is given by $\pi(g)\Phi = \Phi^{l(g)}$.

If $\lambda \in \mathfrak{a}_c^*$, then the joint eigenspace $\mathcal{D}'_\lambda(\Xi_0)$ is a closed subspace of $\mathcal{D}'(\Xi_0)$ invariant under π; we let π_λ denote the restriction of π to $\mathcal{D}'_\lambda(\Xi_0)$.

Proposition 6.2. *Suppose that $\lambda \in \mathfrak{a}_c^*$ is regular. Then the conical distribution Φ_λ in* (50) *is a cyclic vector for the representation π_λ.*

Proof. We need to prove that the linear span of the translates $\Phi_\lambda^{l(g)}$, for all $g \in G_0$, is dense in $\mathcal{D}'_\lambda(\Xi_0)$. Suppose that L belongs to the dual space of $\mathcal{D}'_\lambda(\Xi_0)$. Since $\mathcal{D}'_\lambda(\Xi_0)$ is a closed subspace of $\mathcal{D}'(\Xi_0)$, we may extend L to a continuous linear functional on $\mathcal{D}'(\Xi_0)$; then because $\mathcal{D}(\Xi_0)$ is reflexive, there exists a $\varphi \in \mathcal{D}(\Xi_0)$ for which $L(\Phi) = \Phi(\varphi)$ for all $\Phi \in \mathcal{D}'(\Xi_0)$.

Now suppose that $L\big(\Phi_\lambda^{l(g)}\big) = 0$ for all $g \in G_0$. Then by (50),

$$\begin{aligned} L\big(\Phi_\lambda^{l(g)}\big) &= \Phi^{l(g)}(\varphi) \\ &= \Phi\big(\varphi^{l(g^{-1})}\big) \\ &= \sum_{s\in W} \big(\varphi^{l(g^{-1})}\big)^*[m_s M, \lambda] \end{aligned} \tag{64}$$

Put $g = (k, X)$ for some $k \in K$, $X \in \mathfrak{p}$. Then by (9), $l(g)([k_0M, H_0]) = [kk_0M, H_0 + ((kk_0)^{-1} \cdot X)_{\mathfrak{a}}]$. Hence for each $s \in W$,

$$\big(\varphi^{l(g^{-1})}\big)^*[m_sM, \lambda] = \int_{\mathfrak{a}} \varphi[km_sM, H + ((km_s)^{-1} \cdot X)_{\mathfrak{a}}]\, e^{i\lambda(H)}\, dH$$
$$= e^{-i\,B(km_s \cdot A_\lambda, X)}\, \varphi^*[km_sM, \lambda] \tag{65}$$

Since g is arbitrary, we can let $g = (k, k \cdot H')$ for $k \in K$, $H' \in \mathfrak{a}$. Then (64) becomes

$$\sum_{s \in W} e^{-i\,s\lambda(H')}\, \varphi^*[km_sM, \lambda] = 0 \tag{66}$$

Since λ is regular, the functions $e^{-is\lambda}$ $(s \in W)$ are linearly independent, and thus there exist vectors $H_t \in \mathfrak{a}$, for $t \in W$, such that the $w \times w$ matrix $(e^{-is\lambda(H_t)})$ is nonsingular. Replacing H' in (66) by each H_t, the relations

$$\sum_{s \in W} e^{-i\,s\lambda(H_t)}\, \varphi^*[km_sM, \lambda] = 0 \qquad (t \in W)$$

show in particular that $\varphi^*[kM, \lambda] = 0$ for all $k \in K$. From Corollary 3.3, we conclude that $\Phi(\varphi) = 0$ for all $\Phi \in \mathcal{D}'_\lambda(\Xi_0)$. Hence $L = 0$ on $\mathcal{D}'_\lambda(\Xi_0)$, proving the proposition. ■

For $\lambda \in \mathfrak{a}_c^*$, let $\mathcal{K}_\lambda(\Xi_0)$ denote the vector space of all distributions $\Phi \in \mathcal{D}'_\lambda(\Xi_0)$ given by

$$\Phi(\varphi) = \int_{K/M} \varphi^*[kM, \lambda]\, F(kM)\, dk_M \qquad (\varphi \in \mathcal{D}(K/M)) \tag{67}$$

where $F \in L^2(K/M)$. Note that according to Theorem 3.2, the map $F \mapsto \Phi$ is injective from $L^2(K/M)$ to $\mathcal{K}_\lambda(\Xi_0)$. Thus we may endow $\mathcal{K}_\lambda(\Xi_0)$ with a Hilbert space structure, the norm $\|\Phi\|_\lambda$ of Φ above being the L^2 norm of F on K/M. Now a calculation similar to (65) shows that if $g = (k', X')$, then $(\varphi^{l(g^{-1})})^*[kM, \lambda] = e^{-i\,B(k'k \cdot A_\lambda, X')}\, \varphi^*[k'kM, \lambda]$ for any $\varphi \in \mathcal{D}(\Xi_0)$. Thus if Φ is given by (67), we have

$$\big(\pi_\lambda(g)\Phi\big)(\varphi) = \int_{K/M} \varphi^*[k'kM, \lambda]\, e^{-i\,B(k'k \cdot A_\lambda, X')}\, F(kM)\, dk_M$$
$$= \int_{K/M} \varphi^*[kM, \lambda]\, e^{-i\,B(k \cdot A_\lambda, X')}\, F((k')^{-1}kM)\, dk_M \tag{68}$$

From this one sees that $\mathcal{K}_\lambda(\Xi_0)$ is invariant under π_λ. Let π'_λ denote the restriction of π_λ to $\mathcal{K}_\lambda(\Xi_0)$. Equation (68) shows that we may identify π'_λ with the representation of G_0 on $L^2(K/M)$ given by

$$\pi'_\lambda(k', X')F(kM) = e^{-i\,B(k \cdot A_\lambda, X')}\, F((k')^{-1}kM) \tag{69}$$

Thus π'_λ is the representation of G_0 induced from the one-dimensional representation $(m, X) \mapsto e^{-i\,B(A_\lambda, X)}$ of the subgroup $M \ltimes \mathfrak{p} \subset G_0$. The representation π'_λ is unitary if and only if $\lambda \in \mathfrak{a}^*$. If $\lambda \in \mathfrak{a}^*$ is regular, then Mackey's imprimitivity theorem applied to semidirect products says that π'_λ is unitary and irreducible. (See e.g., [16], Chapter III.)

Since the dual transform $\Phi \mapsto R^*\Phi$ commutes with the left action of G_0, Lemma 6.1 shows the Poisson transform P_λ intertwines π'_λ and the left regular representation of G_0 on $\mathcal{E}_\lambda(\mathfrak{p})$.

Now $\mathcal{K}_\lambda(\Xi_0)$ contains a unique K-invariant element (i.e., the distribution Φ in (67) with $F \equiv 1$), so the family π'_λ (for $\lambda \in \mathfrak{a}_c^*$) is the flat analogue of the spherical principal series for the symmetric space G/K.

Theorem 6.3. *Let $\lambda \in \mathfrak{a}_c^*$. Then π'_λ is irreducible if and only if λ is regular.*

Proof. Theorem 6.3 is the flat analogue of Proposition 5.3 in [7], and our proof is adapted from the proof of that result.

We first make the following observation. Let F_0 be the constant function $F_0(kM) \equiv 1$ on K/M. If $\langle\ ,\ \rangle$ denotes the inner product on $L^2(K/M)$, then equation (69) implies that for any $F \in L^2(K/M)$,

$$\langle \pi'_\lambda(g)F_0, F\rangle = \int_{K/M} e^{-i\,B(k\cdot A_\lambda, X)}\,\overline{F(kM)}\,dk_M$$

where $g = (k, X) \in G_0$. Thus F_0 is a cyclic vector for π'_λ if and only if $-\lambda$ is simple. But $-\lambda$ is simple if and only if $-\lambda$ is regular, so F_0 is cyclic if and only if λ is regular.

Now suppose that λ is not regular. Let $\mathcal{N}$ denote the kernel of the Poisson transform P_λ on $L^2(K/M)$. The Schwartz inequality shows that $\mathcal{N}$ is a closed subspace of $L^2(K/M)$. Since P_λ is G_0-equivariant, $\mathcal{N}$ is invariant under π'_λ. Finally, since λ is not simple, $\mathcal{N} \neq \{0\}$, and since $P_\lambda(F_0)(0) = 1$, we have $\mathcal{N} \neq L^2(K/M)$. This shows that π'_λ is not irreducible.

Conversely, suppose that λ is regular. Let V be a nonzero closed π'_λ-invariant subspace of $L^2(K/M)$. Since λ is simple, $P_\lambda(V) \neq \{0\}$, and by the G_0 equivariance, there is an $h \in P_\lambda(V)$ such that $h(0) = 1$. Letting $h = P_\lambda(F)$, where $F \in V$, we obtain $\int_{K/M} F(kM)\,dk_M = 1$. Now $F_0(kM) = \int_K F((k')^{-1}kM)\,dk' = \int_K \pi'_\lambda(k')\,F(k)\,dk'$ for all $kM \in K/M$, so it follows that $F_0 \in V$. But since λ is regular, F_0 is a cyclic vector for π'_λ, so we conclude that $V = L^2(K/M)$. Hence π'_λ is irreducible. ■

Theorem 6.3 now allows to determine the irreducibility of the representation π_λ of G_0 on the eigenspace $\mathcal{D}'_\lambda(\Xi_0)$.

Theorem 6.4. *Let $\lambda \in \mathfrak{a}_c^*$. Then π_λ is irreducible if and only if λ is regular.*

Proof. Suppose first that λ is not regular. Then λ is not simple, so by Lemma 6.1, the dual transform $R^* : \mathcal{D}'_\lambda(\Xi_0) \to \mathcal{E}(\mathfrak{p})$ is not injective. By the G_0-equivariance and continuity of R^*, its kernel $\mathcal{R}$ is thus a nonzero closed invariant subspace of $\mathcal{D}'_\lambda(\Xi_0)$. Moreover by Lemma 6.1, $\mathcal{R} \neq \mathcal{D}'_\lambda(\Xi_0)$, since $P_\lambda(F_0) \neq 0$. This shows that π_λ is not irreducible.

Next assume that λ is regular. Now Corollary 3.3 asserts that we have a bijective linear map $P : T \mapsto \Phi$ from $\mathcal{D}'(K/M)$ onto $\mathcal{D}'_\lambda(\Xi_0)$ given by

$$\Phi(\varphi) = \int_{K/M} \varphi^*[kM, \lambda]\, dT(kM) \qquad (\varphi \in \mathcal{D}(\Xi_0))$$

This map is continuous since it is the adjoint of the continuous map of $\mathcal{D}(\Xi_0)$ onto $\mathcal{D}(K/M)$ given by

$$\varphi \mapsto \varphi^*[\ \cdot\ , \lambda]$$

In particular, this implies that the inclusion map of the Hilbert space $\mathcal{K}_\lambda(\Xi_0)$ into $\mathcal{D}'_\lambda(\Xi_0)$ is continuous.

Now suppose that E is a closed subspace of $\mathcal{D}'_\lambda(\Xi_0)$ invariant under π_λ. Then $E \cap \mathcal{K}_\lambda(\Xi_0)$ is a closed π'_λ invariant subspace of $\mathcal{K}_\lambda(\Xi_0)$. Since π'_λ is irreducible, we must have $E \cap \mathcal{K}_\lambda(\Xi_0) = \mathcal{K}_\lambda(\Xi_0)$ or $E \cap \mathcal{K}_\lambda(\Xi_0) = \{0\}$.

Let us first consider the case $E \cap \mathcal{K}_\lambda(\Xi_0) = \mathcal{K}_\lambda(\Xi_0)$. Then $E \supset \overline{\mathcal{K}_\lambda(\Xi_0)}$, the closure of $\mathcal{K}_\lambda(\Xi_0)$ in $\mathcal{D}'_\lambda(\Xi_0)$. But since P is continuous and surjective, and since $L^2(K/M)$ is dense in $\mathcal{D}'(K/M)$, $\mathcal{K}_\lambda(\Xi_0) = P(L^2(K/M))$ is dense in $\mathcal{D}'_\lambda(\Xi_0) = P(\mathcal{D}'(K/M))$. It follows that $E = \mathcal{D}'_\lambda(\Xi_0)$.

Next we treat the case $E \cap \mathcal{K}_\lambda(\Xi_0) = \{0\}$. We wish to conclude that $E = \{0\}$. For this, we consider the natural representation π of G_0 on $\mathcal{D}'(\Xi_0)$. Now $\mathcal{D}'(\Xi_0)$ is a Montel space, hence is barelled and complete. Thus for any $f \in \mathcal{D}(K)$, we obtain a well-defined continuous linear operator $\pi(f)$ on $\mathcal{D}'(\Xi_0)$ given by

$$\pi(f) = \int_K f(k)\, \pi(k)\, dk$$

Now $\mathcal{D}'_\lambda(\Xi_0)$ is a closed subspace of $\mathcal{D}'(\Xi_0)$ invariant under π, so if we approximate f by step functions, we see that $\pi(f)\,\Phi \in \mathcal{D}'_\lambda(\Xi_0)$ whenever $\Phi \in \mathcal{D}'_\lambda(\Xi_0)$. For the same reason, since E is closed in $\mathcal{D}'_\lambda(\Xi_0)$ (and hence in $\mathcal{D}'(\Xi_0)$), we have $\pi(f)\,\Phi \in E$ whenever $\Phi \in E$.

Let τ denote the natural (left) representation of K on $\mathcal{D}'(K/M)$. Suppose that $\Phi \in \mathcal{D}'_\lambda(\Xi_0)$, so that $\Phi = P(T)$ for some $T \in \mathcal{D}'(K/M)$. Since P is continuous and linear, and commutes with the left action of K, we have

$$\begin{aligned}\pi(f)\,\Phi &= \int_K f(k)\, \pi(k) P(T)\, dk \\ &= P\left(\int_K f(k)\, \tau(k) T\, dk\right)\end{aligned}$$

If $f \in \mathcal{D}(K)$, then $\int_K f(k)\, \tau(k) T\, dk \in \mathcal{D}(K/M)$, so $\pi(f)\,\Phi \in \mathcal{K}_\lambda(\Xi_0)$. Thus if $\Phi \in E$, we must have $\pi(f)\,\Phi = 0$. Since P is injective, this implies that $\int_K f(k)\, \tau(k) T\, dk = 0$. But because f is arbitrary, we obtain $T = 0$, and therefore $\Phi = P(T) = 0$. Hence $E = \{0\}$.

This completes the two cases and shows that π_λ is irreducible when λ is regular. ∎

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Fulton B. Gonzalez
Department of Mathematics
Tufts University
Medford, MA 02155, USA
fulton.gonzalez@tufts.edu

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