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On Invariants of a Set of Elements of a Semisimple Lie Algebra

Ivan Losev

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Abstract. Let G be a complex reductive algebraic group, $\mathfrak{g}$ its Lie algebra and $\mathfrak{h}$ a reductive subalgebra of $\mathfrak{g}$, n a positive integer. Consider the diagonal actions $G:\mathfrak{g}^n, N_G(\mathfrak{h}):\mathfrak{h}^n$. We study a connection between the algebra $\mathbb{C}[\mathfrak{h}^n]^{N_G(\mathfrak{h})}$ and its subalgebra consisting of restrictions to $\mathfrak{h}^n$ of elements of $\mathbb{C}[\mathfrak{g}^n]^G$.
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1. Introduction

Let G be a reductive algebraic group over the field $\mathbb{C}$, $\mathfrak{g}$ its Lie algebra, $\mathfrak{h}$ a reductive algebraic subalgebra of $\mathfrak{g}$ and $\widetilde{H}=N_G(\mathfrak{h})$ the normalizer of $\mathfrak{h}$ in the group G.

Let n be a positive integer. One has the diagonal actions $G:\mathfrak{g}^n, \widetilde{H}:\mathfrak{h}^n$. Consider a subalgebra of $\mathbb{C}[\mathfrak{h}^n]$ whose elements are restrictions of elements of $\mathbb{C}[\mathfrak{g}^n]^G$ to $\mathfrak{h}^n$. We denote this algebra by $\mathbb{C}[\mathfrak{h}^n]^G$. Clearly, $\mathbb{C}[\mathfrak{h}^n]^G\subset\mathbb{C}[\mathfrak{h}^n]^{\widetilde{H}}$. It is interesting to ask how different these two algebras can be.

It is more convenient to translate this question into geometric language. As usual, we denote by $\mathfrak{h}^n/\!/\widetilde{H}$ the categorical quotient for the action $\widetilde{H}:\mathfrak{h}^n$. In other words, $\mathfrak{h}^n/\!/\widetilde{H}=\operatorname{Spec}(\mathbb{C}[\mathfrak{h}^n]^{\widetilde{H}})$. Put $\mathfrak{h}^n/\!/G=\operatorname{Spec}(\mathbb{C}[\mathfrak{h}^n]^G)$. The inclusion $\mathbb{C}[\mathfrak{h}^n]^G\hookrightarrow\mathbb{C}[\mathfrak{h}^n]^{\widetilde{H}}$ induces a morphism of algebraic varieties $\psi_n:\mathfrak{h}^n/\!/H\to\mathfrak{h}^n/\!/G$. The aim of this paper is to answer the following questions: is ψ_n an isomorphism or a birational (or a bijective, or a finite) morphism?

The starting point for our work is E.B. Vinberg's paper [8], where a morphism $\Psi_n:H^n/\!/\widetilde{H}\to H^n/\!/G$ defined analogously to ψ_n was studied (here H is a reductive subgroup of G). The main result of that paper is that Ψ_n is the morphism of normalization for $n\geqslant 2$.

Now we list our main results.

At first, the morphism ψ_n is always finite (Proposition 3.2). If $n>1$ it is also birational (Proposition 3.5). Therefore, ψ_n is the morphism of normalization

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for $n>1$. In general, ψ_1 is not birational. However, the following statement holds

Theorem 1.1. *Suppose that $G=\mathrm{GL}_n$ and $\mathfrak{h}$ is a simple Lie algebra different from $\mathfrak{so}_9,\mathfrak{sp}_8,\mathfrak{so}_{16},\mathfrak{sl}_8,\mathfrak{sl}_9$. Then ψ_1 is birational. If $\mathfrak{h}$ is one of the five algebras listed above, then for some positive integer m there exists an embedding $\mathfrak{h}\hookrightarrow\mathfrak{gl}_m$ such that the corresponding morphism ψ_1 is not birational.*

If ψ_n is bijective, then ψ_1 is also bijective. For some G and $\mathfrak{h}$ the converse is true. To describe such pairs we need the following definition:

Definition 1.2. Let H be a reductive algebraic group and $\mathfrak{h}$ be its Lie algebra. Suppose that $\mathfrak{h}$ does not contain simple ideals isomorphic to E_6,E_7,E_8. Further, suppose that for every ideal $\mathfrak{h}_1\subset\mathfrak{h}$ isomorphic to $\mathfrak{so}_{2k},k>3$, there exists $h\in H$ such that the restriction of $\mathrm{Ad}(h)$ to a simple ideal $\mathfrak{h}_2\subset\mathfrak{h}$ is an outer involutory automorphism of $\mathfrak{h}_2$ if $\mathfrak{h}_2=\mathfrak{h}_1$ and the identity otherwise (the claim of $\mathrm{Ad}(h)|_{\mathfrak{h}_1}$ being an involution is essential only for $\mathfrak{h}_1\cong\mathfrak{so}_8$). Then we say that H is a *group of type I*.

Put $\overline{H}=\widetilde{H}/Z_G(\mathfrak{h})$.

Theorem 1.3. *If $\overline{H}$ is a group of type I and ψ_1 is bijective, then so is ψ_n.*

However, in some cases ψ_2 is not bijective.

Proposition 1.4. *If G is a group of type I, $\mathfrak{h}$ is a simple algebra and $\overline{H}$ is not a group of type I, then ψ_n is not bijective for $n>1$.*

In fact, the condition that $\mathfrak{h}$ is simple can be omitted, but we do not prove the corresponding result.

Suppose now that $G=\mathrm{GL}_m$. It is known from classical invariant theory that the algebra $\mathbb{C}[\mathfrak{g}^n]^G$ is generated by polynomials of the form $f(X_1,\ldots,X_n)=\mathrm{tr}(X_{i_1}X_{i_2}\ldots X_{i_m})$. Clearly, if ψ_n is an isomorphism, then ψ_1 is an isomorphism too. In some cases the converse is also true.

Proposition 1.5. *The algebra $\mathbb{C}[\mathfrak{h}^n]^G$ is generated by elements of the form $f(L)$, where $f\in\mathbb{C}[\mathfrak{h}]^G$ and L is a Lie polynomial on $X_1,\ldots,X_n$.*

Using Proposition 1.5 one can prove that ψ_n is isomorphism if so is ψ_1 for $\mathfrak{h}=\mathfrak{sp}_{2m}$, $\mathfrak{h}=\mathfrak{so}_{2m+1}$, $\mathfrak{h}=\mathfrak{so}_{2m}$ with $\overline{H}\cong\mathrm{Ad}(\mathrm{O}_{2m})$, $\mathfrak{h}=\mathfrak{sl}_m$ with $\overline{H}=\mathrm{Ad}(\mathrm{SL}_m)$.

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2. Index of notation

$\sim_G$	the orbit equivalence relation for an action of a group G.
$dP:\mathfrak{h}\to\mathfrak{g}$	the tangent homomorphism of a homomorphism $P: H\to G$ of algebraic groups.
G_x	the stabilizer of a point x under an action of a group G.
$\mathfrak{g}^\alpha$	the root subspace of a reductive Lie algebra $\mathfrak{g}$ corresponding to a root α.
H°	the connected component of an algebraic group H.
$N_G(H)$	the normalizer of a subgroup H of a group G.
$N_G(\mathfrak{h})$	the normalizer of a subalgebra $\mathfrak{h}\subset\mathfrak{g}$, where $\mathfrak{g}$ is the Lie algebra of G.
$\mathfrak{R}(\mathfrak{h})$	the set of equivalence classes of representations of a reductive Lie algebra $\mathfrak{h}$ (see conventions on homomorphisms of reductive Lie algebras below).
$R(\lambda)$	the representation of a semisimple Lie algebra with highest weight λ.
X^G	the set of fixed points for an action $G:X$.
$X//G$	the categorical quotient for an action of a reductive group G on an affine variety X.
$\overline{Y}$	the closure (with respect to the Zariski topology) of a subset Y of some algebraic variety.
$Z_G(H)$	the centralizer of a subgroup H of a group G.
$Z_G(\mathfrak{h})$	the centralizer of a subalgebra $\mathfrak{h}\subset\mathfrak{g}$.
$\mathfrak{z}_\mathfrak{g}(\mathfrak{h})$	the centralizer of a subalgebra $\mathfrak{h}$ in a Lie algebra $\mathfrak{g}$.
$\mathfrak{z}(\mathfrak{g})$	$:=\mathfrak{z}_\mathfrak{g}(\mathfrak{g})$.
$\Delta(\mathfrak{g})$	the root system of a reductive Lie algebra $\mathfrak{g}$.

All homomorphisms of reductive Lie algebras are assumed to be differentials of some homomorphisms of some reductive algebraic groups.

All topological terms refer to the Zariski topology.

3. Finiteness and birationality of ψ_n in general case

Let $G,\mathfrak{g},\mathfrak{h},\widetilde{H}$ be as above and n be a positive integer. A natural morphism $\Phi_n:\mathfrak{h}^n\to\mathfrak{g}^n//G$ is constant on $\widetilde{H}$-orbits. Therefore there is a natural morphism $\psi_n:\mathfrak{h}^n//\widetilde{H}\to\mathfrak{g}^n//G$. The closure of $\psi_n(\mathfrak{h}^n//\widetilde{H})$ coincides with $\mathfrak{h}^n//G$.

The following proposition is due to Richardson [7].

Proposition 3.1. *Let G be a reductive group, $\mathfrak{g}$ its Lie algebra and n a positive integer. Consider the action $G:\mathfrak{g}^n$ as above. The orbit of an n-tuple $\mathbf{x}=(x_1,\ldots,x_n)\in\mathfrak{g}^n$ is closed (resp. contains 0 in its closure) if and only if the algebraic subalgebra of $\mathfrak{g}$ generated by $x_1,\ldots,x_n$ is reductive (resp. consists of nilpotent elements).*

Proposition 3.2. *The morphism ψ_n is finite.*

Proof. Denote by $\mathcal{N}_G$ (resp. $\mathcal{N}_{\widetilde{H}}$) the null-cone for the action $G : \mathfrak{g}^n$ (resp. $\widetilde{H} : \mathfrak{h}^n$). It follows from Proposition 3.1, that $\mathcal{N}_G \cap \mathfrak{h}^n = \mathcal{N}_{\widetilde{H}}$. Now our statement follows from a version of the Nullstellensatz (see [4], Ch.2, S. 4.3, Thm. 8). ∎

The following statement is an easy corollary of Propositions 3.1,3.2.

Corollary 3.3. *The image of $\mathfrak{h}^n//\widetilde{H}$ in $\mathfrak{g}^n//G$ is closed and so coincides with $\mathfrak{h}^n//G$. Further, $\mathfrak{h}^n//\widetilde{H}$ (respectively, $\mathfrak{h}^n//G$) can be identified with a set of equivalence classes of n-tuples $(x_1,\ldots,x_n) \in \mathfrak{h}^n$ generating a reductive subalgebra of $\mathfrak{h}$ modulo $\widetilde{H}$- (respectively, G-) conjugacy.*

Lemma 3.4. *Any reductive algebraic Lie algebra $\mathfrak{h}$ can be generated by two elements (as an algebraic algebra). If $\mathfrak{h}$ is commutative, then it can be generated by one element.*

Proof. The proof is completely analogous to the proof of Proposition 2 in [8]. ∎

Proposition 3.5. *Suppose $n > 1$. Then ψ_n is birational.*

Proof. The proof is completely analogous to the proof of Theorem 2 in [8]. ∎

Corollary 3.6. *Suppose $n > 1$. Then ψ_n is a normalization.*

Proof. Since $\mathfrak{h}^n//\widetilde{H}$ is normal, this is a direct consequence of Propositions 3.5, 3.2. ∎

4. Birationality of ψ_1 for $G = \mathrm{GL}_n$ and a simple Lie algebra $\mathfrak{h}$

In this section we prove Theorem 1.1. Here we assume that $\mathfrak{h}$ is a simple Lie algebra of rank r. Denote by $\alpha_1,\ldots,\alpha_r$ simple roots of $\mathfrak{h}$, by $\pi_1,\ldots,\pi_r$ the corresponding fundamental weights and by P and Q the weight and the root lattices of $\mathfrak{h}$, respectively.

Lemma 4.1. *Let Δ be an irreducible root system in a real vector space V, W its Weyl group, Q the lattice generated by Δ. Denote by $(\cdot,\cdot)$ a W-invariant scalar product on V. Let $g \in \mathrm{Aut}(Q)$ be an orthogonal linear operator. If $\Delta \neq C_4$, then $g \in \mathrm{Aut}(\Delta)$.*

Proof. It follows from the construction of root systems (see, for example, [1], Chapter 6) that if $\Delta \neq C_l$, then the elements of Δ are all elements of Q satisfying some conditions on their length. For $\Delta = C_l$ the same is true for roots of minimal length. Suppose now that $\Delta = C_l$ and $l \neq 4$. Let g be an element of $\mathrm{O}(V)$ such that the set of elements of Δ of minimal length is invariant under g. Then $g \in W$ and we are done. ∎

Proof of Theorem 1.1. Denote by $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{h}$. The points of $\mathfrak{h}//\widetilde{H} \cong \mathfrak{h}//\overline{H}$ (respectively, $\mathfrak{h}//G$) are in one-to-one correspondence with equivalence classes of semisimple elements of $\mathfrak{h}$ modulo $\overline{H}$- (respectively G-) conjugacy (see Corollary 3.3). If t is an element of $\mathfrak{t}$ in general position, then $gt \in \mathfrak{t}$ for some $g \in G$ implies $g \in N_G(\mathfrak{t})$. Thus, ψ_1 is birational if and only if $N_G(\mathfrak{t})/Z_G(\mathfrak{t}) = N_{\overline{H}}(\mathfrak{t})/Z_{\overline{H}}(\mathfrak{t})$.

Suppose that $\mathfrak{h} \neq \mathfrak{sp}_8, \mathfrak{so}_9, \mathfrak{sl}_8, \mathfrak{sl}_9, \mathfrak{so}_{16}$. Denote by φ an embedding of $\mathfrak{h}$ into $\mathfrak{gl}_m$. We identify $\mathfrak{h}$ and $\varphi(\mathfrak{h})$. Denote by N the image of $N_G(\mathfrak{t})$ in $\mathrm{GL}(\mathfrak{t})$ under a natural homomorphism. It is clear that N contains the Weyl group of $\mathfrak{h}$. We now prove that $N \subset \mathrm{Aut}(\Delta)$.

For $x, y \in \mathfrak{gl}_m$ put $(x, y) = \mathrm{tr}(xy)$. The restriction of $(,)$ to $\mathfrak{t}$ is an N-invariant non-degenerate symmetric bilinear form. Its restriction to $\mathfrak{t}(\mathbb{R})$ is a scalar product. Further, we notice that the lattice X generated by the weights of φ is invariant under the action of N.

Obviously, $Q \subset X \subset P$. It follows from Lemma 4.1, that if $X = Q$ and $N \not\subset \mathrm{Aut}(\Delta)$, then $\Delta = C_4$. Suppose now that $X = P$. Then the dual root lattice $Q^\vee$ is invariant under the action of N on $\mathfrak{t}$. Lemma 4.1 implies that if $N \not\subset \mathrm{Aut}(\Delta)$, then $\Delta = B_4$.

Suppose now that $X \neq Q, P$. Then the group P/Q is not prime. Tables in [5] imply that $\Delta = A_l$, where $l+1$ is not prime, or $\Delta = D_l$. If $\Delta \neq A_7, A_8, D_8$, then every element of P, whose length is equal to the length of a root, is a root itself. Indeed, the set of all such elements forms a root system, whose rank equals that of Δ. It remains to notice that A_l is not a subsystem in D_l for $l > 3$. So if $\Delta \neq A_7, A_8, D_8$, then $N \subset \mathrm{Aut}(\Delta)$.

The system of weights of the representation φ is invariant under $N \subset \mathrm{Aut}(\Delta)$. Thus, N coincides with the image of $N_{\widetilde{H}}(\mathfrak{t})$ in $\mathrm{GL}(\mathfrak{t}^*)$. So we are done.

Now we construct embeddings of $\mathfrak{h} = \mathfrak{sp}_8, \mathfrak{so}_9, \mathfrak{sl}_8, \mathfrak{sl}_9, \mathfrak{so}_{16}$, for which ψ_1 is not birational.

Suppose $\mathfrak{h} = \mathfrak{sp}_8$. Let $\varphi : \mathfrak{h} \to \mathfrak{gl}_{14}$ be the irreducible representation with highest weight π_2. Let us choose the an orthonormal basis $\varepsilon_1, \ldots, \varepsilon_4 \in \mathfrak{t}$, so that $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i = 1, 2, 3$, $\alpha_4 = 2\varepsilon_4$. The weights of φ are $\varepsilon_i + \varepsilon_j, i \neq j$, with multiplicity 1 and 0 with multiplicity 2. The stabilizer of this weight system in $\mathrm{GL}(\mathfrak{t}^*)$ is just $\mathrm{Aut}(D_4)$. Therefore, $N_G(\mathfrak{t})/Z_G(\mathfrak{t}) \cong \mathrm{Aut}(D_4)$, while $N_{\overline{H}(\mathfrak{t})}/Z_{\overline{H}}(\mathfrak{t})$ is the Weyl group of Δ and has index 3 in $N_G(\mathfrak{t})/Z_G(\mathfrak{t})$.

The algebras $\mathfrak{h} = \mathfrak{so}_9, \mathfrak{sl}_8, \mathfrak{sl}_9, \mathfrak{so}_{16}$ can be embedded into the exceptional algebras $\mathfrak{f} = F_4, E_7, E_8, E_8$, respectively, as regular subalgebras. Suppose that $\varphi : \mathfrak{h} \hookrightarrow \mathfrak{gl}_n$ is the composition of this embedding and some embedding $\rho : \mathfrak{f} \hookrightarrow \mathfrak{gl}_n$. Analogously to the case $\mathfrak{h} = \mathfrak{sp}_8$ one can show that $N_{\overline{H}}(\mathfrak{t})/Z_{\overline{H}}(\mathfrak{t})$ is not equal to $N_G(\mathfrak{t})/Z_G(\mathfrak{t})$, because the latter group is the Weyl group of $\mathfrak{f}$. This completes the proof of the theorem. ∎

5. The algebra $\mathbb{C}[\mathfrak{h}]^H$

It is known (see [1], Chapter 8,§8) that for a connected semisimple group H the vector space $\mathbb{C}[\mathfrak{h}]^H$ is generated by polynomials $\mathrm{tr}(\rho(x)^n)$, where ρ runs over the set of all representations of $\mathfrak{h}$. In this section we generalize this result for arbitrary reductive groups H.

Proposition 5.1. *Let H be an arbitrary reductive group. The vector space $\mathbb{C}[\mathfrak{h}]^H$ is generated by* $\mathrm{tr}(\rho(x)^n)$*, where ρ runs over the set of all representations of H.*

Proof. The proof from [1] for a semisimple connected group H quoted above can be generalized to the case when H is connected but not necessarily semisimple in a straightforward way. The generalization is straightforward.

Now we consider the general case. We have

$$\mathbb{C}[\mathfrak{h}]^H = \{ \sum_{h \in H/H^\circ} h.f \mid f \in \mathbb{C}[\mathfrak{h}]^{H^\circ} \}.$$

Therefore, the vector space $\mathbb{C}[\mathfrak{h}]^H$ is generated by elements of the form

$$\sum_{h \in H/H^\circ} h.\,\mathrm{tr}(\rho(x)^n), \tag{1}$$

where ρ is a representation of H°. Denote by $\widetilde{\rho}$ the representation of H induced from ρ. The corresponding representation of $\mathfrak{h}$ is given by $\widetilde{\rho} = \sum_{h \in H/H^\circ} h.\rho$. The polynomial (1) is just $\mathrm{tr}(\widetilde{\rho}(x)^n)$. ■

Let I be a set of representations of H. Denote by $\mathbb{C}[\mathfrak{h}^n]^I$ the subalgebra of $\mathbb{C}[\mathfrak{h}^n]^H$ generated by polynomials of the form $\mathrm{tr}(\rho(x)^m)$, where $\rho \in I$. When $I = \{\rho\}$, we write $\mathbb{C}[\mathfrak{h}^n]^\rho$ instead of $\mathbb{C}[\mathfrak{h}^n]^{\{\rho\}}$.

It is known from the classical invariant theory (see, for example, [6]) that if $H = \mathrm{GL}_m, \mathrm{SL}_m, \mathrm{O}_m, \mathrm{Sp}_{2m}$, then $\mathbb{C}[\mathfrak{h}^n]^H = \mathbb{C}[\mathfrak{h}^n]^\rho$, where ρ is the tautological representation of the group H. Now let H be one of the exceptional simple groups G_2, F_4, E_6, E_7, E_8 and ρ be the simplest (=non-trivial irreducible of minimal dimension) representation of the group H. It was shown by several authors (see [2] for references) that $\mathbb{C}[\mathfrak{h}]^H = \mathbb{C}[\mathfrak{h}]^\rho$.

6. Linear equivalence of embeddings

Let G, H be reductive algebraic groups and $\mathfrak{g}, \mathfrak{h}$ be the Lie algebras of G and H, respectively. We say that two homomorphisms $P_1, P_2 : H \to G$ are *equivalent* (or, more precisely, *G-equivalent*) if there exists $g \in G$ such that $gP_1(x)g^{-1} = P_2(x)$ for all $x \in H^\circ$. Further, we say that P_1, P_2 are *linearly equivalent* (or *linearly G-equivalent*) if for every representation $P : G \to \mathrm{GL}(V)$ the representations $P \circ P_1, P \circ P_2$ are $\mathrm{GL}(V)$-equivalent. It is obvious that equivalent homomorphisms are linearly equivalent. Homomorphisms $\rho_1, \rho_2 : \mathfrak{h} \to \mathfrak{g}$ are said to be G-equivalent (resp., linearly G-equivalent) if the exist equivalent (resp., linearly equivalent) homomorphisms $P_1, P_2 : H \to G$ with $dP_1 = \rho_1, dP_2 = \rho_2$.

Let $P : H \to G$ be a homomorphism, $\rho : \mathfrak{h} \to \mathfrak{g}$ its tangent homomorphism. Denote by ψ_n^ρ the natural morphism from $\mathfrak{h}^n/\!/H$ to $\rho(\mathfrak{h})^n/\!/G$.

The set $\rho(\mathfrak{h})^n/\!/G$ can be identified with the set of equivalence classes $(\rho(x_1), \ldots, \rho(x_n))$, where $x_1, \ldots, x_n$ generate a reductive subalgebra of $\mathfrak{h}$, modulo G-conjugacy. Therefore, Lemma 3.4 implies that $\psi_n^\rho, n > 1$, is bijective if and only if for every reductive Lie algebra $\mathfrak{f}$ and embeddings $\rho_1, \rho_2 : \mathfrak{f} \to \mathfrak{h}$ the following condition is fulfilled:

if $\rho \circ \rho_1, \rho \circ \rho_2$ are G-equivalent, then ρ_1, ρ_2 are H-equivalent.

Similarly, we get the following

Proposition 6.1. *Let $H, G, \mathfrak{h}, \mathfrak{g}, \rho$ be as above. The following conditions are equivalent:*

(i) The map ψ_1^ρ is injective.

(ii) For every pair (x_1, x_2) of semisimple elements of $\mathfrak{h}$ if $\rho(x_1) \sim_G \rho(x_2)$, then $x_1 \sim_H x_2$.

(iii) For any diagonalizable Lie algebra $\mathfrak{t}$ and embeddings $\rho_1, \rho_2 : \mathfrak{t} \hookrightarrow \mathfrak{h}$ if ρ_1, ρ_2 are H-equivalent, then $\rho \circ \rho_1, \rho \circ \rho_2$ are G-equivalent.

The following proposition is a generalization of Theorem 1.1. from [3].

Proposition 6.2. *Let $G, \mathfrak{h}, \mathfrak{g}$ be as above, $\mathfrak{t} \subset \mathfrak{h}$ be a Cartan subalgebra, $P_1, P_2 : H \to G$ be some homomorphisms, $\rho_1 = dP_1, \rho_2 = dP_2$. The following conditions are equivalent*

(i) ρ_1 and ρ_2 are linearly G-equivalent.

(ii) $\rho_1|_\mathfrak{t}$ and $\rho_2|_\mathfrak{t}$ are G-equivalent.

Proof. *(ii)* $\Rightarrow$ *(i)*. Replacing ρ_1 with $\mathrm{Ad}(g) \circ \rho_1, g \in G$, if necessary, we may assume that $\rho_1|_\mathfrak{t} = \rho_2|_\mathfrak{t}$. The required result follows from the fact that a representation of $\mathfrak{h}$ is uniquely determined by the collection of its weights and their multiplicities.

(i) $\Rightarrow$ *(ii)*. We may assume that $\mathfrak{h} = \mathfrak{t}$. By Proposition 6.1 it is enough to prove the following statement:

Let x_1, x_2 be semisimple elements of $\mathfrak{g}$ such that for every representation $P : G \to \mathrm{GL}(V)$ matrices $\rho(x_1), \rho(x_2)$ are conjugate, where $\rho = dP$. Then x_1, x_2 are conjugate (with respect to the adjoint action of G).

We see that $\mathrm{tr}(\rho(x_1)^n) = \mathrm{tr}(\rho(x_2)^n)$ for all $P : G \to \mathrm{GL}(V)$. It follows from Proposition 5.1 that $f(x_1) = f(x_2)$ for any $f \in \mathbb{C}[\mathfrak{g}]^G$. Since x_1, x_2 are semisimple, we have $x_1 \sim_G x_2$. ■

Remark 6.3. Let I be a set of representations of G such that polynomials $\mathrm{tr}(dP(x)^n), P \in I$, generate the algebra $\mathbb{C}[\mathfrak{g}]^G$. It follows from the previous proof that $\rho_1|_\mathfrak{t}, \rho_2|_\mathfrak{t}$ are H-equivalent if and only if the representations $dP \circ \rho_1, dP \circ \rho_2$ are $\mathrm{GL}(V)$-equivalent for any representation $P \in I$.

Proposition 6.4. *Let $G, \mathfrak{g}, H, \mathfrak{h}$ be such as in Proposition 6.1, $P : H \to G$ a homomorphism, $\rho = dP$, $\mathfrak{f}$ a reductive Lie algebra, and $\rho_1, \rho_2 : \mathfrak{f} \to \mathfrak{h}$ embeddings. Suppose that the equivalent conditions of Proposition 6.1 are fulfilled and ρ_1, ρ_2 are G-equivalent. If $\mathfrak{f} = \mathfrak{sl}_2$ or $\rho_1(\mathfrak{f}), \rho_2(\mathfrak{f})$ are regular subalgebras of $\mathfrak{h}$, then ρ_1 and ρ_2 are H-equivalent.*

Proof. Suppose $\mathfrak{f} = \mathfrak{sl}_2$. Let (e, h, f) be a standard basis of $\mathfrak{f}$, i.e. $[h, e] = 2e, [h, f] = -2f, [e, f] = h$. Since $\rho_1(h) \sim_G \rho_2(h)$ we see that $\rho_1(h) \sim_H \rho_2(h)$. We may assume that $\rho_1(h) = \rho_2(h)$. There exists $g \in Z_H(\rho_1(h))$ such that $\mathrm{Ad}(g) \circ \rho_1 = \rho_2$ (see, for example, [1], Ch.8, §11).

Now suppose that $\rho_1(\mathfrak{f})$ and $\rho_2(\mathfrak{f})$ are regular subalgebras of $\mathfrak{h}$. Denote by $\mathfrak{s}, \mathfrak{t}$ Cartan subalgebras of $\mathfrak{f}, \mathfrak{h}$, respectively. There exists an element $h \in H$ such that $\mathrm{Ad}(h)\rho_1|_{\mathfrak{s}} = \rho_2|_{\mathfrak{s}}$. Therefore, the proof reduces to the case when $\rho_1|_{\mathfrak{s}} = \rho_2|_{\mathfrak{s}}$. Since all Cartan subalgebras of $\mathfrak{z}_{\mathfrak{g}}(\rho_1(\mathfrak{s}))$ are $Z_G(\rho_1(\mathfrak{s}))$-conjugate one also may assume that $\mathfrak{t}$ normalizes $\rho_1(\mathfrak{f}), \rho_2(\mathfrak{f})$. Let $\alpha \in \Delta(\mathfrak{f})$ and $e_\alpha \in \mathfrak{f}^\alpha$ be nonzero. Since $\rho_1(\mathfrak{f}), \rho_2(\mathfrak{f})$ are regular subalgebras we have $\rho_1(e_\alpha) = c_\alpha \rho_2(e_\alpha)$. There exists $t \in \rho_1(\mathfrak{s})$ such that $\exp(\mathrm{ad}\, t) \circ \rho_1 = \rho_2$, see [1], Ch.8,§5. ∎

7. Bijectivity of $\psi_2 : \mathfrak{h}^2//\overline{H} \to \mathfrak{h}^2//G$

Theorem 7.1. *Let $\mathfrak{h}$ be a simple Lie algebra, H an algebraic group with Lie algebra $\mathfrak{h}$. The following conditions are equivalent:*

(i) $\mathfrak{h} = \mathfrak{sl}_n, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n+1}, G_2, F_4$ *or* $\mathfrak{h} = \mathfrak{so}_{2n}, n > 3$, *and the group* $\mathrm{Ad}(H)$ *contains an involutive outer automorphism of the algebra* $\mathfrak{h}$.

(ii) For any reductive algebraic Lie algebra $\mathfrak{f}$ and any pair of linearly H-equivalnent homomorhisms $\rho_1, \rho_2 : \mathfrak{f} \to \mathfrak{h}$ the homomorphisms ρ_1 and ρ_2 are H-equivalent.

Theorem 7.1 is proved in Sections 8-11.

Proof of Theorem 1.3. It is enough to prove that for every reductive Lie algebra $\mathfrak{f}$ and every embeddings $\rho_1, \rho_2 : \mathfrak{f} \to \mathfrak{h}$ if ρ_1, ρ_2 are G-equivalent then they are H-equivalent.

It follows from the bijectivity of ψ_1 that $\rho_1|_{\mathfrak{t}}, \rho_2|_{\mathfrak{t}}$ are H-equivalent. It is enough to prove that the latter implies ρ_1, ρ_2 are H-equivalent. One may assume that $\rho_1|_{\mathfrak{t}} = \rho_2|_{\mathfrak{t}}$. Let $\mathfrak{h} = \mathfrak{z}(\mathfrak{h}) \oplus \mathfrak{h}_1 \oplus \ldots\ldots \oplus \mathfrak{h}_k$, where $\mathfrak{h}_i, i = 1, \ldots, k$, is a simple noncommutative ideal. Now it is enough to consider the case $\overline{H} = H_1 \times \ldots \times H_k$, where $H_i = \mathrm{O}_{2m}$ if $\mathfrak{h}_i = \mathfrak{so}_{2m}, m > 3$, $H_i = \mathrm{Int}(\mathfrak{h}_i)$ otherwise. Denote by π_i the projection from $\mathfrak{h}$ to $\mathfrak{h}_i$. It is enough to prove that there is an element $h_k \in H_k$ such that $\mathrm{Ad}(h_k) \circ \pi_k \circ \rho_1 = \pi_k \circ \rho_2$. By Proposition 6.2, homomorphisms $\pi_k \circ \rho_1, \pi_k \circ \rho_2$ are linearly H-equivalent. It remains to use Theorem 7.1. ∎

Proposition 7.2. *Let G be a reductive algebraic group, $\mathfrak{g}$ the Lie algebra of G. There exists a representation $P : G \to \mathrm{GL}(V)$ such that $\rho = dP$ is an embedding and $\psi_1^\rho : \mathfrak{g}//G \to \rho(\mathfrak{g})//\,\mathrm{GL}(V)$ is injective.*

Proof. We need the following lemma

Lemma 7.3. *Let $\rho_1 : \mathfrak{g} \to \mathfrak{gl}(U_1), \rho_2 : \mathfrak{g} \to \mathfrak{gl}(U_2)$ be representations, $n = \dim U_1 + 1$. Put $U = U_1 \oplus nU_2$, $\rho = \rho_1 + n\rho_2$. For any two semisimple elements $x, y \in \mathfrak{g}$ if the matrices $\rho(x), \rho(y)$ are similar, then for $i = 1, 2$ the matrices $\rho_i(x), \rho_i(y)$ are similar.*

Proof of Lemma 7.3. Denote by $\lambda_1, \dots, \lambda_k$ the different eigenvalues of matrices $\rho(x), \rho(y)$ and by $m_1, \dots, m_k$ their multiplicities. Both $\rho_1(x)$ and $\rho_1(y)$ (respectively, $\rho_2(x), \rho_2(y)$) have eigenvalues $\lambda_1, \dots, \lambda_k$ with multiplicities $n\{\frac{m_1}{n}\}, \dots, n\{\frac{m_k}{n}\}$ (respectively, $[\frac{m_1}{n}], \dots, [\frac{m_k}{n}]$). Therefore, $\rho_i(x), \rho_i(y)$ are similar for $i = 1, 2$. ■

It follows from Proposition 5.1 that there exist representations $P_i : G \to \mathrm{GL}(U_i), i = 1, \dots, m$, such that the algebra $\mathbb{C}[\mathfrak{g}]^G$ is generated by polynomials of the form $\mathrm{tr}(dP_i(x)^n)$.

For $i = 1, \dots, m$ we define a positive integer n_i, a vector space $\widetilde{U}_i$ and representation $\widetilde{P}_i : G \to \mathrm{GL}(\widetilde{U}_i)$ by formulas

$$n_1 = 1, \widetilde{U}_1 = U_1, \widetilde{P}_1 = P_1.$$

$$n_i = \dim \widetilde{U}_{i-1} + 1, \widetilde{U}_i = \widetilde{U}_{i-1} \oplus n_i U_i, \widetilde{P}_i = \widetilde{P}_{i-1} + n_i P_i.$$

By Lemma 7.3, for any semisimple $x, y \in \mathfrak{g}$ if the matrices $d\widetilde{P}_m(x), d\widetilde{P}_m(y)$ are similar, then for $i = 1, \dots, m$ the matrices $dP_i(x), dP_i(y)$ are similar. Thus, $f(x) = f(y)$ for all $f \in \mathbb{C}[\mathfrak{g}]^G$ and so $x \sim_G y$. It follows from Proposition 6.1 that $\psi_1^{dP_m}$ is bijective. ■

Proof. First, suppose that $G = \mathrm{GL}_m$. Let $\mathfrak{f}$ be a reductive Lie algebra, $\rho_1, \rho_2 : \mathfrak{f} \hookrightarrow \mathfrak{h}$ linearly H-equivalent but not equivalent embeddings. Such ρ_1, ρ_2 exist by Theorem 7.1. If ψ_2 is bijective, then ρ_1, ρ_2 are H-equivalent. Contradiction.

Now we consider the general case. By Proposition 7.2, there exists a homomorphism $P : G \to \mathrm{GL}(V)$ such that $\rho = dP$ is an embedding and ψ_1^ρ is bijective. Since ψ_1^ρ is injective, it follows that $\overline{G} := N_{\mathrm{GL}(V)}(\rho(\mathfrak{g}))/Z_{\mathrm{GL}(V)}(\rho(\mathfrak{g}))$ is a group of type I. Therefore, by Theorem 1.3, $\psi_2^\rho : \mathfrak{g}^2 /\!/ G \to \mathfrak{g}^2 /\!/ \mathrm{GL}(V)$ is bijective. Denote by $\widetilde{\rho}$ the embedding $\mathfrak{h} \hookrightarrow \mathfrak{g}$. Since ψ_2^ρ is bijective, the group $N_{\mathrm{GL}(V)}(\rho(\mathfrak{h}))/Z_{\mathrm{GL}(V)}(\rho(\mathfrak{h})) \subset \mathrm{Aut}(\mathfrak{h})$ coincides with $\overline{H}$. Now it follows from the first part of the proof that the map $\psi_2^{\rho\circ\widetilde{\rho}}$ is not bijective. But $\psi_2^{\rho\circ\widetilde{\rho}} = \psi_2^\rho \circ \psi_2$. Therefore, ψ_2 is not bijective. ■

8. Cases $\mathfrak{h} = \mathfrak{sl}_n, \mathfrak{sp}_{2n}, \mathfrak{so}_{2n+1}$

We show that linearly H-equivalent reductive embeddings are H-equivalent for every group H with Lie algebra $\mathfrak{h}$.

Let $\mathfrak{f}$ be a reductive Lie algebra and $\rho_1, \rho_2 : \mathfrak{f} \hookrightarrow \mathfrak{h}$ be linearly H-equivalent embeddings. We must prove that ρ_1, ρ_2 are H-equivalent. By Proposition 6.2, we may assume that $\rho_1|_\mathfrak{t} = \rho_2|_\mathfrak{t}$, where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{f}$. Now it is enough to show that ρ_1, ρ_2 are H°-equivalent. Denote by ρ the tautological representation of $\mathfrak{h}$. Recall that $\mathbb{C}[\mathfrak{h}^2]^\rho = \mathbb{C}[\mathfrak{h}^2]^{H^\circ}$. It follows that the embeddings ρ_1 and ρ_2 are H°-equivalent.

9. Case $\mathfrak{h} = \mathfrak{so}_{2n}, n > 3$

First suppose that $\mathrm{Ad}(H)$ contains an involutive outer automorphism of $\mathfrak{h}$. Then linearly H-equivalent reductive embeddings are H-equivalent. One can prove this analogously to the previous section using the fact that $\mathbb{C}[\mathfrak{h}^n]^\rho = \mathbb{C}[\mathfrak{h}^n]^H$, where ρ is the tautological representation of $\mathfrak{h}$, $H = \mathrm{O}_{2n}$.

Now suppose that $\mathrm{Ad}(H) = \mathrm{Int}(\mathfrak{h})$. Denote by τ the tautological representation of $\mathfrak{h}$ and by θ any outer involutive automorphism of $\mathfrak{h}$. For the proof of the next proposition see [3], Theorem 1.4.

Proposition 9.1. *Let $\mathfrak{f}$ be a reductive Lie algebra and $\rho : \mathfrak{f} \to \mathfrak{h}$ be an embedding. Suppose that*

1. *The representation $\tau \circ \rho : \mathfrak{f} \to \mathfrak{gl}_{2n}$ has zero weight.*
2. *All irreducible components of $\tau \circ \rho$ have even dimension.*

Then the embeddings $\rho, \theta \circ \rho$ are linearly H-equivalent but not equivalent.

Now it is enough to construct an embedding $\rho : \mathfrak{f} \hookrightarrow \mathfrak{h}$ (or representation $\tau \circ \rho$) satisfying both conditions of Proposition 9.1. Denote by ρ_1 the adjoint representation of $\mathfrak{sl}_3$ (of dimension 8), by ρ_2 the exterior square of the tautological representation of $\mathfrak{so}_5$ (of dimension 10) and by ρ_0 the tautological representation of $\mathfrak{so}_4$. For $n = 2k$ we put $\mathfrak{f} = \mathfrak{sl}_3 \oplus \mathfrak{so}_4^{k-4}$, $\tau \circ \rho = \rho_1 \oplus (k-4)\rho_0$. For $n = 2k+1$ put $\mathfrak{f} = \mathfrak{so}_5 \oplus \mathfrak{so}_4^{k-4}, \tau \circ \rho = \rho_2 \oplus (k-4)\rho_0$.

It remains to consider the case $\mathfrak{h} = \mathfrak{so}_8, |\mathrm{Ad}(H)/\mathrm{Int}(\mathfrak{h})| = 3$. It is enough to prove that $\rho_1, \theta \circ \rho_1 : \mathfrak{sl}_3 \hookrightarrow \mathfrak{so}_8$ are not equivalent.

Assume the converse. There exists an $h \in H$ such that $(\mathrm{Ad}(h)\theta) \circ \rho_1 = \rho_1$. The order of the image of $\mathrm{Ad}(h)\theta$ in the group $\mathrm{Aut}(\mathfrak{h})/\mathrm{Int}(\mathfrak{h})$ is 2. This contradicts Proposition 9.1.

10. Case $\mathfrak{h} = E_l, l = 6, 7, 8$

There exists a Levi subalgebra $\mathfrak{l} \subset \mathfrak{h}$ isomorphic to $\mathfrak{so}_{10} \times \mathbb{C}^{l-5}$. Put $\mathfrak{f} = \mathfrak{so}_5 \times \mathbb{C}^{l-5}$. Let ρ^1, ρ^2 be embeddings of $\mathfrak{f}$ into $\mathfrak{l}$ satisfying the following conditions

1. $\rho^1|_{\mathfrak{z}(\mathfrak{f})} = \rho^2|_{\mathfrak{z}(\mathfrak{f})}$ is an isomorphism of $\mathfrak{z}(\mathfrak{f})$ and $\mathfrak{z}(\mathfrak{l})$.
2. $\rho^1|_{\mathfrak{so}_5} = \rho_2, \rho^2|_{\mathfrak{so}_5} = \theta \circ \rho_2$, where ρ_2 is the exterior square of the tautological representation of $\mathfrak{so}_5$, θ is an involutive outer automorphism of $\mathfrak{so}_{10}$.

Since $\rho_2, \theta \circ \rho_2 : \mathfrak{so}_5 \hookrightarrow \mathfrak{so}_{10}$ are linearly SO_{10}-equivalent, we see that ρ^1, ρ^2 are linearly $\mathrm{Int}(\mathfrak{h})$-equivalent.

Assume that there exists $h \in \mathrm{Ad}(H)$ such that $h \circ \rho^1 = \rho^2$. Denote by L' the connected subgroup of H with Lie algebra $[\mathfrak{l}, \mathfrak{l}]$. It is well known that the centralizer of an algebraic subtorus in a connected reductive algebraic group is connected. By Proposition 9.1, $h|_{[\mathfrak{l},\mathfrak{l}]} \notin \mathrm{Ad}(L')$. Since $h \in Z_H(\mathfrak{z}(\mathfrak{l}))$, we have $h \notin \mathrm{Int}(\mathfrak{h})$. Thus, $\mathfrak{h} = E_6, \mathrm{Ad}(H) = \mathrm{Aut}(\mathfrak{h})$.

Denote by $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{l}$. One may assume that $\mathfrak{t} \cap [\mathfrak{l}, \mathfrak{l}]$ is θ-invariant and that θ acts on $\mathfrak{t} \cap [\mathfrak{l}, \mathfrak{l}]$ by a reflection. The centralizer of $\rho_2(\mathfrak{so}_5)$ in SO_{10} coincides with the center of SO_{10}. Hence, $\mathrm{Ad}(h)|_{[\mathfrak{l},\mathfrak{l}]} = \theta$ and $\mathrm{Ad}(h)|_{\mathfrak{t}}$ is a reflection. The subgroup of $N_H(\mathfrak{t})$ generated by all reflections is the Weyl group of $\mathfrak{h}$. Therefore, $h \in \mathrm{Int}(\mathfrak{h})$. Contradiction.

11. Cases $\mathfrak{h} = G_2, F_4$

Since the algebra $\mathfrak{h}$ has no outer automorphisms, one may assume that H is connected.

If $\mathfrak{h} = G_2$ the statement of Theorem 7.1 follows from Proposition 6.4. In the sequel we consider the case $\mathfrak{h} = F_4$.

Let $\mathfrak{f}$ be a reductive Lie algebra, $\rho_1, \rho_2 : \mathfrak{f} \hookrightarrow \mathfrak{h}$ linearly H-equivalent embeddings. We must prove that ρ_1, ρ_2 are equivalent.

Assume the converse. It follows from Proposition 6.4 that $\operatorname{rank} \mathfrak{f} < 4$.

Lemma 11.1. *Let H be a reductive algebraic group, $\mathfrak{f}$ a reductive Lie algebra such that $\mathfrak{f} \cong \mathfrak{s} \oplus \mathfrak{f}_1$, where $\mathfrak{s}, \mathfrak{f}_1$ are ideals of $\mathfrak{f}$ and $\operatorname{rank} \mathfrak{s} = 1$. Suppose $\rho_1, \rho_2 : \mathfrak{f} \to \mathfrak{h}$ are linearly H-equivalent embeddings.*

(1) Then there exists an $h \in H$ such that $\operatorname{Ad}(h) \circ \rho_1$ and ρ_2 coincide on $\mathfrak{s}$.

(2) Suppose that ρ_1, ρ_2 coincide on $\mathfrak{s}$. Then $\rho_1|_{\mathfrak{f}_1}, \rho_2|_{\mathfrak{f}_1}$ are linearly $Z_H(\rho_1(\mathfrak{s}))$-equivalent.

(3) Under the assumptions of (2) ρ_1, ρ_2 are H-equivalent if and only if $\rho_1|_{\mathfrak{f}_1}, \rho_2|_{\mathfrak{f}_2}$ are $Z_H(\rho_1(\mathfrak{s}))$-equivalent.

Proof. The assertion of part (3) is obvious. Propositions 6.2 and 6.4 imply the statements (1),(2) for $\mathfrak{s} \cong \mathbb{C}$ and the statement (1) for $\mathfrak{s} = \mathfrak{sl}_2$, respectively.

Now one may assume that $\mathfrak{s} \cong \mathfrak{sl}_2$ and $\rho_1|_{\mathfrak{s}} = \rho_2|_{\mathfrak{s}}$. Denote by x a general semisimple element of $\mathfrak{f}_1$. By Proposition 6.2, it is enough to show that there exists a $g \in Z_H(\rho_1(\mathfrak{s}))$ such that $\operatorname{Ad}(g)\rho_1(x) = \rho_2(x)$. Denote by e, h, f a standard basis of $\rho_1(\mathfrak{s})$. Since ρ_1, ρ_2 are linearly H-equivalent, there exists a $g_1 \in Z_H(h)$ such that $\operatorname{Ad}(g_1)\rho_1(x) = \rho_2(x)$. Analogously to the proof of Proposition 6.4, there exists a $g_2 \in Z_H(\rho_2(x))$ such that $\operatorname{Ad}(g_2)\operatorname{Ad}(g_1)\rho_1, \rho_2$ coincide on $\mathfrak{s}$. The element $g = g_2 g_1$ has the required properties. ■

Suppose that $\mathfrak{f}$ is not simple. Then $\mathfrak{f}$ satisfies the conditions of Lemma 11.1. It is easy to see that $Z_H(\rho_1(\mathfrak{s}))$ is a group of type I. The rank of the semisimple part of $Z_H(\rho_1(\mathfrak{s}))$ is less then 4. It follows from results of Sections 8,9, that if ρ_1, ρ_2 are linearly equivalent, then they are equivalent (see the proof of Theorem 1.3) . So, we may assume that $\mathfrak{f}$ is simple. Proposition 6.4 implies that $\operatorname{rank} \mathfrak{f} > 1$.

Now we prove that neither of the subalgebras $\rho_1(\mathfrak{f}), \rho_2(\mathfrak{f}) \subset \mathfrak{h}$ is regular. Assume the converse. To be definite, let $\rho_1(\mathfrak{f})$ be a regular subalgebra of $\mathfrak{h}$. By Proposition 6.4, $\rho_2(\mathfrak{f})$ is not regular. Since the representations $\operatorname{ad} \circ \rho_1, \operatorname{ad} \circ \rho_2$ are equivalent, we obtain that $\dim \mathfrak{z}_{\mathfrak{h}}(\rho_2(\mathfrak{f})) = \dim \mathfrak{z}_{\mathfrak{h}}(\rho_1(\mathfrak{f})) \geqslant 4 - \operatorname{rank} \mathfrak{f}$. Since for any reductive Lie algebra $\mathfrak{s}$ the number $\dim \mathfrak{s} - \operatorname{rank} \mathfrak{s}$ is even, $\operatorname{rank} \mathfrak{z}_{\mathfrak{h}}(\rho_2(\mathfrak{f})) \leqslant \operatorname{rank} \mathfrak{z}_{\mathfrak{h}}(\rho_1(\mathfrak{f})) - 2$. Therefore, $\operatorname{rank} \mathfrak{z}_{\mathfrak{h}}(\rho_2(\mathfrak{f})) = 0$. Contradiction.

It follows from [3], Table 25, that every simple subalgebra in $\mathfrak{h}$ of rank greater then 1 is contained in a maximal regular subalgebra. There are three maximal regular subalgebras $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3 \subset \mathfrak{h}$, $\mathfrak{h}_1 \cong \mathfrak{so}_9, \mathfrak{h}_2 \cong \mathfrak{sp}_6 \times \mathfrak{sl}_2, \mathfrak{h}_3 \cong \mathfrak{sl}_3 \times \mathfrak{sl}_3$. See, for example, [3], Table 12, (there is a mistake in this table: the subalgebra of F_4 isomorphic to $\mathfrak{sl}_4 \times \mathfrak{sl}_2$ is not maximal, it is contained in $\mathfrak{h}_1$). One may assume that $\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3$ contain a Cartan subalgebra $\mathfrak{t} \subset \mathfrak{h}$.

All simple subalgebras of rank greater then 1 in $\mathfrak{sp}_6$ are regular. All embeddings $\rho : \mathfrak{f} \hookrightarrow \mathfrak{h}_1$ such that the subalgebra $\rho(\mathfrak{f})$ is not regular are listed up to $\mathrm{Int}(\mathfrak{h}_1)$-conjugacy in the following table:

$\mathfrak{f}$	ρ
$\mathfrak{sl}_4$	$R(\pi_2) \oplus 3R(0)$
$\mathfrak{so}_7$	$R(\pi_3) \oplus R(0)$
G_2	$R(\pi_1) \oplus 2R(0)$
$\mathfrak{so}_5$	$R(\pi_2) \oplus R(\pi_2) \oplus R(0)$
$\mathfrak{sl}_3$	$\mathrm{ad} \oplus R(0)$

In the second column we list the linear representations in $\mathbb{C}^9$ corresponding to ρ. They determine ρ up to $\mathrm{Int}(\mathfrak{h}_1)$-conjugacy.

Since all simple subalgebras of rank greater then 1 in $\mathfrak{h}_3$ are isomorphic to $\mathfrak{sl}_3$, we see that $\mathfrak{f} \cong \mathfrak{sl}_3$. Denote by ρ_1 the embedding of $\mathfrak{sl}_3$ into $\mathfrak{so}_9$ listed in the table. It follows from Table 25 in [3] that the restriction of the simplest representation of $\mathfrak{h}$ to $\rho_1(\mathfrak{f})$ is isomorphic to $3\,\mathrm{ad} \oplus 2R(0)$.

It is easy to see that $N_H(\mathfrak{h}_3)/N_H(\mathfrak{h}_3)^\circ$ is a group of order 2. The group $N_H(\mathfrak{h}_3)$ contains an outer automorphism of $\mathfrak{h}_3$ which acts on $\mathfrak{t}$ by multiplication by -1.

Now we introduce some notation. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{h}$, $\varepsilon_i, i = 1, \ldots, 4$, its orthonormal basis, so that

$$\Delta(\mathfrak{h}) = \Big\{ \pm\varepsilon_i \pm \varepsilon_j, i \neq j, \pm\varepsilon_i, \frac{\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4}{2} \Big\}.$$

Put $\alpha_1 = (\varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2, \alpha_2 = \varepsilon_4, \alpha_3 = \varepsilon_3 - \varepsilon_4, \alpha_4 = \varepsilon_2 - \varepsilon_3$. This is a set of simple roots of $\mathfrak{h}$. Denote by h_1, h_2 simple coroots of $\mathfrak{f}$.

A set of simple roots for $\mathfrak{h}_3$ is $(\varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4)/2, \alpha_1, \alpha_3, \alpha_4$. Highest weights for the restriction of the simplest representation of $\mathfrak{h}$ into $\mathfrak{h}_3$ are ε_1, $(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon_4)/2, (\varepsilon_1 + \varepsilon_2 + \varepsilon_3 - \varepsilon_4)/2$.

There are two equivalence classes of the embeddings of $\mathfrak{sl}_3$ into $\mathfrak{h}_3$ up to $N_H(\mathfrak{h}_3)$-conjugation. Only one of these embeddings is GL_{26}-equivalent to ρ_1, namely the one with $h_1 \mapsto \varepsilon_1 + 2\varepsilon_2 + \varepsilon_4,\ h_2 \mapsto \varepsilon_1 - \varepsilon_2 - 2\varepsilon_4$. We denote this embedding by ρ_2.

It remains to prove that $\rho_1, \rho_2 : \mathfrak{f} \hookrightarrow \mathfrak{h}$ are H-equivalent. Since an outer automorphism of $\mathfrak{f}$ is contained in both $N_H(\rho_1(\mathfrak{f})), N_H(\rho_2(\mathfrak{f}))$, it is enough to show that the subalgebras $\rho_1(\mathfrak{f}), \rho_2(\mathfrak{f})$ are H-conjugate.

Denote by $\mathfrak{h}_4$ the regular subalgebra of $\mathfrak{h}$ corresponding to the set of roots of maximal length. The subalgebra $\mathfrak{h}_4$ is isomorphic to $\mathfrak{so}_8$ and is contained in $\mathfrak{h}_1$. One may assume that $\rho_1(\mathfrak{f}) \subset \mathfrak{h}_4$. It is clear that $N_H(\mathfrak{t}) \subset N_H(\mathfrak{h}_4)$. Therefore $\mathrm{Ad}(N_H(\mathfrak{h}_4)) = \mathrm{Aut}(\mathfrak{h}_4)$. It follows from the description of automorphisms of finite order of simple Lie algebras (see, for example, [5], Ch. 4, §4) that there exists a $t \in N_H(\mathfrak{h}_4)$ such that $\mathrm{Ad}_{\mathfrak{h}_4} t$ is an element of order 3 and $\mathfrak{h}_4^t = \rho_1(\mathfrak{f})$. The centralizer $Z_H(\mathfrak{h}_4)$ is a finite group of order 2. Since $t^3 \in Z_H(\mathfrak{h}_4)$, it follows that $\mathrm{Ad}(t)$ has finite order (3 or 6). $\mathfrak{h}^t$ is a regular subalgebra of rank 4 in $\mathfrak{h}$ and contains $\rho_1(\mathfrak{h})$. Thus $\mathfrak{h}^t$ is not contained in a subalgebra conjugate to $\mathfrak{h}_2$.

Let us note that every proper subalgebra of $\mathfrak{h}_1$ containing $\rho_1(\mathfrak{f})$ coincides with $\mathfrak{h}_4$ or $\rho_1(\mathfrak{f})$. Indeed, let $\widetilde{\mathfrak{f}}$ be such a subalgebra. Since the centralizer of $\mathfrak{f}$ in $\mathfrak{so}_9$ is trivial, we obtain that $\widetilde{\mathfrak{f}}$ is semisimple. The algebra $\mathfrak{so}_9$ does not contain a subalgebra isomorphic to $\mathfrak{sl}_3 \times \mathfrak{sl}_3$. Hence $\mathfrak{f}$ is simple. If $\mathfrak{f}$ is not conjugate to $\mathfrak{so}_8$, then $\mathfrak{f}$ is not regular. It follows from the previous table that $\widetilde{\mathfrak{f}} \cong \mathfrak{so}_7$ or $\mathfrak{f}$. A subalgebra of $\mathfrak{so}_7$ isomorphic to $\mathfrak{sl}_3$ is unique up to conjugation and its centralizer is non-trivial. It follows that $\widetilde{\mathfrak{f}} = \mathfrak{f}$.

Since the order of t is divisible by 3, $\mathfrak{h}^t$ is not conjugate to $\mathfrak{h}_4$. Thus, $\mathfrak{h}^t$ is not contained in a subalgebra conjugate to $\mathfrak{h}_1$.

It follows that $\rho_1(\mathfrak{f})$ is contained in a subalgebra conjugate to $\mathfrak{h}_3$. This completes the proof.

12. The algebra $\mathbb{C}[\mathfrak{h}^n]^{\mathrm{GL}_m}$

Proof of Proposition 1.5. It is known from the classical invariant theory that the algebra $\mathbb{C}[\mathfrak{h}^n]^G$ is generated by the polynomials $\mathrm{tr}(X_{i_1}X_{i_2}\ldots X_{i_k})$. Denote by A the subalgebra of the tensor algebra $T\mathfrak{h}^*$ generated by the elements of the form

$$g(L_1, \ldots, L_d), \tag{2}$$

where $g \in S^d(\mathfrak{h}^*)^G$, L_i are Lie polynomials in $X_1, \ldots, X_k$.

Put $f_k = \mathrm{tr}(X_1 \ldots X_k)$. Using the polarization, we reduce the required statement to the following one:

$f_k \in A$ for every positive integer k.

The proof of the last statement is by induction on k. The case $k = 1$ is trivial. Now assume that we are done for $k < l$.

The symmetric group S_l acts on the space $(\mathfrak{h}^*)^{\otimes l}$ by permuting the factors in a tensor product. It is clear that $A \cap (\mathfrak{h}^*)^{\otimes l}$ is invariant under this action. For every transposition $\sigma_i = (i, i+1)$ one has

$$(\sigma_i f_l)(X_1, \ldots, X_l) - f_l(X_1, \ldots, X_l) = \mathrm{tr}(X_1 \ldots [X_i, X_{i+1}] \ldots X_k).$$

Therefore, $\sigma_i f_l - f_l \in A$. It follows that $f_l - \sigma f_l \in A$ for every $\sigma \in S_l$. Hence, $f_l \in A$ if and only if

$$\frac{1}{l!}\sum_{\sigma \in S_l} \sigma f_l \in A.$$

But the latter is an element of $S^l(\mathfrak{h}^*)^G$ and lies in A by definition. ■

Corollary 12.1. *Let $G = \mathrm{GL}_m$, $\mathfrak{h}$ be a subalgebra of $\mathfrak{g} = \mathfrak{gl}_m$, $\overline{H} = N_G(\mathfrak{h})/Z_G(\mathfrak{h})$. Suppose that $\psi_1 : \mathbb{C}[\mathfrak{h}]^G \to \mathbb{C}[\mathfrak{h}]^{\overline{H}}$ is an isomorphism and $(\mathfrak{h}, \overline{H})$ is one of the following pairs: $(\mathfrak{sl}_k, \mathrm{Ad}(\mathrm{SL}_k))$, $(\mathfrak{so}_k, \mathrm{Ad}(\mathrm{O}_k))$, $(\mathfrak{sp}_{2k}, \mathrm{Ad}(\mathrm{Sp}_{2k}))$. Then ψ_n is an isomorphism.*

Proof. Let ρ be a representation $\mathfrak{h} \hookrightarrow \mathfrak{g}$ and ι be the tautological representation of $\mathfrak{h}$. Proposition 1.5 implies $\mathbb{C}[\mathfrak{h}^n]^\rho = \mathbb{C}[\mathfrak{h}^n]^\iota$. It follows from the classical invariant theory that $\mathbb{C}[\mathfrak{h}^n]^\iota = \mathbb{C}[\mathfrak{h}^n]^{\overline{H}}$ and we are done. ■

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Ivan Losev
Dept. of Math., MIT
77 Massachusetts Avenue
Cambridge, MA 02139, USA
ivanlosev@math.mit.edu

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