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On "Axiom III" of Hilbert's Foundation of Geometries via Transformation Groups

Athanassios Strantzalos and Polychronis Strantzalos*

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Abstract. In 1902, D. Hilbert presented a foundation of classical plane geometries based on three topological axioms concerning a group G of homeomorphisms of the real plane. The third of these axioms required essentially that the action of G on the plane be 2-closed, thus ensuring a kind of compatibility between the topological and the geometrical (in Klein's spirit) structures of the plane. In the present paper we show that the 2-closed actions on noncompact, connected, locally connected and locally compact spaces are essentially restrictions in dense (eventually not strict) subgroups of groups acting properly on the considered spaces. Generalizing Hilbert's setting, we define the notion of a "q-closed geometry" on non-compact and orientable 2-manifolds of finite genus, we determine the manifolds admitting such geometries and we describe the q-closed geometries on them; among which are the classical ones on the plane[1].
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1. Introduction

Shortly after his celebrated classical foundation of geometries in Euclid's spirit, D. Hilbert published, in 1902, another foundation by transformation groups ([10, Anhang IV]). Among the three purely topological axioms of Hilbert (cf. [10, pp. 181-185], and Remark 5.1 below), "Axiom III" postulates a kind of compatibility between the usual topological structure of the plane and its geometrical structure in the framework of Kleinian Geometries, and leads to the class of "q-closed" actions, namely those satisfying this Axiom (cf. Definition 1.1). Numerous references to Hilbert's paper of 1902 in the years following its publication exhibited the significance of this paper for the evolution of the modern theory of topological

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transformation groups, and the role of "Axiom III" concerning proper actions (cf. [15]).

The purpose of this paper is to characterize the q-closed actions on non compact, connected, locally compact, and locally connected spaces (cf. Theorem 1.2 and Corollary 1.3), and to indicate the applicability of the corresponding results in the topological characterization (in the spirit of Hilbert's 5^{th} problem) of the full groups of symmetries of geometrical structures (cf. 7). So, throughout this paper, X denotes a space as above, while the space of the continuous selfmappings and the group of homeomorphisms of the space X will be denoted by $C(X,X)$ and $H(X)$, respectively, both equipped with the topology of the point-wise convergence, in accordance to Hilbert's framework.

Definition 1.1. A group $G < H(X)$ or its action on X is called q-closed for $q \in \mathcal{N}$ (the set of natural numbers), if the following condition is satisfied: If $x_i^k \to y^k$ for $k = 1, \ldots, q$ and there exist $g_i \in G$ with $g_i x_i^k \to z^k$, then there is a $g \in G$ such that $gy^k = z^k$.

The term "q-closed" action was first introduced in [7, p. 420]. Hilbert's "Axiom III" postulated the "3-closedness" of the action of the founding group. In [6, Cor. 2, p. 17] it is shown that 2-closedness suffices.

As remarked in [6, Th. 12], for the axiomatic framework and the special cases considered there, G is closed in $H(X)$. By an example due to H. Abels (cf. Example 4.1), it is shown that, in our more general framework, "Axiom III" alone does not guarantee the closedness of G in $H(X)$. So, the following main result of the paper cannot be improved.

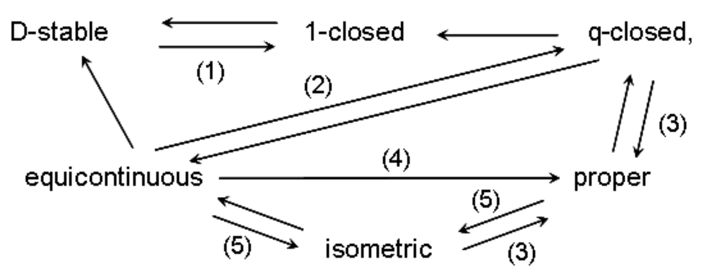
Theorem 1.2. *Let X be a non compact, connected, locally compact, and locally connected space and G a subgroup of $H(X)$ endowed with the topology of the point-wise convergence. Then*

(a) *If G is 2-closed, then $\bar{G}$, the closure of G in $C(X,X)$, is also a 2-closed subgroup of $H(X)$.*

(b) *G is closed in $H(X)$ and 2-closed if and only if it acts properly on X, in which case it is q-closed for any $q \in \mathcal{N}$.*

(c) *If G is q-closed for $q \geq 2$, then a subgroup H of G is q-closed if and only if $\bar{H} = H \cdot (\bigcap_{k=1}^{q} \bar{H}_{x_k})$ holds for every $x_1, \ldots, x_q \in X^q$ (e.g., if H is closed in G).*

This theorem indicates that the restriction to 1- and 2-closed actions (G, X) covers the essential part of the corresponding theory. These actions fit in the following diagram:



Here **(1)** means “each orbit $G(x)$ is a closed subspace of X” (cf. Proposition 3.1), **(2)** “$\bar{G} = G \cdot (\bigcap_{i=1}^{q} \bar{G}_{x_i})$ holds for every $(x_1, \ldots, x_q) \in X^q$” (cf. Theorem 1.2(c)), **(3)** “G is closed in $H(X)$” (cf. Theorem 1.2(b)), **(4)** “G is locally compact”, and **(5)** “holds for metrizable spaces”.

Corollary 1.3. *Let X be a non compact, locally compact, connected, and locally connected metrizable space. The subgroup H of $H(X)$ is q-closed for $q \geq 2$ iff there exists an admissible metric d on X such that H is a subgroup of the group, $I_d(X)$, of the d-isometries of X and $\overline{H} = H \cdot (\bigcap_{i=1}^{q} \overline{H}_{x_i})$ holds for every $(x_1, \ldots, x_q) \in X^q$.*

The above theorem and known results enable the determination of the non compact and orientable 2-manifolds of finite genus admitting “q-closed geometries” (cf. Definition 5.2, 6 and 7 below), which can be regarded as a generalization of Hilbert’s plane geometries founded in [10, Anhang IV]. Moreover, we describe the 2-closed geometries on the involved 2-manifolds (cf. 7), among which there are the classical geometries on the plane founded by Hilbert. Since the definition of a “q-closed geometry” is purely topological, the application of our results provides a characterization of the corresponding groups of isometries in the spirit of Hilbert’s 5^{th} problem: in Hilbert’s words the results of his paper of 1902 “provide a partial answer” to this problem concerning the Lie groups of the isometries of the classical geometries (cf. [15, 6]).

2. Basic notions and notation

We always consider Hausdorff spaces. We recall:

Definition 2.1. Given an action (G, X), for $x \in X$, we denote by $G(x)$ its orbit, and we let, as usual:

$$L(x) = \{y \in X : \exists g_i \in G \text{ with } g_i \to \infty \text{ such that } g_i x \to y\},$$
$$J(x) = \{y \in X : \exists x_i \to x \text{ and } g_i \in G \text{ with } g_i \to \infty \text{ such that } g_i x_i \to y\}.$$

Here $w_i \to \infty$ with $w_i \in W$ means that the net w_i does not have any limit point in W.

Definition 2.2. **(a)** The action (G, X) is *D-stable* if $\overline{G(x)} = G(x) \cup J(x)$ holds for every $x \in X$.
(b) The action is *proper* if $J(x) = \emptyset$ holds for every $x \in X$.

We remark that in equicontinuous actions $J(x) = L(x)$ holds for every $x \in X$, and that a group acting properly on a locally compact space X is necessarily locally compact and closed in $H(X)$ (cf., for instance, [16, p. 320]).

Definition 2.3. For a locally compact space Y, we shall denote by Y^+ the end-point compactification of Y, that is the maximal one with totally disconnected remainder. This compactification can be defined as the quotient space of the Stone-Čech compactification, ßY, of Y such that the equivalence classes are the points of Y and the (connected) components of ß$Y - Y$.

Since the space $Y^+ - Y$ of the ends of Y is totally disconnected, every end has a neighborhood basis in Y^+ consisting of sets U with boundaries ∂U lying in Y.
If Y is moreover connected, every action (G, Y) can be (continuously) extended to an action (G, Y^+) (cf. [1, 2.3]).
$\Re$ will denote the topological real line or the topological group of the real numbers.

3. The 1-closed actions

Proposition 3.1. *An action (G, Y) is 1-closed if and only if it is D-stable and its orbits are closed subsets of Y. Equivalently: if and only if the orbit space of the action is Hausdorff.*

Proof. Let the action be 1-closed. If $y_i \to y$ and $g_i y_i \to \bar{y} \in Y$ for $g_i \in G$, there exists a $g \in G$ such that $\bar{y} = gy \in G(y)$, from which follows that $J(y) \subset G(y)$, therefore that the orbits are closed subsets of Y. The converse can be proved analogously by distinguishing the cases $g_i \to g_0$, or $g_i \to \infty$. The last assertion follows from the fact that an orbit space is Hausdorff if and only if $J(x) \subset G(x)$ holds for every $x \in X$. ∎

Remark 3.2. **(a)** There exist D-stable actions such that their orbits are not closed subsets of the underlying space, as the example of a minimal flow (e.g. an irrational flow on the torus) shows; these actions are, therefore, not 1-closed.
(b) Every transitive action of a subgroup of $H(X)$ on X is 1-closed, and *there exist transitive actions that are not 2-closed*, as the following example shows: On $\Re^2$ consider an $\Re$-action without fixed points and with the following face portrait: The parallels to the y-axis through the points of the x-axis with integer coordinates are orbits tending to infinity alternatively for positive and negative times. The strip between two successive such orbits is filled by orbits homeomorphic to parabolas. Because of this, the boundary orbits of a strip are contained in the J-sets of each other. Let $\{r_n\}$, with $r_n \to \infty$, be a sequence in the acting group $\Re$ and let $\{x_n\}$ be a sequence in a strip with $x_n \to x$ and $r_n x_n \to y$, where x and y are points of the different boundaries of a strip. It follows that $r_n \to \infty$ in $H(\Re^2)$,

which implies that $r_n \to \infty$ in every closed subgroup of $H(\Re^2)$ containing the r_n's. Let G be the connected subgroup of $H(\Re^2)$ generated by the above $\Re$-group and an $\Re$-group acting on $\Re^2$ by translations with orbits parallel to the x-axis. This group acts transitively on $\Re^2$ (for instance, via the horizontal translations and the vertical orbits). The action of $\bar{G}$, the closure of G in $H(\Re^2)$, on $\Re^2$ is not 2-closed, because it is not proper (cf. Theorem 1.2(b)): As mentioned above, y is contained in the J-set of x with respect to the action $(\bar{G}, \Re^2)$. For the same reason G, acting transitively, is 1-closed, but not 2-closed (cf. Theorem 1.2(a)).

Examples 3.3. The structure of non minimal D-stable flows on non compact and orientable 2-manifolds of finite genus, M, is described in [3], where it is shown that manifolds of this kind admitting D-stable flows must have genus at most 1 [3, Th. 4.3]. Moreover, it is shown that $M = B \cup P \cup R$ holds, where B is the set of the (at most two) fixed points which are local centers (: surrounded by periodic orbits) [3; Th. 4.4], P is the (open) set of the periodic orbits, and R is the set of the orbits that are homeomorphic to $\Re$. Concerning a connected component, C, of R, it is shown that the restriction of the flow on it is a parallelizable dynamical system [3, Prop. 3.11], and that, by contracting C to one of its orbits, we obtain a new D-stable flow on M [3, Prop. 4.2]. If the flow is not parallelizable, a non compact orbit tends in positive and negative time to the same end of M [3, Lemma 3.3], and the closure of a component C in M^+, the end-point compactification of M, has a neighborhood basis consisting of invariant open sets with one or two periodic orbits as boundary [3, Cor. 3.12].

These results indicate that *the following examples*, which shall be used in 6 and 7 below, *exhaust the 1-closed flows on the M's with at least one orbit homeomorphic to* $\Re$. The following examples (1) – (4) explain, to some extent, the structure of the "1-closed geometries" on the M's, which are defined by an action (G, M) with at least one non compact orbit (cf. the proof of Corollary 6.5(b)):

(1) On $M = \Re^2$ there exist three types of 1-closed flows:

(a) If $B = \emptyset$, then $P = \emptyset$, therefore the flow is parallelizable, by [3, Th. 3.4]. Hence, the corresponding action is proper and, by Theorem 1.2(b), it is also 2-closed.

(b) If B consists of one point, this point is surrounded by periodic orbits that fill up an open (subset, homeomorphic to a) halfplane, while the flow on the other halfplane is parallelizable. Because of the periodic orbits approaching it, the boundary orbit of the parallelizable part of the flow coincides with its J-set. Therefore the corresponding action is not proper. This, the fact that the acting group, $\Re$ is closed in $H(M)$ (due, for instance, to the parallelizable part of the flow), and Theorem 1.2(b) show that the action, being not prorer, is not 2-closed.

(c) If B consists of two points, then to each fixed point corresponds, as before, a region of periodic orbits surrounding it, the two regions are disjoint, and the flow in the intermediate unbounded strip is parallelizable. As in (b), the corresponding action is not 2-closed.

(2) On $M = \Re \times S^1$ (the plane but one point) the following types of 1-closed flows can occur:

(a) The parallelizable flow. As in (1)(a) above, the corresponding action is also 2-closed.

(b) If the flow is not parallelizable and has no fixed points, then its face portrait is as follows: There exist one or two different connected components of R corresponding to different ends of M; the orbits of such a component tend positively and negatively to the corresponding end; the flow outside R has only periodic orbits. The corresponding action is not 2-closed.

(c) Let the flow have (at most two) fixed points. A fixed point of the flow has an invariant neighborhood U consisting of this point and periodic orbits (cf. [3, Lemma 3.2]). A connected component of the boundary of U in M^+ consists of an end of M and one orbit in a connected component of R, every point of which tends positively and negatively to this end. With this modification, the possible face portraits of the flow correspond to the face portraits in (b). The corresponding action is not 2-closed, too.

(3) To exhaust the M's of genus 0, we have to consider the case $M = \Re \times S^1 - E$, where E is a compact and totally disconnected subset of $\Re \times S^1$ (: a subspace of the space of the ends of M). To every $e \in E$ corresponds a connected component C of the set R, which consists of one orbit [3, Cor. 3.10], or has non empty interior, and e is the only "point at infinity" of C [3, Cor. 3.13]. The complement of the union of the C's is an invariant subset of M with periodic orbits. These actions are also not 2-closed.

(4) Finally, the face portraits of the 1-closed flows *with at least one non compact orbit* on the manifolds under consideration with genus 1, namely on 2-manifolds M with M^+ the torus, can be obtained from (2)(b) and (3) by identifying the boundary components of a bounded copy of the cylinder $\Re \times S^1$. To this, remark that here case (2)(a) is excluded, because otherwise we would have $M = \Re \times S^1$ [3, 3.4], and case (2)(c) is also excluded: The existence of a fixed point of the flow implies that the underlying manifold has genus 0 [3, Th. 4.4].

4. The 2-closed actions

4.1. As we remarked before, if the three axioms in Hilbert's foundation of the classical geometries of the plane are satisfied, the 2-closed acting group is closed in $H(\Re^2)$. However, if X is connected, locally compact and locally connected and the action (G, X) is 2-closed without any additional requirements, then G is not necessarily closed in $H(X)$, as the following shows.

Example 4.1. (proposed by H. Abels): Let P be a tree with initial point p such that p is the starting point of exactly one edge, and every other node is the common point of exactly three edges. On P we consider the usual tree-metric. The group of isometries $I(P)$ acts properly on P [16, 4.2], therefore it is compact because of the fixed point p. Let $x_m^k \to x_0^k$ and $f_m x_m^k \to y_0^k$ in P for $f_m \in I(P)$ and $k = 1, \ldots, q$. First we assume that none of the x_0^k is a node, and denote by b_k, c_k the nodes of the edge containing x_0^k and by z_k, v_k those of the edge containing y_0^k. We may assume $f_m b_k \to z_k$ and $f_m c_k \to v_k$. Since only constant sequences of edges are convergent, we conclude $f_m b_k = z_k$ and $f_m c_k = v_k$, therefore $f_m x_0^k = y_0^k$, because $f_m x_0^k \to y_0^k$ and the f_m's are isometries. The case in which some of the x_0^k are nodes can be treated likewise. So, we have proved that every subgroup of $I(P)$ is q-closed for every $q \in \mathcal{N}$. On the other hand, it is easily seen that $I(P)$ has infinitely many elements. Let F be a subgroup of $I(P)$ generated by a subset of $I(P)$ with infinite countable elements. Then F has countably many elements. Its closure $\bar{F}$ in $I(P)$ is an infinite compact group, therefore it has the cardinality of the continuum. Thus, F is not closed in $I(P)$, hence not even in $H(P)$.

This example shows that assertion (b) in Theorem 1.2 cannot be improved. In order to formulate general results, it is reasonable to embed a 2-closed action in the end-point (instead, as is usual, in the one-point) compactification of the considered space. For the proof of our main result we shall use the following lemmas, resemblant to [6, Th. 10] and [16, Claim in the proof of Th. 3.1].

Lemma 4.2. *Let Y be a non compact, connected, locally compact and locally connected space and (G,Y) a 2-closed action. If $y_i^k \to y_0$ in Y for $k=1,2$ and $g_i \in G$ with $g_i y_i^k \to z^k \in Y^+$, then $z^1 = z^2$.*

Proof. Let $z^1 \neq z^2$ and U_1, U_2 be corresponding neighborhoods of these points in Y^+ with $\partial U_1, \partial U_2 \subset Y$ and $\overline{U_1} \cap \overline{U_2} = \emptyset$. Let $\{V_a : a \in A\}$ be a neighborhood basis of y_0 consisting of connected sets. Then, there are $w_a^k \in V_a$ and $g_a \in \{g_i : i \in I\}$ such that $g_a w_a^k \in \partial U_k$ for $k = 1,2$. Therefore $w_a^k \to y_0$, and we may assume $g_a w_a^k \to w_0^k \in \partial U_k$ for $k=1,2$, because of the compactness of the boundaries in Y^+. From this and the 2-closedness follows that there must be a $g \in G$ such that $g y_0 = w_0^k$ for $k = 1,2$, a contradiction. ■

Lemma 4.3. *Let (G,Y) be as in Lemma 4.2. Then, G is equicontinuous with respect to the uniform structure induced on Y by the one on Y^+.*

Proof. We recall that the sets of the form $B^+ = \bigcup_{m=1}^r (U_m^+ \times U_m^+)$, *where* $\{U_m^+ : m = 1, \ldots, r\}$ defines an open covering of Y^+, constitute a fundamental set of entourages for the uniform structure of Y^+. The corresponding sets $B = \bigcup_{m=1}^r (U_m \times U_m)$, where $U_m = U_m^+ \cap Y$, constitute a fundamental set of entourages for the induced uniform structure of Y. If G fails to be equicontinuous, then there should exist a $y_0 \in Y$ and a B such that, for every V_a as in the preceding proof, we could find a $w_a \in V_a$ and a $g_a \in G$ with $(g_a w_a, g_a y_0) \notin B$. So, we may assume $g_a w_a \to w^1 \in Y^+$ and $g_a y_0 \to w^2 \in Y^+$ with $w^1 \neq w^2$. This contradicts Lemma 4.2, because $w_a \to y_0$ and (trivially) $y_0 \to y_0$. ■

Remark 4.4. The arguments in the preceding proof show that G is equicontinuous with respect to the uniform structure induced on Y by the uniform structure of any compactification of it with totally disconnected remainder, e.g. by that of its one-point compactification. This will be used in the proof of Theorem 7.2.

Lemma 4.5. *Let X be locally compact and locally connected, and (G,X) be 2-closed. For $F \subset G$ we let $K(F) = \{x \in X : F(x) = \{fx : f \in F\}$ is relatively* *compact}. Then, $K(F)$ is open and closed in X.*

Proof. Let $K(F)$ be not open. Then, there exist $x_i \in X - K(F)$ such that $x_i \to x_0 \in K(F)$. Let A be an open and relatively compact neighborhood of the compact set $\overline{F(x_0)}$. Then, there are $f_i \in F$ such that $f_i x_i \in X - \overline{A}$. We may assume $f_i x_i \to y^1 \in X^+ - A$ and $f_i x_0 \to y^2 \in \overline{F(x_0)} \subset A$, which implies $y^1 \neq y^2$. This contradicts Lemma 4.2, because $x_i \to x_0$ and $x_0 \to x_0$. For the closedness of $K(F)$, let $\{d_p : p \in P\}$ be a family of bounded pseudometrics defining the uniform structure $\mathcal{U}$ that is induced on X by the uniform structure of X^+ (cf. the proof of Lemma 4.3). Setting $d_p^*(x,y) = \sup\{d_p(gx, gy) : g \in G\}$, we obtain a new family $\{d_p^* : p \in P\}$ of pseudometrics on X, such that every $g \in G$ is a d_p^*-isometry. Therefore, G is uniformly equicontinuous with respect to the uniform structure $\mathcal{U}^*$ defined by this new family. Since G is equicontinuous with respect to $\mathcal{U}$ and $d_p(x,y) \leq d_p^*(x,y)$, $\mathcal{U}^*$ gives the topology of X. Moreover, the entourages

$$E^* = \{(x,y) : (gx, gy) \in E \forall g \in G\} for E \in \mathcal{U}$$

are symmetric and form a fundamental set for $\mathcal{U}^*$ with the property $(g \times g)E^* = E^*$ for $g \in G$. Let $x_i \in K(F)$ with $x_i \to x_0$ in X, and E^* be such that $E^*(x_0) = \{x \in X : (x_0, x) \in E^*\}$ is a relatively compact neighborhood of x_0. Let W^* be an element of the preceding fundamental set of $\mathcal{U}^*$ such that $W^* \circ W^* \circ W^* \subset E^*$ and $(x_0, x_0) \in W^*$. Since $\overline{F(x_0)}$ is compact, we may assume $\overline{F(x_0)} \subset \bigcup_{m=1}^{t} E^*(f_m x_0)$ for $f_m \in F$. We have $(f_m x_0, f_m x_{i_0}) \in (f_m \times f_m)W^* = W^*$. For $f \in F$ there is some $m \in \{1, \ldots, t\}$ with $f x_{i_0} \in W^*(f_m x_{i_0})$, i.e. $(f_m x_{i_0}, f x_{i_0}) \in W^*$. On the other hand, we have $(f x_{i_0}, f x_0) \in (f \times f)W^* = W^*$. The previous relations and the choice of W^* give $(f_m x_0, f x_0) \in E^*$, which means $f x_0 \in E^*(f_m x_0)$.
Therefore,

$$F(x_0) \subset \bigcup\nolimits_{m=1}^{t} E^*(f_m x_0) = \bigcup\nolimits_{m=1}^{t} f_m(E^*(x_0)),$$

from which follows that $x_0 \in K(F)$. Thus $K(F)$ is closed. ∎

Proof of Theorem 1.2. Let $\bar{g} \in \bar{G} \subset C(X,X)$ and $g_i \in G$ with $g_i \to \bar{g}$, i.e. $g_i x \to \bar{g}(x)$ for every $x \in X$. For given x, we may assume $g_i^{-1}\bar{g}(x) \to z \in X^+$. For $y_i = g_i x \to \bar{g}(x)$, we obtain $g_i^{-1} y_i = x \to x$. From this and Lemma 4.2 follows $z = x \in X$. Let U be a relatively compact neighborhood of x. Then $g_i^{-1}\bar{g}(x) \in U$ for $i \succ i_0$. Setting $F = \{g_i^{-1} : i \succ i_0\}$, we have $\bar{g}(x) \in K(F)$. From this and Lemma 4.4 follows $X = K(F)$, which means that $F(x)$ is relatively compact for every $x \in X$ and, by Ascoli's Theorem, that F is relatively compact in $C(X,X)$. Hence, there is a subnet $\{g_j\}$ of $\{g_i\}$ such that $g_j^{-1} \to h \in C(X,X)$. We claim that $h = \bar{g}^{-1}$, therefore that $\bar{g} \in H(X)$ (: the equicontinuity of G, guaranteed by Lemma 4.3, implies the continuity of $\bar{g}$ and $\bar{g}^{-1}$). We have already shown that $g_j^{-1}\bar{g}(x) \to x$, while $g_j^{-1}\bar{g}(x) \to h(\bar{g}(x))$. So, $x = h \circ \bar{g}(x)$ holds for every $x \in X$. On the other hand, the equicontinuity of G implies that $g_j^{-1}x \to h(x)$ is equivalent to $g_j h(x) \to x$. From this follows that $x = \bar{g} \circ h(x)$, as required. Thus, $\bar{G}$ is a subgroup of the group $H(X)$.
We now show that $\bar{G}$ is 2-closed: Let $x_i^k \to x^k$ and $\bar{g}_i x_i^k \to y^k$ for $\bar{g}_i \in \bar{G}$ and $k = 1, 2$. Also let $g_j^i \to \bar{g}_i$ for $g_j^i \in G$. Choosing neighborhood bases

$\{U_a^1 : a \in A\}, \{U_b^2 : b \in B\}, \{V_c^1 : c \in C\}, \{V_d^2 : d \in D\}$ of y^1, y^2, x^1, x^2 respectively, since $g_j^i x_i^k \to \bar{g}_i x_i^k \to y^k$, we may assume that $g_j^i x_i^1 \in U_a^1$, $x_i^1 \in V_c^1$ and $g_j^i x_i{}^2 \in U_b^2$, $x_i^2 \in V_d^2$ hold for $i \succ i_0 \equiv i(U_a^1, U_b^2, V_c^1, V_d^2)$. Using this notation, and applying a kind of a diagonal procedure, we can construct (1) the set $\{s = s(U_a^1, U_b^2, V_c^1, V_d^2) : a \in A, b \in B, c \in C, d \in D\}$ that is directed, as usual, according to the natural direction of the neighborhoods involved, and (2) nets $\{g_s\} \subset G$, $x_s^1 \to x^1$, $x_s^2 \to x^2$ such that $g_s x_s^1 \to y^1$ and $g_s x_s^2 \to y^2$. Since G is 2-closed, there is some $g \in G \subset \bar{G}$ such that $gx^k = y^k$ for $k = 1, 2$. Thus, $\bar{G}$ is 2-closed.

(b) Let $x_i \to x$ and $g_i x_i \to y$ in X for $g_i \in G$. We have to prove that $g_i \to \infty$ in G is excluded. As in the proof of (a), using a relatively compact neighborhood U of y, we conclude that $F^{-1} = \{g_i : i \succ i_0\}$ is relatively compact in $C(X, X)$. Hence, there is a subnet $\{g_j\}$ of $\{g_i\}$, such that $g_j \to h \in C(X, X)$.

By (a), we have $h \in H(X)$ and, by the closedness of G, $h \in G$, as required. It follows that G acts properly. On the other hand, let $x_i^k \to x^k$ and $g_i x_i^k \to y^k$ for $k = 1, \ldots, q$. Since the action is proper, we may assume that $g_i \to g_0$, otherwise there would exist non-empty J-sets. By the properness, G is closed in $H(X)$, therefore $g_0 \in G$, from which follows $g_0 x^k = y^k$ for $k = 1, \ldots, q$ and every $q \in \mathcal{N}$.
(c) By assumption and (a), we have $\bar{G} < H(X)$, hence, by (b), the group $\bar{H}$, being a closed subgroup of $\bar{G}$, acts properly. Let H be q-closed, $g \in \bar{H}$ and $h_i \to g$ for $h_i \in H$. Assume $x_i^k \to x^k$ for $k = 1, \ldots, q$. Then $h_i x_i^k \to gx^k$. By the q-closedness of H, there is some $h \in H$ such that $hx^k = gx^k$, i.e. $h^{-1}g \in \bar{H}_{x^k}$ for all k, which means $g \in H \cdot (\bigcap_{k=1}^q \bar{H}_{x^k})$. On the other hand, let $x_i^k \to x^k$ and $h_i x_i^k \to y^k$. Since $\bar{G}$, hence $\bar{H}$, acts properly on X, we may assume $h_i \to g \in \bar{H}$ which implies $gx^k = y^k$, where $g = h \cdot f$ for some $h \in H$ and $f \in \bigcap_{k=1}^q \bar{H}_{x^k}$. It follows that $y^k = gx^k = h \cdot fx^k = hx^k$. ■

Remark 4.6. **(a)** Assertion (c) of Theorem 1.2 indicates that q-closedness for $q \geq 2$ does not necessarily respect restrictions of the action on dense subgroups, unless the isotropy groups are sufficiently "big", as it is the case in Example 4.1. So, proper dense subgroups of groups acting properly with trivial isotropy groups cannot be q-closed. For example, the restriction of the action of the q-closed group $\Re$ of translations on a line to the group $\mathcal{Q}$ of the rational numbers is not 2-closed: If a and b are irrational numbers with irrational difference and p_n, q_n for $n \in \mathcal{N}$ are rational numbers with $p_n \to a$ and $q_n \to b$, then the rational translations $q_n - p_n$ send p_n to q_n, but there is no rational translation sending a to b.

(b) According to Theorem 1.2(b), we are allowed to think of 2-closedness as *"pre-properness"* and may replace properness in already known structure theorems appropriately formulated. For example, the existence of a 2-closed action (G, X) with $\bar{G}$ non compact implies remarkable restrictions on X, as the following results indicate, where $\bar{G}$ replaces the properly acting group G in [1] and [2]:

Corollary 4.7. *Let X be as in Theorem 1.2 and G be 2-closed, such that $\bar{G}$ is not compact. Then X has 1, 2 or infinitely many ends. If, moreover, $\bar{G}$ is connected, then $X = \Re^k \times Y$ holds, provided (according to Iwasawa's Theorem, cf. [13, 4.13, Th., p. 188]) $\bar{G}$ is homeomorphic to $\Re^k \times K$, where Y is a global*

K-slice of the action $(\bar{G}, X)$ on which the maximal compact subgroup$K$ of $\bar{G}$ acts effectively.

The proof is an immediate consequence of Theorem 1.2(b), [1, Satz A] and [2, 0.1].

Proof of Corollary 1.3. The proper actions on metrizable spaces like our X are exactly the closed subgroups of the groups of isometries with respect to admissible metrics on X (cf., for instance, [16, 4.2]). This and Theorem 1.2(c) complete the proof. ∎

5. The q-closed geometries

In this section we use the characterizations of the 1- and 2-closed actions proved so far in order to provide topological foundations of "q-closed geometries" on orientable and non compact 2-manifolds of finite genus (see Definition 5.2). Our axioms are similar to those in Hilbert's paper of 1902 as concerns either their function (see Definition 5.2(I) and (II) and Remark 5.3) or their formulation (see Definition 5.2(III)). The classical plane geometries are included in our "2-closed geometries" (cf. Theorem 7.2 and Remark 7.3(b)).

Remark 5.1. Hilbert's foundation, published in 1902, is concerned with the common part of the Euclidean and the Hyperbolic Plane Geometries. The foundation itself is based on a group, G, of homeomorphisms of $\Re^2$ and has a purely topological character, a novelty in its time, where the transformations related to foundations of geometries were supposed to be analytic. In order for the paper not to be lengthy, we shall restrict ourselves to the following remarks about the three axioms in Hilbert's important paper [10, Appendix IV], to which we refer, that initiated the theory of topological transformation groups and its connection to geometry. These remarks intend to indicate the relation of our requirements to Hilbert's axioms.

In modern terminology, Hilbert's Axioms may be formulated as follows:

Axiom I: *Every homeomorphism of G preserves the orientation.* This axiom is responsible, in Hilbert's paper, for the indirect consequence that G lies in the connected component of the identity of the group of Euclidean isometries (if it contains an abelian normal subgroup) or of the hyperbolic isometries. Our requirement (I) in Definition 5.2, stating that the founding group is connected, has its origin in this fact, which, in connection with Axiom II, leads to the conclusion that G is, in fact, the full connected component of the group of either the Euclidean or the hyperbolic isometries.

Axiom II: *For $x, y \in \Re^2$ with $x \neq y$, the orbit $G_x(y)$ has infinitely many points.* This axiom is crucial in proving that the G_x's are the *maximal* possible compact isotropy groups: They are isomorphic to the group S^1 of plane rotations. Hilbert used the fact that the isotropy groups contain the "halfrotations" in order to construct the "lines" of the geometries (cf. [10, pp. 215–227]). This construction is the essential step for the conclusion of Hilbert's paper: The constructed "lines" together with the notion of "betweeness", defined via their parametrization, satisfy the axioms of Hilbert's classical foundation; therefore the three topological axioms provide a foundation of the classical plane geometries. This function of Hilbert's Axiom II led to our requirement in Definition 5.2(II) (cf. also Remark 5.3(b)).

Axiom III: *The group G is 3-closed.* As remarked in the introduction, 2-closedness suffices in the framework of Hilbert's foundations. Because of this, we shall require 2-closedness instead of 3-closedness. The procedures in Hilbert's paper exhibit Axiom III as the most functional one.

Definition 5.2. A *q-closed geometry* for $q \in \mathcal{N}$ on an orientable and non-compact 2-manifold M of finite genus is defined by a group G of homeomorphisms of M satisfying the following three conditions:

(I) The group G is connected.

(II) There exists at least one *group of lines*, i.e., a connected and non compact subgroup F of G, which acts non transitively on M, and has the property that there is at least one point $p \in M$ such that the *van Dantzig - van der Waerden condition* is satisfied on the orbit $F(p)$: If $f_n \to \infty$ in F, then $f_n p \to \infty$ in M.

(III) The action (G, M) is q-closed.

Remark 5.3. **(a)** According to the results of the preceding sections, we are mainly interested in the cases $q = 1$ and 2.

(b) We call a group F as in (II) a *group of lines*, because its orbits will play the role of lines of the occurring geometry. So, our condition (II) is analogous to Hilbert's Axiom II in that it produces "lines". It is more substantial than Hilbert's Axiom II, but it refers only to one F-orbit, while Hilbert's Axiom II refers to all the points of the plane. There may be more than one groups of lines, e.g. the 1-parameter subgroups of the connected component of the Lie group of isometries of the Euclidean Geometry on the plane.

(c) The van Dantzig - van der Waerden condition in Definition 5.2(II) is an abstraction of the transitive $\Re$-action by translations on an ordinary line of the plane. It is equivalently formulated for the first time in [5, p. 375] as a property of the action of the group $I_d(X)$ of the d-isometries on a locally compact, separable and connected metric space (X, d). This property played a crucial role in [5] to prove that $I_d(X)$ is a locally compact topological group.

Problems 5.4. From the above definition arise the following questions concerning orientable and non compact 2-manifolds of finite genus, which will be treated in the sequel:

(1) Which groups of lines actually occur for q-closed geometries on the M's? (cf. Theorem 6.1)

(2) Which M's admit q-closed geometries? (cf. Corollary 6.5(a))

(3) Which are (up to conjugation) the connected subgroups of $H(M)$ that define 1- or 2-closed geometries on the admissible M's? (cf. 7)

6. The groups of lines

In this section we intent to prove the following.

Theorem 6.1. *Let F be a group of lines of a q-closed geometry on an orientable and non compact 2-manifold M of finite genus. Then, F is (topologically) isomorphic to the group $\Re$, the dynamical system (F, M) is D-stable, equivalent to a C^∞-differentiable one, and such that its orbits are closed subsets of M. In particular, the orbit $F(p)$ (cf. Definition 5.2(II)) is homeomorphic to $\Re$.*

To this end, we need the following three Lemmas, where we shall use the assumptions and the notation of the above theorem.

Lemma 6.2. *If G defines a q-closed geometry on M and F is a group of lines of it, then:*
(a) *F is a closed subgroup of G, which is therefore non compact, and $F(p)$ is a closed subset of M.*
(b) *F is a Lie group homeomorphic to $\Re \times K$ where K is a maximal compact subgroup of it.*
(c) *The action (F, M) is q-closed.*

Proof. **(a)** Let $f_i \in F$ and $f_i \to g \in G - F$. Then, $f_i \to \infty$ in F, while $f_i p \to gp \in M$, which contradicts Definition 5.2(II). Thus F is closed in G, which is therefore non compact, because F is non compact. Since the action (F, M) satisfies the van Dantzig - van der Waerden condition, the natural map $F/F_p \to F(p)$ is a homeomorphism, and the isotropy group F_p is compact. Moreover, the orbit $F(p)$ is a closed subset of M, because the set $L(p)$ of its limit points in M is empty.
(b) So, $F(p)$ is locally compact. It follows that F/F_p is locally compact, too, hence that F is a locally compact group. Since it is also connected and non-compact (cf. (a)), by Iwasawa's Theorem, it is homeomorphic to $\Re^n \times K$, where n is a positive integer. Here K is a (possibly empty) maximal compact and connected subgroup of F. Let $K \equiv \{1_M\} \times K$, where 1_M denotes the identity of M, be non-trivial. Since the action (F, M) is not transitive, the codimension of the orbits of the action (K, M^+) is lower than 2. Thus, by [14, Cor. in p. 1, (iii)], K is a compact and connected Lie group. It follows that F is locally euclidean, hence a Lie group. Taking into account the arguments and the notation in the proof of (a), and the fact that the maximal compact subgroups of F are conjugate to each other, we may assume that the compact group F_p is contained in K. From this and the 1-dimensionality of $F(p)$ (because it is connected and locally compact, and F does not act transitively on M), follows $n = 1$, as required.
(c) The assertion is a consequence of (a), Definition 5.2(III), and Theorem 1.2(c). ∎

Concerning the maximal compact subgroup K of F, we have:

Lemma 6.3. ***(a)*** *If M^+ is aspherical, or $M^+ - M$ consists of more than two points, then K is trivial.*
(b) *Let $M^+ = S^2$ and $M^+ - M$ have at most two points. If the case $K \neq \emptyset$ can occur, then a subgroup of K isomorphic to S^1 acts on $M = \Re^2$ with exactly one fixed point surrounded by periodic orbits, or on $M = \Re \times S^1$ with periodic orbits.*

Proof. **(a)** By Lemma 6.2(b), if K is non-trivial, it contains a subgroup isomorphic to S^1. Assume that M^+ is aspherical. Since $M^+ - M$ is a non empty and totally disconnected set, it consists of fixed points of the action (S^1, M^+); therefore, by [4, Cor 5.3], this action is trivial, a contradiction.
Assume now that $M^+ = S^2$ and $M^+ - M$ has more than two points. Then, if e is an end-point of M, the action $(S^1, M^+ - \{e\} = \Re^2)$ has at least one 1-dimensional orbit. Therefore, by [13, 6.5, Th. in p. 252], it has exactly one fixed point surrounded by periodic orbits. This contradicts the assumption about the existence of at least two fixed points of the action $(S^1, M^+ - \{e\})$, which are end-points of M.
(b) There are two subcases: $M = S^2 - \{e\} = \Re^2$ and $M = S^2 - \{e_1, e_2\} = \Re \times S^1$. In the first case, if we assume that K is non-trivial, the preceding arguments show that its subgroup S^1 can act on M as the isotropy group of its unique fixed point. In the second case, e_2 will be the unique fixed point of the action $(S^1, M^+ - \{e_1\} = \Re^2)$, therefore S^1 can act on M with periodic orbits. ■

Proposition 6.4. *A maximal compact subgroup K of F is always trivial.*

Proof. Let K be non trivial. According to Lemma 6.3(a), we may restrict ourselves to the cases $M = \Re^2$ and $M = \Re \times S^1$. By Lemma 6.2(b), F is homeomorphic to $\Re \times K$ and K has a subgroup isomorphic to S^1, which, as in the proof of Lemma 6.3(a), acts on M with periodic orbits that surround a fixed point (an end in the case $M = \Re \times S^1$). So, the point p from Definition 5.2(II) is contained in the interior of such periodic orbits. On the other hand, because of the van Dantzig - van der Waerden condition, the connected orbit $F(p)$ is non compact. Therefore, intersects periodic orbits of the form $(\{1_M\} \times S^1)(x) \subset F(x)$. Let fp, for some $f \in F$, be a point of such an intersection. Then $(\{1_M\} \times S^1)(fp)$ is a simple closed curve contained in $F(p)$, which contains also the "line" $(\Re \times \{1_M\})(p)$. Therefore, $F(p)$ is 2-dimensional, hence an open subset of M. Since, by Lemma 6.2(a), $F(p)$ is also closed in M, the action (F, M) would be transitive, which contradicts Definition 5.2(II). Thus K must be trivial. ■

Proof of Theorem 6.1. From Lemma 6.2(b) and Proposition 6.4 follows that F is isomorphic to the topological group $\Re$. By Lemma 6.2(c), the dynamical system (F, M) is D-stable, therefore the orbit $F(p)$, being non compact, is homeomorphic to $\Re$. For the same reason, the F-orbits are closed subsets of M (cf. Proposition 3.1), and the non compact orbit $F(p)$ is homeomorphic to $\Re$. It remains to prove the C^∞-differentiability of (F, M): The end-point compactification M^+ of M is an orientable and compact 2-manifold such that $M^+ - M$, the set of the ends of M, is totally disconnected [3, p. 620]. The action (F, M) has an extension (F, M^+). By the "Smoothing Theorem" in [8, p. 17], a dynamical system on a compact 2-manifold is topologically equivalent to a C^∞-differentiable one, if it has simple minimal sets. We recall that a minimal set of a non-transitive dynamical system on a 2-manifold is textit{simple} if it consists of a fixed point or a periodic orbit. Non-simple minimal sets on a compact manifold contain Poisson-stable points (i.e., points x for which $x \in L^+(x) \cap L^-(x)$ holds). According to [3, Cor 3.1], the Poisson-stable points of a D-stable dynamical system on a 2-manifold of finite genus are either fixed or periodic. This, and the fact that the ends of M are fixed points of the action (F, M^+), because F is connected, implies that the minimal sets of this action are simple, and the theorem is proved. ■

Corollary 6.5. ***(a)*** *The orientable and non-compact 2-manifolds of finite genus admitting q-closed geometries are exactly those of genus at most one.*
(b) *The face portraits of the actions of the groups of lines for the q-closed geometries on orientable and non compact 2-manifolds of finite genus are described in Examples 3.3.*

Proof. The assertion in (a) is an immediate consequence of Theorem 6.1 and [3, Th. 4.3], where it is shown that an orientable and non compact 2-manifold of finite genus admitting a non minimal D-stable flow has genus at most 1. The assertion in (b) follows from the fact that Examples 3.3 exhaust the D-stable dynamical systems on the M's with at least one non compact orbit (corresponding to the orbit $F(p)$). ■

7. The "minimal" 1- and the "maximal" 2-closed geometries

7.1. The 1- but not 2-closed groups. As we saw in Remark 3.2(b), there exist transitive actions of groups on M defining 1- but not 2-closed geometries. Then, the connected subgroups of $H(M)$ containing such a group would also define an 1- but not 2-closed geometry. So, it is reasonable to deal with "*minimal*" subgroups of $H(M)$ defining 1- but not 2-closed geometries on M. Here the "*minimality*" is defined by requiring that the corresponding group has no strict subgroups determining 1- but not 2-closed geometries. It is clear that these minimal groups correspond to the various groups of lines that do not act properly. By Corollary 6.5(b), the face portraits of the corresponding actions are described in Examples 3.3, excluding the cases of the parallelizable flows. The non minimal 1- but not 2-closed groups correspond to those connected subgroups of $H(M)$ that contain one of these groups of lines and act transitively on M. A typical example of such a transitive action is described in Remark 3.2(b).
We add the following information: If M is different from $\Re^2$ or $\Re \times S^1$, or it has more than two ends, then every group of lines F defines an 1- but not 2-closed geometry. To see this, check that, by Theorem 6.1 and Remark 6.2(a), F is isomorphic to $\Re$ and closed in $H(M)$. It cannot be 2-closed; otherwise, by Theorem 1.2(b), it should act properly, from which follows that M should have an $\Re$-factor, contrary to our assumptions.

Remark 7.1. Since a group of lines acts non transitively, the axiom that "*two different points are joined by a unique line*" is not satisfied in an 1- but not 2-closed geometry defined by a minimal group. It can be satisfied in a 2-closed geometry, if the group G is sufficiently "rich" of 1-parameter subgroups acting properly, e.g. if G is the connected component of the groups of isometries of the classical geometries on the plane.

7.2. The 2-closed geometries. By Theorem 1.2(a) and (b), we may regard the 2-closed groups as dense subgroups of groups $G < H(X)$ acting properly on M. It is, therefore, reasonable to study the "*maximal*" subgroups of $H(M)$ defining 2-closed geometries on M. A 2-closed group G is "*maximal*", if there is no subgroup of $H(X)$ strictly containing G and defining a 2-closed geometry on M. By Theorem 1.2(a), the maximal groups are necessarily closed in $H(M)$. Thus, by Theorem 1.2(b), a maximal group is q-closed for every $q \in \mathcal{N}$. If we know the maximal groups, then any other 2-closed group can be obtained taking into account Theorem 1.2(c).

Theorem 7.2. *$\Re^2$ and $\Re \times S^1$ are the only orientable and non-compact 2-manifolds of finite genus admitting 2-closed geometries. On $\Re^2$ there are, up to conjugation in $H(\Re^2)$, two maximal groups defining 2-closed geometries, namely the connected components of the groups of isometries of either the Euclidean or the Hyperbolic Geometries of $\Re^2$. The maximal subgroups of $H(\Re \times S^1)$ define 2-closed geometries on $\Re \times S^1$ that are Riemannian and such that their groups of isometries are isomorphic to $\Re \times S^1$.*

Proof. By Theorem 1.2(a) and (b), we may assume that the maximal group G defining a 2-closed geometry on M acts properly on M. The group of lines $F = \Re$ (cf. Theorem 6.1) is, by Lemma 6.2(a), a closed subgroup of G, therefore the action (F, M) is proper. So, according to [1, 0.1], M is homeomorphic to $\Re \times S$ for suitable S. Since M is a 2-dimensional manifold, S is a 1-dimensional manifold (cf. [9, Ch. VII, 1.6]), hence $S = \Re$ or S^1.

Let $M = \Re^2$. According to Remark 4.4., G acts equicontinuously on M with respect to the uniform structure induced on $\Re^2$ by that of the one-point compactification, S^2, of $\Re^2$. By [6, 2, Th. 1], the maximal connected subgroups of $H(\Re^2)$ acting equicontinuously on $\Re^2$ with respect to this uniform structure are (up to conjugation in $H(\Re^2)$ the connected components of the group of isometries of either the Euclidean or the Hyperbolic Geometries, which, thus, are the only maximal groups defining 2-closed geometries on $\Re^2$.

Now let $M = \Re \times S^1$. The group G is locally compact (: it acts properly on M), connected (by Definition 5.2.(I)) and non compact (by Lemma 6.2(a)); therefore, it is homeomorphic to $\Re^n \times K$. Since the action (G, M) is proper, its orbits, $G(x)$, are homeomorphic to G/G_x, and the isotropy groups, G_x, are compact; it follows $n = 1$. Regarding $F = \Re \times \{1_M\}$ as a group of lines, we conclude, as before, that the (maximal) K-factor of G acts on the S^1-factor of M, from which follows $K = S^1$. Thus, we have $G = \Re \times S^1$. Since, the actions of $F = \Re \times \{1_M\}$ and $\{1_M\} \times K$ on M are differentiable (by Theorem 6.1 and Lemma 6.3(b), respectively), the action (G, M) is C^∞-differentiable. Moreover, G acts properly on M. So, we may assume that it acts by Riemannian isometries (cf. [12, Ch. I, 4, Th. 2]). Since M is not simply connected, the dimension of its group of Riemannian isometries is less than 3 (cf., for instance, [11, Ch. 2, Th. 3.1]). Therefore, $G = \Re \times S^1$ is the maximal Lie group that can occur as connected Lie group of Riemannian isometries on M. ∎

Remark 7.3. **(a)** Our requirements in Definition 5.2., in conjunction with the maximality of the corresponding groups, *topologically* characterize the geometries mentioned in the above theorem and their Lie groups of isometries in the spirit the of Hilbert's 5^{th} problem.
(b) The considerations in this section show that among the 2-closed geometries on M, only in the case $M = \Re^2$ isotropy groups isomorphic to S^1 can occur, as in the framework of Hilbert's foundation of the classical geometries by virtue of his Axiom II. This axiom is responsible for the "maximal" isotropy groups that can occur in Hilbert's framework; a feature corresponding to the maximality of the groups defining 2-closed geometries on M, among which are the classical geometries on the plane.

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Athanassios Strantzalos
Department of Mathematics
National and Kapodistrian
University of Athens
Panepistimiopolis
Athens, GR-15784, Greece
astrantzalos@gmail.com

Polychronis Strantzalos
Depertment of Mathematics
National and Kapodistrian
University of Athens
Panepistimiopolis
Athens, GR-15784, Greece
pstrantz@math.uoa.gr

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