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A Quantum Type Deformation of the Cohomology Ring of Flag Manifolds

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Abstract. Let $q_1, \dots, q_n$ be some variables and consider the ring $K := \mathbb{Z}[q_1, \dots, q_n]/(\prod_{i=1}^n q_i)$. We show that there exists a K-bilinear product $\star$ on $H^*(F_n;\mathbb{Z}) \otimes K$ which is uniquely determined by some quantum cohomology like properties (most importantly, a degree two relation involving the generators and an analogue of the flatness of the Dubrovin connection). Then we prove that $\star$ satisfies the Frobenius property with respect to the Poincaré pairing of $H^*(F_n;\mathbb{Z})$; this leads immediately to the orthogonality of the corresponding Schubert type polynomials. We also note that if we pick $k \in \{1, \dots, n\}$ and we formally replace q_k by 0, the ring $(H^*(F_n;\mathbb{Z}) \otimes K, \star)$ becomes isomorphic to the usual small quantum cohomology ring of F_n, by an isomorphism which is described precisely.
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1. Introduction

We consider the complex flag manifold

$$F_n = \{V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset \mathbb{C}^n \mid V_k \text{ is a } k\text{-dimensional linear subspace of } \mathbb{C}^n\}.$$

In the following, the coefficient ring for cohomology will be always $\mathbb{Z}$, unless otherwise specified. For every $k \in \{1, \dots, n-1\}$ we consider the tautological vector bundle $\mathcal{V}_k$ over F_n and the cohomology class $y_k = -c_1(\det \mathcal{V}_k) \in H^2(F_n)$. It is known that the cohomology classes $y_1, \dots, y_{n-1}$ generate the ring $H^*(F_n)$. To describe the ideal of relations, it is convenient to consider the classes $x_k := y_k - y_{k-1}$, $1 \le k \le n$ (where we assign $y_0 = y_n := 0$). If $Y_1, \dots, Y_{n-1}$ are some variables, we set

$$X_k := Y_k - Y_{k-1},$$

where $Y_0 = Y_n := 0$. We denote by $S(X_1, \dots, X_n)$ the ideal of $\mathbb{Z}[X_1, \dots, X_n]$ generated by the symmetric polynomials with zero constant term. By a theorem

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of Borel, there is a ring isomorphism

$$H^*(F_n) \simeq \mathbb{Z}[Y_1, \dots, Y_{n-1}]/S(X_1, \dots, X_n). \tag{1}$$

Here y_i is mapped to the coset of Y_i, for all $i \in \{1, \dots, n-1\}$.

The small quantum cohomology ring $qH^*(F_n)$ is a deformation with $n-1$ parameters of $H^*(F_n)$. A theorem of Givental and Kim [5] describes $qH^*(F_n)$ as a quotient of a certain polynomial ring by the ideal generated by the integrals of motions of the open Toda lattice. In this paper we consider the "periodic" version of the latter (quotient) ring: that is, the ideal of relations is generated by the integrals of motion of the *periodic* Toda lattice. We address the following question: does this new ring arise in the spirit of Givental and Kim from a deformation of $H^*(F_n)$? We construct such a deformation, give a list of properties that characterizes it uniquely, and then study it briefly.

Let us be more precise. We consider n variables $q_1, \dots, q_n$ and set

$$K := \mathbb{Z}[q_1, \dots, q_n]/(\prod_{i=1}^{n} q_i).$$

The main result of this paper is as follows.

Theorem 1.1. *There exists a product $\star$ on $H^*(F_n) \otimes K$ which is uniquely determined by the following properties:*

(i) *$\star$ is K-bilinear*

(ii) *$\star$ preserves the grading induced by the usual grading of $H^*(F_n)$ combined with $\deg q_j = 4$, $1 \leq j \leq n$*

(iii) *$\star$ is a deformation of the usual product, in the sense that if we formally replace all q_j by 0, we obtain the usual (cup-)product on $H^*(F_n)$*

(iv) *$\star$ is commutative*

(v) *$\star$ is associative*

(vi) *we have the relation*

$$\sum_{1 \leq i < j \leq n} x_i \star x_j + \sum_{j=1}^{n} q_j = 0$$

(vii) *the coefficients $(y_i \star a)_d \in H^*(F_n)$ of $q^d := q_1^{d_1} \dots q_n^{d_n}$ in $y_i \star a$ satisfy*

$$(d_i - d_n)(y_j \star a)_d = (d_j - d_n)(y_i \star a)_d,$$

for any $d = (d_1, \dots, d_n) \geq 0$, $a \in H^*(F_n)$, and $1 \leq i, j \leq n-1$.

The product $\star$ is constructed in Section 3. The main input consists of the conservation laws of the periodic quantum Toda lattice (cf. [6], [9]). From the construction we can see that we have a ring isomorphism

$$(H^*(F_n) \otimes K, \star) \simeq \mathbb{Z}[Y_1, \dots, Y_{n-1}, q_1, \dots, q_n]/(\mathcal{R}_1, \dots, \mathcal{R}_n), \tag{2}$$

where $\mathcal{R}_1, \ldots, \mathcal{R}_n$ are essentially the integrals of motion of the periodic Toda lattice. They are described explicitly by equation (6) below. We note at this point that $\mathcal{R}_n = \pm \prod_{i=1}^n q_i$. This explains why the "quantum parameter" ring of our deformation of $H^*(F_n)$ is no longer a polynomial ring, like for $qH^*(F_n)$, but rather the somewhat awkward ring $\mathbb{Z}[q_1, \ldots, q_n]/(\prod_{i=1}^n q_i)$.

We will also prove some properties of the product $\star$. The first one involves the Poincaré pairing $(\ ,\ )$ on $H^*(F_n)$; we actually extend it to a K-bilinear form on $H^*(F_n) \otimes K$.

Theorem 1.2. *The product $\star$ satisfies the following Frobenius type property:*

$$(a \star b, c) = (a, b \star c)$$

for any $a, b, c \in H^*(F_n)$.

We also consider representatives of Schubert cohomology classes in $H^*(F_n)$ via the isomorphism given by (2), which are polynomials in $Y_1, \ldots, Y_{n-1}, q_1, \ldots, q_n$. Theorem 1.2 will allow us to prove that these polynomials satisfy a certain orthogonality relation, similar to the one satisfied by the quantum Schubert polynomials of [2] and [8] (see Proposition 4.1 below).

Finally, we consider the ring obtained from $(H^*(F_n) \otimes K, \star)$ by formally setting a certain q_k to zero. More precisely, we denote by (q_k) the set of all elements of $H^*(F_n) \otimes K$ that are multiples of q_k and take the quotient

$$\begin{aligned}(H^*(F_n) \otimes K)/(q_k) &= H^*(F_n) \otimes \mathbb{Z}[q_1, \ldots, q_{k-1}, q_{k+1}, \ldots, q_n] \\ &= \mathbb{Z}[x_1, \ldots, x_n, q_1, \ldots, q_{k-1}, q_{k+1}, \ldots, q_n]/S(x_1, \ldots, x_n).\end{aligned}$$

Here $S(x_1, \ldots, x_n)$ denotes the ideal of $\mathbb{Z}[x_1, \ldots, x_n]$ which is generated by the symmetric polynomials with zero constant term. The space $(H^*(F_n) \otimes K)/(q_k)$ can be equipped with the product induced by $\star$. The resulting ring is isomorphic to the actual small quantum cohomology ring $qH^*(F_n)$: the following theorem describes this ring isomorphism. We recall (cf. e.g. [4]) that the ring $qH^*(F_n)$ consists of the space

$$H^*(F_n) \otimes \mathbb{Z}[Q_1, \ldots, Q_{n-1}] = \mathbb{Z}[X_1, \ldots, X_n, Q_1, \ldots, Q_{n-1}]/S(X_1, \ldots, X_n),$$

which is equipped with a certain (quantum) product.

Theorem 1.3. *Fix $k \in \{1, \ldots, n\}$. The ring $((H^*(F_n) \otimes K)/(q_k), \star)$ is isomorphic to $qH^*(F_n)$. The isomorphism is the map from*

$$\mathbb{Z}[X_1, \ldots, X_n, Q_1, \ldots, Q_{n-1}]/S(X_1, \ldots, X_n)$$

to

$$\mathbb{Z}[x_1, \ldots, x_n, q_1, \ldots, q_{k-1}, q_{k+1}, \ldots, q_n]/S(x_1, \ldots, x_n)$$

which is given by

$$X_i \mapsto x_{k-i+1}, 1 \le i \le n, \quad Q_j \mapsto q_{k-j}, 1 \le j \le n-1,$$

where the indices are evaluated modulo n.

Remarks. 1. A characterization of the quantum cohomology ring $qH^*(F_n)$, similar in spirit to the one given by Theorem 1.1, has been obtained by the author in [11]; see also Theorem 2.1 below.

2. The product $\star$ considered in this paper could also be relevant in the context of the small quantum cohomology of the infinite dimensional flag manifold $\mathcal{F}\ell^{(n)}$. The ring $qH^*(\mathcal{F}\ell^{(n)})$ is conjecturally defined and investigated by Guest and Otofuji in [7] and the author of the present paper in [10]. More precisely, in the latter paper we have considered a product $\circ$ on $H^*(\mathcal{F}\ell^{(n)};\mathbb{R})\otimes\mathbb{R}[q_1,\ldots,q_n]$ which satisfies certain natural properties similar to (i)-(vii) in Theorem 1.1 above; we have also explained why a quantum product $\circ$ resulting from Gromov-Witten invariants of $\mathcal{F}\ell^{(n)}$ should satisfy these properties. However, it is still an open question whether such a product exists. As long as nobody gave a rigorous definition (in terms of stable curves) of $qH^*(\mathcal{F}\ell^{(n)})$, it seems natural to attempt to construct it by other means, for instance with the methods of this paper. In fact, the ring we are constructing and studying here is essentially the quotient of $qH^*(\mathcal{F}\ell^{(n)})$ by the product $\prod_{i=1}^n q_i$.

3. Let G be a compact simple simply connected Lie group and $T\subset G$ a maximal torus. It is likely that some of the results of this paper can be extended to the flag manifold G/T, where G is of type other than A. For instance, if $\ell := \dim T$, we expect that an extension of $H^*(G/T;\mathbb{R})$ with $\ell+1$ parameters can be constructed with the methods used in Section 3 below. More precisely, the ring of "quantum parameters" would be in this case $K=\mathbb{R}[q_1,\ldots,q_{\ell+1}]/(q_1^{m_1}\cdots q_\ell^{m_\ell}q_{\ell+1})$. The numbers $m_1,\ldots,m_\ell$ arise from the expansion

$$\alpha_0^\vee = m_1\alpha_1^\vee+\ldots+m_\ell\alpha_\ell^\vee,$$

where $\{\alpha_1,\ldots,\alpha_\ell\}$ is a simple root system of G, α_0 the highest root, and the superscript $\vee$ indicates the corresponding coroot. However, it should be noted that the construction could only work for G of certain types (e.g. A, B or C), for exactly the same reason as in [10]. We will probably explore this more general situation elsewhere.

The paper is organized as follows: First we prove Theorem 1.3 and the uniqueness part of Theorem 1.1. After that we prove the existence part of the latter theorem, which uses a presentation of the ring $(H^*(F_n)\otimes K,\star)$ in terms of generators and relations. Finally we prove the Frobenius property and the orthogonality of the Schubert type polynomials. Such polynomials are determined explicitly in the case $n=3$.

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2. Uniqueness of the product $\star$

The goals of this section are: show that there exists *at most* one product $\star$ with the properties (i)-(vii) in Theorem 1.1; prove Theorem 1.3. The main instrument is the following result, which is a particular case of [11, Theorem 1.1].

Theorem 2.1. ([11]) *Let $Y_1, \dots, Y_{n-1}, Q_1, \dots, Q_{n-1}$ be some variables and set $X_j := Y_j - Y_{j-1}$ for $1 \le j \le n$, where $Y_0 = Y_n := 0$. Denote by $S(X_1, \dots, X_n)$ the ideal of $\mathbb{Z}[Y_1, \dots, Y_{n-1}]$ generated by the symmetric polynomials with zero constant term in $X_1, \dots, X_n$. There exists a product $\circ$ on*

$$(\mathbb{Z}[Y_1, \dots, Y_{n-1}]/S(X_1, \dots, X_n)) \otimes \mathbb{Z}[Q_1, \dots, Q_{n-1}]$$

which is uniquely determined by the following properties:

(i) $\circ$ *is* $\mathbb{Z}[Q_1, \dots, Q_{n-1}]$*-bilinear*

(ii) $\deg([f] \circ [g]) = \deg[f] + \deg[g]$, *for any* $f, g \in \mathbb{Z}[Y_1, \dots, Y_{n-1}]$ *which are homogeneous (the brackets* $[\ ]$ *indicate the coset modulo* $S(X_1, \dots, X_n)$ *and the grading is given by* $\deg Y_i := 2$, $\deg Q_i := 4$*)*

(iii) $\circ$ *is a deformation of the canonical product on* $\mathbb{Z}[Y_1, \dots, Y_{n-1}]/S(X_1, \dots, X_n)$

(iv) $\circ$ *is commutative*

(v) $\circ$ *is associative*

(vi) we have

$$\sum_{1 \le i < j \le n} [X_i] \circ [X_j] + \sum_{i=1}^{n-1} Q_i = 0$$

(vii) we have

$$Q_j \frac{\partial}{\partial Q_j}([Y_i] \circ [f]) = Q_i \frac{\partial}{\partial Q_i}([Y_j] \circ [f])$$

for any $1 \le i, j \le n-1$ *and any* $f \in \mathbb{Z}[Y_1, \dots, Y_{n-1}]$.

The product $\circ$ mentioned in this theorem is induced by the small quantum cohomology ring $qH^*(F_n)$ via the Borel isomorphism (1).

Let now $\star$ be a product which satisfies the assumptions (i)-(vii) in Theorem 1.1. The ring $(H^*(F_n) \otimes K, \star)$ is generated by $y_1, \dots, y_{n-1}, q_1, \dots, q_n$. Consequently, as a K-algebra, it is generated by $y_1, \dots, y_{n-1}$. Let us consider the Schubert basis $\{\sigma_w \mid w \in S_n\}$ of $H^*(F_n)$. Here S_n is the symmetric group and σ_w are cosets in $\mathbb{Z}[x_1, \dots, x_n]/S(x_1, \dots, x_n)$ of the Schubert polynomials (cf. e.g. [3, Ch. 10]). We also consider the expansion

$$y_i \star \sigma_w = \sum_{v \in S_n} \omega_i^{vw} \sigma_v \tag{3}$$

where $\omega_i^{vw} \in K$. We will show that for any i, v, and w, the coefficient ω_i^{vw} is prescribed.

To this end, we first consider the quotient

$$\begin{aligned}(H^*(F_n) \otimes K)/(q_n) &= H^*(F_n) \otimes \mathbb{Z}[q_1, \dots, q_{n-1}] \\ &= \mathbb{Z}[x_1, \dots, x_n, q_1, \dots, q_{n-1}]/S(x_1, \dots, x_n)\end{aligned}$$

and equip it with the product induced by $\star$. Let us denote this new product by $\star_n$. It is commutative, associative, and satisfies the obvious grading condition. Moreover, for any $a \in H^*(F_n)$ and any $d = (d_1, \ldots, d_{n-1}, 0) \geq 0$ we have

$$d_i(y_j \star_n a)_d = d_j(y_i \star_n a)_d,$$

for all $1 \leq i, j \leq n-1$. This implies that

$$q_i \frac{\partial}{\partial q_i}(y_j \star_n a) = q_j \frac{\partial}{\partial q_j}(y_i \star_n a),$$

for all $i, j \in \{1, \ldots, n-1\}$ and all $a \in H^*(F_n)$. Assumption (vi) in Theorem 1.1 implies that

$$\sum_{1 \leq i < j \leq n} x_i \star_n x_j + \sum_{i=1}^{n-1} q_i = 0.$$

From Theorem 2.1 we deduce that $\star_n = \circ$, via the identifications

$$y_j = Y_j, \quad q_j = Q_j, \quad 1 \leq j \leq n-1.$$

This implies Theorem 1.3 for $k = n$: we also use the fact that the map

$$X_i \mapsto X_{n-i+1}, 1 \leq i \leq n, \quad Q_j \mapsto Q_{n-j}, 1 \leq j \leq n-1$$

is an automorphism of $qH^*(F_n)$. Another consequence of the fact that $\star_n = \circ$ concerns the coefficient ω_i^{vw} in the expansion given by equation (3): namely, the expression $\omega_i^{vw}|_{q_n=0}$ is prescribed.

Let us now consider the quotient

$$\begin{aligned}(H^*(F_n) \otimes K)/(q_1) &= H^*(F_n) \otimes \mathbb{Z}[q_2, \ldots, q_n] \\ &= \mathbb{Z}[y_1, \ldots, y_{n-1}, q_2, \ldots, q_n]/S(x_1, \ldots, x_n)\end{aligned}$$

with the product induced by $\star$. We denote it by $\star_1$. Assumption (vii) in Theorem 1.1 implies that for any $a \in H^*(F_n)$ and any $d = (0, d_2, \ldots, d_n)$ we have

$$d_n[(y_1 - y_j) \star_1 a]_d = d_j(y_1 \star_1 a)_d, \tag{4}$$

as well as

$$d_i[(y_1 - y_j) \star_1 a]_d = d_j[(y_1 - y_i) \star_1 a]_d, \tag{5}$$

for all $2 \leq i, j \leq n-1$. Let us replace

$$y_1 = Y_1, y_1 - y_{n-1} = Y_2, y_1 - y_{n-2} = Y_3, \ldots, y_1 - y_2 = Y_{n-1},$$

and

$$q_n = Q_1, q_{n-1} = Q_2, \ldots, q_2 = Q_{n-1}.$$

As usual, we set $X_j := Y_j - Y_{j-1}$, for $1 \leq j \leq n$, where $Y_0 = Y_n := 0$. Then we have

$$X_i = x_{2-i}$$

for all $1 \leq i \leq n$, where the indices are evaluated modulo n. This implies that

$$H^*(F_n) = \mathbb{Z}[Y_1, \ldots, Y_{n-1}]/S(X_1, \ldots, X_n).$$

This also implies that

$$\sum_{1 \leq i < j \leq n} X_i \star_1 X_j + \sum_{i=1}^{n-1} Q_i = 0.$$

From equations (4) and (5) we deduce that

$$Q_i \frac{\partial}{\partial Q_i}(Y_j \star_1 a) = Q_j \frac{\partial}{\partial Q_j}(Y_i \star_1 a)$$

for all $1 \leq i, j \leq n-1$. All the hypotheses of Theorem 2.1 are verified by $\star_1$. Consequently, we have $\star_1 = \circ$. Thus, for any $1 \leq i \leq n-1$ and any $v, w \in S_n$, the expression $\omega_i^{vw}|_{q_1=0}$ is prescribed.

A similar reasoning can be made when we set q_k to zero for an arbitrary k. It is obvious that if $i \in \{1, \ldots, n-1\}$ and $v, w \in S_n$, there exists *at most* one $\omega_i^{vw} \in K$ with all $\omega_i^{vw}|_{q_1=0}, \ldots, \omega_i^{vw}|_{q_n=0}$ prescribed. This proves the uniqueness part of Theorem 1.1. Theorem 1.3 follows also.

3. Construction of the product $\star$

The construction is related to the periodic Toda lattice of type A. The following determinant expansion arises when describing the conserved quantities of this integrable system (see the introduction of [7] and the references therein).

$$\det\left[\begin{pmatrix} Y_1 & q_1 & 0 & 0 & \ldots & 0 & -z \\ -1 & Y_2 - Y_1 & q_2 & 0 & \ldots & 0 & 0 \\ 0 & -1 & Y_3 - Y_2 & q_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & 0 & -1 & Y_{n-1} - Y_{n-2} & q_{n-1} \\ q_n/z & 0 & \ldots & 0 & 0 & -1 & -Y_{n-1} \end{pmatrix} + \mu I_n\right] \quad (6)$$
$$= \sum_{i=0}^{n} \mathcal{R}_{i-1}\mu^{n-i} + \mathcal{R}_n \frac{1}{z} - z.$$

We can easily see that $\mathcal{R}_{-1} = 1$, $\mathcal{R}_0 = 0$,

$$\mathcal{R}_1 = \sum_{1 \leq i < j \leq n} (Y_i - Y_{i-1})(Y_j - Y_{j-1}) + \sum_{j=1}^{n} q_j$$

(where $Y_0 = Y_n := 0$), and

$$\mathcal{R}_n = (-1)^{n-1} q_1 \ldots q_n.$$

We consider the variables $t_1, \ldots, t_{n-1}, q_n$ and the differential operators

$$\partial_i^n := \frac{\partial}{\partial t_i} - q_n \frac{\partial}{\partial q_n},$$

$1 \le i \le n-1$. Then we consider the ring

$$\begin{aligned}\mathcal{D}^h &:= \mathbb{C}[t_1, \ldots, t_{n-1}, e^{t_1}, \ldots, e^{t_{n-1}}, q_n, h\partial_1^n, \ldots, h\partial_{n-1}^n, h] \\ &= \mathbb{C}[\{t_i\}, \{e^{t_i}\}, q_n, \{h\partial_i^n\}, h],\end{aligned}$$

where h is a formal variable. This ring is obviously non-commutative. It is graded with respect to

$$\deg t_i = 0, \quad \deg \partial_i^n = 0, \quad \deg h = 2, \quad \deg e^{t_i} = 4, \quad \deg q_n = 4,$$

for $1 \le i \le n-1$. The elements of $\mathcal{D}^h$ should be regarded as differential operators on the space $\mathbb{C}[t_1, \ldots, t_{n-1}, e^{t_1}, \ldots, e^{t_{n-1}}, q_n]$. In each $\mathcal{R}_k$ we formally replace as follows:

- Y_i by $h\partial_i^n$, $1 \le i \le n-1$,
- q_i by e^{t_i}, for $1 \le i \le n-1$ (q_n remains unchanged).

The resulting element of $\mathcal{D}^h$ will be denoted by D_k^h. By considering the expansion of the determinant given by equation (6) we can see that each D_k^h is a linear combination of differential operators which are represented by monomials of the form $(h\partial/\partial t_1)^{a_1} \ldots (h\partial/\partial t_{n-1})^{a_{n-1}} (hq_n\partial/\partial q_n)^{a_n} (e^{t_1})^{b_1} \ldots (e^{t_{n-1}})^{b_{n-1}} q_n^{b_n}$, where $a_k b_k = 0$ for all $1 \le k \le n$. This means that D_k^h can be obtained by simply making the replacements indicated above in the determinant given by (6) and then expanding the determinant without any concern about the lack of commutativity of the ring $\mathcal{D}^h$.

We also consider the quotient ring

$$M^h := \mathcal{D}^h / \langle D_1^h, \ldots, D_{n-1}^h, e^{t_1} \cdots e^{t_{n-1}} q_n \rangle, \tag{7}$$

where the brackets $\langle\ \rangle$ denote the left ideal. We mention that

$$\partial_i^n (e^{t_1} \cdots e^{t_{n-1}} q_n) = 0, \ \forall i \in \{1, \ldots, n-1\},$$

thus $e^{t_1} \cdots e^{t_{n-1}} q_n$ is a central element of $\mathcal{D}^h$.

We will need the following theorem, which is a straightforward consequence of [6, Lemma 3.5].

Theorem 3.1. ([6]) *The ring*

$$\mathbb{C}[e^{t_1}, \ldots, e^{t_{n-1}}, q_n, h\partial_1^n, \ldots, h\partial_{n-1}^n, h] / \langle D_1^h, \ldots, D_{n-1}^h \rangle$$

regarded as a left $\mathbb{C}[\{e^{t_i}\}, q_n, h]$*-module has dimension* $n!$. *More precisely, any* $\mathbb{Z}$*-linear basis* $\{[f_w] \mid w \in S_n\}$ *of* $H^*(F_n)$, *with* $f_w \in \mathbb{Z}[Y_1, \ldots, Y_{n-1}]$, *induces the basis of the previously mentioned module which consists of the cosets of the polynomial expressions* $f_w(h\partial_1, \ldots, h\partial_{n-1})$, $w \in S_n$.

Let us fix a $\mathbb{Z}$-linear basis $\{[f_w] \mid w \in S_n\}$ of $H^*(F_n)$ as in the previous theorem. We may assume that $f_{\tau_{i,i+1}} = Y_i$, for all $1 \leq i \leq n-1$, where $\tau_{i,i+1}$ is the transposition of i and $i+1$ in S_n (for instance, we can take the Schubert basis). We choose an ordering of S_n (i.e. of the elements of the basis) such that if $l(v) < l(w)$ then v comes before w. In this way, the basis $\{[f_w] \mid w \in S_n\}$ of $H^*(F_n)$ consists of $s_0 = 1$ elements of degree 0, followed by s_1 elements of degree 2, $\ldots$, followed by $s_m = 1$ elements of degree $2m := \dim F_n$. Linear endomorphisms of $H^*(F_n) \otimes \mathbb{C}$ can be identified with matrices of size $n! \times n!$ whose entries are in $\mathbb{C}$. Any such matrix appears as a block matrix of the type $A = (A_{\alpha\beta})_{1\leq\alpha,\beta\leq m}$. We say that a block matrix $A = (A_{\alpha\beta})_{1\leq\alpha,\beta\leq m}$ is r*-triangular* if $A_{\alpha\beta} = 0$ for all α, β with $\beta - \alpha < r$. We will use the splitting of a block matrix $A = (A_{\alpha\beta})_{1\leq\alpha,\beta\leq m}$ as

$$A = A^{[-m]} + \ldots + A^{[-1]} + A^{[0]} + A^{[1]} + \ldots + A^{[m]} \tag{8}$$

where each block matrix $A^{[r]}$ is r-diagonal.

We set

$$P_w := f_w(h\partial_1^n, \ldots, h\partial_{n-1}^n),$$

where $w \in S_n$. By Theorem 3.1, there exist uniquely determined polynomials $(\Omega_i^h)_{vw} \in \mathbb{C}[e^{t_1}, \ldots, e^{t_{n-1}}, q_n, h]$ such that

$$h\partial_i P_w = \sum_{v\in S_n} (\Omega_i^h)_{vw} P_v \text{ modulo } \langle D_1^h, \ldots, D_{n-1}^h \rangle, \tag{9}$$

for all $1 \leq i \leq n-1$ and all $w \in S_n$. We note that each $(\Omega_i^h)_{vw}$ is a homogeneous polynomial with respect to the grading given by

$$\deg h = 2, \quad \deg e^{t_i} = 4, 1 \leq i \leq n-1, \quad \deg q_n = 4.$$

We consider the matrices

$$\Omega_i^h := ((\Omega_i^h)_{vw})_{v,w\in S_n}, \quad \omega_i := \Omega_i^h \text{ modulo } h.$$

Let us also denote by $\mathbb{C}[\{t_i\}, \{e^{t_i}\}, q_n]^{(1)}$ the subspace of $\mathbb{C}[\{t_i\}, \{e^{t_i}\}, q_n]$ spanned by the non-constant monomials of type $t_1^{k_1} \cdots t_{n-1}^{k_{n-1}} e^{m_1 t_1} \cdots e^{m_{n-1} t_{n-1}} q_n^{m_n}$ with the property that if $k_1 + \ldots + k_{n-1} \geq 1$ then $m_1 \cdots m_n \neq 0$. The following result plays a central role in this section.

Proposition 3.2. *There exists a matrix* $U = (U_{vw})_{v,w\in S_n}$ *of the form*

$$U = (I + hV_1 + h^2 V_2 + \ldots + h^{m-2} V_{m-2}) V_0, \tag{10}$$

with the following properties

(a) V_0 *is a block matrix whose diagonal is* I*, such that* $V_0 - I$ *is* 2*-triangular and all entries of* V_0 *which are not on the diagonal are elements of the space* $\mathbb{C}[t_1, \ldots, t_{n-1}, e^{t_1}, \ldots, e^{t_{n-1}}, q_n]^{(1)}$

(b) for any $1 \leq j \leq m-2$*, the block matrix* V_j *is* $(j+2)$*-triangular and its entries are elements of the space* $\mathbb{C}[t_1, \ldots, t_{n-1}, e^{t_1}, \ldots, e^{t_{n-1}}, q_n]^{(1)}$

(c) each entry U_{vw} *is a polynomial in* $t_1,\dots,t_{n-1},e^{t_1},\dots,e^{t_{n-1}},q_n,h$*, homogeneous with respect to* $\deg t_i=0, \deg h=2, \deg e^{t_i}=\deg q_n=4$*, such that*

$$\deg(U_{vw})+\deg P_v=\deg P_w$$

(d) we have

$$\Omega_i^h U + h\partial_i^n(U) = UV_0^{-1}\omega_i V_0, \tag{11}$$

for all $1\le i\le n-1$*.*

In the following we will show how to construct the product $\star$ by using this proposition (its proof will be done after that). For any $w\in S_n$, set

$$\hat{P}_w := \sum_{v\in S_n} U_{vw}P_v,$$

which is an element of $\mathcal{D}^h$. Its coset modulo $\langle D_1^h,\dots,D_{n-1}^h, e^{t_1}\cdots e^{t_{n-1}}q_n\rangle$ is

$$[\hat{P}_w] := \sum_{v\in S_n} \bar{U}_{vw}[P_v], \tag{12}$$

where

$$\bar{U}_{vw} := U_{vw} \mod e^{t_1}\cdots e^{t_{n-1}}q_n.$$

Due to Proposition 3.2, $\bar{U}_{vw}$ is actually in $\mathbb{C}[e^{t_1},\dots,e^{t_{n-1}},q_n,h]$ (i.e. it is free of $t_1,\dots,t_{n-1}$). On the other hand, equation (9) implies that in M^h we have

$$h\partial_i^n[P_w] = \sum_{v\in S_n} (\bar{\Omega}_i^h)_{vw}[P_v],$$

where we have denoted

$$(\bar{\Omega}_i^h)_{vw} := (\Omega_i^h)_{vw} \mod e^{t_1}\cdots e^{t_{n-1}}q_n.$$

From equation (12) we deduce

$$h\partial_i^n[\hat{P}_w] = \sum_{v\in S_n} (\hat{\Omega}_i^h)_{vw}[\hat{P}_v], \tag{13}$$

where the matrix $\hat{\Omega}_i^h := ((\hat{\Omega}_i^h)_{vw})_{v,w\in S_n}$ is given by

$$\hat{\Omega}_i^h = \bar{U}^{-1}\bar{\Omega}_i^h\bar{U} + h\bar{U}^{-1}\partial_i^n(\bar{U}). \tag{14}$$

Here we have used the elementary fact that if $f\in\mathbb{C}[t_1,\dots,t_{n-1},e^{t_1},\dots,e^{t_{n-1}},q_n]$ then we have the equality

$$\partial_i^n\cdot f = \partial_i^n(f) + f\cdot\partial_i^n$$

as differential operators. Now from equation (11) we deduce that

$$\hat{\Omega}_i^h = \bar{V}_0^{-1}\bar{\omega}_i\bar{V}_0, \tag{15}$$

where

$$\bar{V}_0 := V_0 \text{ modulo } e^{t_1} \cdots e^{t_{n-1}} q_n,$$

and

$$\bar{\omega}_i := \omega_i \text{ modulo } e^{t_1} \cdots e^{t_{n-1}} q_n.$$

Indeed, on the one hand, (11) implies that

$$(\Omega_i^h \text{ mod})(U \text{ mod}) + h(\partial_i^n(U)) \text{ mod} = (U \text{ mod})(V_0 \text{ mod})^{-1}(\omega \text{ mod})(V_0 \text{ mod}),$$

where we have used the abbreviation

$$\text{mod} := \text{modulo } e^{t_1} \cdots e^{t_{n-1}} q_n;$$

on the other hand, we have

$$\partial_i^n(U) \text{ mod} = \partial_i^n(U \text{ mod}) = \partial_i^n(\bar{U}).$$

The cosets $[f_w(Y_1, \ldots, Y_{n-1})]_q$, $w \in S_n$, are a basis of

$$\mathbb{C}[Y_1, \ldots, Y_{n-1}, q_1, \ldots, q_n]/(\mathcal{R}_1, \ldots, \mathcal{R}_n) \tag{16}$$

over $K \otimes \mathbb{C}$. Here we have denoted by $[\ ]_q$ the cosets modulo $(\mathcal{R}_1, \ldots, \mathcal{R}_n)$. Another basis is given by the cosets of

$$\hat{f}_w := \sum_{v \in S_n} (\bar{V}_0)_{vw} f_v, \tag{17}$$

$w \in S_n$ (this follows from Proposition 3.2 (a)). In $(\bar{V}_0)_{vw}$ we have replaced $e^{t_1}, e^{t_2}, \ldots, e^{t_{n-1}}$ by $q_1, q_2, \ldots,$ respectively q_{n-1}. The map

$$\Phi : H^*(F_n) \otimes K \otimes \mathbb{C} \to \mathbb{C}[Y_1, \ldots, Y_{n-1}, q_1, \ldots, q_n]/(\mathcal{R}_1, \ldots, \mathcal{R}_n)$$

given by $[f_w] \mapsto [\hat{f}_w]_q$, for all $w \in S_n$, is an isomorphism of $K \otimes \mathbb{C}$-modules. We define a product $\star$ on $H^*(F_n) \otimes K \otimes \mathbb{C}$ by

$$[f_v] \star [f_w] := \Phi^{-1}([\hat{f}_v \hat{f}_w]_q),$$

for all $v, w \in S_n$.

Proposition 3.3. *(a) The ring $(H^*(F_n) \otimes K \otimes \mathbb{C}, \star)$ constructed above satisfies conditions (i)-(vii) in Theorem 1.1 with K replaced by $K \otimes \mathbb{C}$.*

(b) The assertion stated in Theorem 1.3 with K replaced by $K \otimes \mathbb{C}$ holds true.

(c) The subspace $H^*(F_n) \otimes K$ of $H^*(F_n) \otimes K \otimes \mathbb{C}$ is closed under $\star$. The ring $(H^*(F_n) \otimes K, \star)$ satisfies conditions (i)-(vii) of Theorem 1.1.

(d) The map Φ induces by restriction an isomorphism of K-modules

$$\Phi : (H^*(F_n) \otimes K, \star) \to \mathbb{Z}[Y_1, \ldots, Y_{n-1}, q_1, \ldots, q_n]/(\mathcal{R}_1, \ldots, \mathcal{R}_n).$$

This is the homomorphism of K-algebras which maps y_i to the coset of Y_i, for all $1 \leq i \leq n-1$.

Proof. (a) Only condition (vii) remains to be checked. First we note that the matrix of multiplication by Y_i on the space described by equation (16) with respect to the K-basis $\{[\hat{f}_w(Y_1,\dots,Y_{n-1})]_q\}_{w\in S_n}$ is $\hat{\Omega}_i^h|_{e^{t_1}=q_1,\dots,e^{t_{n-1}}=q_{n-1}}$ modulo $q_1\cdots q_n$ (this follows from equations (15), (17) and the fact that the matrix of multiplication by Y_i with respect to the K-basis $\{[f_w(Y_1,\dots,Y_{n-1})]_q \mid w\in S_n\}$ is $\bar{\omega}_i$). From equation (13), the identity $\partial_i^n\partial_j^n = \partial_j^n\partial_i^n$ for all $1\le i,j\le n-1$, and the fact that all $\hat{\Omega}_i^h$ are independent of h, we deduce that each entry of the matrix $\partial_i^n(\hat{\Omega}_j^h)-\partial_j^n(\hat{\Omega}_i^h)$ is a multiple of $e^{t_1}\cdots e^{t_{n-1}}q_n$, thus it is equal to zero in K. This implies condition (vii).

(b) The assertion can be proved by using the same method as in Section 2.

(c) The assertion follows immediately from the following two claims.

Claim 1. For any $i\in\{1,\dots,n-1\}$ and any $a\in H^*(F_n)$, the product $y_i\star a$ is in $H^*(F_n)\otimes K$.

Claim 2. Any $a\in H^*(F_n)$ can be written as $a=f(\{y_i\star\},\{q_j\})$, where f is a polynomial in $\mathbb{Z}[\{Y_i\},\{q_j\}]$.

To prove Claim 1, we consider again the Schubert basis $\{\sigma_w \mid w\in S_n\}$ of $H^*(F_n)$. We take $i\in\{1,\dots,n-1\}$ and $w\in S_n$ and consider $\omega_i^{vw}\in K\otimes\mathbb{C}$ given by

$$y_i\star\sigma_w=\sum_{v\in S_n}\omega_i^{vw}\sigma_v.$$

We show that $\omega_i^{vw}\in K$. This is an immediate consequence of the fact that for any $k\in\{1,\dots,n\}$ we have

$$\omega_i^{vw}|_{q_k=0}\in\mathbb{Z}[q_1,\dots,q_{k-1},q_{k+1},\dots,q_n].$$

To prove this, we take into account that, by point (b) above, the ring

$$((H^*(F_n)\otimes K\otimes\mathbb{C})/(q_k),\star)$$

is isomorphic to the quantum cohomology ring of F_n tensored with $\mathbb{C}$. Moreover, $\omega_i^{vw}|_{q_k=0}$ is a coefficient of the expansion of the quantum product of two elements of $H^*(F_n)$ (one of them of degree 2).

To prove Claim 2, we assume that a is homogeneous and we use induction by $\deg a$. If a has degree 0 or 2, the claim is obvious. At the induction step, we assume that the claim is true for any element of $H^*(F_n)$ of degree at most d and show that it is true for any y_ia, where $i\in\{1,\dots,n-1\}$ and $a\in H^*(F_n)$ with $\deg a\le d$. To this end, it is sufficient to note that $y_i\star a-y_ia$ is of the form described in the claim: this follows from Claim 1, point a) (especially conditions (ii) and (iii) in Theorem 1.1), and the induction hypothesis.

(d) We use the definition of $\star$ on $H^*(F_n)\otimes K\otimes\mathbb{C}$ and point c) above. ∎

Now comes the postponed proof of Proposition 3.2.

Proof of Proposition 3.2. We go along the lines of Amarzaya and Guest [1, Section 2]. Namely, for each $i\in\{1,\dots,n-1\}$ we write

$$\Omega_i^h=\omega_i+h\theta_i^{(1)}+h^2\theta_i^{(2)}+\ldots+h^{m-2}\theta_i^{(m-2)},$$

where the matrix ω_i is -1-triangular and $\theta_i^{(k)}$ is $k+1$-triangular, $1\le k\le m-2$.

We have the following splittings (see (8)):

$$\begin{aligned} V_0 =& I + V_0^{[2]} + V_0^{[3]} + \ldots + V_0^{[m]} \\ V_\ell =& V_\ell^{[\ell+2]} + V_\ell^{[\ell+3]} + \ldots + V_\ell^{[m]} \quad (1 \le \ell \le m-2) \\ \omega_i =& \omega_i^{[-1]} + \omega_i^{[0]} + \omega_i^{[1]} + \ldots + \omega_i^{[m]} \\ \theta_i^{(k)} =& \theta_i^{(k),[k+1]} + \theta_i^{(k),[k+2]} + \ldots + \theta_i^{(k),[m]} \quad (1 \le k \le m-1, 1 \le i \le n-1). \end{aligned}$$

The main point is that there exists a total ordering on the matrices $V_\ell^{[r]}$, $r \geq \ell + 2 \geq 2$ in such a way that equation (11) is equivalent to the system

$$\partial_i^n (V_\ell^{[r]}) = \text{ expression involving } V_{\ell'}^{[r']} > V_\ell^{[r]}, \ \forall i \in \{1, \ldots, n-1\} \tag{18}$$

for $r \geq \ell + 2 \geq 2$. In this way we can determine all $V_\ell^{[r]}$ inductively.

The system is compatible, the compatibility condition being

$$\partial_i^n(\Omega_j^h) - \partial_j^n(\Omega_i^h) = \frac{1}{h}[\Omega_j^h, \Omega_i^h],$$

where $1 \leq i, j \leq n-1$, which in turn, follows easily from (9) and the fact that $\partial_i^n \partial_j^n = \partial_j^n \partial_i^n$. It is also important to note that the right hand side of equation (18) is a linear combination of $\theta_i^{(k)}$ and left or right side products of such matrices with one or two $V_{\ell'}^{[j']}$ (see the equation displayed before Definition 2.3 in [1]). By using this we show recursively that all $V_\ell^{[j]}$ have entries in $\mathbb{C}[t_1, \ldots, t_{n-1}, e^{t_1}, \ldots, e^{t_{n-1}}, q_n]^{(1)}$. This follows from the general fact that if $f \in \mathbb{C}[t_1, \ldots, t_{n-1}, e^{t_1}, \ldots, e^{t_{n-1}}, q_n]$ has the property that

$$\partial_i(f) \in \mathbb{C}[t_1, \ldots, t_{n-1}, e^{t_1}, \ldots, e^{t_{n-1}}, q_n]^{(1)}, \text{ for all } 1 \leq i \leq n-1,$$

then $f \in \mathbb{C}[t_1, \ldots, t_{n-1}, e^{t_1}, \ldots, e^{t_{n-1}}, q_n]^{(1)}$. The proposition follows in an elementary way. □

4. Frobenius property and orthogonality

We first prove Theorem 1.2.

Proof of Theorem 1.2. We consider for each $i \in \{1, \ldots, n-1\}$ the K-linear operator $\mathcal{Y}_i$ on $H^*(F_n) \otimes K$ given by

$$(\mathcal{Y}_i(a), b) = (y_i \star b, a),$$

for all $a, b \in H^*(F_n)$.

Now let us consider again the Schubert basis $\{\sigma_w \mid w \in S_n\}$ of $H^*(F_n)$. We know that it satisfies the orhogonality condition

$$(\sigma_v, \sigma_w) = \begin{cases} 0, & \text{if } v \neq w_0 w \\ 1, & \text{if } v = w_0 w \end{cases} \tag{19}$$

where w_0 is the longest element of S_n. (That is, $w_0(i) = n - i + 1$, for all $i \in \{1, \ldots, n\}$.) We deduce easily that the matrix of $\mathcal{Y}_i$ on $H^*(F_n) \otimes K$ with respect to the basis $\{\sigma_{w_0 w} \mid w \in S_n\}$ is the transposed of the matrix of "$y_i\star$" with respect to the basis $(\sigma_w)_{w \in S_n}$. From this we deduce as follows:

1. for any $i, j \in \{1, \ldots, n-1\}$, the operators $\mathcal{Y}_i$ and $\mathcal{Y}_j$ commute with each other,

2. if a is a homogeneous element of $H^*(F_n)$, then $\deg \mathcal{Y}_i(a) = \deg a + 2$ and $\mathcal{Y}_i(a)$ modulo $(q_1, \ldots, q_n)$ is equal to $y_i a$.

We will use the following claim.

Claim 1. For any $a \in H^*(F_n)$ there exists a polynomial $f \in \mathbb{Z}[\{Y_i\}, \{q_j\}]$ with

$$a = f(\{\mathcal{Y}_i\}, \{q_j\}).1.$$

To prove this, we use the same method as for Claim 1 in the proof of Proposition 3.3 (property 2 above is used here).

This allows us to define a product, denoted by $\bullet$, on $H^*(F_n) \otimes K$, as follows. For any $f, g \in \mathbb{Z}[\{Y_i\}, \{q_j\}]$, we set

$$(f(\{\mathcal{Y}_i\}, \{q_j\}).1) \bullet (g(\{\mathcal{Y}_i\}, \{q_j\}).1) := (fg)(\{\mathcal{Y}_i\}, \{q_j\}).1,$$

where ".1" denotes evaluation on the element 1 of $H^*(F_n)$. This product satisfies the hypotheses (i)-(vii) of Theorem 1.1. The first non-obvious one is (vi). This follows from the fact that

$$y_i \bullet y_j = \mathcal{Y}_i \mathcal{Y}_j.1,$$

which implies that

$$(y_i \bullet y_j, a) = (\mathcal{Y}_i \mathcal{Y}_j.1, a) = (y_i \star a, \mathcal{Y}_j.1) = (y_j \star y_i \star a, 1),$$

for all $a \in H^*(F_n)$. We use that $\star$ is associative and satisfies condition (vi).

Let us now check condition (vii). Because $\mathcal{Y}_i.1 = y_i$, from Claim 1 we deduce that

$$y_i \bullet a = \mathcal{Y}_i(a), \tag{20}$$

for all $i \in \{1, \ldots, n-1\}$ and all $a \in H^*(F_n)$. We use again the fact that the matrix of $\mathcal{Y}_i$ on $H^*(F_n) \otimes K$ with respect to the basis $\{\sigma_{w_0 w} \mid w \in S_n\}$ is the transposed of the matrix of "$y_i \star$" with respect to the basis $(\sigma_w)_{w \in S_n}$. Condition (vii) follows. From Theorem 1.1, we deduce that $\bullet$ is the same as $\star$.

Claim 2. We have

$$y_{i_1} \star \ldots \star y_{i_k} = \mathcal{Y}_{i_1} \ldots \mathcal{Y}_{i_k}.1,$$

for all $k \geq 1$ and all $i_1, \ldots, i_k \in \{1, \ldots, n-1\}$.

We prove the claim by induction on k. For $k = 1$ the statement is clear. Assume that it is true for k. For any $i_1, \ldots, i_{k+1} \in \{1, \ldots, n-1\}$ we have

$$\mathcal{Y}_{i_1} \ldots \mathcal{Y}_{i_{k+1}}.1 = y_{i_1} \bullet (\mathcal{Y}_{i_2} \ldots \mathcal{Y}_{i_{k+1}}.1) = y_{i_1} \star (y_{i_2} \star \ldots \star y_{i_{k+1}}),$$

where we have used the induction hypothesis and equation (20).

Since the ring $(H^*(F_n) \otimes K, \star)$ is generated as a K-algebra by $y_1, \ldots, y_{n-1}$, Claim 2 implies immediately that

$$\mathcal{Y}_i(a) = y_i \star a$$

for all $i \in \{1, \dots, n-1\}$ and all $a \in H^*(F_n)$. From the definition of $\mathcal{Y}_i$ we deduce that

$$(y_i \star b, c) = (y_i \star c, b)$$

for all $i \in \{1, \dots, n-1\}$ and all $b, c \in H^*(F_n)$. In turn, this implies

$$(a \star b, c) = (a \star c, b)$$

for all $a, b, c \in H^*(F_n)$. Theorem 1.2 is now proved. □

In the second part of this section we consider again the presentation of $(H^*(F_n) \otimes K, \star)$ in terms of generators and relations given in the previous section. More precisely, let Φ be the isomorphism described by Proposition 3.3 (d). For each $w \in S_n$ we pick a polynomial $f_w \in \mathbb{Z}[\{Y_i\}, \{q_j\}]$ such that

$$\Phi(\sigma_w) = [f_w]_q,$$

where $[\ ]_q$ denotes cosets of polynomials modulo $(\mathcal{R}_1, \dots, \mathcal{R}_n)$ (like in the previous section). For any $f \in \mathbb{Z}[\{Y_i\}, \{q_j\}]$ we consider the decomposition

$$[f]_q = \sum_{w \in S_n} \alpha_w [f_w]_q$$

where $\alpha_w \in K$. Then we set

$$(\!([f]_q)\!) := \alpha_{w_0}$$

where w_0 is the longest element of S_n (see above). If f and g are in $\mathbb{Z}[\{Y_i\}, \{q_j\}]$, we define

$$(\!([f]_q, [g]_q)\!) := (\!([fg]_q)\!).$$

We show that the polynomials f_w, $w \in S_n$, satisfy the following orthogonality relation, similarly to the quantum Schubert polynomials (cf. [2] and [8]).

Proposition 4.1. *We have*

$$(\!([f_v]_q, [f_w]_q)\!) = \begin{cases} 0, & \text{if } v \neq w_0 w \\ 1, & \text{if } v = w_0 w. \end{cases}$$

Proof. We consider the expansion

$$[f_v f_w]_q = \sum_{u \in W} \alpha_u [f_u]_q.$$

Since Φ is a ring isomorphism, this implies that

$$\sigma_v \star \sigma_w = \sum_{u \in W} \alpha_u \sigma_u.$$

We deduce that

$$\alpha_{w_0} = (\sigma_v \star \sigma_w, 1) = (\sigma_v, \sigma_w),$$

where we have used the Frobenius property. Finally, we use the orthogonality relation (19). ■

5. Polynomial representatives of Schubert classes: an example

Let us consider again the presentation of $(H^*(F_n)\otimes K,\star)$ given in Proposition 3.3 (d). As usual in this paper, we set

$$X_i := Y_i - Y_{i-1}, 1\le i\le n, \text{ where } Y_0 = Y_n := 0.$$

In this way $\mathcal{R}_1,\ldots,\mathcal{R}_{n-1}$ defined at the beginning of Section 3 are polynomials in $X_1,\ldots,X_n$; we set

$$\mathcal{R}_0 := X_1+\ldots+X_n.$$

We have the ring isomorphism

$$(H^*(F_n)\otimes K,\star)\simeq \mathbb{Z}[X_1,\ldots,X_n,q_1,\ldots,q_n]/(\mathcal{R}_0,\mathcal{R}_1,\ldots,\mathcal{R}_n), \tag{21}$$

where the classes $x_1 := y_1, x_2 := y_2 - y_1, \ldots, x_{n-1} := y_{n-1}-y_{n-2}, x_n := -y_{n-1}$ correspond to the cosets of $X_1, X_2,\ldots,X_{n-1}$, respectively X_n.

In this section we are concerned with the problem of finding, for an arbitrary $w\in S_n$, of a polynomial $f_w\in\mathbb{Z}[X_1,\ldots,X_n,q_1,\ldots,q_n]$ like in the previous section: its coset corresponds to the Schubert class σ_w via the isomorphism (21). In general, this problem can be solved by using Theorem 1.3 and the knowledge of the quantum cohomology ring of F_n. We will illustrate this idea in the special case of F_3.

The Schubert classes for F_3 are as follows (cf. e.g. [2, Tables 14]):

$$\begin{aligned}
\mathfrak{S}_{123} &= 1\\
\mathfrak{S}_{213} &= x_1\\
\mathfrak{S}_{132} &= x_1+x_2\\
\mathfrak{S}_{231} &= x_1x_2\\
\mathfrak{S}_{312} &= x_1^2\\
\mathfrak{S}_{321} &= x_1^2x_2.
\end{aligned}$$

For each of the classes $\mathfrak{S}_w$ above, we need $f_w\in\mathbb{Z}(X_1,X_2,X_3,q_1,q_2,q_3)$ which is homogeneous relative to $\deg X_i = 2, \deg q_j = 4$, has the same degree as the cohomology class $\mathfrak{S}_w$, has integer coefficients, and satisfies the condition $f_w(x_1\star,\ x_2\star,\ x_3\star,\ q_1,\ q_2,\ q_3) = \mathfrak{S}_w$. For instance, for $\mathfrak{S}_{312}$, we look for $f = X_1^2 + g(q_1,q_2,q_3)$, where g is a homogeneous polynomial of degree 1. The condition which must be satisfied is

$$x_1\star x_1 + g(q_1,q_2,q_3) = x_1^2. \tag{22}$$

We make successively $q_k = 0$, for $k = 1,2,3$.

First we make $q_3 = 0$. Equation (22) implies an identity in $qH^*(F_3)$ obtained by making the replacements prescribed by Theorem 1.3. They are as follows:

$$X_1 := x_1, X_2 := x_2, X_3 = x_3, Q_1 = q_1, Q_2 := q_2.$$

The identity in $qH^*(F_3)$ is

$$X_1\circ X_1 + g(Q_1,Q_2,0) = X_1^2.$$

On the other hand, we know that (see for instance [1, Section 3])

$$X_1 \circ X_1 = Y_1 \circ Y_1 = Y_1^2 + Q_1.$$

Thus, $g(Q_1, Q_2, 0) = -Q_1$, which gives

$$g(q_1, q_2, 0) = -q_1. \tag{23}$$

We make $q_1 = 0$. The replacements are as follows:

$$X_1 := x_1, X_2 := x_3, X_3 := x_2, Q_1 := q_3, Q_2 := q_2.$$

The identity in $qH^*(F_3)$ is

$$X_1 \circ X_1 + g(0, Q_2, Q_1) = X_1^2.$$

Using the same method as above, we deduce

$$g(0, q_2, q_3) = -q_3. \tag{24}$$

Finally, we make $q_2 = 0$. The replacements are

$$X_1 := x_2, X_2 := x_1, X_3 := x_3, Q_1 = q_1, Q_2 = q_3.$$

The identity in $qH^*(F_3)$ is

$$X_2 \circ X_2 + g(Q_1, 0, Q_2) = X_2^2.$$

This time we obtain

$$g(q_1, 0, q_3) = -q_1 - q_3. \tag{25}$$

From equations (23), (24), and (25) we deduce that

$$g(q_1, q_2, q_3) = -q_1 - q_3.$$

Thus, a polynomial representative of $\mathfrak{S}_{312}$ is

$$X_1^2 - q_1 - q_3.$$

Similarly one obtains the desired polynomials for all Schubert classes. They are as follows:

$$\begin{aligned}
\mathfrak{S}_{123} &: 1 \\
\mathfrak{S}_{213} &: X_1 \\
\mathfrak{S}_{132} &: X_1 + X_2 \\
\mathfrak{S}_{231} &: X_1X_2 + q_1 \\
\mathfrak{S}_{312} &: X_1^2 - q_1 - q_3 \\
\mathfrak{S}_{321} &: X_1^2X_2 + q_1X_1 - q_3X_2
\end{aligned}$$

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