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Bounded Simple ($\mathfrak{g}$, sl(2))-modules for rk$\mathfrak{g}$=2

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Abstract. This paper is a continuation of our work [PS2] in which we prove some general results about simple $(\mathfrak{g}, \mathfrak{k})$-modules with bounded $\mathfrak{k}$-multiplicities (or bounded simple $(\mathfrak{g}, \mathfrak{k})$-modules). In the absence of a classification of bounded simple $(\mathfrak{g}, \mathfrak{k})$-modules in general, it is important to understand some special cases as best as possible. Here we consider the case $\mathfrak{k} = \mathrm{sl}(2)$. It turns out that in order for an infinite-dimensional bounded simple $(\mathfrak{g}, \mathrm{sl}(2))$-module to exist, $\mathfrak{g}$ must have rank 2, and, up to conjugation, there are five possible embeddings $\mathrm{sl}(2) \to \mathfrak{g}$ which yield infinite-dimensional bounded simple $(\mathfrak{g}, \mathrm{sl}(2))$-modules.

Our main result is a detailed description of the bounded simple $(\mathfrak{g}, \mathrm{sl}(2))$-modules in all five cases. When $\mathfrak{g} \simeq \mathrm{sl}(2) \oplus \mathrm{sl}(2)$ we reproduce in modern terms some classical results from the 1940's. When $\mathfrak{g} \simeq \mathrm{sl}(3)$ and sl(2) is a principal subalgebra, bounded simple $(\mathrm{sl}(3), \mathrm{sl}(2))$-modules are Harish-Chandra modules and our result singles out all Harish-Chandra modules with bounded sl(2)-multiplicities. A case where the result is entirely new is the case of a principal sl(2)-subalgebra of $\mathfrak{g} = \mathrm{sp}(4)$.
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1. Introduction

The classification of simple Harish-Chandra modules is a celebrated result and there is an extensive literature on the general topic of Harish-Chandra modules, see for instance [KV] and the references therein. Algebraically, Harish-Chandra modules are $(\mathfrak{g}, \mathfrak{k})$-modules for a symmetric subalgebra $\mathfrak{k}$ of a semisimple Lie algebra $\mathfrak{g}$, and in the last decade an intense exploration of more general $(\mathfrak{g}, \mathfrak{k})$-modules for not necessarily symmetric subalgebras $\mathfrak{k}$ has begun, [PS1],[PSZ], [PZ1], [PZ2], [PZ3]. A most notable result in this direction is the classification of simple $(\mathfrak{g}, \mathfrak{k})$-modules of finite type and a generic minimal $\mathfrak{k}$-type carried out in [PZ2]. Nevertheless, the classification problem for $(\mathfrak{g}, \mathfrak{k})$-modules of finite type with an arbitrary minimal $\mathfrak{k}$-type is still open even when $\mathrm{rk}\mathfrak{g} = 2$ and $\mathrm{rk}\mathfrak{k} = 1$.

In the recent paper [PS2] we concentrated on the interesting subclass of bounded $(\mathfrak{g}, \mathfrak{k})$-modules (the definition see in Section 1 below) and proved some

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general results regarding the existence of such modules. In particular, we established sufficient and necessary conditions on a reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$ for the existence of a simple infinite-dimensional bounded $(\mathfrak{g},\mathfrak{k})$-module. If $\mathfrak{k}\simeq \mathrm{sl}(2)$, simple $(\mathfrak{g},\mathrm{sl}(2))$-modules of finite type exist for any simple $\mathfrak{g}$, see for instance [PSZ] or [PZ3]. It turns out however, that bounded $(\mathfrak{g},\mathrm{sl}(2))$-modules are very special and exist only for $\mathrm{rk}\mathfrak{g}=2$. The classification of such modules is rather intriguing as they are the "smallest", and thus highly non-generic, $(\mathfrak{g},\mathrm{sl}(2))$-modules.

This classification is carried out in the present paper. We show first that, up to conjugation, there are precisely five possibilities for embedding sl(2) into a Lie algebra of $\mathfrak{g}$ of rank 2 so that bounded infinite-dimensional $(\mathfrak{g},\mathrm{sl}(2))$-modules exist: sl(2) as the diagonal subalgebra of $\mathrm{sl}(2)\oplus\mathrm{sl}(2)$, sl(2) as a root subalgebra or a principal sl(2) subalgebra of sl(3), and sl(2) as a root subalgebra corresponding to a short root or as a principal subalgebra of sp(4).

We then give a classification and a detailed description (we compute characters and minimal sl(2)-types) of all bounded $(\mathfrak{g},\mathrm{sl}(2))$-modules. In the case when $\mathfrak{g}\simeq\mathrm{sl}(2)\oplus\mathrm{sl}(2)$ our results are just a modern reproduction of classical results, in all other cases they are new. The most interesting new case is that of a principal sl(2)-subalgebra of sp(4).

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2. General definitions and preliminary results

The ground field is $\mathbb{C}$.

Let $\mathfrak{g}$ be a semisimple (finite-dimensional) Lie algebra and $\mathfrak{k}\subset\mathfrak{g}$ be a reductive in $\mathfrak{g}$ subalgebra. $U(\cdot)$ stands for enveloping algebra, $U=U(\mathfrak{g})$ and Z_U is the center of U. A $(\mathfrak{g},\mathfrak{k})$-*module* M is a $\mathfrak{g}$-module M on which $\mathfrak{k}$ acts locally finitely, i.e. $\dim U(\mathfrak{k})\cdot m<\infty$, $\forall m\in M$. A $(\mathfrak{g},\mathfrak{k})$-module M has *finite type over* $\mathfrak{k}$ if the Jordan–Hölder multiplicity of any fixed simple finite-dimensional $\mathfrak{k}$-module V (such a V is called a $\mathfrak{k}$-*type*) in arbitrary finite-dimensional $\mathfrak{k}$-submodules of M is bounded. A $(\mathfrak{g},\mathfrak{k})$-module is *bounded* if the above multiplicities are bounded by a constant not depending on the $\mathfrak{k}$-type V. A reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}\subset\mathfrak{g}$ is *bounded* if there exists an infinite-dimensional simple bounded $(\mathfrak{g},\mathfrak{k})$-module M. A bounded subalgebra $\mathfrak{k}\subset\mathfrak{g}$ is *strictly bounded* if there is a simple bounded $(\mathfrak{g},\mathfrak{k})$-module M on which no simple ideal of $\mathfrak{g}$ acts locally finitely. The following necessary conditions on a subalgebra $\mathfrak{k}$ to be bounded, or strictly bounded, are proved in [PS2] (Theorem 4.1 and Corollary 4.6).

Theorem 2.1. *Let $\mathfrak{k}$ be a bounded reductive subalgebra of a semisimple Lie algebra $\mathfrak{g}=\oplus_i\mathfrak{g}_i$ ($\mathfrak{g}_i$ being the simple ideals of $\mathfrak{g}$).*

a) If M is a simple bounded $(\mathfrak{g},\mathfrak{k})$-module and the algebra of $\mathfrak{k}$-invariants $\mathfrak{g}_{i_0}^{\mathfrak{k}}$ is not abelian for some i_0, then $M\simeq M_{i_0}\otimes\overline{M_{i_0}}$, where M_{i_0} is a simple *finite-dimensional* $\mathfrak{g}_{i_0}^{\mathfrak{k}}$*-module and* $\overline{M_{i_0}}$ *is a simple bounded* $(\oplus_{i\neq i_0}\mathfrak{g}_i, (\oplus_{i\neq i_0}\mathfrak{g}_i)\bigcap\mathfrak{k})$*-module.*

b) If $r_{\mathfrak{g}}$ *is the half-dimension of a nilpotent orbit of minimal positive dimension in* $\mathfrak{g}$*, then*

$$r_{\mathfrak{g}} \leq b_{\mathfrak{k}}, \tag{1}$$

where $b_{\mathfrak{k}}$ *is the dimension of a Borel subalgebra of* $\mathfrak{g}$*.*

c) If $\mathfrak{k}$ *is strictly bounded, then*

$$\sum_i r_{\mathfrak{g}_i} \leq b_{\mathfrak{k}}.$$

In [PS2] we also established the following sufficient condition for a reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}\subset\mathfrak{g}$ to be bounded. Recall that a finite-dimensional module W over an algebraic group H is *spherical* if a Borel subgroup B_H has an open orbit in W.

Theorem 2.2. *Let* $K\subset G\subset GL(V)$ *be a chain of reductive algebraic groups, and let* $V'\subset V$ *be a 1-dimensional space whose stabilizers in* G *and* K *are parabolic subgroups* $P\subset G$ *and* $Q\subset K$*. Then, if* $(V')^*\otimes(\mathfrak{g}\cdot V'/\mathfrak{k}\cdot V')$ *is a spherical module over a reductive part* Q_0 *of* Q*,* $\mathfrak{k}$ *is a bounded subalgebra of* $\mathfrak{g}$*.*

3. Bounded subalgebras of a rank-2 Lie Algebra

Our main interest in this paper are infinite-dimensional bounded $(\mathfrak{g},\mathfrak{k})$-modules for $\mathfrak{k}\simeq \mathrm{sl}(2)$. Theorem 2.1 implies that if $\mathfrak{k}\simeq\mathrm{sl}(2)$ is a strictly bounded subalgebra of $\mathfrak{g}=\oplus_i\mathfrak{g}_i$, then $\sum_i r_{\mathfrak{g}_i}\leq 2$. This is easily seen to imply $\mathrm{rk}\mathfrak{g}=2$. Therefore, in the rest of the paper we restrict ourselves to the case when $\mathrm{rk}\mathfrak{g}=2$. The following theorem classifies more generally all reductive in $\mathfrak{g}$ bounded subalgebras $\mathfrak{k}\subset\mathfrak{g}$ under the assumption that $\mathrm{rk}\mathfrak{g}=2$.

Theorem 3.1. *Let* $\mathfrak{g}$ *be a semisimple Lie algebra of rank 2 and* $\mathfrak{k}\subset\mathfrak{g}$ *be a reductive in* $\mathfrak{g}$ *bounded subalgebra. The following is a complete list of such pairs up to conjugation by inner automorphisms.*

(1) $\mathfrak{g}\simeq\mathrm{sl}(2)\oplus\mathrm{sl}(2)$*:* $\mathfrak{k}\simeq\mathrm{gl}(2)$ *is a direct sum of a simple ideal and a Cartan subalgebra of the other simple ideal,* $\mathfrak{k}\simeq\mathrm{sl}(2)$ *is a diagonal subalgebra, or* $\mathfrak{k}$ *is any non-trivial toral subalgebra;*

(2) $\mathfrak{g}\simeq\mathrm{sl}(3)$*:* $\mathfrak{k}$ *is a root subalgebra isomorphic to* $\mathrm{sl}(2)$ *or* $\mathrm{gl}(2)$*,* $\mathfrak{k}$ *is a principal* $\mathrm{sl}(2)$*-subalgebra, or* $\mathfrak{k}$ *is a Cartan subalgebra;*

(3) $\mathfrak{g}\simeq\mathrm{sp}(4)$*:* $\mathfrak{k}\simeq\mathrm{sl}(2)\oplus\mathrm{sl}(2)$ *is the subalgebra generated by the long roots,* $\mathfrak{k}\simeq\mathrm{gl}(2)$ *is any root subalgebra,* $\mathfrak{k}\simeq\mathrm{sl}(2)$ *is a root subalgebra corresponding to a short root,* $\mathfrak{k}$ *is a principal* $\mathrm{sl}(2)$*-subalgebra, or* $\mathfrak{k}$ *is a Cartan subalgebra;*

(4) $\mathfrak{g}\simeq G_2$*:* $\mathfrak{k}$ *is any subalgebra containing a Cartan subalgebra, in this case* $\mathfrak{k}\simeq\mathrm{sl}(3)$*,* $\mathfrak{k}\simeq\mathrm{sl}(2)\oplus\mathrm{sl}(2)$*, or* $\mathfrak{k}\simeq\mathrm{gl}(2)$*.*

Proof. The inequality (1) implies that a 1-dimensional toral subalgebra is not bounded in all cases but (1). In (1) any 1-dimensional toral subalgebra $\mathfrak{t}$ is bounded as the outer tensor product of a Verma module over a suitable ideal of $\mathfrak{g}$ with the trivial module of the complementary ideal of $\mathfrak{g}$ is always bounded as a $(\mathfrak{g}, \mathfrak{t})$-module.

Similarly, (1) implies that a Cartan subalgebra is not bounded in G_2. In all other cases it is well known to be bounded, see for instance [F].

If $\mathfrak{k} \simeq \mathrm{sl}(2)$ then $\mathfrak{k}$ is not bounded in G_2 again by (1), and if $\mathfrak{k}$ is an ideal of $\mathfrak{g} = \mathrm{sl}(2) \oplus \mathrm{sl}(2)$, it is not bounded by Theorem 2.1 a). Furthermore, if $\mathfrak{k} \simeq \mathrm{sl}(2)$ is a root subalgebra of $\mathfrak{g} = \mathrm{sp}(4)$ corresponding to a long root, then $\mathfrak{k}$ is not bounded by Theorem 2.1 a). For the remaining five possible embeddings of sl(2) into a Lie algebra of rank 2, the image $\mathfrak{k}$ is always a bounded subalgebra. This follows for instance from the explicit description of bounded $(\mathfrak{g}, \mathfrak{k})$-modules which we present in Sections 4-7 of this paper.

For any embedding of gl(2) into a Lie algebra $\mathfrak{g}$ of rank 2, $\mathfrak{g} \not\simeq G_2$, any generalized Verma module, corresponding to a parabolic subalgebra $\mathfrak{p}$ which contains the image $\mathfrak{k}$ of gl(2), is a bounded $(\mathfrak{g}, \mathfrak{k})$-module.

Consider next the case $\mathfrak{k} \simeq \mathrm{sl}(2) \oplus \mathrm{sl}(2) \subset \mathfrak{g}$ for $\mathfrak{g} = \mathrm{sp}(4)$ or G_2. Here the pair $(\mathfrak{g}, \mathfrak{k})$ is symmetric. In [V1] and [V2] ladder $(\mathfrak{g}, \mathfrak{k})$-modules are constructed. Fix a Borel subalgebra $\mathfrak{b}_{\mathfrak{k}} \subset \mathfrak{k}$. By definition, a ladder module M has the $\mathfrak{k}$ decomposition $M = \bigoplus_{n \in \mathbb{Z}_{\geq 0}} V_{\mu + n\beta}$, where μ is some integral $\mathfrak{b}_{\mathfrak{k}}$-dominant weight and β is the $\mathfrak{b}_{\mathfrak{k}}$-highest weight of $\mathfrak{g}/\mathfrak{k}$. Clearly, a ladder module is multiplicity-free and hence bounded. Moreover, it remains bounded with respect to any gl(2)-subalgebra of $\mathfrak{k}$. Hence any image of gl(2) in sp(4) or G_2 is bounded.

The only remaining case is $\mathfrak{g} = G_2, \mathfrak{k} \simeq \mathrm{sl}(3)$. To show that $\mathfrak{k}$ is bounded we use Theorem 2.2 with V being the 7-dimensional G_2-module. Then as a $\mathfrak{k}$-module V is isomorphic to $V_{\omega_1} \oplus V_{\omega_1}^* \oplus \mathbb{C}$. One can fix a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ so that there exists a $\mathfrak{b}$-stable one-dimensional subspace $V' \subset V_{\omega_1}^*$. Then $Q_0 \simeq GL(2)$ and

$$(V')^* \otimes (\mathfrak{g} \cdot V'/\mathfrak{k} \cdot V') \simeq \Lambda^2(E) \otimes (E^* \oplus \mathbb{C})$$

where E is the standard $GL(2)$-module. It is easy to check that it is a spherical Q_0-module. ∎

In the rest of this paper $\mathfrak{g}$ will be of rank 2, and $\mathfrak{k}$ will be isomorphic to sl(2). By V_k we denote the $k+1-$dimensional $\mathfrak{k}$-module, and we write $c(M)$ for the $\mathfrak{k}$-character of any $(\mathfrak{g}, \mathfrak{k})$-module M of finite type over $\mathfrak{k}$:

$$c(M) := \sum_{k \geq 0} (\dim M^k) z^k,$$

where $M^k = \mathrm{Hom}_{\mathfrak{k}}(V_k, M)$. By definition, $c(M)$ is a formal power series in z. The *minimal* $\mathfrak{k}$*-type* of M is V_t where $t \in \mathbb{Z}_{\geq 0}$ is minimal with $M^t \neq 0$. A $(\mathfrak{g}, \mathfrak{k})$-module of finite type M is

even (respectively, *odd*) if $M^t = 0$ for all $t \in 1 + 2\mathbb{Z}$ (resp., $t \in 2\mathbb{Z}$).

Let $\mathbb{C}((z))$ be the algebra of Laurent series and $\mathbb{C}((z))'$ be the span of vectors in $\mathbb{C}((z))$ of the form $z^j + z^{-j-2}$ for $j \in \mathbb{Z}$ ($\mathbb{C}((z))'$ is not a subalgebra). Note that $\mathbb{C}((z))'$ is a complement to the subspace $\mathbb{C}[[z]]$ of $\mathbb{C}((z))$. In what follows we denote by π the projection onto the second summand in the direct sum $\mathbb{C}((z)) = \mathbb{C}((z))' \oplus \mathbb{C}[[z]]$, and we set $z^p \otimes z^q := \sum_{0\leq k\leq q} z^{p+q-2k}$ for $p \geq q$ and $z^p \otimes z^q := z^q \otimes z^p$ for $p < q$.

Lemma 3.2.

(a) For any $f(z) \in \mathbb{C}((z))$ and any $j \in \mathbb{Z}$, $\pi(f(z)(z^j + z^{-j})) = \pi(\pi(f(z)(z^j + z^{-j})))$.

(b) For any $(\mathfrak{k}, \mathfrak{k})$-module M of finite type over $\mathfrak{k}$

$$c(M \otimes V_i) = \pi(c(M) \sum_{0\leq k\leq i} z^{i-2k}),$$

for all $i \in \mathbb{N}$.

Proof.

(a) It suffices to check that for any $\psi(z) \in C((z))'$, $\psi(z)(z^j + z^{-j}) \in \mathbb{C}((z))'$, and this is obvious.

(b) It suffices to check that, for any $s \in \mathbb{Z}_{\geq 0}$

$$\pi(z^s \otimes (\sum_{0\leq k\leq i} z^{i-2k})) = \sum_{0\leq k\leq \frac{|i-s|}{2}} z^{s+i-2k},$$

which is also obvious.

■

Finally, by $\Gamma_{\mathfrak{k}}$ we denote the functor of $\mathfrak{k}$-finite vectors:

$$\Gamma_{\mathfrak{k}} : \mathfrak{g} - \text{mod} \rightsquigarrow (\mathfrak{g}, \mathfrak{k}) - \text{mod},$$

$$M \mapsto \{m \in M | \dim(U(\mathfrak{k}) \cdot m) < \infty\}.$$

4. Classification and $\mathfrak{k}$-characters of simple $(\mathrm{sl}(2) \oplus \mathrm{sl}(2), \mathrm{sl}(2))$-modules

Theorem 3.1 singles out the cases when $\mathfrak{k} \simeq \mathrm{sl}(2)$ is a bounded subalgebra of a rank-2 Lie algebra. The simplest case is when $\mathfrak{g} = \mathrm{sl}(2) \oplus \mathrm{sl}(2)$ and $\mathfrak{k} \subset \mathfrak{g}$ is the diagonal subalgebra. Here all simple $(\mathfrak{g}, \mathfrak{k})$-modules are bounded and are moreover multiplicity-free. This follows, for instance, from the algebraic subquotient theorem, see [Dix], Ch. 9. These $(\mathfrak{g}, \mathfrak{k})$-modules are historically among the first examples of $(\mathfrak{g}, \mathfrak{k})$-modules studied. They have been classified already in 1947 by Gelfand and Naimark [GN] and by Bargmann [B], and have been constructed also by Harish-Chandra around the same time, [HC]. A fundamental more modern and much more general reference is the article [BG], where however this explicit example is not written in detail. In the present section we give a quick self-contained description of all simple $(\mathfrak{g}, \mathfrak{k})$-modules based on the approach of [BG].

Lemma 4.1. *Let $\Omega_1, \Omega_2 \in U(\mathfrak{g})$ be the Casimir elements of the two* $\mathrm{sl}(2)$*-direct summands of* $\mathfrak{g}$*, and* $\Omega \in U(\mathfrak{k}) \subset U(\mathfrak{g}) = U$ *be the Casimir element of* $\mathfrak{k}$*. Then* Ω_1, Ω_2 *and* Ω *generate* $U(\mathfrak{g})^{\mathfrak{k}}$*.*

Proof. Straightforward computation. A more general result is proved by F. Knop in [Kn1]. ■

Corollary 4.2. *Every simple* $(\mathfrak{g}, \mathfrak{k})$*-module is multiplicity-free.*

Denote by $\chi(a, b)$ the central character of the Verma $\mathfrak{g}$-module with highest weight (a_1, b_1), where the notation (c, d) is shorthand for the weight $c\omega_{\text{left}} + d\omega_{\text{right}}$, ω_{left} (respectively, ω_{right}) being the fundamental weight of the first (respectively, second) direct summand of $\mathfrak{g}$.

Lemma 4.3. *If* V_n *is the minimal* $\mathfrak{k}$*-type of a simple infinite-dimensional* $(\mathfrak{g}, \mathfrak{k})$*-module* M*, then*

$$c(M) = z^n + z^{n+2} + z^{n+4} + \dots \quad . \tag{2}$$

Proof. To prove (2) it suffices to show that V_n, V_{n+2}, V_{n+4}, etc. are precisely all $\mathfrak{k}$-types of M. The absence of other $\mathfrak{k}$-types follows from the fact that as a $\mathfrak{k}$-module $\mathfrak{g}$ is isomorphic to $V_2 \oplus V_2$, hence when acting by $\mathfrak{g}$ on V_{n+2i} one can only obtain $\mathfrak{k}$-constituents of $(V_2 \oplus V_2) \otimes V_{n+2i}$, i.e. $V_{n+2(i-1)}$, V_{n+2i} and $V_{n+2(i+1)}$. To show that for each $i > 0$ V_{n+2i} is a $\mathfrak{k}$-constituent of M, note that if V_{n+2i} were not a constituent of M, then when acting by $\mathfrak{g}$ on $V_{n+2(i-t)}$ for $t \geq 1$ one would not be able to obtain a constituent of the from $V_{n+2(i+r)}$ for $r \geq 1$. Hence M would turn being finite-dimensional, a contradiction. ■

Lemma 4.4. *Let* M *be a simple* $(\mathfrak{g}, \mathfrak{k})$*-module with minimal* $\mathfrak{k}$*-type* V_0*. Then the central character of* M *equals* $\chi(a, a)$ *for some* $a \in \mathbb{C}$*.*

Proof. Since $\mathfrak{g} \simeq \mathfrak{k} \oplus \mathfrak{k}$, the $\mathfrak{g}$-module $U \otimes_{U(\mathfrak{k})} V_0$ is isomorphic to $U(\mathfrak{k})$. The latter is endowed with a $U \simeq U(\mathfrak{k}) \otimes U(\mathfrak{k})$-module structure via left multiplication by elements of $U(\mathfrak{k}) \otimes 1$ and right multiplication by elements of $1 \otimes U(\mathfrak{k})$. Moreover, the actions of Ω_1 and Ω_2 coincide on $U(\mathfrak{k})$. Since M is a quotient of the $\mathfrak{g}$-module $U(\mathfrak{k})$, the actions of Ω_1 and Ω_2 coincide on M, hence the Lemma. ■

Lemma 4.5. *Let* M *be a simple* $(\mathfrak{g}, \mathfrak{k})$*-module. Then the central character of* M *equals* $\chi(a, a+n)$ *for some* $a \in \mathbb{C}$ *and some* $n \in \mathbb{Z}$*. Moreover, the parity of* n *equals the parity of* k *where* V_k *is the minimal* $\mathfrak{k}$*-type of* M*.*

Proof. Let $\chi(\alpha, \beta)$ be a central character of M and consider the $\mathfrak{g}$-module $M \otimes (V_0 \boxtimes V_k)$, where the $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{k}$-module $V_0 \boxtimes V_k$ is endowed with a $\mathfrak{g}$-module structure via the isomorphism $\mathfrak{g} \simeq \mathfrak{k} \oplus \mathfrak{k}$. Then $\mathrm{Hom}_{\mathfrak{k}}(V_0, M \otimes (V_0 \boxtimes V_k)) \neq 0$, hence a simple subquotient of $M \otimes (V_0 \boxtimes V_k)$ has central character $\chi(a, a)$ for some a. On the other hand, the central characters of all simple subquotients of $M \otimes (V_0 \boxtimes V_k)$ are of the form $\chi(\alpha, \beta - n)$ for n running over the set of weights of V_k. Therefore $\alpha = a$, $\beta - n = a$, i.e. the Lemma follows. ∎

Lemma 4.6. *For any central character χ, up to isomorphism there is at most one infinite dimensional simple $(\mathfrak{g}, \mathfrak{k})$-module with this central character.*

Proof. Let M', M'' be two simple $(\mathfrak{g}, \mathfrak{k})$-modules with central character χ. Then, by Lemma 4.3, for some m $\mathrm{Hom}_{\mathfrak{k}}(V_m, M') = \mathrm{Hom}_{\mathfrak{k}}(V_m, M'') = \mathbb{C}$. Therefore M' and M'' are isomorphic to simple quotients of the $\mathfrak{g}$-module $U \otimes_{Z_U U(\mathfrak{k})} V_m$, where Z_U acts on V_m via the central character χ. The fact that $U^{\mathfrak{k}} \subset Z_U U(\mathfrak{k})$ (Lemma 4.1) implies that $\mathrm{Hom}_{\mathfrak{k}}(V_m, U \otimes_{Z_U U(\mathfrak{k})} V_m) = \mathbb{C}$ for every $m \geq 0$. Hence $U \otimes_{Z_U U(\mathfrak{k})} V_m$ has a unique proper maximal submodule, and in this way also a unique simple quotient. Therefore $M' \simeq M''$. ∎

In the rest of this section we only consider central characters of the form $\chi(a, a-n)$ for $n \in \mathbb{Z}_{\geq 0}$. If $a \in \mathbb{Z}$, we assume in addition that $a \geq 0$ and $a - n \leq 0$. By M_c denote the Verma module over $\mathfrak{k}$ with highest weight $c - 1$. Note that for $a, a-n$ as above, $\mathrm{Hom}_{\mathbb{C}}(M_a, M_{a-n})$ is a $\mathfrak{g}$-module with central character $\chi(a, a-n)$. Define

$$W_{a,a-n} := \Gamma_{\mathfrak{k}}(\mathrm{Hom}_{\mathbb{C}}(M_a, M_{a-n})).$$

Theorem 4.7.

(a) *Fix $a \in \mathbb{C} \backslash \mathbb{Z}_{<0}$ and $n \in \mathbb{Z}_{\geq 0}$ such that $a - n \leq 0$ for integer a. The $\mathfrak{g}$-module $W_{a,a-n}$ is the unique (up to isomorphism) simple infinite-dimensional $(\mathfrak{g}, \mathfrak{k})$-module with central character $\chi(a, a-n)$.*

(b) $c(W_{a,a-n}) = z^n + z^{n+2} + z^{n+4} + \dots$.

Proof. Note that to compute the $\mathfrak{k}$-character of $\Gamma_{\mathfrak{k}}(\mathrm{Hom}_{\mathbb{C}}(M_a, M_{a-n}))$ it suffices to compute $\mathrm{Hom}_{\mathfrak{k}}(V_m, \mathrm{Hom}_{\mathbb{C}}(M_a, M_{a-n}))$ for all $m \in \mathbb{Z}_{\geq 0}$. However,

$$\mathrm{Hom}_{\mathfrak{k}}(V_m, \mathrm{Hom}_{\mathbb{C}}(M_a, M_{a-n})) = \mathrm{Hom}_{\mathfrak{k}}(M_a, M_{a-n} \otimes V_m^*),$$

and

$$\mathrm{Hom}_{\mathfrak{k}}(M_a, M_{a-n} \otimes V_m^*) = \begin{cases} \mathbb{C} & \text{for } m - n \in 2\mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise} \end{cases} .$$

Hence

$$c(W_{a,a-n}) = z^n + z^{n+2} + z^{n+4} + \dots \quad .$$

The simplicity of $W_{a,a-n}$ follows from the observation that if simple, $W_{a,a-n}$ would have a finite-dimensional subquotient, but there is no finite-dimensional $\mathfrak{g}$-module with central character $\chi(a, a-n)$ for $a \in \mathbb{C} \backslash \mathbb{Z}$ or $a = 0$. If $a \in \mathbb{Z}$, the finite-dimensional $\mathfrak{g}$-module with central character $\chi(a, a-n)$ is isomorphic to $V_{a-1} \boxtimes V_{n-a-1}$ whose $\mathfrak{k}$-character is $z^{n-2} + z^{n-4} + \dots + z^{|n-2a-2|}$, and hence it can not be a subquotient of $W_{a,a-n}$. ∎

5. Classification and $\mathfrak{k}$-characters of simple bounded $(\mathrm{sl}(3), \mathrm{sl}(2))$-modules

Throughout this section $\mathfrak{g} = \mathrm{sl}(3)$ and $\mathfrak{k} \simeq \mathrm{sl}(2) \subset \mathfrak{g}$.

5.1. The root case.. In this subsection we fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and simple roots $\alpha_1, \alpha_2 \in \mathfrak{h}^*$ which define a Borel subalgebra $\mathfrak{b}^+ \subset \mathfrak{g}$. We also fix $\mathfrak{k}$ to be the sl(2)-subalgebra generated by the root spaces $\mathfrak{g}^{\pm\alpha_1}$. There are two parabolic subalgebras containing $\mathfrak{k}$ and $\mathfrak{h}$: $\mathfrak{p}^+ := (\mathfrak{h} + \mathfrak{k}) \oplus \mathfrak{g}^{\alpha_2} \oplus \mathfrak{g}^{\alpha_1+\alpha_2}$, $\mathfrak{p}^- := (\mathfrak{h} + \mathfrak{k}) \oplus \mathfrak{g}^{-\alpha_2} \oplus \mathfrak{g}^{-\alpha_1-\alpha_2}$. Note that $\mathfrak{b}^+ \subset \mathfrak{p}^+$ and define $\mathfrak{b}^-$ to be the Borel subalgebra with simple roots $\alpha_1, -\alpha_1 - \alpha_2$. Then $\mathfrak{b}^- \subset \mathfrak{p}^-$. In addition, we fix generators $h_i \in [\mathfrak{g}^{\alpha_i}, \mathfrak{g}^{-\alpha_i}]$ and denote by ω_i, for $i = 1, 2$, the corresponding dual basis of $\mathfrak{h}^*$. Then $\rho_{\mathfrak{b}^+} = \omega_1 + \omega_2$, $\rho_{\mathfrak{b}^-} = \omega_1 - 2\omega_2$.

Lemma 5.1. *Let M be a simple bounded infinite-dimensional $(\mathfrak{g}, \mathfrak{k})$-module. Then $\mathfrak{g}[M] = \mathfrak{p}^\pm$.*

Proof. Since $\mathfrak{h} \subset \mathfrak{g}^{\mathfrak{k}} \oplus \mathfrak{k}$, we have $\mathfrak{h} \subset \mathfrak{g}[M]$. Put $M_0 := \{m \in M | \mathfrak{g}^{\alpha_1} \cdot m = 0\}$ and choose generators x and y of the respective root spaces $\mathfrak{g}^{-\alpha_2}$ and $\mathfrak{g}^{\alpha_1+\alpha_2}$. A straightforward computation shows that for any $i, j \in \mathbb{Z}_{\geq 0}$, $(x^i y^j) \cdot v \in M_0$ if v is any non-zero vector in M_0 such that $h_1 \cdot v = \nu(h_1)v$ for some $\nu \in (\mathfrak{h} \cap \mathfrak{k})^*$. Therefore the assumption that $x, y \notin \mathfrak{g}[M]$ implies that the multiplicity of $V_{\nu+i+j}$ is at least $i + j$, which contradicts the boundedness of M. Hence $\mathfrak{g}^{-\alpha_2} \in \mathfrak{g}[M]$ or $\mathfrak{g}^{\alpha_1+\alpha_2} \in \mathfrak{g}[M]$, and consequently $\mathfrak{g}[M] = \mathfrak{p}^\pm$. ■

Let $F^\pm_{a,b}$ be the simple finite-dimensional $\mathfrak{p}^\pm$-module with $\mathfrak{b}^\pm$-highest weight $a\omega_1 + b\omega_2$. Define $L^\pm_{a,b}$ as the unique simple quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}^\pm)} F^\pm_{a,b}$. Then $L^\pm_{a,b}$ are bounded $(\mathfrak{g}, \mathfrak{k})$-modules, and the existence of an isomorphism $L^\pm_{a,b} \simeq L^\mp_{a',b'}$ implies $\dim L^\pm_{a,b} < \infty$.

Theorem 5.2. *Let, as above, $\mathfrak{k} \simeq \mathrm{sl}(2)$ be a root subalgebra of $\mathfrak{g} = \mathrm{sl}(3)$.*

(a) Any infinite-dimensional bounded $(\mathfrak{g}, \mathfrak{k})$-module is isomorphic either to $L^+_{a,b}$ for $a \in \mathbb{Z}_{\geq 0}$, $b \in \mathbb{C} \backslash \mathbb{Z}_{\geq 0}$ or to $L^-_{a,b}$ for $a \in \mathbb{Z}_{\geq 0}$, $-a - b \in \mathbb{C} \backslash \mathbb{Z}_{\geq 0}$.

(b)

$$c(L^\pm_{a,b}) = 1 + 2z + \dots + az^{a-1} + (a+1)(z^a + z^{a+1} + \dots) \qquad (3)$$

for all $a \geq 0$ and for those b which do not satisfy the conditions $-b \in \mathbb{Z}_{\geq 2}$, $a+b \in \mathbb{Z}_{\geq -1}$ for $L^+_{a,b}$, and respectively the conditions $a+b \in \mathbb{Z}_{\geq 2}$, $-b \in \mathbb{Z}_{\geq -1}$ for $L^-_{a,b}$.

(c) If $-b \in \mathbb{Z}_{\geq 2}$, $a + b \in \mathbb{Z}_{\geq -1}$, then

$$c(L^+_{a,b}) = z^{-b-1} + 2z^{-b} + \dots + (a+b+1)z^{a-1} + (a+b+2)(z^a + z^{a+1} + \dots), \qquad (4)$$

and if $a + b \in \mathbb{Z}_{\geq 2}$, $-b \in \mathbb{Z}_{\geq -1}$, then

$$c(L^-_{a,b}) = z^{a+b-1} + 2z^{a+b} + \dots + (1-b)z^{a-1} + (2-b)(z^a + z^{a+1} + \dots). \qquad (5)$$

Proof. Let M be a simple infinite-dimensional bounded $(\mathfrak{g},\mathfrak{k})$-module. Then, by Lemma 5.1, $\mathfrak{g}[M]=\mathfrak{p}^{\pm}$. If $\mathfrak{g}[M]=\mathfrak{p}^{+}$, let M^{+} be a simple finite-dimensional $\mathfrak{p}^{+}$-submodule of M. Then $M^{+}\simeq F^{+}_{a,b}$ for some $a\in\mathbb{Z}_{\geq 0}$ and some $b\in\mathbb{C}$, and there is an obvious surjection of $\mathfrak{g}$-modules $U(\mathfrak{g})\otimes_{U(\mathfrak{p}^{+})}F^{+}_{a,b}\to M$. Hence M is isomorphic to the unique simple quotient $L^{+}_{a,b}$ of $U(\mathfrak{g})\otimes_{U(\mathfrak{p}^{+})}F^{+}_{a,b}$. However, $L^{+}_{a,b}$ is finite-dimensional iff $b\in\mathbb{Z}_{\geq 0}$, therefore (a) follows for the case when $\mathfrak{g}[M]=\mathfrak{p}^{+}$. The case $\mathfrak{g}[M]=\mathfrak{p}^{-}$ is obtained by replacing b with $-a-b$ which corresponds to the replacement of the simple root α_2 of $\mathfrak{b}^{+}$ by the simple root $-\alpha_1-\alpha_2$ of $\mathfrak{b}^{-}$.

Statements (b) and (c) follow from a non-difficult reducibility analysis for the induced module $U(\mathfrak{g})\otimes_{U(\mathfrak{p}^{\pm})}F^{\pm}_{a,b}$. Note first of all that $\mathrm{ch}_{\mathfrak{k}}(U(\mathfrak{g})\otimes_{U(\mathfrak{p}^{\pm})}F^{\pm}_{a,b})$ is always given by the right-hand side of (3). Indeed as $\mathfrak{k}$-modules $\mathfrak{g}/\mathfrak{p}^{\pm}$ and $F^{\pm}_{a,b}$ are isomorphic respectively to V_1 and V_a, therefore

$$c(U(\mathfrak{g})\otimes_{U(\mathfrak{p}^{\pm})}F^{\pm}_{a,b})=c(S^{\cdot}(V_1)\otimes V_a).$$

A straightforward computation shows that $c(S^{\cdot}(V_1)\otimes V_a)$ is nothing but the right hand side of (3).

We claim now that $U(\mathfrak{g})\otimes_{U(\mathfrak{p}^{\pm})}F^{\pm}_{a,b}$ is irreducible precisely when b does not satisfy the respective conditions stated in (b). Consider first the case of $\mathfrak{p}^{+}$. Then $U(\mathfrak{g})\otimes_{U(\mathfrak{p}^{+})}F^{+}_{a,b}$ is irreducible if and only if there exists $w\in W\backslash W_{\mathfrak{k}}$ such that

$$(w((a+1)\omega_1+(b+1)\omega_2)-(\omega_1+\omega_2))(h_1)\in\mathbb{Z}_{\geq 0} \tag{6}$$

and

$$(w((a+1)\omega_1+(b+1)\omega_2)-(\omega_1+\omega_2))=a\omega_1+b\omega_2-m_1\alpha_1-m_2\alpha_2 \tag{7}$$

for some $m_1,m_2\in\mathbb{Z}_{\geq 0}$. The only non $\mathfrak{b}^{+}$-dominant solution of (6) and (7) is $w=w_{\alpha_1+\alpha_2}$ and $-b\in\mathbb{Z}_{\geq 2}, a+b\in\mathbb{Z}_{\geq -1}$. Moreover, in the latter case $L^{+}_{a,b}\simeq(U(\mathfrak{g})\otimes_{U(\mathfrak{p}^{+})}F^{+}_{a,b})/L^{+}_{-b-2,-a-2}$, where $c(L^{+}_{-b-2,-a-2})$ is given by the right hand side of (3) with a replaced by $-b-2$. An immediate computation shows that $c(L^{+}_{a,b})$ is given in this case by the right hand side of (4), therefore (b) and (c) are proved for the case of $\mathfrak{p}^{+}$. The case of $\mathfrak{p}^{-}$ is obtained by interchanging the parameter b in (4) with $-a-b$. ∎

Corollary 5.3. *Let $\mathfrak{g}$ and $\mathfrak{k}$ be as above.*

(a) The minimal $\mathfrak{k}$-type of a simple bounded infinite-dimensional $(\mathfrak{g},\mathfrak{k})$-module can be arbitrary. The multiplicity of the minimal $\mathfrak{k}$-type is always 1.

(b) The following is a complete list of multiplicity-free simple infinite-dimensional $(\mathfrak{g},\mathfrak{k})$-modules:

- $L^{+}_{0,b}$ *for* $b\in\mathbb{C}\backslash\mathbb{Z}_{\geq 0}$,
- $L^{-}_{0,b}$ *for* $-b\in\mathbb{C}\backslash\mathbb{Z}_{\geq 0}$,
- $L^{+}_{a,b}$ *for* $a+b=-1,\ -b\in\mathbb{Z}_{\geq 2}$,
- $L^{-}_{a,b}$ *for* $b=1,\ a+b\in\mathbb{Z}_{\geq 2}$.

5.2. The principal case.. Let now $\mathfrak{k}$ be a principal sl(2)-subalgebra of $\mathfrak{g} = \mathrm{sl}(3)$. The pair $(\mathfrak{g}, \mathfrak{k})$ is well known to be symmetric and the simple $(\mathfrak{g}, \mathfrak{k})$-modules have been studied extensively, see for instance [Fo] and [Sp]. In principle one should be able to identify all simple bounded modules in the known classification of simple Harish-Chandra modules. However, we propose an alternative approach which leads directly to all bounded simple $(\mathfrak{g}, \mathfrak{k})$-modules and their $\mathfrak{k}$-characters. This is the first case in which the richness of the theory of bounded (generalized) Harish-Chandra modules becomes apparent.

We keep the notations $\mathfrak{h}, \mathfrak{b}^+, \alpha_1, \alpha_2$ from Subsection 5. By $L_{a,b}$ we denote the simple $\mathfrak{g}$-module with $\mathfrak{b}^+$-highest weight $(a-1)\omega_1 + (b-1)\omega_2$, by $V_{p,q}$ we denote the simple finite-dimensional $\mathfrak{g} = \mathrm{sl}(3)$-module with $\mathfrak{b}^+$-highest weight $p\omega_1 + q\omega_2$ $(p, q \in \mathbb{Z}_{\geq 0})$, and $\chi(a, b)$ stands for the central character of $L_{a,b}$. By A we denote the Weyl algebra in the indeterminates t, x, y.

We first describe the primitive ideals of all simple bounded $(\mathfrak{g}, \mathfrak{k})$-modules. Let $\mathrm{GKdim} M$ denote the Gelfand-Kirillov dimension of a $\mathfrak{g}$-module M and X_M denote the associated variety of M.

Lemma 5.4. *Let M be an infinite-dimensional bounded simple $(\mathfrak{g}, \mathfrak{k})$-module. Then* $\mathrm{Ann} M = \mathrm{Ann} L_{a,b}$, *where* $\dim L_{a,b} = \infty$, $a \in \mathbb{Z}_{>0}$, $b \in \mathbb{Z}_{>0}$ *or* $a + b \in \mathbb{Z}_{>0}$.

Proof. By Duflo's Theorem $\mathrm{Ann} M = \mathrm{Ann} L_{a,b}$ for some a, b. By Theorem 4.4 in [PS2], $\mathrm{GKdim} M \leq 2$. Since $\mathrm{GKdim} M \geq \frac{1}{2} \dim X_M$ and $\mathrm{GKdim} L_{a,b} = \frac{1}{2} \dim X_M$, we have $\mathrm{GKdim} L_{a,b} \leq 2$. A straightforward computation shows that this latter condition is equivalent to the condition on (a, b) in the statement of the Lemma. ∎

Let $\mathfrak{B}^\chi_\mathfrak{k}$ be the category of bounded $(\mathfrak{g}, \mathfrak{k})$-modules which afford the central character χ, see [PS2], Section 4.

Corollary 5.5. *If $\mathfrak{B}^\chi_\mathfrak{k}$ is not empty, then $\chi = \chi(u + 1 - n, n + 1)$ for some $n \in \mathbb{Z}_{\geq 0}$, where $u \in \mathbb{C} \backslash \mathbb{Z}_{<n-1}$ or $u = -2$.*

Note that the natural embedding of gl(3) into A maps the center of gl(3) to the line $\mathbb{C}\mathbf{E}$ for $\mathbf{E} := t\partial_t + x\partial_x + y\partial_y$, and that the adjoint action of the central element $\mathbf{E}$ on A defines a $\mathbb{Z}$-grading $A := \bigoplus_{i \in \mathbb{Z}} A_i$. Let $u \in \mathbb{C}$. Define the (associative) algebra D^u as the quotient of A_0 by the ideal generated by $\mathbf{E} - u$. The embedding of $\mathfrak{g} \to A_0$ induces a surjective homomorphism $\gamma_u : U(\mathfrak{g}) \to D^u$. It is not difficult to show that if $u \in \mathbb{Z}$, D^u is isomorphic to the algebra of globally defined differential endomorphisms of the line bundle $\mathscr{O}_{\mathbb{P}^2}(u)$ ($\mathbb{P}^2$ being the projective space with homogeneous coordinates (x, y, z)).

Lemma 5.6. *Consider D^u with its adjoint $\mathfrak{g}$-module structure. Then*

$$D^u \simeq \bigoplus_{m \geq 0} V_{m\rho}.$$

Proof. Let $\mathbb{C} = A^0 \subset A^1 \subset \cdots \subset A$ denote the standard filtration of A. A direct computation shows that as a $\mathfrak{g}$-module A_0^m/A_0^{m-1} is isomorphic to

$$V_{m,0} \otimes V_{0,m} = \oplus_{k=0}^m V_{k\rho}.$$

After factorization by $\mathbf{E} - u$, one obtains

$$(D^u)^m/(D^u)^{m-1} \simeq V_{m\rho}.$$

■

It is not difficult to see that the restriction of γ_u to $U(\mathfrak{k})$ is injective. Slightly abusing notation we identify $U(\mathfrak{k})$ with its image in D^u. We will use the following expression for the standard basis E, H, F of $\mathfrak{k}$:

$$E = t\partial_x + x\partial_y, H = 2t\partial_t - 2y\partial_y, F = 2x\partial_t + 2y\partial_x. \quad (8)$$

Lemma 5.7. *The centralizer of $\mathfrak{k}$ in D^u coincides with the center of $U(\mathfrak{k}) \subset D^u$.*

Proof. As $V_{m\rho}^{\mathfrak{k}} = 0$ for odd m and $V_{m\rho}^{\mathfrak{k}} = \mathbb{C}$ for even m it is clear that the centralizer of $\mathfrak{k}$ in D^u is generated by the quadratic Casimir element $\Omega \in V_{2\rho}^{\mathfrak{k}}$. ■

Corollary 5.8. *Every simple $(D^u, \mathfrak{k})$-module is multiplicity-free. For any non-negative m, there exists at most one (up to isomorphism) simple $(D^u, \mathfrak{k})$-module M with $\mathrm{Hom}_{\mathfrak{k}}(V_m, M) \neq 0$.*

Proof. It is well known that if M is a simple $(\mathfrak{g}, \mathfrak{k})$-module, then $M^V = \mathrm{Hom}_{\mathfrak{k}}(V, M)$ is a simple $U(\mathfrak{g})^{\mathfrak{k}}$-module for every $\mathfrak{k}$-type V, see for instance Lemma 3.3 in [PS2]. Therefore Lemma 5.7 implies the first statement. The proof of the second statement is very similar to the proof of Lemma 4.6. ■

We now introduce the functors

$$\begin{aligned} \mathrm{Ind} : D^u - \mathrm{mod} &\hookrightarrow A - \mathrm{mod} \\ M &\mapsto A \otimes_{A_0} M, \\ \mathrm{Res}_u : A - \mathrm{mod} &\hookrightarrow D^u - \mathrm{mod} \\ M &\mapsto D^u \otimes_{A_0} M. \end{aligned}$$

Obviously, $\mathrm{Res}_u \circ \mathrm{Ind} = \mathrm{id}_{D^u - \mathrm{mod}}$.

Lemma 5.9.

$$\ker \gamma_u = \begin{cases} \mathrm{Ann} L_{u+1,1} = \mathrm{Ann} L_{-u-1,u+2} = \mathrm{Ann} L_{1,-u-2} & \text{for } u \notin \mathbb{Z} \\ \mathrm{Ann} L_{-u-1,u+2} = \mathrm{Ann} L_{1,-u-2} & \text{for } u \in \mathbb{Z}_{\geq -1} \\ \mathrm{Ann} L_{u+1,1} = \mathrm{Ann} L_{-u-1,u+2} & \text{for } u \in \mathbb{Z}_{\leq -2} \end{cases}.$$

Proof. First we prove that $\ker \gamma_u \subset \mathrm{Ann} L_{a,b}$ with a, b as in the statement. Note that $\mathrm{Res}_u(t^u \mathbb{C}[t^{\pm 1}, x, y])$ contains a submodule generated by t^u isomorphic to $L_{u+1,1}$,

$\mathrm{Res}_u(x^u\mathbb{C}[t^{\pm 1},x^{\pm 1},y])/\mathrm{Res}_u(x^u\mathbb{C}[t,x^{\pm 1},y])$ contains a submodule with highest vector $t^{-1}x^{u+1}$ isomorphic to $L_{-u-1,u+2}$ and
$\mathrm{Res}_u(y^u\mathbb{C}[t^{\pm 1},x^{\pm 1},y^{\pm 1}])/(\mathrm{Res}_u(y^u\mathbb{C}[t^{\pm 1},x,y^{\pm 1}])+\mathrm{Res}_u(y^u\mathbb{C}[t,x^{\pm 1},y^{\pm 1}]))$ contains a submodule with highest vector $t^{-1}x^{-1}y^{u+2}$ isomorphic to $L_{1,-u-2}$. Hence $\ker\gamma_u\subset \mathrm{Ann}L_{a,b}$. Next we see from Lemma 5.6 that all proper two-sided ideals of D^u have finite codimension. Thus, $\gamma_u(\mathrm{Ann}L_{a,b})$ is either 0 or has finite codimension in D^u. The latter is impossible because $L_{a,b}$ is infinite-dimensional. Hence $\ker\gamma_u=\mathrm{Ann}L_{a,b}$. ■

Since the eigenvalues of ad_H in $U(\mathfrak{g})$ are all even, every simple $(\mathfrak{g},\mathfrak{k})$-module is either odd or even.

As follows from Lemma 5.9, all simple bounded $(\mathfrak{g},\mathfrak{k})$-modules with central character $\chi(u+1,u)$ are $(D^u,\mathfrak{k})$-modules. This allows us to first classify the simple $(D^u,\mathfrak{k})$-modules and then use translation functors to classify the bounded simple modules with arbitrary possible central character, see Corollary 5.5.

Note that the functor Ind maps $(D^u,\mathfrak{k})$-mod into $\widetilde{(A,\mathfrak{k})}$-mod, the latter being defined as the full subcategory of $A-$ mod consisting of $\mathfrak{k}$-locally finite A-modules with semisimple action of $\mathbf{E}$.

Lemma 5.10. *For any simple $(D^u,\mathfrak{k})$-module M there exists a simple $\widetilde{(A,\mathfrak{k})}$-module $\hat{M}$ with $\mathrm{Res}_u(\hat{M})\simeq M$.*

Proof. Let N be a maximal proper A-submodule of $\mathrm{Ind}(M)$. Then $\mathrm{Res}_u(N)\not\supseteq M$ as M generates $\mathrm{Ind}(M)$. Therefore $\mathrm{Res}_u(N)=0$ and one defines $\hat{M}$ as $\mathrm{Ind}(M)/N$. ■

Set $f:=x^2-2ty$, $\Delta:=\partial_x^2-2\partial_y\partial_t$ and note that $f,\Delta\in A^{\mathfrak{k}}$. For every fixed $p\in\mathbb{C}$,we put $R^p:=f^p\mathbb{C}[t,x,y,f^{-1}]$. Then clearly R^p is an $(A,\tilde{\mathfrak{k}})$-module and $\mathrm{Res}_u(R^p)=0$ if $u-2p\notin\mathbb{Z}$. Otherwise,

$$\mathrm{Res}_u(R^p)=\begin{cases}\mathbb{C}f^{\frac{u}{2}}\oplus f^{\frac{u-2}{2}}\mathcal{H}_2\oplus f^{\frac{u-4}{2}}\mathcal{H}_4\oplus\dots & \text{for } u-2p\in 2\mathbb{Z}\\ \mathbb{C}f^{\frac{u-1}{2}}\mathcal{H}_1\oplus f^{\frac{u-3}{2}}\mathcal{H}_3\oplus f^{\frac{u-5}{2}}\mathcal{H}_4\oplus\dots & \text{for } u-2p\in 2\mathbb{Z}+1,\end{cases}\tag{9}$$

where $\mathcal{H}_n$ denotes the space of homogeneous polynomials of degree n in $\mathbb{C}[t,x,y]$ annihilated by Δ (as a $\mathfrak{k}$-module $\mathcal{H}_n$ is isomorphic to V_{2n}).

Lemma 5.11.
(a) For $u\notin\mathbb{Z}$ and for $u=-1,-2$, $\mathrm{Res}_u(R^{\frac{u}{2}})$ and $\mathrm{Res}_u(R^{\frac{u+1}{2}})$ are simple D^u-modules.
(b) For $u\in 2\mathbb{Z}_{\geq 0}$, $\mathrm{Res}_u(R^{\frac{u+1}{2}})$ is a simple D^u-module and there is an exact sequence

$$0\to V_{u,0}\to \mathrm{Res}_u(R^{\frac{u}{2}})\to I^+_{u,0}\to 0\tag{10}$$

for some simple D^u-module $I^+_{u,0}$.
(c) For $u\in 1+2\mathbb{Z}_{\geq 0}$, $\mathrm{Res}_u(R^{\frac{u}{2}})$ is a simple D^u-module and there is an exact sequence

$$0\to V_{u,0}\to \mathrm{Res}_u(R^{\frac{u+1}{2}})\to I^-_{u,0}\to 0$$

for some simple D^u-module $I^-_{u,0}$.

(d) For $u \in 2\mathbb{Z}_{\le -2}$, $\mathrm{Res}_u(R^{\frac{u}{2}})$ *is a simple* D^u*-module and there is an exact sequence*

$$0 \to I^-_{u,0} \to \mathrm{Res}_u(R^{\frac{u+1}{2}}) \to V_{0,-3-u} \to 0$$

for some simple D^u*-module* $I^-_{u,0}$.

(e) For $u \in 1 + 2\mathbb{Z}_{\le -1}$, $\mathrm{Res}_u(R^{\frac{u+1}{2}})$ *is a simple* D^u*-module and there is an exact sequence*

$$0 \to I^+_{u,0} \to \mathrm{Res}_u(R^{\frac{u}{2}}) \to V_{0,-3-u} \to 0$$

for some simple D^u*-module* $I^+_{u,0}$.

Proof. The isomorphism (9) yields

$$c(\mathrm{Res}_u(R^{\frac{u}{2}})) = 1 + z^4 + z^8 + \dots, \quad c(\mathrm{Res}_u(R^{\frac{u+1}{2}})) = z^2 + z^6 + z^{10} + \dots \quad . \tag{11}$$

Thus, if $\mathrm{Res}_u(R^{\frac{u}{2}})$ (respectively $\mathrm{Res}_u(R^{\frac{u+1}{2}})$) is not simple it has a unique simple finite-dimensional submodule or a unique simple finite-dimensional quotient. By Lemma 5.9 the latter can happen only if $u \in \mathbb{Z}_{\ge 0}$ or $u \in \mathbb{Z}_{\le -3}$. Hence (a).

Let $u \in 2\mathbb{Z}_{\ge 0}$. Then $\mathrm{Res}_u(R^{\frac{u}{2}})$ contains $\mathrm{Res}_u(\mathbb{C}[t,x,y]) \simeq V_{u,0}$ as a finite-dimensional simple submodule, hence (10). The $\mathfrak{g}$-module $\mathrm{Res}_u(R^{\frac{u+1}{2}})$ has the same central character as $\mathrm{Res}_u(R^{\frac{u}{2}})$ and, since $V_{n,0}$ is not a subquotient of $\mathrm{Res}_u(R^{\frac{u+1}{2}})$ by (11), $\mathrm{Res}_u(R^{\frac{u+1}{2}})$ is a simple D^u-module. Hence (b).

As $\Delta(f^{-\frac{1}{2}}) = 0$, $f^{-\frac{1}{2}}$ generates a proper A-submodule $M \subset f^{\frac{1}{2}}\mathbb{C}[t,x,y,f^{-1}]$. A direct computation shows that $\dim \mathrm{Res}_u(M) = \infty$ for any $u \in 1 + 2\mathbb{Z}_{\ge -2}$. Furthermore, the only finite-dimensional module, whose central character coincides with that of D^u is $V_{0,-3-u}$. Therefore one necessarily has

$$0 \to I^+_{u,0} \to \mathrm{Res}_u(R^{\frac{u}{2}}) \to V_{0,-3-u} \to 0$$

where $I^+_{u,0} := \mathrm{Res}_u(M)$. $\mathrm{Res}_u(R^{\frac{u+1}{2}})$ is simple by the same reason as in (b). Hence (e).

(c) and (d) are similar to (b) and (e). ■

For any $u \in \mathbb{C}$ we define now $I^+_{u,0}$ (respectively, $I^-_{u,0}$) as the unique simple infinite-dimensional constituent of $\mathrm{Res}_u(R^{\frac{u}{2}})$ (resp., $\mathrm{Res}_u(R^{\frac{u+1}{2}})$).

Corollary 5.12. *Every simple even infinite-dimensional* $(D^u, \mathfrak{k})$*-module is isomorphic to* $I^{\pm}_{u,0}$.

Proof. For every fixed u and any sufficiently large $m \in 2\mathbb{Z}_{\ge 0}$ (such that V_m is not a $\mathfrak{k}$-type of $V_{u,0}$ or $V_{0,-3-u}$ for $u \in \mathbb{Z}$), Lemma 5.11 implies $\mathrm{Hom}_{\mathfrak{k}}(V_m, I^{\pm}_u) \neq 0$. The statement follows now from Corollary 5.8. ■

Lemma 5.13. *If* $u \notin \frac{1}{2} + \mathbb{Z}$, *then every* $(D^u, \mathfrak{k})$*-module is even.*

Proof. Assume that M is an odd simple $(D^u, \mathfrak{k})$-module and $u \notin \frac{1}{2} + \mathbb{Z}$. Let $\hat{M}$ be as in Lemma 5.10, A_f denote the localization of A in f, $\hat{M}_f = A_f \otimes_A \hat{M}$.

First, we claim that if $u \notin \frac{1}{2} + \mathbb{Z}$, then $\hat{M}_f \neq 0$. Indeed, $\hat{M}_f = 0$ implies that f acts locally nilpotently on $\hat{M}$. Then $M^0 := \ker f$ is a $\mathfrak{k}$-submodule of $\hat{M}$ and a straightforward calculation using (8) shows $\Omega_{|M^0} = 2(\mathbf{E}+3)(\mathbf{E}+2)_{|M^0}$. Thus $\mathrm{Hom}_{\mathfrak{k}}(V_m, M^0) \neq 0$ only if $2(d+3)(d+2) = \frac{m^2}{2} + m$ or equivalently $(d+\frac{5}{2})^2 = (\frac{m+1}{2})^2$, where d is the eigenvalue of $\mathbf{E}$ on M^0. Since $d \in u + \mathbb{Z}$, $u \notin \frac{1}{2} + \mathbb{Z}$ implies $M^0 = 0$.

Our next observation is that $\hat{M}_f$ is an odd $(A, \mathfrak{k})$-module and that t does not act locally nilpotently on $\hat{M}_f$. Indeed, if t acts locally nilpotently, by $\mathfrak{k}$-invariance x and y act locally nilpotenly, and therefore f acts locally nilpotently. Contradiction. Therefore $\hat{M}_f$ is a submodule of its localization in t, $\hat{M}_{f,t}$. Furthermore, for some odd m there exists a non-zero vector $v \in \hat{M}_{f,t}$ such that $H \cdot v = mv$, $E \cdot v = 0$ and $\mathbf{E} \cdot v = uv$. The expressions for E, H and $\mathbf{E}$ imply

$$\partial_t v = \frac{-(u+m/2)ty + mx^2/2}{tf} v, \partial_x v = \frac{(u-m/2)x}{f} v, \partial_y v = \frac{(m/2-u)t}{f} v.$$

Thus, every vector in $\hat{M}_{f,t}$ can be obtained from v by applying elements of $\mathbb{C}[t^{\pm 1}, x, y, f^{-1}]$, i.e. $\hat{M}_{f,t} = \mathbb{C}[t^{\pm 1}, x, y, f^{-1}]v$. It is not difficult to see that $v = t^{\frac{m}{2}} f^{\frac{2u-m}{4}}$ satisfies the above relations. The $A_{f,t}$-module $\mathbb{C}[t^{\pm 1}, x, y, f^{-1}]v$ is simple and free over $\mathbb{C}[t^{\pm 1}, x, y, f^{-1}]$. Hence $\hat{M}_{f,t} \simeq \mathbb{C}[t^{\pm 1}, x, y, f^{-1}]v$ and it is obvious that $\hat{M}_{f,t}$ has no non-zero $\mathfrak{k}$-finite vectors. As we pointed out above, $\hat{M}_f \subset \hat{M}_{f,t}$. Therefore $\hat{M}_f = 0$. ■

We now turn to odd simple $(D^u, \mathfrak{k})$-modules.

Lemma 5.14. *Let $u \in \frac{1}{2} + \mathbb{Z}$. Up to isomorphism, there exists exactly one odd simple $(D^u, \mathfrak{k})$-module $J_{u,0}$. Moreover,*

$$c(J_{u,0}) = \begin{cases} z^{2-2u} + z^{6-2u} + z^{10-2u} + \dots & \text{for } u < 0 \\ z^{4+2u} + z^{8+2u} + z^{12+2u} + \dots & \text{for } u > 0 \end{cases}. \tag{12}$$

Proof. Let $P \subset G = SL(3)$ be the maximal parabolic subgroup whose Lie algebra $\mathfrak{p}$ equals $\mathfrak{b} \oplus \mathfrak{g}^{-\alpha_1}$, $K \subset G$ be the algebraic subgroup with Lie algebra $\mathfrak{k}$, and Z be the closed K-orbit on $G/P \simeq \mathbb{P}^2$. Then $Z \simeq \mathbb{P}^1$ and the embedding $i : Z \to \mathbb{P}^2$ is a Veronese embedding of degree 2. It is not difficult to verify that the relative tangent bundle $\mathcal{T}_P$ of the projection $p : G/B \to G/P$ is a $\mathscr{O}_{G/B}$-submodule of the twisted sheaf of differential operators $\mathscr{D}_{G/B}^{(u+1)\omega_1+\omega_2}$ (the definition of $\mathscr{D}_{G/B}^{\zeta}$ see in [PS2], Section 5). Furthermore, the direct image $p_*(\mathscr{D}_{G/B}^{(u+1)\omega_1+\omega_2}/\mathcal{I}_P)$, where $\mathcal{I}_P$ is the left ideal in $\mathscr{D}_{G/B}^{(u+1)\omega_1+\omega_2}$ generated by $\mathcal{T}_P$, is a well-defined twisted sheaf of differential operators on G/P. We denote this sheaf by $\mathscr{D}_{G/P}^{(u+1)\omega_1+\omega_2}$.

Our next observation is that, similarly to the equivalence of categories $i_\bigstar$ discussed in Section 5 of [PS2], Kashiwara's theorem yields an equivalence of categories

$$i_\bigstar^u : \mathscr{O}_Z(2u) \otimes_{\mathscr{O}_{G/P}} \mathscr{D}_{G/P} \otimes_{\mathscr{O}_{G/P}} \mathscr{O}_Z(-2u) \;-\; \mathrm{mod} \;\to (\mathscr{D}_{G/P}^{(u+1)\omega_1+\omega_2} \;-\; \mathrm{mod}\;)^Z,$$

where $(\mathscr{D}_{G/P}^{(u+1)\omega_1+\omega_2} - \text{mod})^Z$ denotes the full subcategory of $\mathscr{D}_{G/P}^{(u+1)\omega_1+\omega_2}$- mod supported on Z, and $\mathscr{O}_Z(2u)$ is the line bundle on Z with Chern class $2u$. Therefore we can put

$$J_{u,0} := \Gamma(\mathbb{P}^2, i_\star^u \mathscr{O}_Z(2u)).$$

It is clear that $J_{u,0}$ is a $(\mathfrak{g},\mathfrak{k})$-module, and furthermore, using the facts that $\mathcal{N} \simeq \mathscr{O}_Z(4)$ and that $i_\star^u \mathscr{O}_Z(2u)$ has a filtration with successive quotients $\mathscr{O}_Z(2u+4(i+1))$, one easily verifies that $c(J_{u,0})$ is given by the right-hand side of (12). Since there are no finite-dimensional modules with central character $\chi(u+1,1)$ for $u \in \frac{1}{2}+\mathbb{Z}$, $J_{u,0}$ is a simple $\mathfrak{g}$-module.

It remains to prove that every simple odd $(D^u,\mathfrak{k})$-module is isomorphic to $J_{u,0}$ for some $u \in \frac{1}{2}+\mathbb{Z}$. Let M be a simple odd $(D^u,\mathfrak{k})$-module and $\hat{M}$ be a simple $(A,\tilde{\mathfrak{k}})$-module such that $\mathrm{Res}_u(\hat{M}) = M$. Then by the proof of Lemma 5.14 $\hat{M}_f = 0$. For every $\mathfrak{b}_\mathfrak{k}$-highest vector $v \in \mathrm{Res}_u(\hat{M})$ there exists k such that $f^k \cdot v = 0$. Let v have weight m. Then by the relation $(d+\frac{5}{2})^2 = (\frac{m+1}{2})^2$ from the proof of Lemma 5.14, $\frac{m+1}{2} = \pm(u+2k+\frac{5}{2})$, as $\mathbf{E}f^k \cdot v = (2k+u)f^k \cdot v$. Without loss of generality we may assume that m is very large and then $\frac{m+1}{2} = (u+2k+\frac{5}{2})$. Therefore $\mathrm{Hom}_\mathfrak{k}(V_m, M) \neq 0$ implies $m = 2u+4k+4$. Hence if M and M' are two odd $(D^u,\mathfrak{k})$-modules one can find m such that $\mathrm{Hom}_\mathfrak{k}(V_m, M) \neq 0$, $\mathrm{Hom}_\mathfrak{k}(V_m, M') \neq 0$. But then $M \simeq M'$ by Corollary 5.8. ■

Let M be some A-module with semisimple $\mathbf{E}$-action. Consider the $U(\mathfrak{g})$-modules $M^{(n)} := M \otimes S^n(\mathrm{span}\{x,y,t\})$ for $n \in \mathbb{Z}_{\geq 0}$, together with the linear operators

$$\begin{aligned} \bar{d} : M^{(n)} &\to M^{(n-1)} \\ \bar{d} &= t \otimes \partial_t + x \otimes \partial_x + y \otimes \partial_y \\ \bar{\delta} : M^{(n)} &\to M^{(n+1)} \\ \bar{\delta} &= \partial_t \otimes t + \partial_x \otimes x + \partial_y \otimes y. \end{aligned}$$

It is straightforward to check that $\bar{d}$, $\mathbf{E}\otimes 1 - 1 \otimes \mathbf{E}$ and $\bar{\delta}$ form a standard sl(2)-triple. Let $\mathrm{Res}_s(M^{(k)})$ be the eigenspace of the operator $\mathbf{E}\otimes 1 + 1\otimes \mathbf{E}$ in $M^{(k)}$. Then obviously $\bar{d}$ and $\bar{\delta}$ induce operators

$$\begin{aligned} d : \mathrm{Res}_s(M^{(n)}) &\to \mathrm{Res}_s(M^{(n-1)}) \\ \delta : \mathrm{Res}_s(M^{(n-1)}) &\to \mathrm{Res}_s(M^{(n)}), \end{aligned}$$

and elementary sl(2) representation theory implies that if $s \notin \mathbb{Z}$, $s < n-1$ or $s \geq 2n$, then d is surjective, δ is injective, and

$$\mathrm{Res}_s(M^{(n)}) = \ker d \oplus \mathrm{im}\delta. \tag{13}$$

For any $(D^u,\mathfrak{k})$-module M choose a simple $(A,\tilde{\mathfrak{k}})$-module $\hat{M}$ such that $\mathrm{Res}_u(\hat{M}) = M$ (in fact $\hat{M}$ is unique).

Let $T^n(M) := \mathrm{Res}_{u+n}(\hat{M}^{(n)}) \cap \ker d$. If $u \neq -1, 0, \ldots, n-1$, (13) implies

$$c(T^n(M)) = c(\mathrm{Res}_{u+n}(\hat{M}^{(n)})) - c(\mathrm{Res}_{u+n}(\hat{M}^{(n-1)})). \tag{14}$$

Lemma 5.15. *Let M be a bounded simple $(D^u, \mathfrak{k})$-module. Assume that $u \neq -1, 0, \ldots, n-1$. Then $T^n(M)$ is a simple $(\mathfrak{g}, \mathfrak{k})$-module with central character $\chi(u+1-n, n+1)$.*

Proof. Lemma 5.9 implies that M is a $(\mathfrak{g}, \mathfrak{k})$-module with central character $\chi(u+1, 1)$. Therefore $M \otimes S^n(\mathrm{span}\{x, y, t\})$ has constituents with central character $\chi(u+1+n-2k, 1+k)$, $k = 0, \ldots, n$, and $\mathrm{im}\delta$ has constituents with central character $\chi(u+1+n-2k, 1+k)$, $k = 0, \ldots, n-1$. Thus, $T^n(M)$ is a direct summand of $M \otimes S^n(\mathrm{span}\{x, y, t\})$ with central character $\chi(u+1-n, n+1)$.

Our restrictions on u imply that the weights

$$(u+1)\omega_1 + \omega_2 \text{ and } (u-n+1)\omega_1 + (n+1)\omega_2$$

belong to the same Weyl chamber and have the same stabilizer in the Weyl group. Hence, T^n is nothing but the translation functor

$$T^{(u-n+1)\omega_1+(n+1)\omega_2}_{(u+1)\omega_1+\omega_2} : \mathfrak{B}^{\chi(u+1,1)}_{\mathfrak{k}} \to \mathfrak{B}^{\chi(u-n+1,n+1)}_{\mathfrak{k}}.$$

Therefore T^n is an equivalence of categories, in particular $T^n(M)$ is simple. ■

We put for $u \neq -1, 0, \ldots, n-1$

$$I^{\pm}_{u,n} := T^n(I^{\pm}_{u,0}),$$

$$J_{u,n} := T^n(J_{u,0}).$$

Theorem 5.16. *Let M be a simple bounded infinite-dimensional $(\mathfrak{g}, \mathfrak{k})$-module with central character χ. Then*

(a) if $\chi = \chi(u+1-n, n+1)$ for $u \notin \mathbb{Z}$,

$$M \simeq \begin{cases} I^{\pm}_{u,n} & \text{for } u \notin \frac{1}{2} + \mathbb{Z} \\ I^{\pm}_{u,n}, J_{u,n} & \text{for } u \in \frac{1}{2} + \mathbb{Z} \end{cases};$$

(b) if $\chi = \chi(u+1-n, n+1)$ for $u \in \mathbb{Z}_{\geq n}$,

$$M \simeq I^{\pm}_{-n-3,u-n}, I^{\pm}_{u,n};$$

(c) if $\chi = \chi(-1-n, n+1)$,

$$M \simeq I^{\pm}_{-2,n};$$

(d) if $\chi = \chi(0, n+1)$,

$$M \simeq (I^{\pm}_{-2,n})^{\tau},$$

where τ stands for the outer automorphism $\tau(X) = -X^t$ for any $X \in \mathfrak{g}$.

Proof. By Corollary 5.5 every simple bounded $(\mathfrak{g}, \mathfrak{k})$-module has central character χ of the form $\chi(u+1-n, n+1)$ for some $n \in \mathbb{Z}_{\geq 0}$ and some $u \in \{\mathbb{C} \backslash \mathbb{Z}_{<n-1}\} \cup \{-2\}$. Moreover, $T^n = T^{(u-n+1)\omega_1+(n+1)\omega_2}_{(u+1)\omega_1+\omega_2}$ is an equivalence of the categories $\mathfrak{B}^{\chi(u+1,1)}_{\mathfrak{k}}$ and $\mathfrak{B}^{\chi(u+1-n,n+1)}_{\mathfrak{k}}$. If $u \notin \mathbb{Z}$, $\frac{1}{2} + \mathbb{Z}$ then $\mathfrak{B}^{\chi(u+1,1)}_{\mathfrak{k}}$ has two non-isomorphic simple objects, and, if $u \in \frac{1}{2} + \mathbb{Z}$, $\mathfrak{B}^{\chi(u+1,1)}_{\mathfrak{k}}$ has three non-isomorphic simple objects. This implies (a).

If $u \in \mathbb{Z}_{\geq 0}$, $u \geq n$, we have

$$\chi = \chi(u+1-n, n+1) = \chi((-n-3)+1-(u-n), (u-n)+1),$$

hence in this case $\mathfrak{B}^{\chi}_{\mathfrak{k}}$ has 4 non-isomorphic simple objects: $I^{\pm}_{u,n}$ and $I^{\pm}_{-n-3,u-n}$. This proves (b). If $n=-2$, $\mathfrak{B}^{\chi}_{\mathfrak{k}}$ is equivalent to $\mathfrak{B}^{\chi(1,1)}_{\mathfrak{k}}$ and has two simple objects, $I^{\pm}_{-2,n}$, which proves (c). Finally if $u = n-1$, the automorphism τ establishes an equivalence between $\mathfrak{B}^{\chi(0,n+1)}_{\mathfrak{k}}$ and $\mathfrak{B}^{\chi(-1-n,n+1)}_{\mathfrak{k}}$, hence (d). ∎

Lemma 5.17. *For $a \in \mathbb{Z}_{\geq 2}$, define*

$$\mu_n(a,z) := \frac{z^a}{1-z^4} \otimes c(V_{n,0}) - \frac{z^{a-2}}{1-z^4} \otimes c(V_{n-1,0}).$$

For $a \in \mathbb{Z}_{\geq 0}$, define

$$\kappa_n(a,z) := \frac{z^a}{1-z^4} \otimes c(V_{n,0}) - \frac{z^{a+2}}{1-z^4} \otimes c(V_{n-1,0}).$$

Then

$$\mu_{2p}(a,z) = \frac{z^a}{1-z^4} + \frac{z^{a-2}(z^4+z^8+\cdots+z^{4p})}{1-z^2}, \tag{15}$$

$$\mu_{2p+1}(a,z) = \frac{z^a(1+z^4+\cdots+z^{4p})}{1-z^2}, \tag{16}$$

$$\kappa_{2p}(a,z) = \frac{z^a}{1-z^4} + \frac{z^{|a-4|}+\cdots+z^{|a-4p|}}{1-z^2}, \tag{17}$$

$$\kappa_{2p+1}(a,z) = \frac{z^{|a-2|}+\cdots+z^{|a-4p-2|}}{1-z^2}. \tag{18}$$

Proof. Since $V_{n,0} = S^n(V_{1,0})$, and since $S^n(V_{1,0})$ is isomorphic as a $\mathfrak{k}$-module to $S^n(V_2)$, we have

$$c(V_{2p,0}) = 1 + z^4 + \cdots + z^{2p},$$
$$c(V_{2p+1,0}) = z^2 + z^6 + \cdots + z^{2p+2}.$$

Recall that $z^a \otimes z^b = \pi(z^a \sum_{i=0}^{i=b} z^{b-2i})$ (Lemma 3.2,(b)). Therefore

$$\frac{z^a}{1-z^4} \otimes z^{2k} - \frac{z^{a-2}}{1-z^4} \otimes z^{2k-2} = \pi\left(\frac{z^{a-2}(z^2\sum_{i=0}^{i=2k} z^{2k-2i} - z^{-2}\sum_{i=0}^{i=2k-2} z^{2k-2i})}{1-z^4}\right) =$$
$$= \pi\left(\frac{z^{a-2}(z^{2k+2}+z^{2k})}{1-z^4}\right) = \frac{z^{a-2+2k}}{1-z^2}.$$

$$\frac{z^a}{1-z^4} \otimes z^{2k} - \frac{z^{a+2}}{1-z^4} \otimes z^{2k-2} = \pi\left(\frac{z^{a}(\sum_{i=0}^{i=2k} z^{2k-2i} - z^{2}\sum_{i=0}^{i=2k-2} z^{2k-2-2i})}{1-z^4}\right) =$$
$$= \pi\left(\frac{z^a(z^{-2k}+z^{2-2k})}{1-z^4}\right) = \pi\left(\frac{z^{a-2k}}{1-z^2}\right) = \frac{z^{|a-2k|}}{1-z^2}.$$

The above identities imply (15)-(18). ∎

Theorem 5.18.
(a) Let $u \notin \mathbb{Z}$, $\frac{1}{2}+\mathbb{Z}$. Then

$$c(I^+_{u,n}) = \kappa_n(0,z), \quad c(I^-_{u,n}) = \mu_n(2,z).$$

(b) Let $u \in \frac{1}{2}+\mathbb{Z}$. Then

$$\begin{array}{ll} c(J_{u,n}) = \kappa_n(4+2u,z) & \text{for } u \geq -\frac{1}{2}, \\ c(J_{u,n}) = \mu_n(2-2u,z) & \text{for } u \leq -\frac{3}{2}. \end{array}$$

(c) Let $u \in 2\mathbb{Z}_{\geq 0}$. Then

$$\begin{array}{ll} c(I^+_{u,0}) = \frac{z^{2u+4}}{1-z^4}, & c(I^-_{u,0}) = \frac{z^2}{1-z^4}, \\ c(I^+_{u,n}) = \kappa_n(2u+4,z), & c(I^-_{u,n}) = \mu_n(2,z). \end{array}$$

(d) Let $u \in 1+2\mathbb{Z}_{\geq 0}$. Then

$$\begin{array}{ll} c(I^+_{u,0}) = \frac{1}{1-z^4}, & c(I^-_{u,0}) = \frac{z^{2u+4}}{1-z^4}, \\ c(I^+_{u,n}) = \kappa_n(0,z), & c(I^-_{u,n}) = \kappa_n(2u+4,z). \end{array}$$

(e) Let $u \in 2\mathbb{Z}_{\leq -2}$. Then

$$\begin{array}{ll} c(I^+_{u,0}) = \frac{1}{1-z^4}, & c(I^-_{u,0}) = \frac{z^{-2-2u}}{1-z^4}, \\ c(I^+_{u,n}) = \kappa_n(0,z), & c(I^-_{u,n}) = \mu_n(-2-2u,z). \end{array}$$

(f) Let $u \in -1+2\mathbb{Z}_{\leq -1}$. Then

$$\begin{array}{ll} c(I^+_{c,0}) = \frac{z^{-2-2u}}{1-z^4}, & c(I^-_{u,0}) = \frac{z^2}{1-z^4}, \\ c(I^+_{u,n}) = \mu_n(-2-2u,z), & c(I^-_{u,n}) = \mu_n(2,z). \end{array}$$

(g)

$$\begin{array}{l} c(I^+_{-2,n}) = c((I^+_{-2,n})^\tau) = \kappa_n(0,z), \\ c(I^-_{-2,n}) = c((I^-_{-2,n})^\tau) = \mu_n(2,z). \end{array}$$

Proof. Using (14) one obtains the identities

$$\begin{array}{l} c(I^\pm_{u,n}) = c(I^\pm_{u,0} \otimes V_{n,0}) - c(I^\mp_{u+1,0} \otimes V_{n-1,0}), \\ c(J_{u,n}) = c(J_{u,0} \otimes V_{n,0}) - c(J_{u+1,0} \otimes V_{n-1,0}). \end{array} \tag{19}$$

The theorem is a straightforward corollary of (19). Indeed, let us prove (f). In this case

$$c(I^+_{u,0}) = \frac{z^{-2u-2}}{1-z^4}, \quad c(I^+_{u-1,0}) = \frac{z^{-2u-4}}{1-z^4},$$

$$c(I^+_{u,n}) = \frac{z^{-2u-2}}{1-z^4} \otimes c(V_{n,0}) - \frac{z^{-2u-4}}{1-z^4} \otimes c(V_{n-1,0}) = \mu_n(-2-2u,z);$$

$$c(I^-_{u-1,0}) = \frac{z^{-2u-4}}{1-z^4}, \quad c(I^+_{u-1,0}) = \frac{1}{1-z^4},$$

$$c(I^-_{u,n}) = \frac{z^2}{1-z^4} \otimes c(V_{n,0}) - \frac{1}{1-z^4} \otimes c(V_{n-1,0}) = \mu_n(2,z).$$

In all other cases the arguments are similar. ∎

Corollary 5.19.

(a) The minimal $\mathfrak{k}$*-type can be any* V_k *but its multiplicity is always 1.*

(b) For sufficiently large i $c_i(M) = c_{i+4}(M)$ *for any simple bounded* $(\mathfrak{g}, \mathfrak{k})$*-module, and for sufficiently large* j *there are the following* $\mathfrak{k}$*-multiplicities:*

$$c_{4j}(I^{\pm}_{u,2p+1}) = c_{4j+2}(I^{\pm}_{u,2p+1}) = p+1,$$

$$c_{4j}(I^{+}_{u,2p}) = p+1, c_{4j+2}(I^{+}_{u,2p}) = p,$$

$$c_{4j+2}(I^{-}_{u,2p}) = p+1, c_{4j}(I^{-}_{u,2p}) = p,$$

$$c_{4j+1}(J_{u,2p+1}) = c_{4j+3}(J_{u,2p+1}) = p+1,$$

$$c_{4j+2u}(J_{u,2p}) = p, c_{4j+2u+2}(J_{u,2p}) = p+1.$$

(c) The only multiplicity-free simple infinite-dimensional $(\mathfrak{g}, \mathfrak{k})$*-modules are* $I^{\pm}_{u,0}$, $J_{u,0}$, $I^{\pm}_{u,1}$, $J_{u,1}$, $(I^{\pm}_{-2,1})^{\tau}$.

The complete list of multiplicity-free simple $(\mathfrak{g}, \mathfrak{k})$-modules has been first found by Dj. Sijacki, see [S] and the references therein for a historic perspective on this problem.

6. Classification of simple bounded $(\mathrm{sp}(4), \mathrm{sl}(2))$-modules

In this section we classify all simple bounded $(\mathfrak{g}, \mathfrak{k})$-modules, where $\mathfrak{g} = \mathrm{sp}(4)$ and $\mathfrak{k}$ is a principal $\mathrm{sl}(2)$-subalgebra or a $\mathrm{sl}(2)$-subalgebra corresponding to a short root. We fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and write the roots of $\mathfrak{g}$ as $\{\pm 2\epsilon_1, \pm 2\epsilon_2, \pm\epsilon_1 \pm \epsilon_2\}$. Our fixed simple roots are $\epsilon_1 - \epsilon_2, 2\epsilon_2$, and $\rho = 2\epsilon_1 + \epsilon_2$. By e_1, e_2, h_1, h_2, f_1, f_2 we denote the Serre generators of $\mathfrak{g}$ associated to our choice of simple roots, [OV]. We define two $\mathrm{sl}(2)$-subalgebras of $\mathfrak{g}$: one with basis e_1, h_1, f_1 and one with basis $e_1 + 2e_2$, $3h_1 + 4h_2, 3f_1 + 2f_2$. The first one is the root subalgebra corresponding to the simple root $\epsilon_1 - \epsilon_2$, and the second one is a principal $\mathrm{sl}(2)$-subalgebra. In Sections 6 and 7, we denote by $\mathfrak{k}$ any one of these two subalgebras, referring respectively to the *root case* and to the *principal case* when we want to be specific. We set $\mathfrak{b}_{\mathfrak{k}} := \mathfrak{b} \cap \mathfrak{k}$, where $\mathfrak{b}$ is the Borel subalgebra generated by e_1, e_2, h_1, h_2. By $L_{a,b}$ we denote the simple $\mathfrak{b}$-highest weight $\mathfrak{g}$-module with highest weight $a\epsilon_1 + b\epsilon_2 - \rho = (a-2)\epsilon_1 + (b-1)\epsilon_2$, by $V_{a,b}$ we denote the simple finite-dimensional $\mathfrak{g}$-module with highest weight $a\epsilon_1 + b\epsilon_2$, and $\chi(a,b)$ is the central character of $L_{a,b}$.

Lemma 6.1. *Let dim* $L_{a,b} = \infty$ *and* $\mathrm{GKdim} L_{a,b} \leq 2$. *Then* $a > |b|$ *and* $a, b \in \frac{1}{2} + \mathbb{Z}$.

Proof. Let $\lambda = a\epsilon_1 + b\epsilon_2$. If $(\lambda, \alpha) \notin \mathbb{Z}_{>0}$ for all positive roots α, then $L_{a,b}$ is a Verma module and therefore its Gelfand-Kirillov dimension equals 4. If $(\lambda, \check{\alpha}) \in \mathbb{Z}_{>0}$ for exactly one positive root, then one has the following exact sequence

$$0 \to L_{w_\alpha(\lambda)} \to M_\lambda \to L_\lambda \to 0,$$

where w_α denotes the reflection in α. A straightforward computation shows that in this case $\mathrm{GKdim} L_\lambda = 3$. Therefore $\mathrm{GKdim} L_\lambda \leq 2$ implies the existence of two positive roots α and β such that $(\lambda, \check{\alpha}), (\lambda, \check{\beta}) \in \mathbb{Z}_{>0}$. One can see immediately that at least one of these roots, say α, is simple. If N_λ denotes the quotient of M_λ by the submodule generated by a highest vector with weight $w_\alpha(\lambda) - \rho$, then $\mathrm{GKdim} N_\lambda = 3$. The condition $\mathrm{GKdim} L_\lambda \leq 2$ implies the reducibility of N_λ which in turn implies $(\lambda, \check{\gamma}) \in \mathbb{Z}_{>0}$ for the positive root γ orthogonal to α. That leaves only two possibilities for λ: λ is either regular integral or λ satisfies the conditions of the Lemma.

It remains to eliminate the case of a regular integral non-dominant λ. By using the translation functor we may assume without loss of generality that λ belongs to the Weyl group orbit of ρ. That leaves four possibilities for λ: $2\epsilon_1 - \epsilon_2$, $\epsilon_1 - 2\epsilon_2$, $\epsilon_1 + 2\epsilon_2$, $-\epsilon_1 + 2\epsilon_2$. Let $\mathfrak{p}_1$ and $\mathfrak{p}_2$ be the parabolic subalgebras obtained from $\mathfrak{b}$ by joining $\epsilon_2 - \epsilon_1$ and $-2\epsilon_2$ respectively. It is not difficult to verify the existence of embeddings

$$L_{2,-1} \to U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_1)} F^1_{2,1}, \quad L_{1,-2} \to U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_1)} F^1_{2,-1},$$
$$L_{1,2} \to U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_2)} F^2_{2,1}, \quad L_{-1,2} \to U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_2)} F^2_{1,2},$$

where $F^1_{a,b}$ (respectively, $F^2_{a,b}$) is the finite dimensional $\mathfrak{p}_1$-module (resp., $\mathfrak{p}_2$-module) with $\mathfrak{b}$-highest weight $a\epsilon_1 + b\epsilon_2 - \rho$. Therefore the Gelfand-Kirillov dimension of any of the above four simple modules equals the Gelfand-Kirillov dimension of the corresponding parabolically induced module, i.e. 3. The proof is now complete. ∎

Corollary 6.2. *Let M be a simple bounded infinite-dimensional $(\mathfrak{g}, \mathfrak{k})$-module. Then $\mathrm{Ann} M = \mathrm{Ann} L_{a,b}$ for some a, b with $a > |b|$, $a, b \in \frac{1}{2} + \mathbb{Z}$. In particular, $\chi(a,b)$ is the central character of M.*

Proof. By Duflo's theorem, $\mathrm{Ann} M = \mathrm{Ann} L_{a,b}$ for some a, b. It is known that $\frac{1}{2} \dim X_{L_{a,b}} = \mathrm{GKdim} L_{a,b}$, thus $\mathrm{GKdim} M \geq \mathrm{GKdim} L_{a,b}$. On the other hand, $\mathrm{GKdim} M \leq 2 = b_{\mathfrak{k}}$ holds by Theorem 4.4 in [PS2] . Hence $\mathrm{GKdim} L_{a,b} \leq 2$, and Lemma 6.1 applies to $L_{a,b}$. ∎

Corollary 6.3. *Let $a, b \in \frac{1}{2} + \mathbb{Z}$, $a > |b|$. Then $\mathfrak{B}_{\mathfrak{k}}^{\chi(a,b)}$ is equivalent to $\mathfrak{B}_{\mathfrak{k}}^{\chi(\frac{3}{2},\frac{1}{2})}$.*

Proof. It is well known that the categories $U^{\chi(a,b)}$-mod for (a,b) as above are translation-equivalent to the category $U^{\chi(\frac{3}{2},\frac{1}{2})}$-mod. Since the translation functor preserves the subcategories of bounded modules, the categories $\mathfrak{B}_{\mathfrak{k}}^{\chi(a,b)}$ and $\mathfrak{B}_{\mathfrak{k}}^{\chi(\frac{3}{2},\frac{1}{2})}$ are equivalent as well. ∎

Our next step is to describe the quotient algebra $U(\mathfrak{g})/\mathrm{Ann} L_{\frac{3}{2},\frac{1}{2}}$. In this section we denote by A the Weyl algebra in two variables, i.e. the algebra of differential operators acting in $\mathbb{C}[x,y]$. We introduce a $\mathbb{Z}_2$-grading, $A := A_0 \oplus A_1$, by putting $\deg x = \deg y = \deg \partial_x = \deg \partial_y := \bar{1} \in \mathbb{Z}_2$. It is well known that there exists a surjective algebra homomorphism

$$\kappa : U(\mathfrak{g}) \to A_0$$

such that

$$\kappa(e_1) = x\partial_y, \quad \kappa(e_2) = \frac{y^2}{2}, \quad \kappa(f_1) = y\partial_x, \quad \kappa(f_2) = -\frac{\partial_y^2}{2},$$

$$\kappa(h_1) = x\partial_x - y\partial_y, \quad \kappa(h_2) = y\partial_y + \frac{1}{2}.$$

The kernel of κ equals $\mathrm{Ann} L_{\frac{3}{2},\frac{1}{2}}$. Furthermore, $\kappa(\mathfrak{k})$ is spanned by $E := x\partial_y$, $F := y\partial_x$, $H := x\partial_x - y\partial_y$ in the root case, and respectively by $E := x\partial_y + y^2$, $H := 3x\partial_x + y\partial_y + 2$, $F := 3y\partial_x - \partial_y^2$ in the principal case.

The problem of describing all simple modules in $\mathfrak{B}_{\mathfrak{k}}^{\chi(\frac{3}{2},\frac{1}{2})}$ is equivalent to the problem of describing all simple $(A_0, \mathfrak{k})$-modules, i.e. all simple locally $\kappa(\mathfrak{k})$-finite A_0-modules. The following lemma reduces this problem to a classification of all simple $(A, \mathfrak{k})$-modules.

Lemma 6.4. *Every simple $(A, \mathfrak{k})$-module M is a $\mathbb{Z}_2$-graded A-module, i. e. $M = M_0 \oplus M_1$ where M_0 and M_1 are simple $(A_0, \mathfrak{k})$-modules. Furthermore, $M = A \otimes_{A_0} M_0$, and the $\mathbb{Z}_2$-grading on M is unique up to interchanging M_0 with M_1.*

Proof. The element H (as defined above separately for the root case and for the principal case) acts semisimply on M with integer eigenvalues. We define M_0 (respectively, M_1) as the direct sum of H-eigenspaces with even (resp., odd) eigenvalues. It is obvious that $M = M_0 \oplus M_1$, that M_0 and M_1 are simple A_0 modules, and that $M = A \otimes_{A_0} M_0$. Since M_0 and M_1 are non-isomorphic as A_0-modules, the uniqueness follows from the fact that a decomposition of M as an A_0-module into a direct sum of two non-isomorphic A_0-modules is unique. ■

Remark. More generally, if $\mathfrak{k}'$ is a subalgebra of $\mathfrak{g}' = \mathrm{sp}(2m)$ such that the centralizer of $\mathfrak{k}'$ in the Weyl A' algebra of m indeterminates is abelian, every $(A', \mathfrak{k}')$-module is a multiplicity-free $(\mathfrak{g}', \mathfrak{k}')$-module whose primitive ideal is a Joseph ideal. F. Knop has classified all such subalgebras $\mathfrak{k}'$, [Kn2], which makes us optimistic that this idea can eventually lead to a classification of simple bounded $(\mathfrak{g}', \mathfrak{k}')$-modules.

Let $\mathrm{Fou} : A \to A$ be the automorphism defined by

$$\mathrm{Fou}(x) := \partial_x, \quad \mathrm{Fou}(y) := \partial_y, \quad \mathrm{Fou}(\partial_x) := -x, \quad \mathrm{Fou}(\partial_y) := -y$$

If M is an A-module, we denote by M^{Fou} the twist of M by Fou.

Theorem 6.5. *In the root case, any simple $(A, \mathfrak{k})$-module is isomorphic to $\mathbb{C}[x, y]$ or $\mathbb{C}[x, y]^{\mathrm{Fou}}$.*

Proof. Let M be a simple $(A, \mathfrak{k})$-module. Then there exists $0 \neq v \in M$ such that $E \cdot v = 0$, i.e. $x\partial_y \cdot v = 0$. Hence either x or ∂_y act locally nilpotently on M.

Assume first that ∂_y acts locally nilpotently on M. Then $\partial_x \in [\mathfrak{k}, \partial_y]$ also acts locally nilpotenly on M. Let A^+ be the abelian subalgebra in A generated by ∂_x, ∂_y. One can find $0 \neq w \in M$ such that $A^+ \cdot w = 0$, and hence

$$M \cong A \otimes_{A^+} \mathbb{C} \cong \mathbb{C}[x, y].$$

If x acts locally nilpotently on M, one considers M^{Fou} and reduces to the previous case. ■

Corollary 6.6. *In the root case, up to isomorphism, there are exactly four simple* $(\mathfrak{g}, \mathfrak{k})$*-modules with central character* $\chi(\frac{3}{2}, \frac{1}{2})$*. As* $\mathfrak{k}$*-modules two of these modules are isomorphic to*

$$V_0 \oplus V_2 \oplus V_4 \oplus \dots \quad ,$$

and the other two are isomorphic to

$$V_1 \oplus V_3 \oplus V_5 \oplus \dots \quad .$$

Theorem 6.7. *In the principal case, up to isomorphism, there exist exactly two simple* $(A, \mathfrak{k})$*-modules and they have the following* $\mathfrak{k}$*-module decompositions:*

$$V_0 \oplus V_3 \oplus V_6 \oplus V_9 \oplus \dots, \qquad V_1 \oplus V_4 \oplus V_7 \oplus V_{10} \oplus \dots \quad .$$

Proof. Note that $\mathfrak{k}$ is a maximal subalgebra of $\mathfrak{g}$. Hence, every element $g \in \mathfrak{g} \backslash \mathfrak{k}$ acts freely on a simple $(A, \mathfrak{k})$-module M. In particular, x^2 acts freely on M, and therefore x acts freely on M. Let A_x be the localization of A in x, and $M_x := A_x \otimes_A M$. Then $M \subset M_x$. Fix $0 \neq m \in M$ with $E \cdot m = 0$ and $H \cdot m = \lambda m$ for a minimal $\lambda \in \mathbb{Z}_{\geq 0}$. Since $E = x\partial_y + y^2$ and $H = 3x\partial_x + y\partial_y + 2$, we have

$$\partial_y \cdot m = -\frac{y^2}{x} \cdot m, \; \partial_x \cdot m = \left(\frac{y^3}{3x^2} + \frac{\lambda - 2}{3x} \right) \cdot m.$$

Therefore, $M_x = \mathbb{C}[x, x^{-1}, y] \cdot m$. Set

$$u_\lambda := x^{\frac{\lambda-2}{3}} \exp \left(\frac{-y^3}{3x} \right).$$

Then it is easy to see that M_x is isomorphic to $\mathcal{F}_\lambda := \mathbb{C}[x, x^{-1}, y]\, u_\lambda$ and that $\mathcal{F}_\lambda = \mathcal{F}_{\lambda+3}$. Hence, M_x is isomorphic $\mathcal{F}_0, \mathcal{F}_1$ or $\mathcal{F}_2$.

Next we calculate $\Gamma_{\mathfrak{k}}(\mathcal{F}_\lambda)$. Note that the space of $\mathfrak{b}_{\mathfrak{k}}$-singular vectors in $\mathcal{F}_\lambda$ is spanned by the family $u_{\lambda+3k}$, $k \in \mathbb{Z}$ of solutions to the differential equation

$$E \cdot u = x\partial_y(u) + y^2 u = 0.$$

If $\lambda \in \mathbb{Z}_{\geq 0}$, then $F^{\lambda+1} \cdot u_\lambda$ is again a $\mathfrak{b}_{\mathfrak{k}}$-highest vector of weight $-\lambda - 2$. Therefore $F^{\lambda+1} \cdot u_\lambda = c u_{-\lambda-2}$ for some constant c. On the other hand, $u_{-\lambda-2} \in \mathcal{F}_\lambda$ iff $\lambda - (-\lambda - 2) = 2\lambda + 2 \in 3\mathbb{Z}$ or $\lambda = 3k + 2$. Hence $F^{\lambda+1} \cdot u_\lambda = 0$ for $\lambda = 3k$ or $\lambda = 3k + 1$. Thus, $\Gamma_{\mathfrak{k}}(\mathcal{F}_0)$ is generated by u_{3k} for $k \geq 0$, $\Gamma_{\mathfrak{k}}(\mathcal{F}_1)$ is generated by u_{3k+1} for $k \geq 0$, and we have the $\mathfrak{k}$-module decompositions

$$\Gamma_{\mathfrak{k}}(\mathcal{F}_0) \simeq V_0 \oplus V_3 \oplus V_6 \oplus V_9 \oplus \dots, \Gamma_{\mathfrak{k}}(\mathcal{F}_1) \simeq V_1 \oplus V_4 \oplus V_7 \oplus V_{10} \oplus \dots .$$

Let us prove that $\Gamma_{\mathfrak{k}}(\mathcal{F}_0)$ and $\Gamma_{\mathfrak{k}}(\mathcal{F}_1)$ are simple A-modules. Indeed, let N be a proper submodule of $\Gamma_{\mathfrak{k}}(\mathcal{F}_0)$. If $u_\lambda \in N$, then $u_{\lambda+3k} = x^k u_\lambda \in N$ for all positive k. Choose the minimal λ such that $u_\lambda \in N$. Then the quotient module has a decomposition $V_{\lambda-3} \oplus \cdots \oplus V_0$, hence it is finite-dimensional. Since A has no non-zero finite-dimensional modules, this is a contradiction. The case of $\Gamma_{\mathfrak{k}}(\mathcal{F}_1)$ is very similar. In this way we obtain that, if $M_x = \mathcal{F}_0$ or $\mathcal{F}_1$, then M is respectively isomorphic to $\Gamma_{\mathfrak{k}}(\mathcal{F}_0)$ or $\Gamma_{\mathfrak{k}}(\mathcal{F}_1)$.

Finally, we show that $\Gamma_{\mathfrak{k}}(\mathcal{F}_2) = 0$. It is sufficient to check that there is no non-zero $v \in \mathcal{F}_2$ with $F \cdot v = 0$ and

$$H \cdot v = (-3k - 2)\, v \text{ for } k \in \mathbb{Z}_{\geq 0}. \tag{20}$$

Indeed, then v would be a solution of the differential equation

$$3yv_x = v_{yy}.$$

Since $v \in \mathcal{F}_2$,

$$v = g\,(x, y) \exp\left(-\frac{y^3}{3x}\right)$$

for some $g\,(x, y) \in \mathbb{C}\,[x, x^{-1}, y]$ such that

$$3yg_x = g_{yy} - 2\frac{y^2}{x} g_y - 2\frac{y}{x} g.$$

As $g\,(x, y)$ is homogeneous with respect to H, we may assume without loss of generality that

$$g\,(x, y) = \sum_{i=0}^{l} b_i x^{p-i} y^{3i+s},$$

where $s \in \mathbb{Z}_{\geq 0}$, $p \in \mathbb{Z}$, $b_i \in \mathbb{C}$, $b_0 = 1$. The equation on the highest term with respect to x gives the condition

$$\partial_y^2\,(y^s) = 0,$$

or, equivalently, $s = 0, 1$. But $H \cdot g = (3p + s + 2)\, g$, hence $H \cdot v = (3p + s + 2) \cdot v$. Therefore

$$H \cdot v = (3p + 2)\, v \text{ or } H \cdot v = (3p + 3)\, v,$$

and (20) does not hold. ■

Theorem 6.7 together with Lemma 6.4 yield the following.

Corollary 6.8. *In the principal case, up to isomorphism, there are exactly four simple* $(\mathfrak{g}, \mathfrak{k})$*-modules with central character* $\chi(\frac{3}{2}, \frac{1}{2})$*. They have the following* $\mathfrak{k}$*-module decompositions:*

$$V_0 \oplus V_6 \oplus V_{12} \oplus \ldots,\ V_1 \oplus V_7 \oplus V_{13} \oplus \ldots,\ V_3 \oplus V_9 \oplus V_{15} \oplus \ldots,\ V_4 \oplus V_{10} \oplus V_{16} \oplus \ldots. \tag{21}$$

7. $\mathfrak{k}$-characters of simple bounded $(\mathrm{sp}(4),\ \mathrm{sl}(2))$-modules

7.1. The root case.. In this case, the four simple modules of Corollary 6.6 are nothing but the simple highest weight modules $L_{\frac{3}{2},\frac{1}{2}}$, $L_{\frac{3}{2},-\frac{1}{2}}$, and their respective restricted duals $L'_{\frac{3}{2},\frac{1}{2}}$, $L'_{\frac{3}{2},-\frac{1}{2}}$, i.e. the simple $\mathfrak{b}$-lowest weight modules with lowest weights $(-\frac{3}{2},-\frac{1}{2})$ and $(-\frac{3}{2},\frac{1}{2})$. Therefore, by Corollaries 6.2, 6.3 we conclude that all simple bounded $(\mathfrak{g},\mathfrak{k})$-modules are precisely $L_{a,b}$ and the lowest weight modules $L'_{-a,-b}$, where $a > |b| \in \frac{1}{2}+\mathbb{Z}$. Since $c(L_{a,b}) = c(L'_{-a,-b})$, it suffices to compute $c(L_{a,b})$, for a,b as above.

The $\mathfrak{h}$-character of $L_{a,b}$ is given by the formula

$$\mathrm{ch}_{\mathfrak{h}} L_{a,b} = \frac{(x^{a-b}-x^{b-a})(y^{a+b}-y^{-a-b})}{(x-x^{-1})(y-y^{-1})(xy-x^{-1}y^{-1})(x^{-1}y-xy^{-1})}, \tag{22}$$

where $x = e^{\frac{\epsilon_1-\epsilon_2}{2}}$, $y = e^{\frac{\epsilon_1+\epsilon_2}{2}}$. We rewrite (22) as

$$\frac{(x^{a-b}-x^{b-a})(y^{a-b}-y^{b-a})}{(x-x^{-1})(y-y^{-1})} y^{-2}(1-x^2y^{-2})^{-1}(1-x^{-2}y^{-2})^{-1}. \tag{23}$$

Next we note that

$$(1-x^2y^{-2})^{-1}(1-x^{-2}y^{-2})^{-1} = \sum_{k=0}^{\infty} y^{-2k}(x^{2k}+x^{2k-4}+\cdots+x^{-2k}), \tag{24}$$

and use the expression

$$z^k = x^k + x^{k-2} + \cdots + x^{-k} = \frac{x^{k+1}-x^{-(k+1)}}{x-x^{-1}}$$

to rewrite the right-hand side of (24) in the form

$$\sum_{k=0}^{\infty} y^{-2k}(z^{2k}-z^{2k-2}+\cdots+(-1)^k) = \frac{1}{1+y^2}\sum_{k=0}^{\infty} z^{2k}y^{-2k}.$$

Now (23) becomes

$$\mathrm{ch}_{\mathfrak{h}} L_{a,b} = z^{a-b-1}\frac{y^{a+b}-y^{-a-b}}{y-y^{-1}}\frac{1}{1+y^2}\sum_{k=0}^{\infty} z^{2k}y^{-2k}.$$

To find the $\mathfrak{k}$-character of $L_{a,b}$, we set $y=1$:

$$c(L_{a,b}) = \frac{a+b}{2} z^{a-b-1} \otimes \sum_{k=0}^{\infty} z^{2k}. \tag{25}$$

Thus, equation (25) implies the following result.

Theorem 7.1.

(a) If $a-b$ is even and $a+b$ is odd, then

$$c(L_{a,b}) = \frac{a+b}{2}(2z + 4z^3 + \cdots + (a-b)z^{a-b-1} + (a-b)z^{a-b+1} + \dots).$$

(b) If $a-b$ is odd and $a+b$ is even, then

$$c(L_{a,b}) = \frac{a+b}{2}(1 + 3z^2 + 5z^4 + \cdots + (a-b)z^{a-b-1} + (a-b)z^{a-b+1} + \dots).$$

(c) In the case (a) the minimal $\mathfrak{k}$-type is V_1 and its multiplicity is $a+b$. In the case (b) the minimal $\mathfrak{k}$-type is V_0 and its multiplicity is $\frac{a+b}{2}$.

(d) For sufficiently large i,

$$c_i(L_{a,b}) = c_{i+2}(L_{a+b}) = \frac{(a^2+b^2)(1+(-1)^{a+b-i})}{4}.$$

(e) $L_{a,b}$ is $\mathfrak{k}$-multiplicity-free if and only if $a = \frac{3}{2}$, hence the only simple multiplicity-free $(\mathfrak{g},\mathfrak{k})$-modules are those with central character $\chi(\frac{3}{2},\frac{1}{2})$, i.e. the four $\mathfrak{g}$-modules from Corollary 6.8.

7.2. The principal case.. We now proceed to calculating the $\mathfrak{k}$-characters of all simple bounded $(\mathfrak{g},\mathfrak{k})$-modules where $\mathfrak{g} = \mathrm{sp}(4)$ and $\mathfrak{k}$ is the principal subalgebra of $\mathfrak{g}$ fixed in Section 6. In this case, let $M^0_{\frac{3}{2},\frac{1}{2}}$ and $M^1_{\frac{3}{2},\frac{1}{2}}$ denote the simple bounded $(\mathfrak{g},\mathfrak{k})$-modules with central character $\chi(\frac{3}{2},\frac{1}{2})$ and respective $\mathfrak{k}$-module decompositions $V_0 \oplus V_6 \oplus V_{12} \oplus \dots$ and $V_1 \oplus V_7 \oplus V_{13} \oplus \dots$. We set $M^s_{a,b} := T^{\frac{3}{2}\epsilon_1+\frac{1}{2}\epsilon_2}_{a\epsilon_1+b\epsilon_2}(M^s_{\frac{3}{2},\frac{1}{2}})$ for $a,b \in \frac{1}{2}+\mathbb{Z}, a > |b|$, $s \in \{0,1\}$, and $M^s_{a,b} := 0$ for $a,b \in \frac{1}{2}+\mathbb{Z}, a \leq |b|$, $s \in \{0,1\}$. By $V_{p,q}$ we denote the simple finite-dimensional $\mathfrak{g} = \mathrm{sp}(4)$-module with $\mathfrak{b}$-highest weight $p\epsilon_1 + q\epsilon_2$ ($p,q \in \mathbb{Z}_{\geq 0}$, $p \geq q$).

Lemma 7.2. *We have*

$$V_{1,0} \otimes M^s_{a,b} \simeq M^s_{a+1,b} \oplus M^s_{a,b+1} \oplus M^s_{a-1,b} \oplus M^s_{a,b-1}, \tag{26}$$

and, for $a \neq |b|+1$,

$$V_{1,1} \otimes M^s_{a,b} \simeq M^s_{a+1,b+1} \oplus M^s_{a,b} \oplus M^s_{a-1,b+1} \oplus M^s_{a+1,b-1} \oplus M^s_{a-1,b-1}. \tag{27}$$

If $a = b+1, b > 0$, then

$$V_{1,1} \otimes M^s_{a,b} \simeq M^s_{a+1,b+1} \oplus M^s_{a+1,b-1} \oplus M^s_{a-1,b-1}, \tag{28}$$

and if $a = -b+1, b < 0$, then

$$V_{1,1} \otimes M^s_{a,b} \simeq M^s_{a+1,b+1} \oplus M^s_{a+1,b-1} \oplus M^s_{a-1,b+1}. \tag{29}$$

Proof. Let us first prove (26). In what follows we use the notation of [PS2], Section 5. Let $\mathscr{M}^s_{a,b} := \mathscr{D}^{a,|b|}_{G/B} \otimes_{U^{\chi(a,b)}} M^s_{a,b}$ be the localization of $M_{a,b}$ on G/B. Then as a sheaf of U-modules $V_{1,0} \otimes \mathscr{M}^s_{a,b}$ has a filtration of length 4 with the following associated factors given in increasing order:

$$\mathscr{O}(-\epsilon_1) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}, \quad \mathscr{O}(-\epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}, \quad \mathscr{O}(\epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}, \quad \mathscr{O}(\epsilon_1) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}.$$

Note that Z_U acts via a character on any of the four associated factors, and that these characters are pairwise distinct. Therefore, as a sheaf of U-modules, $V_{1,0} \otimes \mathscr{M}^s_{a,b}$ is isomorphic to the direct sum

$$\left(\mathscr{O}(-\epsilon_1) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}\right) \oplus \left(\mathscr{O}(-\epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}\right) \oplus \left(\mathscr{O}(\epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}\right) \oplus \left(\mathscr{O}(\epsilon_1) \otimes \mathscr{M}^s_{a,b}\right).$$

Now we calculate $\Gamma(G/B, V_{1,0} \otimes \mathscr{M}^s_{a,b})$. If $a = b+1, b > 0$, then

$$\Gamma(G/B, \mathscr{O}(-\epsilon_1) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}) = \Gamma(G/B, \mathscr{O}(\epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}) = 0$$

as there are no bounded modules with these central characters. Similarly, if $a = -b+1,\ b < 0$, then

$$\Gamma(G/B, \mathscr{O}(-\epsilon_1) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}) = \Gamma(G/B, \mathscr{O}(-\epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}) = 0.$$

In all other cases

$$\Gamma(G/B, \mathscr{O}(\pm\epsilon_1) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}) \simeq M^s_{a\pm 1,b},$$
$$\Gamma(G/B, \mathscr{O}(\pm\epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}) \simeq M^s_{a,b\pm 1}.$$

Thus, (26) is established.

Consider (27). Then as a sheaf of U-modules $V_{1,1} \otimes \mathscr{M}^s_{a,b}$ has a filtration of length 5 with the following associated factors given in increasing order:

$$\mathscr{O}(-\epsilon_1 - \epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}, \quad \mathscr{O}(\epsilon_1 - \epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}, \quad \mathscr{M}^s_{a,b},$$
$$\mathscr{O}(-\epsilon_1 + \epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}, \quad \mathscr{O}(\epsilon_1 + \epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}.$$

Note that Z_U acts via a character on any of the five associated factors, and that these characters are pairwise distinct if $a \neq |b| + 1$. Therefore the proof of (27) is very similar to that of (26).

Let now $a = b + 1$. Then $\mathscr{M}^s_{a,b}$ and $\mathscr{O}(-\epsilon_1 + \epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}$ both afford the central character $\chi(a,b)$. Thus, as a sheaf of U-modules, $V_{1,1} \otimes \mathscr{M}^s_{a,b}$ is isomorphic to the direct sum

$$\left(\mathscr{O}(-\epsilon_1 - \epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}\right) \oplus \left(\mathscr{O}(\epsilon_1 - \epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}\right) \oplus \left(\mathscr{M}^s_{a,b}\right)' \oplus \qquad (30)$$
$$\oplus \left(\mathscr{O}(\epsilon_1 + \epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b}\right),$$

where for $\left(\mathscr{M}^s_{a,b}\right)'$ we have an exact sequence

$$0 \to \mathscr{M}^s_{a,b} \to \left(\mathscr{M}^s_{a,b}\right)' \to \mathscr{O}(-\epsilon_1 + \epsilon_2) \otimes_{\mathscr{O}} \mathscr{M}^s_{a,b} \to 0.$$

We will show that $\Gamma(G/B, (\mathscr{M}^s_{a,b})') = 0$. It suffices to show that the tensor product $V_{1,1} \otimes M^s_{a,b}$ has no simple constituent with central character $\chi(a,b)$. Indeed, from

(26), we see that $V_{1,0} \otimes V_{1,0} \otimes M^s_{a,b}$ has exactly two simple constituents affording the central character $\chi(a,b)$ and that both these constituents are isomorphic to $M^s_{a,b}$. Recall that

$$V_{1,0} \otimes V_{1,0} \cong V_{2,0} \oplus V_{1,1} \oplus V_{0,0}.$$

Clearly, $V_{0,0} \otimes M^s_{a,b} = M^s_{a,b}$. Furthermore, $V_{2,0}$ is the adjoint representation and therefore the very $\mathfrak{g}$-module structure on $M^s_{a,b}$ defines a non-trivial intertwining operator $V_{2,0} \otimes M^s_{a,b} \to M^s_{a,b}$. Thus, $V_{2,0} \otimes M^s_{a,b}$ must have a constituent isomorphic to $M^s_{a,b}$ and consequently $V_{1,1} \otimes M^s_{a,b}$ has no simple constituent affording the central character $\chi(a,b)$. By taking the global sections of the direct sum (30) we obtain (28). The case $a = -b+1$, which leads to (29), is similar. ■

Lemma 7.3. *There is the following $\mathfrak{k}$-module decomposition*

$$M^s_{\frac{3}{2},-\frac{1}{2}} \simeq V_{3+s} \oplus V_{9+s} \oplus V_{15+s} \oplus \dots \quad . \tag{31}$$

Proof. By (26),

$$M^0_{\frac{3}{2},\frac{1}{2}} \otimes V_{1,0} \simeq M^0_{\frac{5}{2},\frac{1}{2}} \oplus M^0_{\frac{3}{2},-\frac{1}{2}}.$$

As a $\mathfrak{k}$-module, $V_{1,0}$ is isomorphic to V_3. Hence $M^0_{\frac{3}{2},\frac{1}{2}} \otimes V_{1,0}$ has a $\mathfrak{k}$-module decomposition

$$2V_3 \oplus V_5 \oplus \dots \quad .$$

Since $\chi(\frac{3}{2},-\frac{1}{2}) = \chi(\frac{3}{2},\frac{1}{2})$, $M^0_{\frac{3}{2},-\frac{1}{2}}$ must have one of the four $\mathfrak{k}$-module decompositions (21), and hence (26) implies (31) for $s=0$. Similarly, $M^1_{\frac{3}{2},\frac{1}{2}} \otimes V_{1,0}$ has the $\mathfrak{k}$-module decomposition $V_2 \oplus 2V_4 \oplus \dots$, which implies (31) for $s=1$. ■

We set now $\varphi^s_{a,b}(z) := c(M^s_{a,b})$ for $a,b \in \frac{1}{2}+\mathbb{Z}, a \geq |b|$, $s \in \{0,1\}$ and extend the definition of $\varphi^s_{a,b}(z)$ to arbitrary pairs $a,b \in \frac{1}{2}+\mathbb{Z}$ by putting

$$\varphi^s_{a,b}(z) = -\varphi^s_{b,a}(z) = -\varphi^s_{-b,-a}(z) = \varphi^s_{-a,-b}(z). \tag{32}$$

Lemma 7.4. *For all $a,b \in \frac{1}{2}+\mathbb{Z}$ and $s \in \{0,1\}$,*

$$\pi(\varphi^s_{a,b}(z^3+z+z^{-1}+z^{-3})) = \varphi^s_{a-1,b} + \varphi^s_{a+1,b} + \varphi^s_{a,b+1} + \varphi^s_{a,b-1}$$

$$\pi(\varphi^s_{a,b}(z^4+z^2+1+z^{-2}+z^{-4})) = \varphi^s_{a+1,b+1} + \varphi^s_{a-1,b+1} + \varphi^s_{a+1,b-1} + \varphi^s_{a-1,b-1} + \varphi^s_{a,b}.$$

(the projection π is introduced in Section 3).

Proof. Both equalities are straightforward corollaries of Lemma 7.2 and Lemma 3.2 (b) if one takes into account the isomorphisms of $\mathfrak{k}$-modules $V_{1,0} \simeq V_3$ and $V_{1,1} \simeq V_4$. ■

We define now $\psi^s_{a,b}(z) \in \mathbb{C}((z))$ via the conditions:

(c1) $\psi^s_{a,b}(z)(z^3+z+z^{-1}+z^{-3}) = \psi^s_{a+1,b}(z) + \psi^s_{a-1,b}(z) + \psi^s_{a,b+1}(z) + \psi^s_{a,b-1}(z)$,

(c2) $\psi^s_{a,b}(z)(z^4+z^2+1+z^{-2}+z^{-4}) = \psi^s_{a+1,b+1}(z) + \psi^s_{a+1,b-1}(z) + \psi^s_{a-1,b+1}(z) + \psi^s_{a-1,b-1}(z) + \psi^s_{a,b}(z)$, (c3) $\psi^s_{a,b}(z) = -\psi^s_{b,a}(z) = -\psi^s_{-b,-a}(z) = \psi^s_{-a,-b}(z)$,

(c4) $\psi^s_{\frac{3}{2},\frac{1}{2}}(z) = \dfrac{z^s}{1-z^6}, \quad \psi^s_{\frac{3}{2},-\frac{1}{2}}(z) = \dfrac{z^{3+s}}{1-z^6}$.

Theorem 7.5. *The Laurent series $\psi^s_{a,b}(z)$ exists and is unique, and* $\psi^s_{a,b}(z) =$

$$\frac{z^{5+s}(z^{3a+b} - z^{a+3b} - z^{-a-3b} + z^{-3a-b}) - z^{6+s}(z^{3a-b} - z^{-a+3b} - z^{a-3b} + z^{-3a+b})}{(1-z^2)^2(1-z^4)(1-z^6)}. \tag{33}$$

Proof. We show first that $\psi^s_{a,b}(z)$ is unique if it exists. By (32) $\psi^s_{a,b}(z)$ is determined by $\psi^s_{a,b}(z)$ for $a > |b|$. Assume, by induction on a, that $\psi^s_{a,b}(z)$ is unique for all $a \leq a_0$, $|b| < a$. Then equation (c1) determines $\psi^s_{a_0+1,b}(z)$, and equation (c2) determines $\psi^s_{a_0+1,a_0}(z)$ and $\psi^s_{a_0+1,a_0+1}(z)$.

To prove the existence of $\psi^s_{a,b}(z)$, it suffices to verify that the right-hand side of (33) satisfies all conditions (c1)-(c4). This is a direct calculation, which is simplified by the observation that both Laurent polynomials

$$z^{3a+b} - z^{a+3b} - z^{-a-3b} + z^{-3a-b},$$

$$z^{3a-b} - z^{-a+3b} - z^{a-3b} + z^{-3a+b}$$

satisfy (c1),(c2) and (c3). The condition (c4) is satisfied only by the entire expression. ∎

Corollary 7.6.

$$\varphi^s_{a,b} = \pi(\psi^s_{a,b}).$$

Corollary 7.7. *Any simple bounded $(\mathfrak{g},\mathfrak{k})$-module is either even or odd. More precisely, $M^s_{a,b}$ is even if $a+b+s$ is even, and $M^s_{a,b}$ is odd if $a+b+s$ is odd.*

In the calculations below we use binomial coefficients $\binom{s}{k}$, for which we always assume $\binom{s}{k} = 0$ if s or k are not integers.

Lemma 7.8.

$$\frac{1}{(1-z^2)^2(1-z^4)(1-z^6)} = \sum_{n=0}^{\infty} \gamma(n) z^{2n},$$

where

$$\gamma(n) := \frac{1}{144}\left[119\binom{n+3}{3} - 179\binom{n+2}{3} + 109\binom{n+1}{3} - 25\binom{n}{3}\right] + \frac{(-1)^n}{16} + \frac{\beta(n)}{9}$$

and

$$\beta(n) := \begin{cases} 0 & n \equiv 1 \pmod 3 \\ 1 & n \equiv 0 \pmod 3 \\ -1 & n \equiv -1 \pmod 3 \end{cases}.$$

Proof. The statement follows from the identity $\frac{1}{(1-z^2)^2(1-z^4)(1-z^6)} =$

$$\frac{1}{(1-z^2)^2(1-z^4)(1-z^6)} = \frac{119-179z^2+109z^4-25z^6}{144(1-z^2)^4} + \frac{1}{16(1+z^2)}$$

$+\frac{1+z^2}{9(1+z^2+z^4)}$. ■

Corollary 7.9. *Let*

$$\begin{aligned}\delta^s_{a,b}(n) \quad = \quad & \gamma\left(\frac{n-(3a+b+5)-s}{2}\right) - \gamma\left(\frac{n-(a+3b+5)-s}{2}\right) - \\ & -\gamma\left(\frac{n-(-a-3b+5)-s}{2}\right) + \gamma\left(\frac{n-(-3a-b+5)-s}{2}\right) - \\ & -\gamma\left(\frac{n-(3a-b+6)-s}{2}\right) + \gamma\left(\frac{n-(-a+3b+6)-s}{2}\right) + \\ & +\gamma\left(\frac{n-(a-3b+6)-s}{2}\right) - \gamma\left(\frac{n-(-3a+b+6)-s}{2}\right).\end{aligned}$$

Then

$$c_i(M^s_{a,b}) = \delta^s_{a,b}(i) - \delta^s_{a,b}(-i-2).$$

Proof. The statement follows directly from Theorem 7.5, Corollary 7.6, and Lemma 7.8. ■

Corollary 7.10. *For any simple bounded* $(\mathfrak{g},\mathfrak{k})$*-module* M, $c_i(M) = c_{i+6}(M)$ *for sufficiently large* $i \in \mathbb{N}$.

Proof. The given $(\mathfrak{g},\mathfrak{k})$-module M is isomorphic to $M^s_{a,b}$ for some $a, b \in \frac{1}{2}+\mathbb{Z}$, $s \in \{0,1\}$. For sufficiently large i, $\delta^s_{a,b}(-i-2) = 0$, hence $c_i(M) = \delta^s_{a,b}(i)$. The explicit formula for $\gamma(i)$ from Lemma 7.8 implies that $\delta^s_{a,b}(i+6n)$ is a polynomial in n. Since this polynomial is a bounded function, it is necessarily a constant. ■

For large enough values of i, Corollary 7.10 enables us to write $c_{\bar{i}}(M^s_{a,b})$, $\bar{i} \in \mathbb{Z}_6$. Here are simple explicit expressions for $c_{\bar{i}}(M^s_{a,b})$.

Theorem 7.11. *Let* $\sigma_{a,b} := \begin{cases} 1 & \textit{if } 3|2a, 3 \nmid 2b \\ -1 & \textit{if } 3|2b, 3 \nmid 2a \\ 0 & \textit{in all other cases} \end{cases}$.

Then

$$c_{\overline{0+s}}(M^s_{a,b}) = \frac{1}{6}(1+(-1)^{a+b})\left(\frac{a^2-b^2}{2} + 2\sigma_{a,b}\right),$$

$$c_{\overline{1+s}}(M^s_{a,b}) = c_{\overline{5+s}}(M^s_{a,b}) = \frac{1}{6}(1-(-1)^{a+b})\left(\frac{a^2-b^2}{2} - \sigma_{a,b}\right),$$

$$c_{\overline{2+s}}(M^s_{a,b}) = c_{\overline{4+s}}(M^0_{a,b}) = \frac{1}{6}(1+(-1)^{a+b})\left(\frac{a^2-b^2}{2} - \sigma_{a,b}\right),$$

$$c_{\overline{3+s}}(M^s_{a,b}) = \frac{1}{6}(1-(-1)^{a+b})\left(\frac{a^2-b^2}{2} + 2\sigma_{a,b}\right).$$

Proof. Let $\{\xi_{\bar{i}}\}_{\bar{i}\in\mathbb{Z}_6}$ denote the standard basis in $\mathbb{C}^6$. Set

$$\overline{\varphi}^s_{a,b} := \sum_{\bar{i}\in\mathbb{Z}_6} c_{\bar{i}}(M^s_{a,b})\xi_{\bar{i}}$$

for $a,b \in \frac{1}{2}+\mathbb{Z},\ a \geq |b|$. Extend $\overline{\varphi}^s_{a,b}$ to all $a,b \in \frac{1}{2}+\mathbb{Z}$ by putting

$$\overline{\varphi}^s_{a,b} = -\overline{\varphi}^s_{b,a} = -\overline{\varphi}^s_{-b,-a} = \overline{\varphi}^s_{-a,-b},$$

and let $S,T : \mathbb{C}^6 \to \mathbb{C}^6$ be the linear operators

$$S(\xi_{\bar{i}}) := 2\xi_{\overline{i+3}} + \xi_{\overline{i+1}} + \xi_{\overline{i-1}}, \quad T(\xi_{\bar{i}}) := 2\xi_{\overline{i+2}} + 2\xi_{\overline{i+4}}.$$

Then $\overline{\varphi}^s_{a,b}$ satisfy the following version of conditions (c1)-(c4):

(c5) $S(\overline{\varphi}^s_{a,b}) = \overline{\varphi}^s_{a+1,b} + \overline{\varphi}^s_{a,b+1} + \overline{\varphi}^s_{a-1,b} + \overline{\varphi}^s_{a,b-1}$,

(c6) $T(\overline{\varphi}^s_{a,b}) = \overline{\varphi}^s_{a+1,b+1} + \overline{\varphi}^s_{a-1,b+1} + \overline{\varphi}^s_{a+1,b-1} + \overline{\varphi}^s_{a-1,b-1}$,

(c7) $\overline{\varphi}^s_{a,b} = -\overline{\varphi}^s_{b,a} = -\overline{\varphi}^s_{-b,-a} = \overline{\varphi}^s_{-a,-b}$,

(c8) $\overline{\varphi}^s_{\frac{3}{2},\frac{1}{2}} = \xi_{\bar{s}}, \quad \overline{\varphi}^s_{\frac{3}{2},-\frac{1}{2}} = \xi_{\overline{3+s}}$.

Denote by ω a primitive sixth root of unity. Then $\{\eta_{\bar{i}} := \sum_{\bar{j}\in\mathbb{Z}_6} \omega^{\bar{i}\bar{j}}\xi_{\bar{j}}\}_{\bar{i}\in\mathbb{Z}_6}$ is an eigenbasis for S and T. Put

$$\eta_{\bar{0},a,b} := \frac{(a^2-b^2)}{2}\eta_{\bar{0}}, \quad \eta_{\bar{3},a,b} := (-1)^{a+b}\frac{(a^2-b^2)}{2}\eta_{\bar{3}},$$

$$\eta_{\bar{2},a,b} := \sigma_{a,b}\eta_{\bar{2}}, \quad \eta_{\bar{4},a,b} := \sigma_{a,b}\eta_{\bar{4}},$$

$$\eta_{\bar{3},a,b} := (-1)^{a+b}\sigma_{a,b}\eta_{\bar{3}}, \quad \eta_{\bar{5},a,b} := (-1)^{a+b}\sigma_{a,b}\eta_{\bar{5}}.$$

Using the identity

$$\sigma_{a,b} = \frac{\omega^{2b}+\omega^{-2b}-\omega^{2a}-\omega^{-2a}}{3},$$

one can easily check that $\eta_{\bar{i},a,b}$ satisfies (c5)-(c7). The linear combination

$$\overline{\varphi}^s_{a,b} = \frac{1}{6}\sum_{\bar{i}\in\mathbb{Z}_6} \omega^{-\bar{i}\bar{s}}\eta_{\bar{i},a,b}$$

satisfies the condition (c8), hence its coefficients in the basis $\{\xi_{\bar{i}}\}$ equal $c_{\bar{i}}\left(M^s_{a,b}\right)$. ■

Corollary 7.12. *The following is a complete list of multiplicity-free simple* $(\mathfrak{g},\mathfrak{k})$*-modules:* $M^s_{\frac{3}{2},\pm\frac{1}{2}}$, $M^s_{\frac{5}{2},\pm\frac{3}{2}}$, $M^s_{\frac{5}{2},\pm\frac{1}{2}}$, $M^s_{\frac{7}{2},\pm\frac{5}{2}}$, $s \in \{0,1\}$.

Proof. A straightforward computation based on Theorem 7.11 shows that $c_{\bar{i}}(M^s_{a,b}) \in \{0,1\}$ for $\bar{i} \in \mathbb{Z}_6$ iff (a,b) is one of the pairs $\left(\frac{3}{2},\pm\frac{1}{2}\right)$, $\left(\frac{5}{2},\pm\frac{3}{2}\right)$, $\left(\frac{5}{2},\pm\frac{1}{2}\right)$, and $\left(\frac{7}{2},\pm\frac{5}{2}\right)$. Then, using Corollary 7.9 one verifies that all modules $M^s_{a,b}$ for (a,b) as above are indeed multiplicity-free. ■

Theorem 7.13.

(a) The minimal $\mathfrak{k}$-type of any even (respectively, odd) bounded simple $(\mathfrak{g},\mathfrak{k})$-module M equals V_0, V_2 or V_4 (resp., V_1 or V_3).

(b) If M is an even (respectively, odd) simple module in $\mathfrak{B}^{\chi(a,b)}$, then $c_0(M)$ (resp., $c_1(M)$) equals $\frac{a\pm b}{6}+\epsilon$ or $\frac{a\pm b}{12}+\epsilon$ (resp., $\frac{a\pm b}{3}+\epsilon$ or $\frac{a\pm b}{6}+\epsilon$) for some ϵ with $|\epsilon|<1$.

Proof. (a) Note that for any bounded $(\mathfrak{g},\mathfrak{k})$-module M, $c_i(M)$ equals the constant term of the Laurent polynomial $z^{-i}(1-z^{2i+2})c(M)$. Hence $c_1(M)+c_3(M)$ equals the constant term in the Laurent expansion of

$$(z^{-1}(1-z^4)+z^{-3}(1-z^8))c(M).$$

A straightforward calculation shows that for $M=M^s_{a,b}$ the latter is nothing but the constant term of the Laurent series

$$\frac{z^{3a+b+2+s}-z^{a+3b+2+s}-z^{-a-3b+2+s}+z^{-3a-b+2+s}-z^{-3a+b+3+s}}{(1-z^2)^3}+$$
$$+\frac{z^{a-3b+3+s}+z^{-a+3b+3+s}-z^{3a-b+3+s}}{(1-z^2)^3}.$$

Using the identity

$$\frac{1}{(1-z^2)^3}=\sum_{n=0}^{\infty}\binom{n+2}{2}z^{2n}, \tag{34}$$

we obtain $c_1(M^s_{a,b})+c_3(M^s_{a,b})$

$$=: d^s_{a,b}=\binom{\frac{-3a-b+2-s}{2}}{2}-\binom{\frac{-a-3b+2-s}{2}}{2}-\binom{\frac{a+3b+2-s}{2}}{2}+\binom{\frac{3a+b+2-s}{2}}{2} \tag{35}$$
$$-\binom{\frac{3a-b+1-s}{2}}{2}+\binom{\frac{-a+3b+1-s}{2}}{2}+\binom{\frac{a-3b+1-s}{2}}{2}-\binom{\frac{-3a+b+1-s}{2}}{2},$$

where we set $\binom{l}{2}:=0$ for $l\notin\mathbb{Z}_{\geq 0}$.

This expression is a piecewise polynomial function which equals identically zero whenever $M^s_{a,b}$ is even, i.e. when $a+b+s$ is even. In fact, the right hand side of (35) turns out to be very simple as an explicit calculation shows that, for $a+b+s$ odd,

$$d^s_{a,b}=\begin{cases}\dfrac{a+(-1)^{s+1}b}{2} & \text{for } a+(-1)^s3b\geq 0\\ a+(-1)^sb & \text{for } a+(-1)^s3b\leq 0\end{cases}. \tag{36}$$

Since $a>|b|$, the right hand side of (36) is never 0, i.e. the minimal $\mathfrak{k}$-type of $M^s_{a,b}$ is V_1 or V_3 whenever $a+b+s$ is odd.

A similar analysis proves that the minimal $\mathfrak{k}$-type of $M^s_{a,b}$ is V_0, V_2, or V_4 whenever $a+b+s$ is even. Indeed, in this case

$$e^s_{a,b}:=c_0(M^s_{a,b})+c_2(M^s_{a,b})+c_4(M^s_{a,b})$$

equals the constant term of the Laurent series

$$(1-z^2)+z^{-2}(1-z^6)+z^{-4}(1-z^{10})c(M) \quad .$$

Using the identity

$$\frac{(1-z^2)+z^{-2}(1-z^6)+z^{-4}(1-z^{10})}{(1-z^2)^2(1-z^4)(1-z^6)} = \frac{1}{8z^4}\left(\frac{7+4z^2+z^4}{(1-z^2)^3}+\frac{1}{(1+z^2)}\right),$$

as well as the identity (34), we calculate

$$\begin{aligned} e^s_{a,b} &= \theta\left(\frac{-3a-b-1-s}{2}\right)-\theta\left(\frac{-a-3b-1-s}{2}\right)- \\ &\quad -\theta\left(\frac{a+3b-1-s}{2}\right)+\theta\left(\frac{3a+b-1-s}{2}\right)- \\ &\quad -\theta\left(\frac{3a-b-2-s}{2}\right)+\theta\left(\frac{-a+3b-2-s}{2}\right)+ \\ &\quad +\theta\left(\frac{a-3b-2-s}{2}\right)-\theta\left(\frac{-3a+b-2-s}{2}\right), \end{aligned}$$

where $\theta(n) := \frac{3}{4}n^2+\frac{3}{2}n+\frac{7}{8}+\frac{(-1)^n}{8}$ for $n \in \mathbb{Z}_{\geq 0}$ and $\theta(n) := 0$ otherwise. Further calculations show:

$$e^s_{a,b} = \begin{cases} \frac{3}{4}\left(a+(-1)^{s+1}b\right)+\frac{(-1)^{\frac{a+(-1)^{s+1}b-1}{2}}}{4} & \text{for } (-1)^s a+3b \geq 0 \\ \frac{3}{2}\left(a+(-1)^s b\right) & \text{for } (-1)^s a+3b \leq 0 \end{cases} \tag{37}$$

under the assumption that $a+b+s$ is even. Since the right-hand side of (37) never equals 0, we obtain that $e^s_{a,b} \neq 0$ under the same assumption. Hence the minimal $\mathfrak{k}$-type of any even simple bounded $(\mathfrak{g},\mathfrak{k})$-module equals V_0, V_2, or V_4.

(b) To compute $c_0(M)$ we use the identity

$$\begin{aligned} \frac{1-z^2}{(1-z^2)^2(1-z^4)(1-z^6)} &= \frac{1}{(1-z^2)(1-z^4)(1-z^6)} \\ &= \frac{47-52z^2+17z^4}{72(1-z^2)^3}+\frac{1}{8(1+z^2)}+\frac{2-z^2-z^4}{9(1-z^6)} \end{aligned}$$

which yields

$$\begin{aligned} c_0(M^s_{a,b}) &= \gamma'\left(\frac{-3a-b-5-s}{2}\right)-\gamma'\left(\frac{-a-3b-5-s}{2}\right)- \\ &\quad -\gamma'\left(\frac{a+3b-5-s}{2}\right)+\gamma'\left(\frac{3a+b-5-s}{2}\right)- \\ &\quad -\gamma'\left(\frac{3a-b-6-s}{2}\right)+\gamma'\left(\frac{-a+3b-6-s}{2}\right)+ \\ &\quad +\gamma'\left(\frac{a-3b-6-s}{2}\right)-\gamma'\left(\frac{-3a+b-6-s}{2}\right), \end{aligned}$$

where

$$\gamma'(n) := \frac{n^2}{12} + \frac{n}{2} + \frac{94}{144} + \frac{(-1)^n}{8} + \frac{\sigma'(n)}{9},$$
$$\sigma'(n) := \begin{cases} -1 & 3 \nmid n \\ 2 & 3 \mid n \end{cases}$$

for $n \in \mathbb{Z}_{\geq 0}$ and $\gamma'(n) = \sigma'(n) := 0$ otherwise. Similarly, using the identity

$$\frac{z^{-1}(1-z^4)}{(1-z^2)^2(1-z^4)(1-z^6)} = z^{-1}\left(\frac{8-7z^2+2z^4}{9(1-z^2)^3} + \frac{1+z^2-2z^4}{9(1-z^6)}\right)$$

we obtain

$$\begin{aligned} c_1(M^s_{a,b}) \;=\; & \gamma''\left(\frac{-3a-b-4-s}{2}\right) - \gamma''\left(\frac{-a-3b-4-s}{2}\right) - \\ & -\gamma''\left(\frac{a+3b-4-s}{2}\right) + \gamma''\left(\frac{3a+b-4-s}{2}\right) - \\ & -\gamma''\left(\frac{3a-b-5-s}{2}\right) + \gamma''\left(\frac{-a+3b-5-s}{2}\right) + \\ & +\gamma''\left(\frac{a-3b-5-s}{2}\right) - \gamma''\left(\frac{-3a+b-5-s}{2}\right), \end{aligned}$$

where

$$\gamma''(n) := \frac{n^2}{6} + \frac{5n}{6} + \frac{8}{9} + \frac{\sigma''(n)}{9},$$
$$\sigma''(n) := \begin{cases} -2 & n = -1 (\mathrm{mod}\ 3) \\ 1 & n \neq -1 (\mathrm{mod}\ 3) \end{cases}$$

for $n \in \mathbb{Z}_{\geq 0}$ and $\gamma''(n) = \sigma''(n) := 0$ otherwise. Using the expressions for $c_0(M^s_{a,b})$ and $c_1(M^s_{a,b})$ we notice that the terms $\frac{(-1)^n}{8} + \frac{\sigma'(n)}{9}$ and $\frac{\sigma''(n)}{9}$ will give a contribution ϵ with $|\epsilon| < 1$. Thus, a direct computation implies

$$c_0(M^s_{a,b}) = \begin{cases} \frac{a+(-1)^s b}{6} + \epsilon & \text{for } a + (-1)^s 3b < 0 \\ \frac{a-(-1)^s b}{12} + \epsilon & \text{for } a + (-1)^s 3b > 0 \end{cases},$$

$$c_1(M^s_{a,b}) = \begin{cases} \frac{a-(-1)^s b}{6} + \epsilon & \text{for } a + (-1)^s 3b > 0 \\ \frac{a+(-1)^s b}{3} + \epsilon & \text{for } a + (-1)^s 3b < 0. \end{cases}$$

■

Corollary 7.14. *For $a \pm b \geq 24$, the minimal $\mathfrak{k}$-type of $M^s_{a,b}$ equals V_0 (respectively, V_1) if $a+b+s$ is odd (resp., even).*

Corollary 7.15. *A simple $(\mathfrak{g}, \mathfrak{k})$-module with minimal $\mathfrak{k}$-type V_i for $i \geq 5$ is unbounded.*

Note that all simple $(\mathfrak{g}, \mathfrak{k})$-modules of finite type over $\mathfrak{k}$ with minimal $\mathfrak{k}$-type V_i for $i \geq 6$ are classified in [PZ2]. In particular it is proved, [PZ2], that if M is a $(\mathfrak{g}, \mathfrak{k})$-module with minimal $\mathfrak{k}$-type V_i for $i \geq 6$, then M is necessarily of finite type over $\mathfrak{k}$ and $c_i(M) = 1$. Recently G. Zuckerman and the first named author have shown that this holds also for $i = 5$, and Theorem 7.13 (b) implies that the statement is false for $i \leq 1$.

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