

On the Singularity of some Special Components of Springer Fibers*

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Abstract. Let $u \in \text{End}(\mathbb{C}^n)$ be nilpotent. The variety of u -stable complete flags is called the Springer fiber over u . Its irreducible components are parameterized by a set of standard Young tableaux. The Richardson (resp. Bala-Carter) components of Springer fibers correspond to the Richardson (resp. Bala-Carter) elements of the symmetric group, through Robinson-Schensted correspondence. Every Richardson component is isomorphic to a product of standard flag varieties. By contrast, the Bala-Carter components are very susceptible to be singular. First, we characterize the singular Bala-Carter components in terms of two minimal forbidden configurations. Next, we introduce two new families of components, wider than the families of Bala-Carter components and Richardson components, and both in duality via the tableau transposition. The components in the first family are characterized by the fact that they have a dense orbit of special type under the action of the stabilizer of u , whereas all components in the second family are iterated fiber bundles over projective spaces.

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1. Introduction

Let V be an n -dimensional \mathbb{C} -vector space and let $u : V \rightarrow V$ be a nilpotent endomorphism. We denote by \mathcal{B} the set of complete flags, that is, chains of vector subspaces $F = (V_0 \subset V_1 \subset \dots \subset V_n = V)$ with $\dim V_i = i$ for all i . Then, \mathcal{B} is an algebraic projective variety. We define

$$\mathcal{B}_u = \{F = (V_0, \dots, V_n) \in \mathcal{B} : u(V_i) \subset V_i \text{ for all } i\},$$

the subset of u -stable flags. Then \mathcal{B}_u is a closed subvariety of \mathcal{B} (in general non-irreducible). The variety \mathcal{B}_u identifies with the fiber over u of the Springer resolution (cf. [14], [15]), it is called Springer fiber. Springer fibers arise in geometric representation theory, in relation with Springer Weyl group representations. The study of their geometry sets quite challenging problems. Among those, we study in this article the question of the singularity of their irreducible components.

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The singularity of the components of \mathcal{B}_u has been studied by only few authors. First it was proved that, in some simple cases depending on the Jordan form of u , every component of \mathcal{B}_u is smooth. J.A. Vargas [17] proved this for u having only one non-trivial Jordan block (this is the so-called hook case). F. Fung [7] established the property for u having two blocks (the so-called two-row case), proving in addition that in this case every component is an iterated fiber bundle of base $(\mathbb{P}^1, \dots, \mathbb{P}^1)$. N. Spaltenstein [12] and J.A. Vargas [17] provided the first example of a singular component, in \mathcal{B}_u , for u having four blocks of lengths $(2, 2, 1, 1)$. G. Pagnon and N. Ressayre [10] constructed a family of smooth components, adjacent to Richardson components.

The present article comes as a continuation of two recent joint works with A. Melnikov. In [4], we complete the picture started by F. Fung and J.A. Vargas: we prove that every component of \mathcal{B}_u is smooth in exactly four cases depending on the Jordan form of u : 1) the hook case; 2) the two-row case; 3) if u has three Jordan blocks, two of arbitrary length and one which is trivial; 4) if u has three blocks of length 2.

In [5], we provide several characterizations of the singular components of a given Springer fiber \mathcal{B}_u , in the special case $u^2 = 0$ (called two-column case). We show in particular that the singular components do not satisfy the Poincaré duality, and thus they are rationally singular. The study of the singularity of components of Springer fibers is the most fruitful in the two-column case. N. Perrin and E. Smirnov [11] proved that, in that case, the components are normal and have rational singularities.

In the present article, we study the singularity of the components of \mathcal{B}_u for u general, but while concentrating on some particular families of components. This article contains two main results.

1) The so-called Bala-Carter components are in duality towards the Richardson components. Whereas Richardson components are always smooth (they are parabolic orbits), Bala-Carter components are in many cases singular. As a first result, we give a necessary and sufficient condition for a Bala-Carter component to be singular. It will follow from the criterion that, among components of Springer fibers, Bala-Carter components are the most susceptible to be singular, in the sense that whenever \mathcal{B}_u has a singular component, it admits one of Bala-Carter type.

2) We introduce a new family of components generalizing the Bala-Carter components, which is in duality towards a family of components generalizing the Richardson components. The generalized Bala-Carter components are those containing a dense orbit of a special type under the action of the stabilizer of u . Then we prove that every generalized Richardson component is an iterated fiber bundle over projective spaces. In the two-column case, every component is generalized Bala-Carter, whereas in the two-row case, every component is generalized Richardson. Then we retrieve in particular the result due to F. Fung.

2. Background and statement of main results

Before stating our results, which will be done in the subsections 2.2 and 2.3, we need to set up the basic background.

2.1. Components of Springer fibers

The variety \mathcal{B}_u is an algebraic projective variety, which is connected but, in general, reducible. Following R. Steinberg [16] and N. Spaltenstein [13], \mathcal{B}_u is equidimensional and its irreducible components are parameterized by a set of standard tableaux. In this subsection, first, we present Spaltenstein’s construction. Then, we recall the definition of two special families of components of \mathcal{B}_u : the Richardson and Bala-Carter components.

2.1.1. The Jordan form $\lambda(u)$ and the Young diagram $Y(u)$

Let $\lambda(u) = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r)$ be the sizes of the Jordan blocks of u . We have $\lambda_1 + \lambda_2 + \dots + \lambda_r = n$, that is, the sequence $\lambda(u)$ is a partition of n . Let $Y(u)$ be the Young diagram of rows of lengths $(\lambda_1, \dots, \lambda_r)$, that is, $Y(u)$ is an array of r left-adjusted rows with the i -th row containing λ_i empty boxes.

Example: $\lambda(u) = (3, 2, 2, 1) \Rightarrow Y(u) =$

 $.$

The diagram $Y(u)$ is a datum equivalent to the sequence $\lambda(u)$, and to the Jordan form of u . Also, the particular Jordan forms invoked in section 1 may be interpreted in terms of the diagram $Y(u)$. In a transparent way, in the hook case (i.e., u has only one non-trivial Jordan block) the diagram has only one row of length ≥ 2 , in the two-row case (i.e., u has two blocks) the diagram has two rows, and respectively, in the two-column case (i.e., $u^2 = 0$) the diagram has two columns.

Let $\lambda^*(u) = (\lambda_1^* \geq \dots \geq \lambda_s^*)$ be the partition of n conjugate of $\lambda(u)$, that is, $\lambda_1^*, \dots, \lambda_s^*$ are the sizes of the columns of the diagram $Y(u)$. The dimension of \mathcal{B}_u has the following expression in terms of $\lambda^*(u)$ (see [13, §II.5]):

$$\dim \mathcal{B}_u = \sum_{j=1}^s \frac{\lambda_j^*(\lambda_j^* - 1)}{2}. \tag{1}$$

2.1.2. The irreducible components $\mathcal{K}^T \subset \mathcal{B}_u$

Recall that a standard Young tableau (in short, standard tableau) is a numbering of $Y(u)$ with the entries $1, \dots, n$, such that the entries increase from left to right along the rows and respectively from top to bottom along the columns. For instance, here is a standard tableau of shape $Y(u)$, where $Y(u)$ comes from the previous example:

$$T =$$

1	3	8
2	5	
4	6	
7		

 $.$

A standard tableau T is in fact equivalent to the datum of the maximal chain of subdiagrams $Y_1^T \subset Y_2^T \subset \dots \subset Y_n^T = Y(u)$, where Y_i^T is the shape of the subtableau of T formed by the entries $1, \dots, i$.

Spaltenstein’s construction of the components of \mathcal{B}_u relies on a partition of \mathcal{B}_u into subsets \mathcal{B}_u^T associated to the T ’s standard. For $F = (V_0, V_1, \dots, V_n) \in \mathcal{B}_u$,

each V_i is u -stable and the restriction $u|_{V_i} \in \text{End}(V_i)$ is nilpotent. Its Jordan form is represented by a Young subdiagram $Y(u|_{V_i}) \subset Y(u)$. This defines a maximal chain of subdiagrams $Y(u|_{V_1}) \subset Y(u|_{V_2}) \subset \dots \subset Y(u|_{V_n}) = Y(u)$. Then we set

$$\mathcal{B}_u^T = \{F \in \mathcal{B}_u : Y(u|_{V_i}) = Y_i^T \ \forall i = 1, \dots, n\}.$$

The variety \mathcal{B}_u is indeed the disjoint union of the subsets \mathcal{B}_u^T . Due to [13, §II.5], each subset \mathcal{B}_u^T is locally closed in \mathcal{B}_u , irreducible and smooth, and $\dim \mathcal{B}_u^T = \dim \mathcal{B}_u$. Therefore, the irreducible components of \mathcal{B}_u are the closures $\mathcal{K}_u^T := \overline{\mathcal{B}_u^T}$ in the Zariski topology, they are parameterized by the standard tableaux of shape $Y(u)$, and we have $\dim \mathcal{K}_u^T = \dim \mathcal{B}_u$ for every T .

Note that, up to isomorphism, \mathcal{K}_u^T depends only on the tableau T . That is why we will drop the index from the notation and write $\mathcal{K}^T = \mathcal{K}_u^T$.

2.1.3. Bala-Carter and Richardson components

A Bala-Carter component of \mathcal{B}_u is associated to a permutation of $\lambda(u)$, that is, a sequence $\pi = (\pi_1, \dots, \pi_r)$ which coincides with $\lambda(u) = (\lambda_1, \dots, \lambda_r)$ up to ordering. A Richardson component is associated to a permutation of the conjugate partition $\lambda^*(u)$. We denote by Λ_u the set of permutations of $\lambda(u)$ and by Λ_u^* the set of permutations of $\lambda^*(u)$.

Observe preliminarily that a flag $F = (V_0, \dots, V_n) \in \mathcal{B}_u$ induces a nilpotent endomorphism $u|_{V_j/V_i} \in \text{End}(V_j/V_i)$ for any $0 \leq i < j \leq n$. Moreover, the map $F \mapsto \text{rank } u|_{V_j/V_i}$ is lower semi-continuous (see [3, Lemma 2.2]).

Let $\pi = (\pi_1, \dots, \pi_r) \in \Lambda_u$. For $j \in \{0, \dots, r\}$, set $i_j = \pi_1 + \dots + \pi_j$. We say that $F \in \mathcal{B}_u$ is π -regular if $u|_{V_{i_j}/V_{i_{j-1}}}$ is regular for all $j = 1, \dots, r$. The set

$$\mathcal{U}_\pi^{\text{BC}} = \{F \in \mathcal{B}_u : F \text{ is } \pi\text{-regular}\}$$

is then an open subset of \mathcal{B}_u . Let $Z_u = \{g \in GL(V) : gug^{-1} = u\}$ be the stabilizer of u . In fact, the set $\mathcal{U}_\pi^{\text{BC}}$ is irreducible, since it is a Z_u -orbit of \mathcal{B}_u (see also section 2.3.1). Hence, its closure in the Zariski topology, denoted by $\mathcal{K}_\pi^{\text{BC}}$, is an irreducible component of \mathcal{B}_u , called a Bala-Carter component (see also [2, §5.10]).

Let $\pi = (\pi_1, \dots, \pi_s) \in \Lambda_u^*$ and, for $j \in \{0, \dots, s\}$, set $i_j = \pi_1 + \dots + \pi_j$. We say that $F \in \mathcal{B}_u$ is π -trivial if $u|_{V_{i_j}/V_{i_{j-1}}} = 0$ for all $j = 1, \dots, s$. The set

$$\mathcal{K}_\pi^{\text{R}} = \{F \in \mathcal{B}_u : F \text{ is } \pi\text{-trivial}\}$$

is then a closed subset of \mathcal{B}_u . In fact, there is a unique partial flag $W_1 \subset W_2 \subset \dots \subset W_s = V$ with $\dim W_j = i_j$ and $u(W_j) \subset W_{j-1}$ for all j , hence, $F \in \mathcal{B}_u$ is π -trivial if and only if $V_{i_j} = W_j$ for all j . Denoting by $\mathcal{B}(W)$ the variety of complete flags of a space W and letting $\mathcal{B}^{(m)} = \mathcal{B}(\mathbb{C}^m)$, we obtain a natural isomorphism

$$\mathcal{K}_\pi^{\text{R}} \cong \mathcal{B}(W_1) \times \mathcal{B}(W_2/W_1) \times \dots \times \mathcal{B}(W_s/W_{s-1}) \cong \mathcal{B}^{(\lambda_1^*)} \times \mathcal{B}^{(\lambda_2^*)} \times \dots \times \mathcal{B}^{(\lambda_s^*)} \quad (2)$$

(see [10, §7]). Therefore, $\mathcal{K}_\pi^{\text{R}}$ is irreducible and of same dimension as \mathcal{B}_u . Hence, it is an irreducible component of \mathcal{B}_u , called a Richardson component.

2.1.4. Duality between Bala-Carter and Richardson components

Let us describe the standard tableaux corresponding to Richardson and Bala-Carter components. Let $\pi = (\pi_1, \dots, \pi_r) \in \Lambda_u$, and let T_π be the tableau obtained as follows: draw an array of r left-adjusted rows, with π_i boxes in the i -th row filled in with the numbers $\pi_1 + \dots + \pi_{i-1} + 1, \dots, \pi_1 + \dots + \pi_i$, and then push to the top the supernumerary boxes in each row, so to obtain a standard tableau of shape $Y(u)$. For example:

$$\pi = (2, 3, 1, 2) \rightarrow \begin{array}{|c|c|c|} \hline 1 & 2 & \\ \hline 3 & 4 & 5 \\ \hline 6 & & \\ \hline 7 & 8 & \\ \hline \end{array} \rightarrow \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline 6 & 8 & \\ \hline 7 & & \\ \hline \end{array} = T_\pi.$$

Then, we can see that $\mathcal{U}_\pi^{\text{BC}} \subset \mathcal{B}_u^{T_\pi}$ (see also section 2.3.1), so that $\mathcal{K}_\pi^{\text{BC}} = \mathcal{K}^{T_\pi}$.

Similarly, for $\pi = (\pi_1, \dots, \pi_s) \in \Lambda_u^*$, let T_π^* be the tableau obtained as follows: draw an array of s top-adjusted columns, with π_i boxes in the i -th column filled in with the numbers $\pi_1 + \dots + \pi_{i-1} + 1, \dots, \pi_1 + \dots + \pi_i$, and push to the left the supernumerary boxes in each column, to obtain a standard tableau of shape $Y(u)$. Then, we have the inclusion $\mathcal{B}_u^{T_\pi^*} \subset \mathcal{K}_\pi^{\text{R}}$, and therefore, $\mathcal{K}_\pi^{\text{R}} = \mathcal{K}^{T_\pi^*}$ (see [10, §7]).

Recall that an element $w \in \mathbf{S}_n$ is called Richardson if it is the longest element of a subgroup of \mathbf{S}_n generated by elementary transpositions, and an element $w' \in \mathbf{S}_n$ is called Bala-Carter if it is of the form $w' = ww_0$ with w Richardson and $w_0 : i \mapsto n - i + 1$. In fact, one can see that the tableau T_π^* is obtained through the Robinson insertion algorithm from a Richardson element $w \in \mathbf{S}_n$, whereas T_π is obtained from a Bala-Carter element $w' \in \mathbf{S}_n$ (cf., [6]).

To u , we may associate a nilpotent $u^* \in \text{End}(V)$ such that u, u^* have conjugate Jordan forms: $\lambda(u^*) = \lambda^*(u)$. We consider the corresponding Springer fiber \mathcal{B}_{u^*} . For T a standard tableau of shape $Y(u)$, denote by T^* its transpose of shape $Y(u^*)$, that is, the i -th column of T^* coincides with the i -th row of T . Thus, to each component $\mathcal{K}^T = \mathcal{K}_u^T \subset \mathcal{B}_u$, it corresponds the component $\mathcal{K}^{T^*} = \mathcal{K}_{u^*}^{T^*} \subset \mathcal{B}_{u^*}$. From the description above, we infer the duality property:

$$\mathcal{K}^T \text{ is Bala-Carter} \Leftrightarrow \mathcal{K}^{T^*} \text{ is Richardson.}$$

2.2. Singularity criterion for Bala-Carter components

It follows from formula (2) that the Richardson components of \mathcal{B}_u are smooth and pairwise isomorphic. On the contrary, the Bala-Carter components of \mathcal{B}_u are not pairwise isomorphic in general, in addition there can be smooth and singular ones. Our first result is a characterization of the singular Bala-Carter components.

We recall two examples of singular components of Springer fibers. The first example is $\mathcal{K}^T \subset \mathcal{B}_u$ for u having the Jordan form $\lambda(u) = (2, 2, 1, 1)$, the second example is $\mathcal{K}^S \subset \mathcal{B}_u$ for u of Jordan form $\lambda(u) = (3, 2, 2)$. These two singular

components are associated to the tableaux

$$T = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 5 \\ \hline 4 & \\ \hline 6 & \\ \hline \end{array} \quad \text{and} \quad S = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 4 & \\ \hline 6 & 7 & \\ \hline \end{array}$$

(see [4] and [17]). Notice that both components are Bala-Carter, associated to the following permutations of the Jordan block size sequences: $(1, 2, 2, 1)$ and $(2, 3, 2)$.

If $\pi = (\pi_1, \dots, \pi_r)$ and $\rho = (\rho_1, \dots, \rho_k)$ are two sequences of integers, say $\pi \geq \rho$ if there are $1 \leq i_1 < \dots < i_k \leq r$ such that $\pi_{i_l} \geq \rho_l$ for every $l \in \{1, \dots, k\}$ (in particular $r \geq k$). Then, our result can be stated:

Theorem 2.1. *Let $\pi \in \Lambda_u$ be a permutation of the Jordan block sizes sequence $\lambda(u)$, and let $\mathcal{K}_\pi^{\text{BC}} \subset \mathcal{B}_u$ be the corresponding Bala-Carter component. Then, $\mathcal{K}_\pi^{\text{BC}}$ is singular if and only if $\pi \geq (1, 2, 2, 1)$ or $\pi \geq (2, 3, 2)$.*

We derive two corollaries from this characterization. First, as we know from [4] that \mathcal{B}_u admits a singular component provided that $\lambda(u) \geq (2, 2, 1, 1)$ or $\lambda(u) \geq (3, 2, 2)$, we see that in this case, we can always find $\pi \in \Lambda_u$ satisfying $\pi \geq (1, 2, 2, 1)$ or $\pi \geq (2, 3, 2)$. That is:

Corollary 2.2. *Whenever the Springer fiber \mathcal{B}_u admits a singular component, it admits a singular component of Bala-Carter type.*

Second, we easily infer from the theorem that in many cases, all the Bala-Carter components of \mathcal{B}_u are singular:

Corollary 2.3. *If $\lambda(u) \geq (2, 2, 2, 2)$ or $\lambda(u) \geq (3, 3, 3)$, then every Bala-Carter component of \mathcal{B}_u is singular.*

It is convenient to represent a sequence $\pi = (\pi_1, \dots, \pi_r) \in \Lambda_u$ by a graph: each number π_i is represented by a chain of $\pi_i - 1$ arcs connecting two by two π_i vertices arranged along a horizontal line, then the graph corresponding to π is obtained by juxtaposing the chains for π_1, \dots, π_r . For instance, $(1, 2, 2, 1)$ and $(2, 3, 2)$ are represented by the graphs

$$\bullet \quad \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \quad \bullet \quad \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \quad \bullet \quad \text{and} \quad \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \quad \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \quad \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \quad \begin{array}{c} \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} \quad \bullet \quad (3)$$

The relation $\pi \geq \rho$ is then translated in terms of an inclusion of graphs: the graph of ρ can be obtained from the one of π by repeating the procedure of deleting either a whole connected component or an extremal vertex of a connected component (together with the corresponding arc). Then, the Bala-Carter component $\mathcal{K}_\pi^{\text{BC}} \subset \mathcal{B}_u$ is singular if and only if the graph representing π contains one of both graphs of formula (3).

2.3. Generalization of Bala-Carter and Richardson components

Let $Z_u = \{g \in GL(V) : gug^{-1} = u\}$ be the stabilizer of u . This is a connected closed subgroup of $GL(V)$, and its natural action on flags leaves \mathcal{B}_u and every component of \mathcal{B}_u invariant. In this subsection, we define a notion of particular orbit of \mathcal{B}_u under the action of Z_u , which we will call Jordan orbit. We point out that a component is Bala-Carter if and only if it contains a dense Jordan orbit of a particular type, called standard. Then we consider the components which contain a (not necessarily standard) Jordan orbit, they provide a generalization of Bala-Carter components. We state two results: the first one characterizes the generalized Bala-Carter components, the second one says that, if \mathcal{K}^T is generalized Bala-Carter, then the component \mathcal{K}^{T*} is an iterated fiber bundle with a sequence of projective spaces as its base.

2.3.1. Jordan orbits

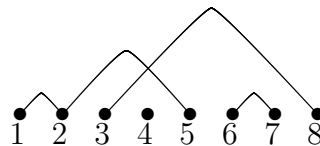
Recall that $\lambda(u) = (\lambda_1, \dots, \lambda_r)$ are the sizes of the Jordan blocks of u . We denote by Π_u the set of partitions of $\{1, \dots, n\}$ into r subsets of cardinality $\lambda_1, \dots, \lambda_r$. Hence, an element $\pi \in \Pi_u$ can be written $\pi = (I_1, \dots, I_r)$ such that

$$I_1 \sqcup \dots \sqcup I_r = \{1, \dots, n\} \quad \text{and} \quad |I_k| = \lambda_k \quad \text{for all } k.$$

We will write $I_1, \dots, I_r \in \pi$. A partition $\pi \in \Pi_u$ can be (and will be) identified to a map $\pi : \{1, \dots, n\} \rightarrow \{1, \dots, n\} \cup \{\emptyset\}$ such that

$$\pi(i) \in \{\emptyset, 1, \dots, i-1\}, \quad \text{and} \quad \pi(i) = \pi(j) \neq \emptyset \Rightarrow i = j.$$

Indeed, assuming $i \in I_k$, let $\pi(i) = \emptyset$ if $i = \min(I_k)$, and let $\pi(i)$ be the predecessor of i in I_k otherwise. Alternatively, we represent π by a pattern (also denoted by π) consisting of n points along a horizontal line which we label from left to right by $1, \dots, n$, with arcs (j, i) for $\pi(i) = j$. For instance, $\pi = (\{1, 2, 5\}, \{3, 8\}, \{6, 7\}, \{4\})$ is represented by



For $\pi \in \Pi_u$, a basis (e_1, \dots, e_n) of V is said to be a π -basis if it satisfies

$$\begin{cases} u(e_i) = e_{\pi(i)} & \text{if } \pi(i) \neq \emptyset, \\ u(e_i) = 0 & \text{otherwise} \end{cases}$$

(this is simply a Jordan basis, numbered according to π). We denote by \mathcal{Z}_π the set of flags which can be written $F = (\langle e_1, \dots, e_i \rangle)_{i=0, \dots, n}$ for some π -basis (e_1, \dots, e_n) . Clearly, $\mathcal{Z}_\pi \subset \mathcal{B}_u$. Notice that the set of π -bases is an orbit of V^n under the action of Z_u . Therefore, \mathcal{Z}_π is a Z_u -orbit of \mathcal{B}_u . We call it a Jordan orbit.

To a partition $\pi \in \Pi_u$, we associate a standard tableau T_π in the following manner. For $i \in \{1, \dots, n\}$, let $c_\pi(i)$ be minimal such that $\pi^{c_\pi(i)}(i) = \emptyset$. Then, let

T_π be the unique standard tableau of shape $Y(u)$ which contains i in its $c_\pi(i)$ -th column for every i . For instance, for π like in the previous example, we have

$$T_\pi = \begin{array}{|c|c|c|} \hline 1 & 2 & 5 \\ \hline 3 & 7 & \\ \hline 4 & 8 & \\ \hline 6 & & \\ \hline \end{array} .$$

For $\pi \in \Pi_u$, it is easy to see that the orbit \mathcal{Z}_π lies in the set $\mathcal{B}_u^{T_\pi}$ (cf. section 2.1.2) and therefore, in the component \mathcal{K}^{T_π} .

Remark 2.4. In general, any Z_u -orbit of \mathcal{B}_u is not a Jordan orbit. Suppose for example $\lambda(u) = (3, 1)$. Let (e, e', e'', f) with $u(e'') = e'$, $u(e') = e$, $u(e) = u(f) = 0$ be a Jordan basis. Then the flag $F = (0 \subset \langle e \rangle \subset \langle e, e' + f \rangle \subset \langle e, e', f \rangle \subset V)$ does not belong to any \mathcal{Z}_π . However, the Jordan orbits are all the Z_u -orbits of \mathcal{B}_u in the two-column case (see [5, §2]).

Now, let us characterize a Bala-Carter component in terms of Jordan orbits.

A partition $\pi \in \Pi_u$ is said to be standard if $\pi(i) \in \{\emptyset, i - 1\}$ for all i (equivalently, $\pi = (I_1, \dots, I_r)$ where the I_j 's are integer intervals). We denote by $\Pi_u^0 \subset \Pi_u$ the subset of standard partitions. The orbit \mathcal{Z}_π associated to a standard partition π is called a standard Jordan orbit. Notice that there is a one-to-one correspondence between the set Λ_u of permutations of the sequence $\lambda(u)$ and the set Π_u^0 of standard partitions: an element $\pi = (\pi_1, \dots, \pi_r) \in \Lambda_u$ can be seen as a standard partition $(I_1, \dots, I_r) \in \Pi_u^0$ where for $j = 1, \dots, r$, we set $I_j = \{\pi_1 + \dots + \pi_{j-1} + 1, \dots, \pi_1 + \dots + \pi_j\}$. The definition of the tableau T_π for $\pi \in \Lambda_u$ in the subsection 2.1.4 is compatible with the previous one given for $\pi \in \Pi_u$ in this subsection.

Let $\pi = (\pi_1, \dots, \pi_r) \in \Lambda_u$ and let us come back to the definition of the Bala-Carter component $\mathcal{K}_\pi^{\text{BC}} \subset \mathcal{B}_u$. For $j = 0, \dots, r$, write $i_j = \pi_1 + \dots + \pi_j$. Then $\mathcal{K}_\pi^{\text{BC}}$ contains as a dense subset the set $\mathcal{U}_\pi^{\text{BC}}$ formed by the flags $F = (V_0, \dots, V_n) \in \mathcal{B}_u$ such that $u|_{V_{i_j}/V_{i_{j-1}}}$ is regular for all j . For such a flag, we find $x_j \in V_{i_j}$ such that

$$x_j, u(x_j), \dots, u^{\pi_j-1}(x_j) \notin V_{i_{j-1}}.$$

Due to the Jordan form of u , we have $V_{i_j} \subset V_{i_{j-1}} + \ker u^{\pi_j}$ for all j , hence we may choose $x_j \in \ker u^{\pi_j}$. For $i_{j-1} < i \leq i_j$, put $e_i = u^{i-j-i}(x_j)$. We necessarily have

$$V_i = V_{i_{j-1}} + \langle e_l : i_{j-1} < l \leq i \rangle.$$

Altogether, e_1, \dots, e_n form a basis of V which satisfies

$$V_i = \langle e_1, \dots, e_i \rangle \quad \forall i \in \{0, \dots, n\},$$

and $\begin{cases} u(e_i) = 0 & \text{if } i = i_j + 1 \text{ for some } j \in \{0, \dots, r - 1\}, \\ u(e_i) = e_{i-1} & \text{otherwise.} \end{cases}$

Thus, this is a π -basis. Conversely, if (e_1, \dots, e_n) is a π -basis, then the flag $(\langle e_1, \dots, e_i \rangle)_{i=0, \dots, n}$ lies in $\mathcal{U}_\pi^{\text{BC}}$. It results $\mathcal{U}_\pi^{\text{BC}} = \mathcal{Z}_\pi$, hence $\mathcal{K}_\pi^{\text{BC}}$ contains a dense standard Jordan orbit. Using the correspondence $\Pi_u^0 \cong \Lambda_u$, we therefore obtain:

Proposition 2.5. *A component $\mathcal{K}^T \subset \mathcal{B}_u$ is Bala-Carter if and only if it contains a dense standard Jordan orbit.*

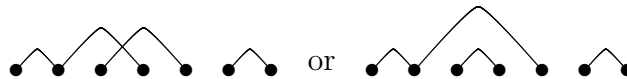
2.3.2. Generalized Bala-Carter components

We consider the components which have the property to contain a dense Jordan orbit \mathcal{Z}_π (not necessarily standard). In particular, due to the above, Bala-Carter components satisfy this property. Our first purpose is to characterize these components.

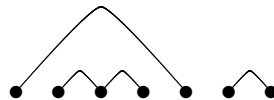
Let $\pi = (I_1, \dots, I_r) \in \Pi_u$ be a partition of $\{1, \dots, n\}$. We say that π has a crossing if there exist $i, j \in \{1, \dots, n\}$ with $\emptyset < \pi(j) < \pi(i) < j < i$ (that is, there is a crossing of two arcs in the graph representing π). We say that $I_j < I_k$ if $\min(I_k) < \min(I_j) \leq \max(I_j) < \max(I_k)$. Then, we define Π_u^1 as the subset of elements $\pi \in \Pi_u$ with no crossings, and satisfying

$$I_j < I_k \Rightarrow |I_j| \geq |I_k|.$$

Recall that Π_u^0 is the set of partitions $\pi \in \Pi_u$ into subsets I_j which are integer intervals, hence in particular $\Pi_u^0 \subset \Pi_u^1$. The elements of Π_u^1 are characterized by the fact that their graph representation has no crossing, and if a chain of arcs lies under another one, then the first chain is the longest. For example, if the graph is



then π does not belong to Π_u^1 . If the graph is



then π lies in Π_u^1 . Notice that these graphs generalize the notion of cup diagrams (see [7], [9], [18]).

We characterize the components with a dense Jordan orbit as follows.

Proposition 2.6. *The Jordan orbit $\mathcal{Z}_\pi \subset \mathcal{B}_u$ has maximal dimension $\dim \mathcal{Z}_\pi = \dim \mathcal{B}_u$ if and only if $\pi \in \Pi_u^1$. In particular, the mapping $\pi \mapsto \mathcal{K}^{T_\pi}$ is a one-to-one correspondence between partitions $\pi \in \Pi_u^1$ and irreducible components of \mathcal{B}_u which contain a dense Jordan orbit.*

The proof is given in section 3.

Let X, B_1, \dots, B_m be algebraic varieties. We recall from [7] the notion of iterated fiber bundle, defined in the following inductive manner. If $m = 1$, then we say that X is an iterated fiber bundle of base B_1 if there is an isomorphism $X \xrightarrow{\sim} B_1$. If $m > 1$, we say that X is an iterated fiber bundle of base (B_1, \dots, B_m) if there is a fiber bundle $X \rightarrow B_m$ whose fiber is an iterated fiber bundle of base (B_1, \dots, B_{m-1}) . For instance $\mathcal{B}^{(m)}$, the variety of complete flags of \mathbb{C}^m , is naturally an iterated fiber bundle of base $(\mathbb{P}^1, \dots, \mathbb{P}^{m-1})$. Then, it follows from formula (2) that every Richardson component of \mathcal{B}_u is an iterated fiber bundle of base $(\mathbb{P}^1, \dots, \mathbb{P}^{\lambda_1^* - 1}, \dots, \mathbb{P}^1, \dots, \mathbb{P}^{\lambda_s^* - 1})$. Or, in other words, if $\mathcal{K}^T \subset \mathcal{B}_u$ is a Bala-Carter component and T^* denotes the transposed tableau of T , then \mathcal{K}^{T^*}

is an iterated fiber bundle of base $(\mathbb{P}^1, \dots, \mathbb{P}^{\lambda_1-1}, \dots, \mathbb{P}^1, \dots, \mathbb{P}^{\lambda_r-1})$ (cf. section 2.1.4).

More generally, we have the following result:

Theorem 2.7. *Let $\lambda(u) = (\lambda_1 \geq \dots \geq \lambda_r)$ be the sizes of the Jordan blocks of u . Let $\mathcal{K}^T \subset \mathcal{B}_u$ be an irreducible component, associated to the tableau T . Let $\mathcal{K}^{T^*} \subset \mathcal{B}_{u^*}$ be the component corresponding to the transposed tableau T^* . If \mathcal{K}^T contains a dense Jordan orbit, then \mathcal{K}^{T^*} is an iterated fiber bundle of base $(\mathbb{P}^1, \dots, \mathbb{P}^{\lambda_1-1}, \dots, \mathbb{P}^1, \dots, \mathbb{P}^{\lambda_r-1})$.*

Remark 2.8. In the two-column case, every component contains a dense Jordan orbit (see [5, §2]). Theorem 2.7 then implies that, in the two-row case, every component is an iterated fiber bundle of base $(\mathbb{P}^1, \dots, \mathbb{P}^1)$ (λ_2 terms). In this manner, we retrieve a property which had been shown directly by F. Fung [7].

2.4. Outline

The remainder of the paper comprises five parts. Section 3 is devoted to the proof of Proposition 2.6. Using counting arguments, we provide an inductive estimation of the dimension of a Jordan orbit \mathcal{Z}_π (formula (6)). By induction, we derive that \mathcal{Z}_π has the same dimension as \mathcal{B}_u if and only if $\pi \in \Pi_u^1$. Proposition 2.6 is obtained from this property.

The proof of Theorem 2.7 is given in section 7. It relies on the description of the standard tableaux associated to generalized Bala-Carter components that Proposition 2.6 provides.

Sections 4–6 are devoted to the proof of Theorem 2.1. In section 4, some preliminary results are provided, which are expressed as inductive criteria of singularity for Bala-Carter components (Corollaries 4.6–4.8). In section 5, we prove the implication (\Leftarrow) of Theorem 2.1, and to this end we construct two families of singular Bala-Carter components and invoke the inductive arguments of the previous section. In section 6, we prove the implication (\Rightarrow) of Theorem 2.1. To do this, again applying the preliminary results provided in section 4, we reduce the problem to check the smoothness of a single type of Bala-Carter components. The proof that this component is indeed smooth (Proposition 6.2) is done by computing.

Notation. We set some conventional notation. We denote by \mathbf{S}_n the group of permutations of $\{1, \dots, n\}$. Let \mathbb{P}^m be the projective space of dimension m (i.e., the variety of one-dimensional vector subspaces of \mathbb{C}^{m+1}). If V is a vector space, let $\text{End}(V)$ be the space of endomorphisms of V and let $GL(V)$ be the group of invertible endomorphisms. If A is a finite set, let $|A|$ denote its cardinality. Moreover, if A is composed of integers, let $\min A$ (resp. $\max A$) be its minimal (resp. maximal) element. If X is an algebraic variety and $x \in X$, let $T_x X$ denote the tangent space of X at the point x . If $Y \subset X$ is a subset, we denote by \overline{Y} its closure in the Zariski topology. Other pieces of notation will be introduced in what follows. The reader can find an index of the notation at the end of the article.

3. Components with a dense Jordan orbit

As in section 2, we fix a nilpotent element $u \in \text{End}(V)$. We denote by $\lambda(u) = (\lambda_1 \geq \dots \geq \lambda_r)$ the sizes of the Jordan blocks of u , by $Y(u)$ the Young diagram of rows of lengths $\lambda_1, \dots, \lambda_r$, and by $\lambda^*(u) = (\lambda_1^* \geq \dots \geq \lambda_s^*)$ the lengths of the columns of $Y(u)$. Let $Z_u = \{g \in GL(V) : gu g^{-1} = u\}$ be the stabilizer of u . The purpose of this section is to prove Proposition 2.6.

3.1. Maximal dimensional Jordan orbits

We consider an element $\pi \in \Pi_u$. Thus π is a partition of $\{1, \dots, n\}$ which can be written $\pi = (I_1 \sqcup \dots \sqcup I_r)$, with $|I_j| = \lambda_j$ for all j . Alternatively, π can be seen as the map $\pi : \{1, \dots, n\} \rightarrow \{\emptyset, 1, \dots, n\}$ with $\pi(i) = \emptyset$ if $i = \min(I_j)$, and $\pi(i)$ is the predecessor of i in I_j if $i \in I_j$, $i \neq \min(I_j)$.

We consider the Z_u -orbit $\mathcal{Z}_\pi \subset \mathcal{B}_u$. In this subsection, we determine under which condition we have $\dim \mathcal{Z}_\pi = \dim \mathcal{B}_u$. More precisely, our purpose is to show:

Proposition 3.1. *We have $\dim \mathcal{Z}_\pi = \dim \mathcal{B}_u$ if and only if $\pi \in \Pi_u^1$.*

We need some preliminary computations. As a first step, we give the dimension of the group Z_u .

Lemma 3.2. *We have $\dim Z_u = \sum_{k=1}^s (\lambda_k^*)^2$.*

Proof. The group Z_u is an open subset of the vector space

$$\mathfrak{Z}_u = \{x \in \text{End}(V) : xu = ux\},$$

hence $\dim Z_u = \dim \mathfrak{Z}_u$. Let W_1, \dots, W_r be the Jordan blocks of u , that is $V = W_1 \oplus \dots \oplus W_r$, $\dim W_j = \lambda_j$ and $W_j = \langle e_j, u(e_j), \dots, u^{\lambda_j-1}(e_j) \rangle$. An element $x \in \mathfrak{Z}_u$ is characterized by the images $x(e_j)$ for $j = 1, \dots, r$, and we necessarily have $x(e_j) \in \ker u^{\lambda_j}$. Conversely, given $f_j \in \ker u^{\lambda_j}$ for every $j \in \{1, \dots, r\}$, there is a unique $x \in \mathfrak{Z}_u$ such that $x(e_j) = f_j$ for all j . It follows:

$$\dim \mathfrak{Z}_u = \sum_{j=1}^r \dim \ker u^{\lambda_j} = \sum_{k=1}^s |\{j : \lambda_j = k\}| \cdot \dim \ker u^k.$$

Note that, for $k = 1, \dots, s$, $\dim \ker u^k = \lambda_1^* + \dots + \lambda_k^*$. In addition, $|\{j : \lambda_j = k\}| = \lambda_k^* - \lambda_{k+1}^*$ (by convention $\lambda_{s+1}^* = 0$). It results that

$$\dim \mathfrak{Z}_u = \sum_{k=1}^s (\lambda_k^* - \lambda_{k+1}^*)(\lambda_1^* + \dots + \lambda_k^*) = \sum_{k=1}^s (\lambda_k^*)^2.$$

This proves the lemma. □

Now, fix a π -basis (e_1, \dots, e_n) and let $F_0 = (\langle e_1, \dots, e_i \rangle)_{i=0, \dots, n} \in \mathcal{Z}_\pi$ be the corresponding adapted flag. Then \mathcal{Z}_π is the Z_u -orbit of F_0 . Let $Z_u^{F_0} = \{g \in Z_u : g(e_i) \in \langle e_1, \dots, e_i \rangle \forall i = 1, \dots, n\}$ be the subgroup of elements fixing F_0 . Then

$$\dim \mathcal{Z}_\pi = \dim Z_u - \dim Z_u^{F_0}. \tag{4}$$

In the next step we determine $\dim Z_u^{F_0}$. Set by convention $\pi(\emptyset) = \emptyset$ and $\emptyset < a$ for all $a \in \{1, \dots, n\}$. For $j = 1, \dots, r$, let $k_j = \max(I_j)$. Define

$$A(\pi) = \{(i, j) \in \{1, \dots, n\} \times \{1, \dots, r\} : \pi^l(i) \leq \pi^l(k_j) \ \forall l \geq 0\}.$$

Then:

Lemma 3.3. *We have $\dim Z_u^{F_0} = |A(\pi)|$.*

Proof. The subgroup $Z_u^{F_0}$ is an open subset of the vector space

$$\mathfrak{Z}_u^{F_0} = \{x \in \mathfrak{Z}_u : x(e_i) \in \langle e_1, \dots, e_i \rangle \ \forall i = 1, \dots, n\},$$

hence $\dim Z_u^{F_0} = \dim \mathfrak{Z}_u^{F_0}$.

An element $x \in \mathfrak{Z}_u^{F_0}$ is determined by the images $x(e_{k_j})$ for $j = 1, \dots, r$. In addition, we must have $x(e_{\pi^l(k_j)}) \in \langle e_i : 1 \leq i \leq \pi^l(k_j) \rangle$ for all $j \in \{1, \dots, r\}$, $l \geq 0$ (setting $e_\emptyset = 0$). Recall that $x(e_{\pi^l(i)}) = xu^l(e_i) = u^l(x(e_i))$ for all $i \in \{1, \dots, n\}$, $l \geq 0$. Hence we must have $x(e_{k_j}) \in \langle e_i : \pi^l(i) \leq \pi^l(k_j) \text{ for all } l \geq 0 \rangle$. Conversely, given $f_j \in \langle e_i : \pi^l(i) \leq \pi^l(k_j) \ \forall l \geq 0 \rangle$ for $j = 1, \dots, r$, there exists a unique $x \in \mathfrak{Z}_u^{F_0}$ such that $x(e_{k_j}) = f_j$ for all j . It follows

$$\dim \mathfrak{Z}_u^{F_0} = \sum_{j=1}^r |\{i \in \{1, \dots, n\} : \pi^l(i) \leq \pi^l(k_j) \ \forall l \geq 0\}| = |A(\pi)|.$$

The lemma is proved. □

Example 3.4. We illustrate the definition of the set $A(\pi)$ by the following example. Consider the partition of $\{1, \dots, 7\}$ into the subsets $I_1 = \{1, 2, 3, 6\}$, $I_2 = \{4, 5, 7\}$. Thus, $\pi(6) = 3$, $\pi(3) = 2$, $\pi(2) = 1$, $\pi(1) = \emptyset$ and $\pi(7) = 5$, $\pi(5) = 4$, $\pi(4) = \emptyset$. Moreover, $k_1 = \max(I_1) = 6$ and $k_2 = \max(I_2) = 7$. The elements of $A(\pi)$ take the form (i, j) with $i \in \{1, \dots, 7\}$ and $j \in \{1, 2\}$. By definition, $(i, 2)$ lies in $A(\pi)$ if $i \leq 7$, $\pi(i) \leq 5$, $\pi^2(i) \leq 4$, $\pi^3(i) = \emptyset$. Thus $(i, 2) \in A(\pi)$ for $i \in \{1, 2, 3, 4, 5, 7\}$, and $(6, 2) \notin A(\pi)$ since $\pi^3(6) = 1 \neq \emptyset$. Similarly, we see that $(i, 1) \in A(\pi)$ for $i \in \{1, 2, 3, 4, 6\}$ and $(5, 1), (7, 1) \notin A(\pi)$.

Next, we establish an inductive estimate of the cardinality of the set $A(\pi)$. We suppose $n \in I_{j_0}$ and we may suppose that $\lambda_{j_0} = |I_{j_0}| > |I_j|$ for $j > j_0$. Let $I'_{j_0} = I_{j_0} \setminus \{n\}$, and let $I'_j = I_j$ for $j \neq j_0$. Let $\lambda'_{j_0} = \lambda_{j_0} - 1$ and let $\lambda'_j = \lambda_j$ for $j \neq j_0$. Consider a nilpotent element $u' \in \text{End}(\mathbb{C}^{n-1})$ of Jordan form $\lambda(u') = (\lambda'_1, \dots, \lambda'_r)$. Then $\pi' = (I'_1, \dots, I'_r)$ belongs to $\Pi_{u'}$, it defines a $Z_{u'}$ -orbit $\mathcal{Z}_{\pi'} \subset \mathcal{B}_{u'}$ and, in addition, it can be seen as a map $\pi' : \{1, \dots, n-1\} \rightarrow \{\emptyset, 1, \dots, n-1\}$ which, in fact, is the restriction of π .

Let $Y(u')$ be the Young diagram associated to $\lambda(u')$, that is, the sizes of its rows are $\lambda'_1, \dots, \lambda'_r$. Let $\lambda^*(u') = (\lambda^*_1, \dots, \lambda^*_s)$ be the sizes of its columns. Setting $l_0 = \lambda_{j_0}$, we then have $\lambda^*_{l_0} = j_0 - 1$ and $\lambda^*_l = \lambda_l$ for $l \neq l_0$. Therefore, using Lemma 3.2 and formula (1), we infer that

$$\dim Z_{u'} = \dim Z_u - 2j_0 + 1 \quad \text{and} \quad \dim \mathcal{B}_{u'} = \dim \mathcal{B}_u - j_0 + 1. \tag{5}$$

We may consider the set $A(\pi')$ relative to π' .

Lemma 3.5. *We have $|A(\pi)| \geq |A(\pi')| + j_0$, where equality holds if and only if there are no $j > j_0$, $i \in I_j$ and $l \geq 0$ such that $\pi^{l+1}(i) < \pi^{l+1}(n) < \pi^l(i) < \pi^l(n)$.*

Proof. For $j = 1, \dots, r$, we write $k'_j = \max(I'_j)$. Notice that we possibly have $I'_{j_0} = \emptyset$. In this case set by convention $k'_{j_0} = \emptyset$. The arguments remain valid in this case. We always have $k'_{j_0} = \pi(n)$.

First, we show that $A(\pi') \subset A(\pi)$. Let $(i, j) \in A(\pi')$, that is, $\pi^l(i) \leq \pi^l(k'_j)$ for all $l \geq 0$. We have to show that $\pi^l(i) \leq \pi^l(k_j)$ for all $l \geq 0$. If $j \neq j_0$, then $k'_j = k_j$ and, as π' is the restriction of π on $\{1, \dots, n-1\}$, the property holds and we infer that $(i, j) \in A(\pi)$. Assume $j = j_0$. Then $k'_j = \pi(n)$ and we have $\pi^l(k'_j) = \pi^{l+1}(n) \leq \pi^l(n)$ for all $l \geq 0$. Hence the property holds, and we get $(i, j) \in A(\pi)$. Finally, we have obtained the desired inclusion $A(\pi') \subset A(\pi)$.

We have $(n, j_0) \in A(\pi) \setminus A(\pi')$. Since $k_j < n$ for $j \neq j_0$, by definition, there cannot be another element of the form (n, j) in $A(\pi)$.

Let $j \in \{1, \dots, j_0 - 1\}$. We have $\pi^{\lambda_j-1}(k_j) > \pi^{\lambda_j-1}(k'_{j_0}) = \emptyset$, hence $(k_j, j_0) \notin A(\pi')$. Hence there is $i_j \in I_j$ minimal such that $(i_j, j_0) \notin A(\pi')$. By minimality, we have $\pi^{l+1}(i_j) \leq \pi^l(k'_{j_0}) = \pi^{l+1}(n)$ for all $l \geq 0$, therefore, $(i_j, j_0) \in A(\pi)$. Thus, for $i \in I_j$, we have $(i, j_0) \in A(\pi') \cap A(\pi)$ for $i < i_j$, $(i_j, j_0) \in A(\pi) \setminus A(\pi')$, and necessarily $(i, j_0) \notin A(\pi)$ for $i > i_j$ (because otherwise, we would have $(i_j, j_0) \in A(\pi')$).

At this stage, we have obtained that $|A(\pi)| \geq |A(\pi')| + j_0$, with equality if and only if there are no $j > j_0$ and $i \in I_j$ such that $(i, j_0) \in A(\pi) \setminus A(\pi')$. It remains to show that this condition is equivalent to the condition in the statement of the lemma. Let us show this equivalence.

(\Leftarrow) Suppose that there are $j > j_0$ and $i \in I_j$ such that

$$(i, j_0) \in A(\pi) \setminus A(\pi').$$

As $(i, j_0) \notin A(\pi')$, there is $l \geq 0$ such that $\pi^l(i) > \pi^l(k'_{j_0})$, that is $\pi^l(i) > \pi^{l+1}(n)$. As $(i, j_0) \in A(\pi)$, we have $\pi^l(n) \geq \pi^l(i)$ and $\pi^{l+1}(n) \geq \pi^{l+1}(i)$. In fact, $\pi^l(n) \neq \pi^l(i)$, because both are $\neq \emptyset$. Moreover, $\pi^l(i) \neq \emptyset$ implies $\pi^{l+1}(n) \neq \emptyset$, by the definition of π and because $|I_j| < |I_{j_0}|$, hence it also comes $\pi^{l+1}(n) \neq \pi^{l+1}(i)$. Thus, we have found $j > j_0$, $i \in I_j$ and $l \geq 0$ such that $\pi^{l+1}(i) < \pi^{l+1}(n) < \pi^l(i) < \pi^l(n)$.

(\Rightarrow) Suppose there are $j > j_0$, $i \in I_j$, $l \geq 0$ with $\pi^{l+1}(i) < \pi^{l+1}(n) < \pi^l(i) < \pi^l(n)$. In particular $(i, j_0) \notin A(\pi')$. Then, there is $i_j \in I_j$ minimal such that $(i_j, j_0) \notin A(\pi')$. As above, the minimality implies $(i_j, j_0) \in A(\pi)$. Thus, we have found $j > j_0$ and $i_j \in I_j$ such that $(i_j, j_0) \in A(\pi) \setminus A(\pi')$. \square

Now we are ready to prove Proposition 3.1.

Proof of Proposition 3.1. We reason by induction on $n \geq 1$ with immediate initialization for $n = 1$. Suppose that the property holds for $n - 1 \geq 1$ and prove it for n . Combining the relations (4) and (5) and Lemmas 3.3, 3.5, we obtain

$$\dim \mathcal{B}_u - \dim \mathcal{Z}_\pi \geq \dim \mathcal{B}_{u'} - \dim \mathcal{Z}_{\pi'} \quad (6)$$

with equality if and only if there are no $j > j_0$, $i \in I_j$ and $l \geq 0$ such that $\pi^{l+1}(i) < \pi^{l+1}(n) < \pi^l(i) < \pi^l(n)$.

Let us interpret the last relation whenever it holds for some $j > j_0$, $i \in I_j$, $l \geq 0$. If $\pi^{l+1}(i) > \emptyset$, then it implies that π has a crossing. If $\pi^{l+1}(i) = \emptyset$, then

$\pi^l(i) = \min(I_j)$ and we have $I_j < I_{j_0}$ though $|I_j| < |I_{j_0}|$. In both cases, the relation implies $\pi \notin \Pi_u^1$.

Suppose $\pi \in \Pi_u^1$. Then, the equality holds in relation (6). Let us show that $\pi' \in \Pi_{u'}^1$. It is immediate that π' has no crossing. Suppose $I'_j < I'_l$ with $j, l \in \{1, \dots, r\}$ and let us show $|I'_j| \geq |I'_l|$. If $j_0 \notin \{j, l\}$, then this follows from the fact that $I'_j = I_j$, $I'_l = I_l$ and $\pi \in \Pi_u^1$. We cannot have $I'_{j_0} < I'_l$, otherwise there is $i \in I'_l$ with $\pi(i) < \pi(n) < i < n$ and π has a crossing. If $I'_j < I'_{j_0}$, then $I_j < I_{j_0}$ and we derive $|I'_j| = |I_j| \geq |I_{j_0}| > |I'_{j_0}|$. Finally we have proved $\pi' \in \Pi_{u'}^1$. By induction hypothesis, we have $\dim \mathcal{Z}_{\pi'} = \dim \mathcal{B}_{u'}$. Therefore, we get $\dim \mathcal{Z}_{\pi} = \dim \mathcal{B}_u$.

Conversely, suppose $\dim \mathcal{Z}_{\pi} = \dim \mathcal{B}_u$. Then, necessarily, the equality holds in relation (6), and we have $\dim \mathcal{Z}_{\pi'} = \dim \mathcal{B}_{u'}$. The latter fact implies $\pi' \in \Pi_{u'}^1$ by induction hypothesis. Because π' has no crossing, a crossing of π , if it exists, is of the form $\emptyset < \pi(i) < \pi(n) < i < n$. It follows from the equality condition in (6) that $i \in I_j$ with $j < j_0$. There is $l \geq 1$ with $\pi^{l+1}(n) < \pi(i) < \pi^l(n) < i$. In the case $\pi^{l+1}(n) \neq \emptyset$, we get that π' has a crossing. In the case $\pi^{l+1}(n) = \emptyset$, we get $I'_{j_0} < I'_j$, though we have $|I'_{j_0}| < |I'_j|$. In both cases, this contradicts the fact that $\pi' \in \Pi_{u'}^1$. Thus, we obtain that π has no crossing. Next, we show: $I_j < I_l \Rightarrow |I_j| \geq |I_l|$. In the case $j_0 \notin \{j, l\}$, this follows from the fact that $I'_j = I_j$, $I'_l = I_l$ and $\pi' \in \Pi_{u'}^1$. Note that we have $I_{j_0} \not< I_j$ for all j . Thus, it remains to suppose $I_j < I_{j_0}$ and then to show that $|I_j| \geq |I_{j_0}|$, that is $j < j_0$. If $I'_j \not< I'_{j_0}$, then there is $i \in I_j$ such that $\pi(i) < \pi(n) < i < n$. If $I'_j < I'_{j_0}$, then the fact that $\pi' \in \Pi_{u'}^1$ implies $|I'_j| \geq |I'_{j_0}|$, hence there are $i \in I_j$ and $l \geq 1$ such that $\emptyset = \pi^{l+1}(i) < \pi^{l+1}(n) < \pi^l(i) = \min(I_j) < \pi^l(n)$. In both cases, using the equality condition in (6), we infer that $j < j_0$. Finally, we have shown $\pi \in \Pi_u^1$. The proof of the proposition is then complete. \square

3.2. Proof of Proposition 2.6

The first part of Proposition 2.6 is provided by Proposition 3.1. It remains to show the second part, that is: the mapping $\pi \mapsto \mathcal{K}^{T_\pi}$ provides a one-to-one correspondence between elements $\pi \in \Pi_u^1$ and components of \mathcal{B}_u which contain a dense Jordan orbit.

First, suppose $\pi \in \Pi_u^1$. From section 2.3.1, we know that $\mathcal{Z}_{\pi} \subset \mathcal{K}^{T_\pi}$. Then, by Proposition 3.1, we have that \mathcal{Z}_{π} is indeed a dense Jordan orbit of \mathcal{K}^{T_π} . Hence, the mapping is well defined.

Conversely, suppose that the component \mathcal{K}^T contains a dense Jordan orbit. Recall that $\dim \mathcal{K}^T = \dim \mathcal{B}_u$, hence this orbit has the same dimension as \mathcal{B}_u . By Proposition 3.1, it is of the form \mathcal{Z}_{π} with $\pi \in \Pi_u^1$. As \mathcal{Z}_{π} is dense in \mathcal{K}^{T_π} , we get $\mathcal{K}^T = \mathcal{K}^{T_\pi}$. Hence, the mapping is surjective.

Let $\pi, \pi' \in \Pi_u^1$ and suppose that $\mathcal{K}^{T_\pi} = \mathcal{K}^{T_{\pi'}}$. Thus, $\overline{\mathcal{Z}_{\pi}} = \overline{\mathcal{Z}_{\pi'}}$, which implies $\mathcal{Z}_{\pi} = \mathcal{Z}_{\pi'}$ since two different orbits cannot have the same closure. Let $\pi(j) = i \in \{\emptyset, 1, \dots, j-1\}$, and let $F = (V_0, \dots, V_n) \in \mathcal{Z}_{\pi}$. Then i is minimal such that $u(V_j) \subset V_i + u(V_{j-1})$. Therefore, $\mathcal{Z}_{\pi} = \mathcal{Z}_{\pi'}$ implies $\pi = \pi'$. Hence, the mapping is injective. The proof of Proposition 2.6 is then complete.

4. Inductive properties

Let $\pi = (I_1, \dots, I_r) \in \Pi_u$ and let \mathcal{Z}_π be the corresponding Jordan orbit. Following section 2.3.1, the partition π can be seen as a map $\pi : \{1, \dots, n\} \rightarrow \{\emptyset, 1, \dots, n\}$ such that for $i \in I_j$ we set $\pi(i) = \emptyset$ if $i = \min(I_j)$ and $\pi(i)$ to be the predecessor of i in I_j otherwise. Alternatively, π is represented by a graph with n vertices labeled by $1, \dots, n$ displayed along a horizontal line, and with an arc between i, j if $i = \pi(j)$.

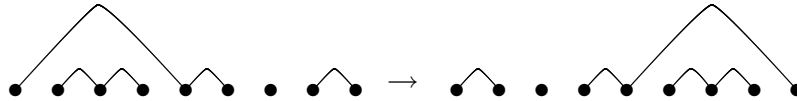
We suppose in addition $\pi \in \Pi_u^1$. That is: π has no crossing, and if $I_j < I_k$ (i.e., $\min(I_k) < \min(I_j) \leq \max(I_j) < \max(I_k)$), then $|I_j| \geq |I_k|$. Due to Proposition 2.6, we know that the orbit \mathcal{Z}_π is dense in the component \mathcal{K}^{T_π} . In this section, our purpose is to establish some inductive singularity criteria for \mathcal{K}^{T_π} . We consider π' obtained from π either by removing an extremal vertex (1 or n) together with the corresponding arc, or by removing a whole connected component (from the point of view of the graph representation). In both cases we show that the singularity of the component $\mathcal{K}^{T_{\pi'}}$ associated to π' implies the singularity of \mathcal{K}^{T_π} .

We start by pointing out a symmetry property: if the graph of $\tilde{\pi}$ is the mirror of the graph of π , then both components $\mathcal{K}^{T_\pi}, \mathcal{K}^{T_{\tilde{\pi}}}$ are isomorphic.

4.1. Symmetry

For $j = 1, \dots, r$, we write $\tilde{I}_j = \{n - i + 1 : i \in I_j\}$. Then $|\tilde{I}_j| = |I_j|$ for all j , and $\tilde{\pi} := (\tilde{I}_1, \dots, \tilde{I}_r) \in \Pi_u$ is another partition of $\{1, \dots, n\}$. The graph representation of $\tilde{\pi}$ is symmetric to the graph representation of π (that is, it is its mirror reflection). Example:

$$\pi = \{1, 5, 6\} \sqcup \{2, 3, 4\} \sqcup \{8, 9\} \sqcup \{7\} \rightarrow \tilde{\pi} = \{4, 5, 9\} \sqcup \{6, 7, 8\} \sqcup \{1, 2\} \sqcup \{3\}$$



As we suppose $\pi \in \Pi_u^1$, it is clear that $\tilde{\pi} \in \Pi_u^1$. Thus, the partition $\tilde{\pi}$ defines a component $\mathcal{K}^{T_{\tilde{\pi}}} \subset \mathcal{B}_u$ which is the closure of the Jordan orbit $\mathcal{Z}_{\tilde{\pi}}$. We show:

Proposition 4.1. *The components \mathcal{K}^{T_π} and $\mathcal{K}^{T_{\tilde{\pi}}}$ are isomorphic.*

Proof. Let \tilde{V} be the vector space of linear forms $\phi : V \rightarrow \mathbb{C}$, and let $\tilde{u} : \tilde{V} \rightarrow \tilde{V}$, $\phi \mapsto \phi \circ u$ be the dual map of u . Then \tilde{u} is nilpotent and has the same Jordan form as u . Let $\mathcal{B}_{\tilde{u}}$ be the variety of \tilde{u} -stable flags of \tilde{V} . If $W \subset V$ is a subspace, we set $W^\perp = \{\phi \in \tilde{V} : \phi(w) = 0 \forall w \in W\}$. Then, W is u -stable if and only if W^\perp is \tilde{u} -stable, and the map $\Phi : \mathcal{B}_u \rightarrow \mathcal{B}_{\tilde{u}}$, $(V_0, \dots, V_n) \mapsto (V_n^\perp, \dots, V_0^\perp)$ is an isomorphism of algebraic varieties. In particular, $\Phi(\mathcal{K}^{T_\pi}) \subset \mathcal{B}_{\tilde{u}}$ is an irreducible component.

Let $Z_{\tilde{u}} = \{g \in GL(\tilde{V}) : g\tilde{u}g^{-1} = \tilde{u}\}$ be the stabilizer of \tilde{u} . We denote by $\tilde{\mathcal{Z}}_{\tilde{\pi}} \subset \mathcal{B}_{\tilde{u}}$ the Jordan $Z_{\tilde{u}}$ -orbit associated to $\tilde{\pi}$, and by $\tilde{\mathcal{K}}^{T_{\tilde{\pi}}} \subset \mathcal{B}_{\tilde{u}}$ the irreducible component associated to the tableau $T_{\tilde{\pi}}$, which is actually the closure of $\tilde{\mathcal{Z}}_{\tilde{\pi}}$. It

is easy to see that $\Phi(\mathcal{Z}_\pi) = \tilde{\mathcal{Z}}_{\tilde{\pi}}$. Thus, $\Phi(\mathcal{K}^{T_\pi}) = \tilde{\mathcal{K}}_{\tilde{\pi}}$ so that Φ restricts to an isomorphism from \mathcal{K}^{T_π} to $\tilde{\mathcal{K}}^{T_{\tilde{\pi}}}$.

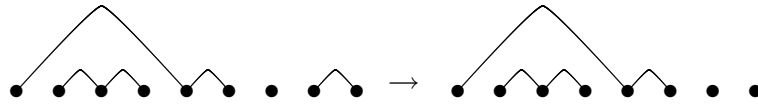
We distinguish between $\tilde{\mathcal{K}}^{T_{\tilde{\pi}}} \subset \mathcal{B}_{\tilde{u}}$, the component associated to the standard tableau $T_{\tilde{\pi}}$ in $\mathcal{B}_{\tilde{u}}$, and $\mathcal{K}^{T_{\tilde{\pi}}}$, the component associated to the same tableau, but in \mathcal{B}_u . A component only depends up to isomorphism on the standard tableau it is associated to, hence these two components are actually isomorphic. We deduce that \mathcal{K}^{T_π} and $\mathcal{K}^{T_{\tilde{\pi}}}$ are isomorphic. \square

Remark 4.2. We can see that the tableau $T_{\tilde{\pi}}$ is in fact the image of the tableau T_π by the classical Schützenberger involution.

4.2. Removing an extremal point

There is $j_0 \in \{1, \dots, r\}$ such that $n \in I_{j_0}$. We may assume that $\lambda_{j_0} = |I_{j_0}| > |I_j|$ for all $j > j_0$. As in section 3.1, let $\lambda'_{j_0} = \lambda_{j_0} - 1$ and let $\lambda'_j = \lambda_j$ for $j \neq j_0$, and consider a nilpotent element $u' \in \text{End}(\mathbb{C}^{n-1})$ of Jordan form $\lambda(u') = (\lambda'_1, \dots, \lambda'_r)$. Let $I'_{j_0} = I_{j_0} \setminus \{n\}$, and let $I'_j = I_j$ for $j \neq j_0$, then $\pi' := (I'_1, \dots, I'_r)$ belongs to $\Pi_{u'}$ and it defines a $Z_{u'}$ -orbit $\mathcal{Z}_{\pi'} \subset \mathcal{B}_{u'}$. At the level of the graph representation, the graph of π' is obtained from the graph of π by removing the last vertex together with the possible upcoming arc, for instance:

$$\pi = \{1, 5, 6\} \sqcup \{2, 3, 4\} \sqcup \{8, 9\} \sqcup \{7\} \rightarrow \pi' = \{1, 5, 6\} \sqcup \{2, 3, 4\} \sqcup \{8\} \sqcup \{7\}$$



It is clear that $\pi' \in \Pi_{u'}^1$, hence the orbit $\mathcal{Z}_{\pi'}$ is dense in the component $\mathcal{K}^{T_{\pi'}} \subset \mathcal{B}_{u'}$. We have:

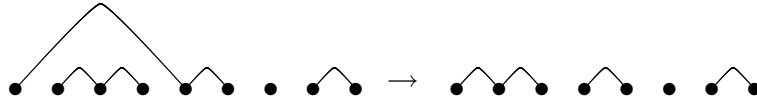
Proposition 4.3. *If the component $\mathcal{K}^{T_{\pi'}}$ is singular, then \mathcal{K}^{T_π} is singular. Suppose moreover that $n \in I_{j_0}$ with $|I_{j_0}| = \max\{|I_j| : 1 \leq j \leq r\}$, then, $\mathcal{K}^{T_{\pi'}}$ is singular if and only if \mathcal{K}^{T_π} is singular.*

Proof. Notice that it follows from the definition of the tableaux $T_\pi, T_{\pi'}$ in section 2.3.1 that $T_{\pi'}$ is obtained from T_π simply by deleting the box n . Moreover, under the assumption $|I_{j_0}| = \max\{|I_j| : 1 \leq j \leq r\}$, n lies in the last column of T_π . Then the result follows from [4, Theorem 2.1], which says more generally that, if T is a standard tableau and T' is the subtableau obtained by deleting n , then the singularity of the component $\mathcal{K}^{T'}$ implies the singularity of \mathcal{K}^T , and moreover, if n lies in the last column of T , then the singularity of $\mathcal{K}^{T'}$ is equivalent to the singularity of \mathcal{K}^T . \square

We have just studied the situation where we remove from π the last number n , the situation where we remove the first number 1 is quite similar. Let $j_1 \in \{1, \dots, r\}$ be such that $1 \in I_{j_1}$ and we may suppose $\lambda_{j_1} = |I_{j_1}| > |I_j|$ for $j > j_1$. Let $\lambda''_{j_1} = \lambda_{j_1} - 1$ and $\lambda''_j = \lambda_j$ for $j \neq j_1$. Fix $u'' \in \text{End}(\mathbb{C}^{n-1})$ nilpotent of Jordan form $\lambda(u'') = (\lambda''_1, \dots, \lambda''_r)$. Let $I''_{j_1} = \{i - 1 : i \in I_{j_1} \setminus \{1\}\}$ and $I''_j = \{i - 1 : i \in I_j\}$ for $j \neq j_1$, then $\pi'' = (I''_1, \dots, I''_r)$ belongs to $\Pi_{u''}$ and it

defines a $Z_{u''}$ -orbit $\mathcal{Z}_{\pi''} \subset \mathcal{B}_{u''}$. At the level of graphs, the graph of π'' is obtained from the graph of π by removing the first vertex and the possible upcoming arc.

$$\pi = \{1, 5, 6\} \sqcup \{2, 3, 4\} \sqcup \{8, 9\} \sqcup \{7\} \rightarrow \pi'' = \{1, 2, 3\} \sqcup \{4, 5\} \sqcup \{7, 8\} \sqcup \{6\}$$



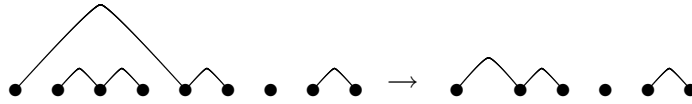
Clearly $\pi'' \in \Pi_{u''}^1$, hence $\mathcal{Z}_{\pi''}$ is dense in the component $\mathcal{K}^{T_{\pi''}} \subset \mathcal{B}_{u''}$. Combining Propositions 4.1 and 4.3, we derive:

Proposition 4.4. *If the component $\mathcal{K}^{T_{\pi''}}$ is singular, then $\mathcal{K}^{T_{\pi}}$ is singular. Suppose moreover $1 \in I_{j_1}$ with $|I_{j_1}| = \max\{|I_j| : 1 \leq j \leq r\}$, then, $\mathcal{K}^{T_{\pi''}}$ is singular if and only if $\mathcal{K}^{T_{\pi}}$ is singular.*

4.3. Removing a connected component

We consider the situation of a partition obtained from π by removing a term in the partition. We suppose $\pi = (I_1, \dots, I_r) \in \Pi_u$ with $|I_j| = \lambda_j$ for all j . Fix $\hat{j} \in \{1, \dots, r\}$ and fix a nilpotent endomorphism $\hat{u} \in \text{End}(\mathbb{C}^{n-\lambda_{\hat{j}}})$ of Jordan form $\lambda(\hat{u}) = (\lambda_1, \dots, \lambda_{\hat{j}-1}, \lambda_{\hat{j}+1}, \dots, \lambda_r)$. Write $\{1, \dots, n\} \setminus I_{\hat{j}} = \{k_1 < \dots < k_{n-\lambda_{\hat{j}}}\}$. For $j \neq \hat{j}$, let $\hat{I}_j = \{i : k_i \in I_j\}$. Then $\hat{\pi} = (\hat{I}_1, \dots, \hat{I}_{\hat{j}-1}, \hat{I}_{\hat{j}+1}, \dots, \hat{I}_r)$ is an element of $\Pi_{\hat{u}}$. At the level of graphs, the graph of $\hat{\pi}$ is obtained from the graph of π by removing a connected component, the one which corresponds to the subset $I_{\hat{j}}$. For example:

$$\pi = \{1, 5, 6\} \sqcup \underbrace{\{2, 3, 4\}}_{\text{to remove}} \sqcup \{8, 9\} \sqcup \{7\} \rightarrow \hat{\pi} = \{1, 2, 3\} \sqcup \{5, 6\} \sqcup \{4\}$$



As we suppose $\pi \in \Pi_u^1$, it is clear that $\hat{\pi} \in \Pi_{\hat{u}}^1$. Thus, the partition $\hat{\pi}$ defines a component $\mathcal{K}^{T_{\hat{\pi}}} \subset \mathcal{B}_{\hat{u}}$ which is the closure of the Jordan orbit $\mathcal{Z}_{\hat{\pi}}$. We show:

Proposition 4.5. *If the component $\mathcal{K}^{T_{\hat{\pi}}}$ is singular, then $\mathcal{K}^{T_{\pi}}$ is singular.*

Proof. Suppose that $\mathcal{K}^{T_{\pi}}$ is smooth and let us show that $\mathcal{K}^{T_{\hat{\pi}}}$ is smooth. Recall that π can be seen as a map $\pi : \{1, \dots, n\} \rightarrow \{\emptyset, 1, \dots, n\}$. We fix a π -basis (e_1, \dots, e_n) , that is $u(e_i) = e_{\pi(i)}$ for all i , with by convention $e_{\emptyset} = 0$. Then $W_j = \langle e_i : i \in I_j \rangle$ (for $j = 1, \dots, r$) are the Jordan blocks of u . For $t \in \mathbb{C}^*$, we consider $h_t : V \rightarrow V$ defined by $(h_t)|_{W_j} = \text{id}_{W_j}$ for $j \neq \hat{j}$ and $(h_t)|_{W_{\hat{j}}} = t \cdot \text{id}_{W_{\hat{j}}}$. Then, $H = \{h_t : t \in \mathbb{C}^*\}$ is a subtorus of rank one of the group Z_u . Therefore, it acts on \mathcal{B}_u and stabilizes its irreducible components, in particular it stabilizes $\mathcal{K}^{T_{\pi}}$. As we suppose that $\mathcal{K}^{T_{\pi}}$ is smooth, we know that the fixed point set $(\mathcal{K}^{T_{\pi}})^H = \{F \in \mathcal{K}^{T_{\pi}} : h_t F = F \ \forall t \in \mathbb{C}^*\}$ is a smooth subvariety (cf., [1]). To prove the proposition, it is then sufficient to show that $\mathcal{K}^{T_{\hat{\pi}}}$ is isomorphic to a connected component of $(\mathcal{K}^{T_{\pi}})^H$. This is what we do in the following.

Let $\hat{V} = \bigoplus_{j \neq \hat{j}} W_j$ and write \hat{u} the restriction of u to \hat{V} (this accords with the previous notation \hat{u}). Let $\mathcal{B}_{\hat{u}}$ be the variety of \hat{u} -stable complete flags of \hat{V} . We construct a map $\Phi : \mathcal{B}_{\hat{u}} \rightarrow \mathcal{B}_u$ as follows. For $i = 0, \dots, n$, set $a_i = |\{1, \dots, i\} \cap I_{\hat{j}}|$. Write $I_{\hat{j}} = \{\hat{k}_1 < \dots < \hat{k}_{\lambda_{\hat{j}}}\}$, and for $a \in \{0, \dots, \lambda_{\hat{j}}\}$ set $W_{\hat{j},a} = \langle e_{\hat{k}_1}, \dots, e_{\hat{k}_a} \rangle$. Hence, $(W_{\hat{j},0} \subset W_{\hat{j},1} \subset \dots \subset W_{\hat{j},\lambda_{\hat{j}}})$ is the unique u -stable complete flag of $W_{\hat{j}}$. Define Φ by

$$\Phi : \mathcal{B}_{\hat{u}} \rightarrow \mathcal{B}_u, (\hat{V}_0, \dots, \hat{V}_{n-\lambda_{\hat{j}}}) \mapsto (\hat{V}_{i-a_i} \oplus W_{\hat{j},a_i} : i = 0, \dots, n).$$

The map Φ is well defined and algebraic, and its image lies in the fixed point set $(\mathcal{B}_u)^H$ for the action of H on \mathcal{B}_u . Notice that a flag $F = (V_0, \dots, V_n) \in (\mathcal{B}_u)^H$ satisfies $V_i = (V_i \cap \hat{V}) \oplus (V_i \cap W_{\hat{j}})$ for all i . Then, the image of Φ can be characterized as follows:

$$\begin{aligned} \Phi(\mathcal{B}_{\hat{u}}) &= \{F = (V_0, \dots, V_n) \in (\mathcal{B}_u)^H : \dim V_i \cap W_{\hat{j}} = a_i \ \forall i = 0, \dots, n\} \\ &= \{F \in (\mathcal{B}_u)^H : \dim V_i \cap W_{\hat{j}} \leq a_i \text{ and } \dim V_i \cap \hat{V} \leq i - a_i \ \forall i = 0, \dots, n\} \\ &= \{F \in (\mathcal{B}_u)^H : \dim V_i \cap W_{\hat{j}} \geq a_i \text{ and } \dim V_i \cap \hat{V} \geq i - a_i \ \forall i = 0, \dots, n\}. \end{aligned}$$

Hence $\Phi(\mathcal{B}_{\hat{u}})$ is open and closed in $(\mathcal{B}_u)^H$, and the map Φ is an open and closed immersion.

The intersection $\Phi(\mathcal{B}_{\hat{u}}) \cap \mathcal{K}^{T_\pi}$ is also open and closed in $(\mathcal{K}^{T_\pi})^H$, hence it is a union $C_1 \sqcup \dots \sqcup C_m$ of connected components of $(\mathcal{K}^{T_\pi})^H$. It is easy to see that, by construction, we have the inclusion $\Phi(\mathcal{Z}_{\hat{\pi}}) \subset \mathcal{Z}_\pi$, which implies $\Phi(\mathcal{K}^{T_{\hat{\pi}}}) \subset (\mathcal{K}^{T_\pi})^H$. As $\Phi(\mathcal{K}^{T_{\hat{\pi}}})$ is irreducible, we have $\Phi(\mathcal{K}^{T_{\hat{\pi}}}) \subset C_l$ for some $l \in \{1, \dots, m\}$. Since $\Phi(\mathcal{K}^{T_{\hat{\pi}}})$ has the same dimension as $\Phi(\mathcal{B}_{\hat{u}})$ and since C_l is irreducible (because $(\mathcal{K}^{T_\pi})^H$ is smooth), we actually have $\Phi(\mathcal{K}^{T_{\hat{\pi}}}) = C_l$. Therefore, Φ restricts to an isomorphism between the component $\mathcal{K}^{T_{\hat{\pi}}}$ and a connected component of $(\mathcal{K}^{T_\pi})^H$. The proof of the proposition is then complete. \square

4.4. At the level of Bala-Carter components

In the previous subsections, we have provided inductive criteria for the singularity of an irreducible component of the form $\mathcal{K}^{T_\pi} \subset \mathcal{B}_u$ with $\pi \in \Pi_u^1$, that is, more general than a component of Bala-Carter type. In this subsection, we translate the previous criteria for a Bala-Carter component.

We consider an element $\pi \in \Lambda_u$, that is, a sequence $\pi = (\pi_1, \dots, \pi_r)$ such that the numbers π_1, \dots, π_r coincide up to ordering with the Jordan block sizes $\lambda_1, \dots, \lambda_r$. This sequence defines the Bala-Carter component $\mathcal{K}_\pi^{\text{BC}}$. According to subsection 2.3.1, π can be seen as an element of the set Π_u^0 , that we also denote by π , and the component $\mathcal{K}_\pi^{\text{BC}}$ then coincides with the component \mathcal{K}^{T_π} . Through this identification, we derive the following corollaries from Propositions 4.1–4.5.

Nota: a sequence of nonnegative integers defines a Bala-Carter component in the appropriate Springer fiber. In the following statements, we do not precise the underlying Springer fibers in which the components are imbedded.

Corollary 4.6. *Let $\tilde{\pi} = (\pi_r, \pi_{r-1}, \dots, \pi_1)$. Then the Bala-Carter components $\mathcal{K}_\pi^{\text{BC}}$ and $\mathcal{K}_{\tilde{\pi}}^{\text{BC}}$ are isomorphic.*

Corollary 4.7. *Let $\pi' = (\pi_1, \dots, \pi_{r-1}, \pi_r - 1)$. If the component $\mathcal{K}_{\pi'}^{\text{BC}}$ is singular, then $\mathcal{K}_{\pi}^{\text{BC}}$ is singular. Suppose moreover $\pi_r = \max\{\pi_j : j = 1, \dots, r\}$, then $\mathcal{K}_{\pi'}^{\text{BC}}$ is singular if and only if $\mathcal{K}_{\pi}^{\text{BC}}$ is singular.*

Let $\pi'' = (\pi_1 - 1, \pi_2, \dots, \pi_r)$. Likewise, if $\mathcal{K}_{\pi''}^{\text{BC}}$ is singular, then $\mathcal{K}_{\pi}^{\text{BC}}$ is singular. Moreover, if $\pi_1 = \max\{\pi_j : j = 1, \dots, r\}$, then $\mathcal{K}_{\pi''}^{\text{BC}}$ is singular if and only if $\mathcal{K}_{\pi}^{\text{BC}}$ is singular.

Corollary 4.8. *Let $\hat{j} \in \{1, \dots, r\}$ and $\hat{\pi} = (\pi_1, \dots, \pi_{\hat{j}-1}, \pi_{\hat{j}+1}, \dots, \pi_r)$. If the component $\mathcal{K}_{\hat{\pi}}^{\text{BC}}$ is singular, then $\mathcal{K}_{\pi}^{\text{BC}}$ is singular.*

5. Singular Bala-Carter components

Let $\pi \in \Lambda_u$, that is, a sequence $\pi = (\pi_1, \dots, \pi_r)$ which coincides up to ordering with the sequence $\lambda(u) = (\lambda_1 \geq \dots \geq \lambda_r)$ of the Jordan block sizes of u . We associate to π the Bala-Carter component $\mathcal{K}_{\pi}^{\text{BC}} \subset \mathcal{B}_u$. The purpose of this section is to show the implication (\Leftarrow) of Theorem 2.1, namely:

Proposition 5.1. *If $\pi \geq (1, 2, 2, 1)$ or $\pi \geq (2, 3, 2)$, then $\mathcal{K}_{\pi}^{\text{BC}}$ is singular.*

First, we construct two families of singular Bala-Carter components. Then, we derive the proposition by using the results in the previous section.

5.1. Singular components of the form $\mathcal{K}_{\pi}^{\text{BC}}$ with $\pi = (1, p, q, 1)$

In [17], J.A. Vargas gives an example of a singular component, which is $\mathcal{K}_{\pi}^{\text{BC}}$ for $\pi = (1, 2, 2, 1)$. Here, we provide a family of singular components which generalizes this example:

Lemma 5.2. *Let $p \geq 2$, $q \geq 2$, and $\pi = (1, p, q, 1)$. Then the component $\mathcal{K}_{\pi}^{\text{BC}}$ is singular.*

Proof. Due to Corollary 4.6, we may assume $p \geq q$. Moreover we may assume $p > 2$, otherwise $\pi = (1, 2, 2, 1)$ and we already know that the component $\mathcal{K}_{\pi}^{\text{BC}}$ is singular by Vargas’s result. The component lies in \mathcal{B}_u where $u \in \text{End}(\mathbb{C}^{p+q+2})$ has Jordan form $\lambda(u) = (p, q, 1, 1)$. Due to formula (1), we have $\dim \mathcal{K}_{\pi}^{\text{BC}} = \dim \mathcal{B}_u = q + 5$.

We fix a Jordan basis (e_1, \dots, e_{p+q+2}) such that u acts on the basis according to the following picture

$$\begin{array}{cccccccccccc}
 0 & \leftarrow & e_1 & \leftarrow & e_3 & \leftarrow & e_4 & \cdots & \leftarrow & e_{p-1} & \leftarrow & e_p & \leftarrow & e_{p+q+1} \\
 0 & \leftarrow & e_2 & \leftarrow & e_{p+3} & \leftarrow & \cdots & \leftarrow & e_{p+q} & \leftarrow & e_{p+q+2} & & & \\
 0 & \leftarrow & e_{p+1} & & & & & & & & & & & \\
 0 & \leftarrow & e_{p+2} & & & & & & & & & & &
 \end{array}$$

Note that, if $q = 2$, then the sequence $(e_{p+3}, \dots, e_{p+q})$ disappears from the second

line, and the picture becomes

$$\begin{array}{cccccccccccc}
0 & \leftarrow & e_1 & \leftarrow & e_3 & \leftarrow & e_4 & \cdots & \leftarrow & e_{p-1} & \leftarrow & e_p & \leftarrow & e_{p+3} \\
0 & \leftarrow & e_2 & \leftarrow & e_{p+4} & & & & & & & & & & \\
0 & \leftarrow & e_{p+1} & & & & & & & & & & & & \\
0 & \leftarrow & e_{p+2} & & & & & & & & & & & &
\end{array}$$

We consider the flag $F_0 = (\langle e_1, \dots, e_i \rangle)_{i=0, \dots, p+q+2}$ adapted to the basis, and the purpose is to show that F_0 is a singular point of the component $\mathcal{K}_\pi^{\text{BC}}$. To do this, we first check that F_0 lies in $\mathcal{K}_\pi^{\text{BC}}$ and then we construct $q+6$ linearly independent vectors of the tangent space of $\mathcal{K}_\pi^{\text{BC}}$ at the point F_0 .

Consider $B = \{g \in GL(V) : ge_i \in \langle e_i, \dots, e_{p+q+2} \rangle\}$ the subgroup of lower triangular matrices in the basis. Then the orbit $\Omega = BF_0$ is an affine open neighborhood of F_0 in the variety of complete flags of \mathbb{C}^{p+q+2} . Consider $U \subset B$ the subgroup of unipotent matrices and $\mathfrak{n} = \{g - I : g \in U\}$ its Lie algebra. The map $\varphi : \mathfrak{n} \rightarrow \Omega$, $x \mapsto (I + x)F_0$ is an isomorphism of affine varieties and we consider the subvariety $\varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}}) \subset \mathfrak{n}$. For $1 \leq k < l \leq p + q + 2$, let $E_{l,k} \in \mathfrak{n}$ be the canonical basic element $E_{l,k}(e_k) = e_l$ and $E_{l,k}(e_i) = 0$ for $i \neq k$. The elements $E_{l,k}$, for $1 \leq k < l \leq p + q + 2$, form a basis of \mathfrak{n} .

For $\underline{t} = (t_1, \dots, t_7) \in \mathbb{C}^7$, we consider the element $x_{\underline{t}} \in \mathfrak{n}$ defined as follows:

$$\begin{aligned}
x_{\underline{t}}(e_1) &= t_1(e_2 + t_2(e_{p+1} + t_3e_{p+2})), \\
x_{\underline{t}}(e_2) &= t_2(e_{p+1} + t_3e_{p+2}), \quad x_{\underline{t}}(e_i) = 0 \text{ for } i = 3, \dots, p-1, \\
x_{\underline{t}}(e_p) &= t_2t_4(e_{p+1} + t_3e_{p+2}), \\
x_{\underline{t}}(e_{p+1}) &= \begin{cases} t_3(e_{p+2} + t_5(e_{p+2} + t_6(e_{p+q+1} - t_4e_{p+3}))) & \text{if } q > 2, \\ t_3(e_{p+2} + t_5(e_{p+2} + t_6(e_{p+3} - t_4e_{p+4}))) & \text{if } q = 2, \end{cases} \\
x_{\underline{t}}(e_{p+2}) &= \begin{cases} t_6(e_{p+q+1} - t_4e_{p+3}) & \text{if } q > 2, \\ t_6(e_{p+3} - t_4e_{p+4}) & \text{if } q = 2, \end{cases} \\
x_{\underline{t}}(e_i) &= 0 \text{ for } i = p+3, \dots, p+q, \\
x_{\underline{t}}(e_{p+q+1}) &= t_7e_{p+q+2} \quad \text{and} \quad x_{\underline{t}}(e_{p+q+2}) = 0.
\end{aligned}$$

The map $\mathbb{C}^7 \rightarrow \mathfrak{n}$, $\underline{t} \mapsto x_{\underline{t}}$ is well defined and algebraic. Let T_π be the standard tableau associated to π in the sense of section 2.1.4, then the component $\mathcal{K}_\pi^{\text{BC}}$ is the closure of the subset $\mathcal{B}_u^{T_\pi}$ for the Zariski topology. A straightforward computation shows that $x_{\underline{t}} \in \varphi^{-1}(\Omega \cap \mathcal{B}_u^{T_\pi})$ whenever $t_3t_5t_6t_7 \neq 0$ (if $q > 2$) or $t_3t_5t_6(t_4+t_7) \neq 0$ (if $q = 2$). Hence $\varphi(x_{\underline{t}}) \in \Omega \cap \mathcal{K}_\pi^{\text{BC}}$ for all $\underline{t} \in \mathbb{C}^7$. In particular, $F_0 = \varphi(0) \in \mathcal{K}_\pi^{\text{BC}}$.

For $i \in \{1, \dots, 7\}$, let $\underline{t}^{(i)} = (t_1, \dots, t_7)$ with $t_i = t$ and $t_j = 0$ for $j \neq i$. Then the curve $\{x_{\underline{t}^{(i)}} : t \in \mathbb{C}\}$ lies in $\varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}})$ and its tangent vector at $t = 0$ is an element of $T_0 \varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}})$, the tangent space of $\varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}})$ at 0. Making i run over $\{1, \dots, 7\}$, we get:

$$E_{2,1}, E_{p+1,2}, E_{p+2,p+1}, E_{p+q+1,p+2}, E_{p+q+2,p+q+1} \in T_0 \varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}}).$$

The curves $\{x_{\underline{t}} : \underline{t} = (0, t, 0, 1, 0, 0, 0), t \in \mathbb{C}\}$, $\{x_{\underline{t}} : \underline{t} = (0, 0, 0, 1, 0, t, 0), t \in \mathbb{C}\}$ and $\{x_{\underline{t}} : \underline{t} = (0, 0, 1, 0, -1, t, 0), t \in \mathbb{C}\}$ also lie in $\varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}})$ and pass through 0 for $t = 0$. Considering their tangent vectors at $t = 0$, we derive

$$E_{p+1,p}, E_{l,p+2}, E_{p+q+1,p+1} \in T_0 \varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}})$$

(with $l = p + 3$ for $q > 2$ and $l = p + 4$ for $q = 2$). Altogether, we get 8 linearly independent tangent vectors.

If $q = 2$, then we obtain

$$\dim T_{F_0} \mathcal{K}_\pi^{\text{BC}} = \dim T_0 \varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}}) \geq 8 > 7 = \dim \mathcal{K}_\pi^{\text{BC}}$$

and we are done. If $q > 2$, then we construct some additional elements in the tangent space.

Fix $k \in \{3, \dots, q\}$. For $t \in \mathbb{C}$, there is a unique element $g_t \in Z_u$ such that

$$g_t(e_{p+q+1}) = \begin{cases} e_{p+q+1} + te_{p+k+1} & \text{if } k < q, \\ e_{p+q+1} + te_{p+q+2} & \text{if } k = q, \end{cases}$$

$$g_t(e_i) = e_i \text{ for } i \in \{p+1, p+2, p+q+2\}.$$

As $F_0 \in \mathcal{K}_\pi^{\text{BC}}$, we get $g_t F_0 \in \mathcal{K}_\pi^{\text{BC}}$ for all $t \in \mathbb{C}$. In fact we see that $g_t F_0 = \varphi(y_t)$ with $y_t \in \mathfrak{n}$ defined by

$$y_t = t(\delta_{k=p} E_{2,1} + (E_{p+k,p} + \dots + E_{p+3,p-k+3}) + \delta_{k=q} E_{p+q+2,p+q+1}),$$

where $\delta_{i=j} = 1$ for $i = j$ and $\delta_{i=j} = 0$ otherwise. We infer that

$$E_{p+k,p} + \dots + E_{p+3,p-k+3} \in T_0 \varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}}) \quad \forall k \in \{3, \dots, q\}.$$

These tangent vectors are linearly independent of the previous ones we constructed. Altogether we have $q + 6$ tangent vectors. It results:

$$\dim T_{F_0} \mathcal{K}_\pi^{\text{BC}} = \dim T_0 \varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}}) \geq q + 6 > q + 5 = \dim \mathcal{K}_\pi^{\text{BC}}.$$

Therefore, F_0 is a singular point of the component $\mathcal{K}_\pi^{\text{BC}}$. □

5.2. Singular components of the form $\mathcal{K}_\pi^{\text{BC}}$ with $\pi = (2, p, 2)$

In [4, §2.3], another example of a singular Bala-Carter component is given, and this is $\mathcal{K}_\pi^{\text{BC}}$ for $\pi = (2, 3, 2)$. More generally, we show:

Lemma 5.3. *Let $p \geq 3$ and $\pi = (2, p, 2)$. Then, the component $\mathcal{K}_\pi^{\text{BC}}$ is singular.*

Proof. The arguments are of the same type as in the proof of Lemma 5.2. We may suppose that $p > 3$, since otherwise $\pi = (2, 3, 2)$ and the singularity of $\mathcal{K}_\pi^{\text{BC}}$ follows from [4, §2.3]. The considered component $\mathcal{K}_\pi^{\text{BC}}$ lies in \mathcal{B}_u for $u \in \text{End}(\mathbb{C}^{p+4})$ nilpotent of Jordan form $\lambda(u) = (p, 2, 2)$. Due to formula (1), $\dim \mathcal{K}_\pi^{\text{BC}} = \dim \mathcal{B}_u = 6$.

We fix a Jordan basis (e_1, \dots, e_{p+4}) , on which u acts as follows:

$$\begin{array}{cccccccc} 0 & \leftarrow & e_1 & \leftarrow & e_4 & \leftarrow & e_5 & \cdots & \leftarrow & e_{p+1} & \leftarrow & e_{p+4} \\ 0 & \leftarrow & e_2 & \leftarrow & e_{p+2} & & & & & & & \\ 0 & \leftarrow & e_3 & \leftarrow & e_{p+3} & & & & & & & \end{array}$$

We consider the adapted flag $F_0 = (\langle e_1, \dots, e_i \rangle)_{i=0, \dots, p+4}$. Our purpose is to show that F_0 is a singular point of the component $\mathcal{K}_\pi^{\text{BC}}$. To do this, as in the proof of

Lemma 5.2, we show that F_0 lies in $\mathcal{K}_\pi^{\text{BC}}$, and we construct more than 6 linearly independent vectors of the tangent space $T_{F_0} \mathcal{K}_\pi^{\text{BC}}$.

As in the proof of Lemma 5.2, we consider the group $B = \{g \in GL(V) : ge_i \in \langle e_i, \dots, e_{p+4} \rangle \forall i\}$ and the orbit $\Omega = BF_0$, which is an open affine neighborhood of F_0 in the variety of complete flags of \mathbb{C}^{p+4} . We consider the subgroup $U \subset B$ of unipotent matrices, and its Lie algebra $\mathfrak{n} = \{g - I : g \in U\}$. The map $\varphi : \mathfrak{n} \rightarrow \Omega, x \mapsto (I + x)F_0$ is an isomorphism of algebraic varieties. For $1 \leq k < l \leq p + 4$, let $E_{l,k}(e_k) = e_l$ and $E_{l,k}(e_i) = 0$ for $i \neq k$, then the elements $E_{l,k}$ (for all $1 \leq k < l \leq p + 4$) form the canonical basis of \mathfrak{n} .

For $\underline{t} = (t_1, \dots, t_6) \in \mathbb{C}^6$, we consider $x_{\underline{t}} \in \mathfrak{n}$ defined as follows:

$$\begin{aligned} x_{\underline{t}}(e_1) &= t_1(e_2 + t_2e_3), \\ x_{\underline{t}}(e_2) &= t_2e_3 + t_3(e_3 + t_4t_5(e_4 + t_1(e_{p+2} + t_2e_{p+3}))), \\ x_{\underline{t}}(e_3) &= t_4t_5(e_4 + t_1(e_{p+2} + t_2e_{p+3})), \\ x_{\underline{t}}(e_i) &= 0 \text{ for } i = 4, \dots, p, \\ x_{\underline{t}}(e_{p+1}) &= t_4(e_{p+2} + t_2e_{p+3}), \\ x_{\underline{t}}(e_{p+2}) &= t_2e_{p+3} + t_6(e_{p+3} - t_1t_5e_{p+4}), \\ x_{\underline{t}}(e_{p+3}) &= -t_1t_5e_{p+4}, \\ x_{\underline{t}}(e_{p+4}) &= 0. \end{aligned}$$

The map $\mathbb{C}^6 \rightarrow \mathfrak{n}, \underline{t} \mapsto x_{\underline{t}}$ is well defined and algebraic. The component $\mathcal{K}_\pi^{\text{BC}}$ is the closure of the subset $\mathcal{B}_u^{T_\pi}$, where T_π is the standard tableau associated to the sequence π (see section 2.1.4). It is straightforward to see that $x_{\underline{t}} \in \varphi^{-1}(\Omega \cap \mathcal{B}_u^{T_\pi})$ whenever $t_1t_3t_4t_5t_6 \neq 0$. Hence $x_{\underline{t}} \in \varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}})$ for all $\underline{t} \in \mathbb{C}^6$. In particular, $F_0 = \varphi(0) \in \mathcal{K}_\pi^{\text{BC}}$.

For $i \in \{1, \dots, 6\}$ and $t \in \mathbb{C}$, let $\underline{t}^{(i)} = (t_1, \dots, t_6)$ with $t_i = t$ and $t_j = 0$ for $j \neq i$. The curves $\{x_{\underline{t}^{(i)}} : t \in \mathbb{C}\}$ (for $i = 1, \dots, 6$) lie in $\varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}})$, and their tangent vectors at $t = 0$ provide elements

$$E_{2,1}, E_{3,2}, E_{p+2,p+1}, E_{p+3,p+2} \in T_0 \varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}}).$$

The curves $\{x_{\underline{t}} : \underline{t} = (t, -1, 1, 0, 0, 1), t \in \mathbb{C}\}$, $\{x_{\underline{t}} : \underline{t} = (0, -1, 1, t, 0, 1), t \in \mathbb{C}\}$ and $\{x_{\underline{t}} : \underline{t} = (t, 0, 0, 0, 1, 0), t \in \mathbb{C}\}$ also pass through 0 at $t = 0$. Considering their tangent vectors at $t = 0$, we get:

$$E_{3,1}, E_{p+3,p+1}, E_{p+4,p+3} \in T_0 \varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}}).$$

Altogether, we obtain that 7 linearly independent elements lie in the tangent space $T_0 \varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}})$. It results:

$$\dim T_{F_0} \mathcal{K}_\pi^{\text{BC}} = \dim T_0 \varphi^{-1}(\Omega \cap \mathcal{K}_\pi^{\text{BC}}) \geq 7 > 6 = \dim \mathcal{K}_\pi^{\text{BC}}.$$

Therefore, F_0 is a singular point of $\mathcal{K}_\pi^{\text{BC}}$. □

5.3. Proof of Proposition 5.1

Consider a sequence $\pi = (\pi_1, \dots, \pi_r) \in \Lambda_u$ such that

$$\pi \geq (1, 2, 2, 1) \quad \text{or} \quad \pi \geq (2, 3, 2).$$

First, assume $\pi \geq (1, 2, 2, 1)$, that is, there are $1 \leq i_1 < i_2 < i_3 < i_4 \leq r$ such that $\pi_{i_1}, \pi_{i_4} \geq 1$, $\pi_{i_2}, \pi_{i_3} \geq 2$. Set $\hat{\pi} = (\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4})$ and $\hat{\pi}' = (1, \pi_{i_2}, \pi_{i_3}, 1)$. Using Lemma 5.2, we get that the Bala-Carter component $\mathcal{K}_{\hat{\pi}'}$ associated to $\hat{\pi}'$ is singular. Using Corollary 4.7, we infer that $\mathcal{K}_{\hat{\pi}}^{\text{BC}}$ is singular. Finally, applying Corollary 4.8, it follows that the component $\mathcal{K}_{\pi}^{\text{BC}}$ is singular.

Next, assume $\pi \geq (2, 3, 2)$. Hence, there are $1 \leq i_1 < i_2 < i_3 \leq r$ such that $\pi_{i_1}, \pi_{i_3} \geq 2$, $\pi_{i_2} \geq 3$. Similarly as above, we set $\hat{\pi} = (\pi_{i_1}, \pi_{i_2}, \pi_{i_3})$ and $\hat{\pi}' = (2, \pi_{i_2}, 2)$. By Lemma 5.3, the component $\mathcal{K}_{\hat{\pi}'}$ associated to $\hat{\pi}'$ is singular. Corollary 4.7 then implies that $\mathcal{K}_{\hat{\pi}}^{\text{BC}}$ is singular. Finally, using Corollary 4.8, we infer that the component $\mathcal{K}_{\pi}^{\text{BC}}$ is singular.

In each case, we obtain that the component $\mathcal{K}_{\pi}^{\text{BC}}$ is singular. The proof of Proposition 5.1 is then complete.

6. Smooth Bala-Carter components

As in the previous section, we consider a sequence $\pi = (\pi_1, \dots, \pi_r) \in \Lambda_u$, and its corresponding Bala-Carter component $\mathcal{K}_{\pi}^{\text{BC}} \subset \mathcal{B}_u$. Our purpose is now to establish the implication (\Rightarrow) of Theorem 2.1, namely:

Proposition 6.1. *If $\mathcal{K}_{\pi}^{\text{BC}}$ is singular, then $\pi \geq (1, 2, 2, 1)$ or $\pi \geq (2, 3, 2)$.*

We prove the contraposal: suppose $\pi \not\geq (1, 2, 2, 1)$ and $\pi \not\geq (2, 3, 2)$, and let us show that $\mathcal{K}_{\pi}^{\text{BC}}$ is smooth. Of course, if π is of the form $\pi = (\pi_1)$ (one term in the sequence), we know that the component is smooth (it is a point). More generally, if π is of the form $\pi = (1, \dots, 1, p, 1, \dots, 1)$ (the component is of hook type) or $\pi = (p, q)$ (the component is of two-row type), then we know that it is smooth (cf. [7], [17]). Taking also into account Corollary 4.6, the situations which remain to be analyzed are the following two ones:

- a) $\pi = (p, 1, \dots, 1, 2, 1, \dots, 1, q)$, and b) $\pi = (1, \dots, 1, p, 1, \dots, 1, q)$.

In both situations, $p, q \geq 2$ are arbitrary, the number of 1's is arbitrary (can be zero). Consider situation a), and let us show that the component is smooth in this case. First suppose $p = q = 2$: applying [5, Theorem 1.2], we obtain that the component $\mathcal{K}_{\pi}^{\text{BC}}$ is smooth in this case. Next, applying Corollary 4.7, we derive that $\mathcal{K}_{\pi}^{\text{BC}}$ is smooth for $p, q \geq 2$ arbitrary.

It remains to show that $\mathcal{K}_{\pi}^{\text{BC}}$ is smooth, whenever π accords with situation b). Again, applying Corollary 4.7, we see that it is sufficient to treat the case where $p = q$. Applying Corollary 4.8, we see that we may suppose in addition that the number of 1's on the left of p coincides with the number of 1's on the right and that this number is nonzero. Therefore, it remains to show the following

Proposition 6.2. *Let $p \geq 2$ be arbitrary, let $m \geq 1$. Set*

$$\pi = (\underbrace{1, \dots, 1}_m, p, \underbrace{1, \dots, 1}_m).$$

Then, the Bala-Carter component $\mathcal{K}_{\pi}^{\text{BC}}$ is smooth.

The remainder of this section is devoted to the proof of Proposition 6.2.

6.1. Setting

In [4, §4.3–4.5], it is proved that the component $\mathcal{K}_\pi^{\text{BC}}$ is smooth, for $\pi = (1, p, p)$. The proof of Proposition 6.2 we give here will follow the same pattern.

In the remainder of the section, we consider $V = \mathbb{C}^{2p+2m}$ and $u \in \text{End}(V)$ nilpotent, of Jordan form $\lambda(u) = (p, p, 1, \dots, 1)$ (with $p \geq 2$, $m \geq 1$). Let $\pi = (1, \dots, 1, p, 1, \dots, 1, p)$ be as in the statement of Proposition 6.2, and let $\mathcal{K}_\pi^{\text{BC}} \subset \mathcal{B}_u$ be the corresponding Bala-Carter component. In general, the irreducible components of \mathcal{B}_u are parameterized by standard tableaux and we denote by \mathcal{K}^T the component associated to the tableau T (cf. section 2.1.2). In particular, the component $\mathcal{K}_\pi^{\text{BC}}$ coincides with \mathcal{K}^{T_π} , where T_π is the following standard tableau

$$T_\pi = \begin{array}{cccccc} & 1 & m+2 & m+3 & \cdots & m+p \\ & \vdots & & & & \\ & m+1 & & & & \\ m+p+1 & & & & & \\ & \vdots & & & & \\ 2m+p+1 & & & & & \end{array}$$

(cf. section 2.1.4). The component admits $\mathcal{B}_u^{T_\pi}$ as a dense open subset (cf. section 2.1.2). Due to formula (1),

$$\dim \mathcal{K}_\pi^{\text{BC}} = \dim \mathcal{B}_u = \frac{(2m+2)(2m+1)}{2} + p - 1 = 2m^2 + 3m + p.$$

6.2. Special flags $F_{\underline{d}}$

In this subsection, the purpose is to produce a set of special elements $F_{\underline{d}} \in \mathcal{B}_u$ with the property that it will be sufficient to check the smoothness of the component $\mathcal{K}_\pi^{\text{BC}}$ at these particular points.

We fix a basis (e_1, \dots, e_{2p+2m}) of V , on which u acts as follows:

$$\begin{aligned} e_1, \dots, e_{2m+2} &\in \ker u, \\ u(e_{2m+3}) &= e_1, \quad u(e_{2m+4}) = e_2 \quad \text{and} \quad u(e_{2m+i}) = e_{2m+i-2} \quad \forall i \in \{5, \dots, 2p\}. \end{aligned}$$

We can also describe the action of u on the basis with the following tableau τ_0 :

$$\tau_0 = \begin{array}{cccccc} & 1 & 2m+3 & 2m+5 & \cdots & 2m+2p-1 \\ & 2 & 2m+4 & 2m+6 & \cdots & 2m+2p \\ & 3 & & & & \\ & \vdots & & & & \\ & 2m+2 & & & & \end{array}.$$

We have $u(e_i) = 0$ if i lies in the first column of τ_0 , and $u(e_i) = e_j$, where j is the number on the left of i in τ_0 , otherwise.

We consider tableaux which are obtained by permuting the entries of τ_0 . For a permutation $\sigma \in \mathbf{S}_{2p+2m}$, we denote by $\sigma(\tau_0)$ the tableau obtained from τ_0

after replacing each entry i by σ_i . The tableau $\sigma(\tau_0)$ is said to be row-increasing if its entries increase from left to right along the rows. Let $\mathbf{S}_u = \{\sigma \in \mathbf{S}_{2p+2m} : \sigma^{-1}(\tau_0) \text{ is row-increasing}\}$.

For $\sigma \in \mathbf{S}_{2p+2m}$ we consider the flag $F_\sigma = (\langle e_{\sigma_1}, \dots, e_{\sigma_i} \rangle)_{i=0, \dots, n}$. It is easy to see that $F_\sigma \in \mathcal{B}_u$ if and only if $\sigma \in \mathbf{S}_u$. We have (cf. [3, Lemma 1.3]):

Lemma 6.3. *An irreducible component $\mathcal{K}^T \subset \mathcal{B}_u$ is smooth if and only if every flag of the form F_σ (with $\sigma \in \mathbf{S}_u$) lying in \mathcal{K}^T is a smooth point.*

According to this lemma, to determine if a component is smooth, it is sufficient to study the smoothness of the special points of the form F_σ . In the case of the component $\mathcal{K}_\pi^{\text{BC}}$ which we are concerned with, the set of special points to consider can be reduced.

Let D denote the set of tuples

$\underline{d} = (d_1 < \dots < d_{m+1})$ with $d_i \in \{m + 2, \dots, m + p + i\}$ for all i , and such that $|\{p + m, \dots, p + 2m + 1\} \cap \{d_1, \dots, d_{m+1}\}| \geq m$. For $\underline{d} \in D$, we define a flag $F_{\underline{d}} \in \mathcal{B}_u$ as follows. Write $\{m + 2, \dots, 2p + 2m\} \setminus \{d_1, \dots, d_{m+1}\} = \{i_3, \dots, i_{2p}\}$ with $i_3 < \dots < i_{2p}$. Let $\tau_{\underline{d}}$ be the following tableau:

$$\tau_{\underline{d}} = \begin{array}{cccccc} 1 & i_3 & i_5 & \cdots & i_{2p-1} & \\ \vdots & i_4 & i_6 & \cdots & i_{2p} & \\ m+1 & & & & & \\ d_1 & & & & & \\ \vdots & & & & & \\ d_{m+1} & & & & & \end{array} .$$

The tableau $\tau_{\underline{d}}$ is row-increasing, hence it can be written $\tau_{\underline{d}} = \sigma_{\underline{d}}^{-1}(\tau_0)$ with $\sigma_{\underline{d}} \in \mathbf{S}_u$. We define $F_{\underline{d}} = F_{\sigma_{\underline{d}}}$. Then, we have:

Lemma 6.4. *The component $\mathcal{K}_\pi^{\text{BC}}$ is smooth if and only if every flag $F_{\underline{d}}$ (with $\underline{d} \in D$) such that $F_{\underline{d}} \in \mathcal{K}_\pi^{\text{BC}}$ is a smooth point of $\mathcal{K}_\pi^{\text{BC}}$.*

Proof. The implication (\Rightarrow) is immediate. To prove the second implication, we rely on Lemma 6.3. Recall that $Z_u = \{g \in GL(V) : gug^{-1} = u\}$ is the stabilizer of u . Note that, if $F, F' \in \mathcal{K}_\pi^{\text{BC}}$ are such that $F' \in \overline{Z_u F}$ is a smooth point of $\mathcal{K}_\pi^{\text{BC}}$, then F is also a smooth point of $\mathcal{K}_\pi^{\text{BC}}$. Then, to complete our proof, it is sufficient to show that the closure of the Z_u -orbit of every flag of the form F_σ (with $\sigma \in \mathbf{S}_u$) lying in $\mathcal{K}_\pi^{\text{BC}}$ contains a flag of the form $F_{\underline{d}}$ (with $\underline{d} \in D$).

Let $g_t \in GL(V)$ ($t \in \mathbb{C}$) be defined by $g_t(e_2) = e_2 + te_1$, $g_t(e_i) = e_i + te_{i-1}$ for $i \in \{2m + 4, \dots, 2m + 2p\}$ even, and $g_t(e_i) = e_i$ otherwise. We have $g_t \in Z_u$. Notice that $\lim_{t \rightarrow \infty} g_t F_\sigma = F_{\sigma'}$ where $\sigma' \in \mathbf{S}_u$ is such that

$$\sigma'_i{}^{-1} = \min\{\sigma_i^{-1}, \sigma_{i+1}^{-1}\}, \quad \sigma'_{i+1}{}^{-1} = \max\{\sigma_i^{-1}, \sigma_{i+1}^{-1}\}$$

for all $i \in \{1, 2m + 3, \dots, 2m + 2p - 1\}$ odd, and

$$\sigma'_i{}^{-1} = \sigma_i^{-1} \quad \text{for } i \in \{3, \dots, 2m + 2\}.$$

Let $g'_t \in GL(V)$ ($t \in \mathbb{C}$) be defined by $g'_t(e_{2m+3}) = e_{2m+3} + te_2$, $g'_t(e_i) = e_i + te_{i-1}$ for $i \in \{2m+5, \dots, 2m+2p-1\}$ odd, and $g'_t(e_i) = e_i$ otherwise. Likewise, $g'_t \in Z_u$ and $\lim_{t \rightarrow \infty} g'_t F_\sigma = F_{\sigma'}$ for $\sigma' \in \mathbf{S}_u$ such that

$$\sigma_2'^{-1} = \min\{\sigma_2^{-1}, \sigma_{2m+3}^{-1}\}, \quad \sigma_{2m+3}'^{-1} = \max\{\sigma_2^{-1}, \sigma_{2m+3}^{-1}\},$$

$$\sigma_i'^{-1} = \min\{\sigma_i^{-1}, \sigma_{i+1}^{-1}\}, \quad \sigma_{i+1}'^{-1} = \max\{\sigma_i^{-1}, \sigma_{i+1}^{-1}\}$$

for all $i \in \{2m+4, \dots, 2m+2p-2\}$ even, and

$$\sigma_i'^{-1} = \sigma_i^{-1} \text{ for } i \in \{1, 3, 4, \dots, 2m+2, 2m+2p\}.$$

For $1 \leq i < j \leq 2m+2$ such that $(i, j) \neq (1, 2)$, let $h_t^{(i,j)} = h_t \in GL(V)$ ($t \in \mathbb{C}$) be defined by $h_t(e_j) = e_j + te_i$ and $h_t(e_l) = e_l$ for $l \neq j$. Then $h_t \in Z_u$, and $\lim_{t \rightarrow \infty} h_t F_\sigma = F_{\sigma'}$ with $\sigma' \in \mathbf{S}_u$ such that

$$\sigma_i'^{-1} = \min\{\sigma_i^{-1}, \sigma_j^{-1}\}, \quad \sigma_j'^{-1} = \max\{\sigma_i^{-1}, \sigma_j^{-1}\}, \quad \text{and } \sigma_l'^{-1} = \sigma_l^{-1} \text{ for } l \notin \{i, j\}.$$

We assume $F_\sigma \in \mathcal{K}_\pi^{\text{BC}}$. Combining the operations $(\lim_{t \rightarrow \infty} g_t)$, $(\lim_{t \rightarrow \infty} g'_t)$ and $(\lim_{t \rightarrow \infty} h_t^{(i,j)})$, we transform F_σ into a flag $F_{\sigma'}$ with σ' such that

$$\sigma_2'^{-1} < \sigma_i'^{-1} < \sigma_{i+1}'^{-1} \text{ for all } i \in \{2m+3, \dots, 2m+2p-1\},$$

$$\sigma_1'^{-1} < \sigma_2'^{-1} < \dots < \sigma_{2m+2}'^{-1}.$$

In particular, $F_{\sigma'} \in \overline{Z_u F_\sigma}$. As we suppose $F_\sigma \in \mathcal{K}_\pi^{\text{BC}}$, we get $F_{\sigma'} \in \mathcal{K}_\pi^{\text{BC}}$. Let us show that $\sigma' = \sigma_{\underline{d}}$ for some $\underline{d} \in D$, which will complete the proof.

Denote $c_i = \sigma_i'^{-1}$ for $i \in \{1, \dots, m+1\}$, $d_i = \sigma_i'^{-1}$ for $i \in \{m+2, \dots, 2m+2\}$, and $i_j = \sigma_{2m+j}'^{-1}$ for $j \in \{3, \dots, 2p\}$. The tableau $\sigma'^{-1}(\tau_0)$ is as follows:

$$\sigma'^{-1}(\tau_0) = \begin{array}{cccccc} c_1 & i_3 & i_5 & \cdots & i_{2p-1} & \\ \vdots & i_4 & i_6 & \cdots & i_{2p} & \\ c_{m+1} & & & & & \\ d_1 & & & & & \\ \vdots & & & & & \\ d_{m+1} & & & & & \end{array} .$$

Moreover, we have $c_1 < \dots < c_{m+1} < d_1 < \dots < d_{m+1}$ and $c_2 < i_3 < \dots < i_{2p}$.

Following section 2.3.1, the sequence $\pi = (\pi_1, \dots, \pi_r)$ induces a standard partition of $\{1, \dots, n\}$, which we also denote by π , and the component $\mathcal{K}_\pi^{\text{BC}}$ is the closure of the standard orbit \mathcal{Z}_π . By definition of the orbit \mathcal{Z}_π , any flag $F = (V_0, \dots, V_{2m+2p}) \in \mathcal{Z}_\pi$ satisfies

$$\begin{aligned} &V_{m+1} \subset \ker u, \\ &\dim V_{m+p+i} \cap \ker u \geq m+1+i \quad \forall i \in \{1, \dots, m+1\}, \\ &\text{and } V_{2m+p+1} \subset u^{-1}(V_{m+p-1}). \end{aligned}$$

By lower semi-continuity of the map $F \mapsto \text{rank } u|_{V_j/V_i}$, these three properties are satisfied more generally for all $F \in \mathcal{K}_\pi^{\text{BC}}$. In particular, $F_{\sigma'} = (V'_0, \dots, V'_{2m+2p})$ satisfies them. As $V'_{m+1} \subset \ker u$, using the definition of the flag $F_{\sigma'}$, we get

$$|\{c_1, \dots, c_{m+1}\} \cap \{1, \dots, m+1\}| = \dim V'_{m+1} \cap \ker u = m+1,$$

hence $c_i = i$ for all i . Similarly, for all $i \in \{1, \dots, m + 1\}$ we get

$$m + 1 + |\{d_1, \dots, d_{m+1}\} \cap \{1, \dots, m + p + i\}| = \dim V'_{m+p+i} \cap \ker u \geq m + 1 + i,$$

hence $d_i \leq m + p + i$. Moreover, as $V'_{2m+p+1} \subset u^{-1}(V'_{m+p-1})$, any Jordan block of the induced map $u|_{V'_{2m+p+1}/V'_{m+p-1}} = 0$ has size 1, which implies that the entries $m + p, \dots, 2m + p + 1$ lie in different rows of the tableau $\sigma'^{-1}(\tau_0)$. In particular, at most 2 among these $m + 2$ numbers can lie in the first two rows of the tableau, therefore $|\{d_1, \dots, d_{m+1}\} \cap \{m + p, \dots, 2m + p + 1\}| \geq m$. It follows $\underline{d} \in D$, and finally $\sigma' = \sigma_{\underline{d}}$. The proof of the lemma is now complete. \square

6.3. Proof of Proposition 6.2

According to Lemma 6.4, to show that the component $\mathcal{K}_\pi^{\text{BC}}$ is smooth, it is sufficient to check the smoothness at the points of the form $F_{\underline{d}}$. Thus, Proposition 6.2 follows from the next proposition.

Proposition 6.5. *For all $\underline{d} \in D$, we have $F_{\underline{d}} \in \mathcal{K}_\pi^{\text{BC}}$, and moreover $F_{\underline{d}}$ is a smooth point of $\mathcal{K}_\pi^{\text{BC}}$.*

Our purpose is then to establish Proposition 6.5. To do this, we employ the same technique as in the proof of [4, Proposition 4.3]. Let us outline our proof. For $\underline{d} \in D$, recall that $F_{\underline{d}} = (\langle e_{\sigma_{\underline{d}}(1)}, \dots, e_{\sigma_{\underline{d}}(i)} \rangle)_{i=0, \dots, 2m+2p}$, where the basis (e_1, \dots, e_{2p+2m}) and the permutation $\sigma_{\underline{d}} \in \mathbf{S}_{2p+2m}$ have been introduced in section 6.2. We consider the Borel subgroup

$$B = \{g \in GL(V) : ge_{\sigma_{\underline{d}}(i)} \in \langle e_{\sigma_{\underline{d}}(i)}, \dots, e_{\sigma_{\underline{d}}(2p+2m)} \rangle \quad \forall i\}$$

and the orbit $\Omega_{\underline{d}} = BF_{\underline{d}}$. The orbit $\Omega_{\underline{d}}$ is an open subset of the variety of complete flags of V , and it is isomorphic to an affine space: for $F = (V_0, \dots, V_{2p+2m}) \in \Omega_{\underline{d}}$, there is a unique basis $(\eta_1, \dots, \eta_{2p+2m})$ such that $V_i = \langle \eta_1, \dots, \eta_i \rangle$ and

$$\eta_i = e_{\sigma_{\underline{d}}(i)} + \sum_{j=i+1}^{2m+2p} \phi_{i,j} e_{\sigma_{\underline{d}}(j)} \tag{7}$$

for some $\phi_{i,j} \in \mathbb{C}$. The maps $F \mapsto \phi_{i,j}$ are algebraic and the product map $F \mapsto (\phi_{i,j})_{1 \leq i < j \leq 2p+2m}$ is an isomorphism from $\Omega_{\underline{d}}$ to the affine space $\mathbb{C}^{(p+m)(2p+2m-1)}$. Then, we construct a closed immersion $\Phi_{\underline{d}} : \mathbb{C}^{2m^2+3m+p} \rightarrow \Omega_{\underline{d}}$ with the following two properties:

- (A) For $\underline{t} = (t_1, t_2, \dots) \in \mathbb{C}^{2m^2+3m+p}$ with $t_i \neq 0$ for all i , one has $\Phi_{\underline{d}}(\underline{t}) \in \mathcal{B}_u^{T_\pi}$.
- (B) $\Phi_{\underline{d}}(0, \dots, 0) = F_{\underline{d}}$.

By (A), we get that $\Phi_{\underline{d}}$ is an isomorphism on a locally closed subset of the component $\mathcal{K}_\pi^{\text{BC}}$. Since $\dim \mathcal{K}_\pi^{\text{BC}} = 2m^2 + 3m + p$ (see section 6.1), the image of $\Phi_{\underline{d}}$ is actually an open subset of $\mathcal{K}_\pi^{\text{BC}}$. By (B), we get that the flag $F_{\underline{d}}$ lies in the image of $\Phi_{\underline{d}}$, hence it lies in $\mathcal{K}_\pi^{\text{BC}}$ and is a smooth point.

To lead our construction, we need the preliminary construction summed up in the following lemma, which we quote from [4, §4.4].

Lemma 6.6. For $\underline{t} = (t_3, \dots, t_{p+1}) \in \mathbb{C}^{p-1}$, there are vectors v_1, \dots, v_{2p} depending algebraically on \underline{t} , with the following properties:

- (a) $v_1 = e_1, v_2 = e_2, v_i - e_{2m+i} \in \langle e_{2m+i+1}, \dots, e_{2p+2m} \rangle$ for all $i \in \{3, \dots, 2p\}$.
- (b) $\langle v_1, \dots, v_i \rangle$ is u -stable for all $i \in \{1, \dots, 2p\}$.
- (c) For $i \in \{3, \dots, p+1\}$, $u(v_i) = v_{i-2} + t_i v_{i-1}$. In particular $v_i \in \ker u^{i-1}$, and if $t_j \neq 0$ for all j , then $v_i \notin \ker u^{i-2}$.
- (d) Denoting $t'_1 = t'_2 = 0$ and $t'_j = t'_{j-2} + t_j$ for $j \in \{3, \dots, p+1\}$, we have

$$v_i - e_{2m+i} - t'_i e_{2m+i+1} \in \langle e_{2m+i+2}, \dots, e_{2p+2m} \rangle \quad \forall i \in \{3, \dots, p+1\},$$

$$v_i - e_{2m+i} - t'_{2p-i+2} e_{2m+i+1} \in \langle e_{2m+i+2}, \dots, e_{2p+2m} \rangle \quad \forall i \in \{p+2, \dots, 2p-1\}.$$

- (e) If $\underline{t} = (0, \dots, 0)$, then $v_i = e_{2m+i}$ for all $i \in \{3, \dots, 2p\}$.

Proof. We consider the subspace generated by the basis vectors $e_1, e_2, e_{2m+3}, \dots, e_{2m+2p}$. The action of u on these vectors is represented by the following picture:

$$\begin{array}{ccccccccccc} 0 & \leftarrow & e_1 & \leftarrow & e_{2m+3} & \leftarrow & e_{2m+5} & \leftarrow & \dots & \leftarrow & e_{2m+2p-1} \\ 0 & \leftarrow & e_2 & \leftarrow & e_{2m+4} & \leftarrow & e_{2m+6} & \leftarrow & \dots & \leftarrow & e_{2m+2p}. \end{array}$$

In particular, note that u restricts to an isomorphism from the subspace $E_1 := \langle e_{2m+3}, \dots, e_{2m+2p} \rangle$ onto $E_2 := \langle e_1, e_2, e_{2m+3}, \dots, e_{2m+2p-2} \rangle$. Let $\check{u} : E_2 \rightarrow E_1$ be its inverse, that is $\check{u}(e_i) = e_{2m+i+2}$ for $i \in \{1, 2\}$ and $\check{u}(e_i) = e_{i+2}$ for $i \in \{2m+3, \dots, 2m+2p-2\}$. We then put $v_1 = e_1, v_2 = e_2$, and by induction $v_i = \check{u}(v_{i-2} + t_i v_{i-1})$ for all $i \in \{3, \dots, p+1\}$.

For $i \in \{p+1, \dots, 2p\}$, we define vectors $(v_j^{(i)})_{1 \leq j \leq i}$ in the following manner. First, we set $v_j^{(p+1)} = v_j, \forall j \in \{1, \dots, p+1\}$. Then for $i \geq p+2$, we set $v_1^{(i)} = e_1, v_2^{(i)} = e_2$, and by induction: $v_j^{(i)} = \check{u}(v_{j-2}^{(i-1)})$ if $j \geq 3$. For $i \in \{p+2, \dots, 2p\}$, we put $v_i = v_i^{(i)}$.

The fact that the vectors v_i , for $i = 1, \dots, 2p$, are well defined and satisfy properties (a)–(e) of the lemma then follows from [4, Lemmas 4.3-4.4]. \square

We are ready to prove Proposition 6.5.

Proof of Proposition 6.5.

The constructions will rely on the Jordan basis (e_1, \dots, e_{2p+2m}) introduced in section 6.2. We also consider the subspaces $E = \langle e_1, e_2 \rangle, F = \langle e_3, \dots, e_{m+1} \rangle$ and $G = \langle e_{m+2}, \dots, e_{2m+2} \rangle$. Let $\mathfrak{n}_F \subset \text{End}(F)$ be the subspace of strictly lower triangular maps, i.e., $X \in \text{End}(F)$ such that $Xe_i \in \langle e_{i+1}, \dots, e_{m+1} \rangle$ for all $i = 3, \dots, m+1$. Let $\mathfrak{n}_G \subset \text{End}(G)$ be the subspace of strictly lower triangular maps. Let $\mathcal{L}(E, F), \mathcal{L}(F, G)$ be the spaces of linear maps from E to F and from F to G , respectively.

Let $\underline{d} = (d_1, \dots, d_{m+1}) \in D$ and let $F_{\underline{d}}$ be the corresponding flag. The purpose of the construction is to associate to a tuple of variables $\underline{t} \in \mathbb{C}^{2m^2+3m+p}$ a certain basis $(\eta_1, \dots, \eta_{2p+2m})$ of V satisfying relation (7), such that the map $\Phi_{\underline{d}} : \underline{t} \mapsto (\langle \eta_1, \dots, \eta_i \rangle)_{i=0, \dots, 2m+2p}$ will fulfill the properties (A) and (B) above. We distinguish two situations:

- 1) $d_1 \geq p + m$,
- 2) $d_1 < p + m$.

We describe the tuple of variables in each case. In case (1), the tuple of variables \underline{t} is taken of the form

$$\underline{t} = (X, Y, H, K, s_1, t_3, \dots, t_{p+1}, y_1, y'_1, \dots, y_{m+1}, y'_{m+1}),$$

where $X \in \mathfrak{n}_F$, $Y \in \mathfrak{n}_G$, $H \in \mathcal{L}(E, F)$, $K \in \mathcal{L}(F, G)$ and $s_1, t_l, y_j, y'_j \in \mathbb{C}$. In case (2), the tuple \underline{t} is taken of the form

$$\underline{t} = (X, Y, H, K, s_1, t_3, \dots, t_{d_1-m+1}, t_{d_1-m+3}, \dots, t_{p+2}, y_1, y'_1, \dots, y_{m+1}, y'_{m+1}),$$

where X, Y, H, K are as above, $s_1, t_l, y_j, y'_j \in \mathbb{C}$. In fact, in all cases:

$$\underline{t} \in \mathfrak{n}_F \times \mathfrak{n}_G \times \mathcal{L}(E, F) \times \mathcal{L}(F, G) \times \mathbb{C}^{p+2(m+1)},$$

and it is straightforward to check that the space $\mathfrak{n}_F \times \mathfrak{n}_G \times \mathcal{L}(E, F) \times \mathcal{L}(F, G) \times \mathbb{C}^{p+2(m+1)}$ has dimension $2m^2 + 3m + p$, that is, the same dimension as the component $\mathcal{K}_\pi^{\text{BC}}$.

In case (2), we set in addition $t_{d_1-m+2} = -y_1 y'_1$. In both cases, we consider the vectors v_1, \dots, v_{2p} associated to the tuple (t_3, \dots, t_{p+1}) in the sense of Lemma 6.6. We put $f_i = e_i + X e_i + K e_i$ for $i \in \{3, \dots, m+1\}$ and $f_{d_j} = e_{m+1+j} + Y e_{m+1+j}$ for $j \in \{1, \dots, m+1\}$. We also put $f_1 = e_1 + s_1 e_2 + (I + X + K) H e_1$ and $f_2 = e_2 + (I + X + K) H e_2$, where I is the identity endomorphism of V . In both cases, put

$$\eta_i = f_i \text{ for all } i \in \{3, \dots, m+1\}.$$

We will define in each case the remaining vectors $\eta_1, \eta_2, \eta_{m+2}, \dots, \eta_{2p+2m}$. It will follow from the definition that $\eta_i - e_{\sigma_{\underline{d}}(i)} \in \langle e_{\sigma_{\underline{d}}(i+1)}, \dots, e_{\sigma_{\underline{d}}(2p+2m)} \rangle$ for all i , and the so-obtained map $\Phi_{\underline{d}} : \mathbb{C}^{2m^2+3m+p} \rightarrow \Omega_{\underline{d}}$, $\underline{t} \mapsto (\langle \eta_1, \dots, \eta_i \rangle)_{i=0, \dots, 2p+2m}$ will be algebraic. In addition, we will verify the following four properties:

- a) $\Phi_{\underline{d}}(\mathbb{C}^{2m^2+3m+p}) \subset \mathcal{B}_u$;
- b) the map $\Phi_{\underline{d}}$ is a closed immersion;
- c) $\Phi_{\underline{d}}(\underline{t}) \in \mathcal{B}_u^{T_\pi}$ whenever all numbers t_l, y_j, y'_j are nonzero;
- d) $\Phi_{\underline{d}}(0, \dots, 0) = F_{\underline{d}}$.

As a conclusion, $\Phi_{\underline{d}}(\mathbb{C}^{2m^2+3m+p}) \cong \mathbb{C}^{2m^2+3m+p}$ is an open neighborhood of the flag $F_{\underline{d}}$ in the component $\mathcal{K}_\pi^{\text{BC}}$, therefore $F_{\underline{d}}$ is a smooth point of $\mathcal{K}_\pi^{\text{BC}}$: the proof will be complete.

(1) We suppose $d_1 \geq p + m$.

By definition of the set D , the tuple \underline{d} satisfies $d_1 < d_2 < \dots < d_{m+1} \leq 2m + p + 1$. Hence, there is a unique $k \in \{0, \dots, m+1\}$ such that $p + m + k \notin \{d_1, \dots, d_{m+1}\}$. Consequently the indices d_1, \dots, d_{m+1} and the indices i_3, \dots, i_{2p} involved in the definition of $\sigma_{\underline{d}}$ are given by:

$$d_j = p + m + j - 1 \quad \forall j \in \{1, \dots, k\} \text{ and } d_j = p + m + j \quad \forall j \in \{k + 1, \dots, m + 1\},$$

$$i_l = m - 1 + l \quad \forall l \in \{3, \dots, p\}, \quad i_{p+1} = p + m + k, \text{ and } i_l = 2m + l \quad \forall l \in \{p + 2, \dots, 2p\}.$$

The indices are organized as follows:

$$i_3 < \dots < i_p < d_1 < \dots < d_k < i_{p+1} < d_{k+1} < \dots < d_{m+1} < i_{p+2} < \dots < i_{2p}.$$

For $l \in \{1, \dots, p\}$, $j \in \{1, \dots, m+1\}$, we define a number $x_{l,j}$ by setting $x_{p,j} = y_j$ and by induction $x_{l,j} = -t_{l+2}x_{l+1,j}$ for $l = 2, \dots, p-1$, and $x_{1,j} = -(t_3 - s_1)x_{2,j}$. We put

$$\begin{aligned} \eta_1 &= f_1 + \sum_{j=1}^{m+1} x_{1,j} f_{d_j}, & \eta_2 &= f_2 + \sum_{j=1}^{m+1} x_{2,j} f_{d_j}, \\ \eta_{i_l} &= v_l + \sum_{j=1}^{m+1} x_{l,j} f_{d_j} \quad \forall l \in \{3, \dots, p\}, \\ \eta_{d_j} &= f_{d_j} + y'_j v_{p+1} \quad \forall j \in \{1, \dots, k\}, \\ \eta_{i_{p+1}} &= v_{p+1} + \sum_{j=k+1}^{m+1} y'_j f_{d_j}, \\ \eta_{d_j} &= f_{d_j} \quad \forall j \in \{k+1, \dots, m+1\}, \quad \text{and} \quad \eta_{i_l} = v_l \quad \forall l \in \{p+2, \dots, 2p\}. \end{aligned}$$

Let us show properties a) – d).

a) Note that for $i \in \{1, \dots, m+1, d_{k+1}, \dots, d_{m+1}\}$, we have $\eta_i \in \ker u$, which implies $u(\eta_i) \in \langle \eta_1, \dots, \eta_i \rangle$. Using Lemma 6.6 (c) and the definition of $x_{l,j}$, we have:

$$u(v_3) = v_1 + t_3 v_2 = (e_1 + s_1 e_2 + \sum_{j=1}^{m+1} x_{1,j} f_{d_j}) + (t_3 - s_1)(e_2 + \sum_{j=1}^{m+1} x_{2,j} f_{d_j}),$$

hence $u(\eta_{i_3}) = u(v_3) \in \langle \eta_1, \dots, \eta_{m+1} \rangle$. Similarly, for $l \in \{4, \dots, p+1\}$:

$$u(v_l) = v_{l-2} + t_l v_{l-1} = (v_{l-2} + \sum_{j=1}^{m+1} x_{l-2,j} f_{d_j}) + t_l (v_{l-1} + \sum_{j=1}^{m+1} x_{l-1,j} f_{d_j}),$$

hence $u(\eta_{i_l}) = u(v_l) \in \langle \eta_1, \dots, \eta_{i_l} \rangle$ for $l \in \{4, \dots, p\}$, and $u(\eta_i) \in \mathbb{C}u(v_{p+1}) \subset \langle \eta_1, \dots, \eta_i \rangle$ for $i \in \{d_1, \dots, d_k, i_{p+1}\}$. Notice that $v_1, \dots, v_{p+1} \in \langle \eta_1, \dots, \eta_{2m+p+1} \rangle$. Then, for $l \in \{p+2, \dots, 2p\}$, we get $u(\eta_{i_l}) = v_{l-2} + t_l v_{l-1} \in \langle \eta_1, \dots, \eta_{i_l} \rangle$. Therefore, we have $u(\eta_i) \in \langle \eta_1, \dots, \eta_i \rangle$ for all i . It follows $\Phi_{\underline{d}}(t) \in \mathcal{B}_u$.

b) We show that $\Phi_{\underline{d}}$ is a closed immersion, and to do this, we show that the algebra morphism $\Phi_{\underline{d}}^* : \mathbb{C}[\phi_{i,j} : 1 \leq i < j \leq 2p+2m] \rightarrow \mathbb{C}[X, Y, H, K, s_1, t_i, y_j, y'_j]$ associated to $\Phi_{\underline{d}}$ is surjective. The functions $\phi_{i,j}$ are those involved in the expression of the basis $\eta_1, \dots, \eta_{2p+2m}$ according to formula (7). Then, we see that the coefficients of the matrices H, X, K, Y are recovered by considering $\phi_{i,j}$ for $i \in \{1, 2\}$, $j \in \{3, \dots, m+1\}$, or $i, j \in \{3, \dots, m+1\}$, or $i \in \{3, \dots, m+1\}$, $j \in \{d_1, \dots, d_{m+1}\}$, or $i, j \in \{d_1, \dots, d_{m+1}\}$, respectively. Therefore, they belong to $\text{Im } \Phi_{\underline{d}}^*$. We have $s_1 = \phi_{1,2} \in \text{Im } \Phi_{\underline{d}}^*$. By Lemma 6.6 (d), we have $t'_l = \phi_{i_l, i_{l+1}} \in \text{Im } \Phi_{\underline{d}}^*$ for all $l \in \{3, \dots, p+1\}$. Due to the definition of t'_l given in Lemma 6.6, we infer that $t_l \in \text{Im } \Phi_{\underline{d}}^*$ for all $l \in \{3, \dots, p+1\}$. Considering ϕ_{i_p, d_j} , we see that $y_j \in \text{Im } \Phi_{\underline{d}}^*$ for all j . We have $y'_j = \phi_{d_j, i_{p+1}} \in \text{Im } \Phi_{\underline{d}}^*$ for $j \leq k$, and moreover considering ϕ_{i_{p+1}, d_j} we see that $y'_j \in \text{Im } \Phi_{\underline{d}}^*$ for all $j > k$. Finally, we obtain that $\Phi_{\underline{d}}^*$ is surjective.

c) Suppose that all numbers $t_3, \dots, t_{p+1}, y'_1$ are nonzero, and let us show that $\Phi_{\underline{d}}(\underline{t}) \in \mathcal{B}_u^{T_\pi}$. Let u_i denote the restriction of u to the subspace $\langle \eta_1, \dots, \eta_i \rangle$, and let $\lambda(u_i)$ be the sequence of its Jordan block sizes. First, notice that $\eta_1, \dots, \eta_{m+1} \in \ker u$, hence for all $i \in \{1, \dots, m+1\}$, $\lambda(u_i) = (1, \dots, 1)$. Next, we have either $u^{p-1}(\eta_{m+p}) = y'_1 u^{p-1}(v_{p+1})$ (if $k > 0$) or $u^{p-1}(\eta_{m+p}) = u^{p-1}(v_{p+1})$ (if $k = 0$). By Lemma 6.6 (c), $u^{p-1}(v_{p+1}) \neq 0$. It follows $\lambda(u_{m+p}) = (p, 1, \dots, 1)$ and necessarily $\lambda(u_i) = (i - m, 1, \dots, 1)$ for all $i \in \{m+2, \dots, m+p\}$. Also, as $f_{d_1}, \dots, f_{d_{m+1}} \in \ker u$, we have

$$\lambda(u_i) = (p, 1, \dots, \dots, 1) \text{ for all } i \in \{m+p+1, \dots, 2m+p+1\}.$$

Finally, as $\lambda(u_{2m+2p}) = \lambda(u) = (p, p, 1, \dots, 1)$, we derive $\lambda(u_i) = (p, i - 2m - p, 1, \dots, 1)$ for all $i \in \{2m+p+2, \dots, 2m+2p\}$. For all i , the sequence $\lambda(u_i)$ indeed coincides with the lengths of the rows of the subtableau of T_π of entries $1, \dots, i$. Therefore, $\Phi_{\underline{d}}(\underline{t}) \in \mathcal{B}_u^{T_\pi}$.

d) follows from Lemma 6.6 (e) and the definition of the vectors η_i .

(2) We suppose $d_1 < p + m$.

We always have $d_1 \geq m + 2$. In particular, in this case, we have $p \geq 3$. Due to the definition of D , we have $\{d_2, \dots, d_{m+1}\} \subset \{p+m, \dots, p+2m+1\}$, so that there are exactly two numbers $h, k \in \{0, \dots, m+1\}$ with $h < k$ such that $p+m+h, p+m+k \notin \{d_1, \dots, d_{m+1}\}$. Then, the indices d_j and i_l are given by

$$d_j = p+m+j-2 \quad \forall j \in \{2, \dots, h+1\}, \quad d_j = p+m+j-1 \quad \forall j \in \{h+2, \dots, k\},$$

$$d_j = p+m+j \quad \forall j \in \{k+1, \dots, m+1\},$$

and, letting $d_1 = m + c$,

$$i_l = m - 1 + l \quad \forall l \in \{3, \dots, c\}, \quad i_l = m + l \quad \forall l \in \{c+1, \dots, p-1\},$$

$$i_p = m + p + h, \quad i_{p+1} = m + p + k, \quad i_l = 2m + l \quad \forall l \in \{p+2, \dots, 2p\}.$$

The indices are then organized as follows:

$$i_3 < \dots < i_c < d_1 < i_{c+1} < \dots < i_{p-1} < d_2 < \dots < d_{h+1} < i_p < d_{h+2} < \dots$$

$$< d_k < i_{p+1} < d_{k+1} < \dots < d_{m+1} < i_{p+2} < \dots < i_{2p}.$$

For $l \in \{1, \dots, c\}$, we define a number x_l by setting $x_c = y_1$, and by induction $x_l = -t_{l+2}x_{l+1}$ for $l \in \{2, \dots, c-1\}$, and $x_1 = -(t_3 - s_1)x_2$. Then, we put:

$$\eta_1 = f_1 + x_1 f_{d_1}, \quad \eta_2 = f_2 + x_2 f_{d_1},$$

$$\eta_{i_l} = v_l + x_l f_{d_1} \quad \forall l \in \{3, \dots, c\}, \quad \eta_{d_1} = f_{d_1} + y'_1 v_{c+1},$$

$$\eta_{i_l} = v_l + t_{l+2} v_{l+1} \quad \forall l \in \{c+1, \dots, p-1\},$$

$$\eta_{d_j} = f_{d_j} + y_j v_p + y'_j v_{p+1} \quad \forall j \in \{2, \dots, h+1\},$$

$$\eta_{i_p} = v_p + t_{p+2} v_{p+1} + \sum_{j=h+2}^{m+1} y_j f_{d_j},$$

$$\eta_{d_j} = f_{d_j} + y'_j v_{p+1} \quad \forall j \in \{h+2, \dots, k\}, \quad \eta_{i_{p+1}} = v_{p+1} + \sum_{j=k+1}^{m+1} y'_j f_{d_j},$$

$$\eta_{d_j} = f_{d_j} \quad \forall j \in \{k+1, \dots, m+1\}, \quad \text{and} \quad \eta_{i_l} = v_l \quad \forall l \in \{p+2, \dots, 2p\}.$$

We show properties a) – d).

a) Let us check that $u(\eta_i) \in \langle \eta_1, \dots, \eta_i \rangle$ for all i . First, notice that $\eta_1, \dots, \eta_{m+1} \in \ker u$. Similarly as in case (1), using Lemma 6.6 (c), we have:

$$u(v_3) = v_1 + t_3v_2 = (e_1 + s_1e_2 + x_1f_{d_1}) + (t_3 - s_1)(e_2 + x_2f_{d_1}) \in \langle \eta_1, \dots, \eta_{m+1} \rangle,$$

and for $l \in \{4, \dots, c + 1\}$:

$$u(v_l) = v_{l-2} + t_lv_{l-1} = (v_{l-2} + x_{l-2}f_{d_1}) + t_l(v_{l-1} + x_{l-1}f_{d_1}) \in \langle \eta_i : 1 \leq i \leq i_{l-1} \rangle.$$

Using that $x_c = y_1$ and $t_{c+2} = t_{d_1-m+2} = -y_1y'_1$, we get:

$$u(v_{c+2}) = v_c + t_{c+2}v_{c+1} = (v_c + x_cf_{d_1}) - y_1(f_{d_1} + y'_1v_{c+1}) \in \langle \eta_i : 1 \leq i \leq d_1 \rangle.$$

Moreover, $u(v_l) = v_{l-2} + t_lv_{l-1} = \eta_{i_{l-2}}$ for $l \in \{c + 3, \dots, p + 1\}$. Then, using in addition that $u(f_{d_j}) = 0$ for all j , we get: $u(\eta_i) \in \langle \eta_1, \dots, \eta_i \rangle$ for all $i \in \{d_1, \dots, d_{m+1}, i_3, \dots, i_{p+1}\}$. Observe that $v_1, \dots, v_{p+1} \in \langle \eta_1, \dots, \eta_{i_{p+2-1}} \rangle$. Applying Lemma 6.6 (b), we derive that $u(\eta_i) \in \langle \eta_1, \dots, \eta_i \rangle$ also for $i \in \{i_{p+2}, \dots, i_{2p}\}$, which completes the proof of property a).

b) To show that the map $\Phi_{\underline{d}}$ is a closed immersion, as in case (1), we show that its associated algebra morphism $\Phi_{\underline{d}}^*$ is surjective. Exactly as in case (1), we obtain that the coefficients of the matrices X, Y, H, K lie in the image of $\Phi_{\underline{d}}^*$. We have $s_1 = \phi_{1,2} \in \text{Im } \Phi_{\underline{d}}^*$. Let t'_l be the numbers introduced in Lemma 6.6 (d). For $l \in \{3, \dots, p\}$, we get $t'_l = t'_{2p+2-(2p-l+2)} = \phi_{i_{2p-l+2}, i_{2p-l+3}}$ by applying Lemma 6.6 (d). Similarly, we have $t'_{p+1} = \phi_{i_{p+1}, i_{p+2}}$ and $t'_p + t_{p+2} = \phi_{i_p, i_{p+1}}$. Then, in view of the definition of the numbers t'_l , we infer that $t_l \in \text{Im } \Phi_{\underline{d}}^*$ for all $l \in \{3, \dots, p+2\}$. We see that $x_l \in \text{Im } \Phi_{\underline{d}}^*$ for all $l \in \{1, \dots, c\}$, by considering ϕ_{1,d_1} , ϕ_{2,d_1} and ϕ_{i_l,d_1} for $l \geq 3$. Thus in particular $y_1 = x_c \in \text{Im } \Phi_{\underline{d}}^*$. Also, $y'_1 = \phi_{d_1, i_{c+1}} \in \text{Im } \Phi_{\underline{d}}^*$. We have $y_j = \phi_{d_j, i_p} \in \text{Im } \Phi_{\underline{d}}^*$ for all $j \in \{2, \dots, h + 1\}$, while for $j > h + 1$ we see that $y_j \in \text{Im } \Phi_{\underline{d}}^*$ by considering ϕ_{i_p, d_j} . Similarly, for $j \in \{2, \dots, m + 1\}$, we get that $y'_j \in \text{Im } \Phi_{\underline{d}}^*$ by considering either $\phi_{d_j, i_{p+1}}$ (if $j \leq k$) or ϕ_{i_{p+1}, d_j} (for $j > k$). Finally, we have shown that $\Phi_{\underline{d}}^*$ is surjective.

c) Suppose that all numbers $t_3, \dots, t_{p+2}, y_1, y'_1, y'_2$ are nonzero, and let us show that $\Phi_{\underline{d}} \in \mathcal{B}_u^{T_\pi}$. As in case (1), we denote by u_i the restriction of u to the subspace $\langle \eta_1, \dots, \eta_i \rangle$, and by $\lambda(u_i)$ its Jordan form. As in case (1), since $\eta_1, \dots, \eta_{m+1} \in \ker u$, we have $\lambda(u_i) = (1, \dots, 1)$ for all $i \in \{1, \dots, m+1\}$. We have $u^{p-1}(v_{p+1}) \neq 0$ by Lemma 6.6 (c), which implies $u^{p-1}(\eta_{m+p}) \neq 0$, therefore $\lambda(u_i) = (i-m, 1, \dots, 1)$ for all $i \in \{m + 2, \dots, m + p\}$. Notice that $\ker u \subset \langle \eta_1, \dots, \eta_{2m+p+1} \rangle$, and since $\dim \ker u = 2m + 2$, we have necessarily $\lambda(u_{2m+p+1}) = (p, 1, \dots, \dots, 1)$ (with $2m + 1$ terms 1) and consequently $\lambda(u_i)$ is of the form $(p, 1, \dots, 1)$ for all $i \in \{m + 2, \dots, 2m + p + 1\}$. Finally, as in case (1), since $\lambda(u_{2p+2m}) = \lambda(u) = (p, p, 1, \dots, 1)$, we derive that $\lambda(u_i) = (p, i - 2m - p, 1, \dots, 1)$ for all $i \in \{2m + p + 2, \dots, 2p + 2m\}$. For all i , the sequence $\lambda(u_i)$ coincides with the lengths of the rows of the subtableau of T_π of entries $1, \dots, i$. Therefore, $\Phi_{\underline{d}} \in \mathcal{B}_u^{T_\pi}$.

d) follows from Lemma 6.6 (e) and the definition of the vectors $\eta_1, \dots, \eta_{2p+2m}$.

The proof of Proposition 6.5 is then complete. □

7. Components as iterated fiber bundles over projective spaces

Recall that, for a standard tableau T , we denote by T^* its transpose (that is, the rows of T^* coincide with the columns of T). To the tableaux T and T^* , we associate the components \mathcal{K}^T and \mathcal{K}^{T^*} , imbedded in the appropriate Springer fibers \mathcal{B}_u and \mathcal{B}_{u^*} . In this section, we consider the situation where \mathcal{K}^T contains a dense Jordan orbit, and try to derive properties for \mathcal{K}^{T^*} . The purpose of the section is to show Theorem 2.7, which states that in this situation, \mathcal{K}^{T^*} is an iterated fiber bundle with a sequence of projective spaces as its base. The proof is done by induction, and uses the combinatorial description of components with a dense Jordan orbit, provided by Proposition 2.6.

7.1. Concatenation of standard tableaux

In this subsection, we show a preliminary result, saying that if the standard tableau T is obtained as the concatenation (which we will call the sum) of two standard tableaux T_1, T_2 , then the component \mathcal{K}^T is isomorphic to the product of the components $\mathcal{K}^{T_1}, \mathcal{K}^{T_2}$. We start with the definition of the sum of two standard tableaux.

Let Y_1, Y_2 be two Young diagrams with n_1 and n_2 boxes, respectively. We define the sum $Y = Y_1 + Y_2$ as the Young diagram with $n_1 + n_2$ boxes such that for all j , the length of the j -th row of Y is the sum of the lengths of the j -th rows of Y_1 and Y_2 . Now, let T_1, T_2 be two standard tableaux of shape Y_1 and Y_2 respectively. We define the sum $T = T_1 + T_2$ as the standard tableau of shape Y such that for all j , the j -th row of T contains the entries of the j -th row of T_1 , and the entries of the j -th row of T_2 increased by n_1 . For instance:

$$\text{If } T_1 = \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline 4 & \\ \hline 5 & \\ \hline \end{array} \quad \text{and} \quad T_2 = \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array}, \quad \text{then} \quad T = T_1 + T_2 = \begin{array}{|c|c|c|c|} \hline 1 & 2 & 6 & 8 \\ \hline 3 & 7 & 9 & \\ \hline 4 & & & \\ \hline 5 & & & \\ \hline \end{array}.$$

To the tableaux T_1, T_2, T , we associate respective components $\mathcal{K}^{T_1}, \mathcal{K}^{T_2}, \mathcal{K}^T$ imbedded in the appropriate Springer fibers.

Proposition 7.1. *Suppose $T = T_1 + T_2$. Then, $\mathcal{K}^T \cong \mathcal{K}^{T_1} \times \mathcal{K}^{T_2}$.*

Proof. Let $V = \mathbb{C}^{n_1+n_2}$, and let $u \in \text{End}(V)$ be nilpotent of Jordan form $Y(u) = Y$, so that the component \mathcal{K}^T is imbedded in the Springer fiber \mathcal{B}_u . For a u -stable subspace $W \subset V$, we denote by $Y(u|_W)$ and $Y(u|_{V/W})$ the Young diagrams representing the Jordan forms of the maps induced by u on W and V/W , respectively. Let $\mathcal{G}_m(V)$ be the set of m -dimensional u -stable subspaces of V . Our first step is to show that the set

$$\mathcal{A} = \{W \in \mathcal{G}_{n_1}(V) : Y(u|_W) = Y_1 \text{ and } Y(u|_{V/W}) = Y_2\}$$

is a single point.

We show this by induction on $n_1 + n_2$ with immediate initialization if $n_1 + n_2 = 0$. Assume $n_1 + n_2 > 0$. As in the proof of Proposition 4.1, we

denote by \tilde{V} the space of linear forms $\phi : V \rightarrow \mathbb{C}$ and by $\tilde{u} : \tilde{V} \rightarrow \tilde{V}$ the dual nilpotent map of u . If $W \subset V$ is a subspace, let $W^\perp = \{\phi \in \tilde{V} : W \subset \ker \phi\}$ be its dual space. Let $\tilde{\mathcal{A}} = \{W \in \mathcal{G}_{n_2}(\tilde{V}) : Y(\tilde{u}|_W) = Y_2 \text{ and } Y(\tilde{u}|_{\tilde{V}/W}) = Y_1\}$. Notice that $W \in \mathcal{A}$ if and only if $W^\perp \in \tilde{\mathcal{A}}$. Therefore, up to considering $\tilde{\mathcal{A}}$ instead of \mathcal{A} , we may assume that the length of the first column of Y_1 is bigger than or equal to the length of the first column of Y_2 . Thus, the lengths of the first columns of Y_1 and Y are equal, which implies that every $W \in \mathcal{A}$ satisfies $\dim \ker(u|_W) = \dim \ker u$, hence $\ker u \subset W$. Let $r = \dim \ker u$. Let Y'_1 be the Young diagram obtained from Y_1 by deleting the first column. Let $V' = V/\ker u$ and let $u' \in \text{End}(V')$ be the nilpotent map induced by u . Thus $Y(u') = Y'_1 + Y_2$. Let $\mathcal{A}' = \{W \in \mathcal{G}_{n_1-r}(V') : Y(u'|_W) = Y'_1 \text{ and } Y(u'|_{V'/W}) = Y_2\}$. It is easy to see that $W \in \mathcal{A}$ if and only if $W/\ker u \in \mathcal{A}'$. By induction hypothesis, \mathcal{A}' is single point. It results that \mathcal{A} is a single point, as it was claimed.

Denote by W_1 the unique element of \mathcal{A} .

For $l \in \{1, 2\}$ and $j \in \{1, \dots, n_l\}$, let $Y_j^{T_l}$ be the shape of the subtableau of T_l of entries $1, \dots, j$. Alternatively, for $0 \leq i < j \leq n_1 + n_2$, let $Y_{j/i}^T$ be the shape of the rectification by jeu de taquin of the skew subtableau of T of entries $i+1, \dots, j$ (we refer to [6] for the definition of jeu de taquin). By [8, Theorem 3.3], the set $\{F = (V_0, \dots, V_{n_1+n_2}) \in \mathcal{K}^T : Y(u|_{V_j/V_i}) = Y_{j/i}^T\}$ is a nonempty open subset of the component \mathcal{K}^T . On the other hand, by definition of T , we have $Y_{j/0}^T = Y_j^{T_1}$ for all $j \in \{1, \dots, n_1\}$ and $Y_{j/n_1}^T = Y_{j-n_1}^{T_2}$ for all $j \in \{n_1 + 1, \dots, n_1 + n_2\}$. It follows that $\{F \in \mathcal{K}^T : V_{n_1} \in \mathcal{A}\}$ is a nonempty open subset of \mathcal{K}^T , and we infer that $V_{n_1} = W_1$ for all $(V_0, \dots, V_{n_1+n_2}) \in \mathcal{K}^T$.

Let $u_1 = u|_{W_1}$ and $u_2 = u|_{V/W_1}$ be the nilpotent maps induced by u . Let \mathcal{B}_1 (resp. \mathcal{B}_2) be the variety of complete flags of W_1 (resp. of V/W_1), and let $\mathcal{B}_{u_1} \subset \mathcal{B}_1$ (resp. $\mathcal{B}_{u_2} \subset \mathcal{B}_2$) be the subvariety of u_1 -stable (resp. u_2 -stable) flags. Thus the components $\mathcal{K}^{T_1}, \mathcal{K}^{T_2}$ are imbedded in \mathcal{B}_{u_1} and \mathcal{B}_{u_2} respectively. The map

$$\Phi : \{F \in \mathcal{B}_u : V_{n_1} = W_1\} \rightarrow \mathcal{B}_{u_1} \times \mathcal{B}_{u_2}, \quad (V_0, \dots, V_{n_1+n_2}) \mapsto ((V_j)_{j=0}^{n_1}, (V_j/W_1)_{j=n_1}^{n_2})$$

is an isomorphism of algebraic varieties. Hence, it induces an isomorphism of \mathcal{K}^T onto its image. Let Φ_1, Φ_2 be the projections of Φ on both terms. We have

$$\{F \in \mathcal{K}^T : Y(u|_{V_j}) = Y_j^{T_1}, \forall j = 1, \dots, n_1\} \subset \Phi_1^{-1}(\mathcal{K}^{T_1})$$

while $\{F \in \mathcal{K}^T : Y(u|_{V_j}) = Y_j^{T_1}, \forall j = 1, \dots, n_1\}$ is a nonempty open subset of \mathcal{K}^T . Therefore, $\Phi_1(\mathcal{K}^T) \subset \mathcal{K}^{T_1}$. Similarly, we have

$$\{F \in \mathcal{K}^T : Y(u|_{V_j/V_{n_1}}) = Y_{j-n_1}^{T_2}, \forall j = n_1 + 1, \dots, n_1 + n_2\} \subset \Phi_2^{-1}(\mathcal{K}^{T_2})$$

while $\{F \in \mathcal{K}^T : Y(u|_{V_j/V_{n_1}}) = Y_{j-n_1}^{T_2}, \forall j = n_1 + 1, \dots, n_1 + n_2\}$ is a nonempty open subset of \mathcal{K}^T . Therefore, $\Phi_2(\mathcal{K}^T) \subset \mathcal{K}^{T_2}$. It results $\Phi(\mathcal{K}^T) \subset \mathcal{K}^{T_1} \times \mathcal{K}^{T_2}$. The product $\mathcal{K}^{T_1} \times \mathcal{K}^{T_2}$ is an irreducible variety, and due to formula (1), it has the same dimension as \mathcal{K}^T . Hence, we actually have the equality $\Phi(\mathcal{K}^T) = \mathcal{K}^{T_1} \times \mathcal{K}^{T_2}$. We have finally obtained that Φ restricts to an isomorphism between \mathcal{K}^T and the product $\mathcal{K}^{T_1} \times \mathcal{K}^{T_2}$. The proof of the proposition is then complete. \square

7.2. Proof of Theorem 2.7

The proof relies on Proposition 7.1 and the following additional preliminary result (see [5, Proposition 6.1]).

Lemma 7.2. *Let T be a standard tableau of entries $1, \dots, n$, and let T' be the subtableau of entries $1, \dots, n - 1$. Let $\mathcal{K}^T, \mathcal{K}^{T'}$ be the components corresponding to T, T' in the appropriate Springer fibers. Assume that n lies in the last column of T . Let k denote the length of the last column of T . Then \mathcal{K}^T is a locally trivial fiber bundle of base the projective space \mathbb{P}^{k-1} , of fiber isomorphic to $\mathcal{K}^{T'}$.*

Proof of Lemma 7.2. Let s be the nilpotent index of u (which coincides with the number of columns of T). We then have $k = \dim(V/\ker u^{s-1})$. In the case where n lies in the last column of T , it follows from the definition of \mathcal{K}^T that $\ker u^{s-1} \subset V_{n-1}$ for all $F = (V_0, \dots, V_n) \in \mathcal{K}^T$. Then, we consider the map

$$\Phi : \mathcal{K}^T \rightarrow \mathbb{P}^{k-1}, \quad (V_0, \dots, V_n) \mapsto V_{n-1}/\ker u^{s-1}$$

and we show that it is a locally trivial fiber bundle of fiber $\mathcal{K}^{T'}$ (see [5, Proposition 6.1], or [4, Theorem 2.1]). □

Now, let us prove Theorem 2.7. We fix $V = \mathbb{C}^n$ and a nilpotent endomorphism $u \in \text{End}(V)$. Let $\lambda(u) = (\lambda_1, \dots, \lambda_r)$ be the sizes of its Jordan blocks and let $Y(u)$ be the corresponding Young diagram (i.e., of rows of sizes $\lambda_1, \dots, \lambda_r$). Let $\lambda^*(u) = (\lambda_1^*, \dots, \lambda_s^*)$ be the conjugate partition of n . Let $u^* \in \text{End}(V)$ be nilpotent, such that u, u^* have conjugate Jordan forms, hence $\lambda(u^*) = \lambda^*(u)$. Let $Y(u^*)$ be the Young diagram corresponding to u^* . If T is a standard tableau of shape $Y(u)$, then its transposed tableau T^* has shape $Y(u^*)$. Hence, to the irreducible component $\mathcal{K}^T \subset \mathcal{B}_u$, we associate the component $\mathcal{K}^{T^*} \subset \mathcal{B}_{u^*}$. We suppose that \mathcal{K}^T contains a dense Jordan orbit. Our purpose is to show that \mathcal{K}^{T^*} is an iterated fiber bundle of base $(\mathbb{P}^1, \dots, \mathbb{P}^{\lambda_1-1}, \dots, \mathbb{P}^1, \dots, \mathbb{P}^{\lambda_r-1})$. We reason by induction on n , with immediate initialization for $n = 1$. Suppose the property holds until the rank $n - 1 \geq 1$ and let us show it for n .

By Proposition 2.6, the tableau T is of the form $T = T_\pi$ for some $\pi \in \Pi_u^1$. Set $\pi = (I_1, \dots, I_r)$, where $|I_j| = \lambda_j$ and $I_1 \sqcup \dots \sqcup I_r = \{1, \dots, n\}$. We distinguish two cases, depending on whether $1, n$ belong to the same subset I_j or not.

(1) Suppose there is $j \in \{1, \dots, r\}$ such that $1, n \in I_j$.

Then, we have $I_k < I_j$ for all $k \neq j$ (see section 2.3.2). By definition of the set Π_u^1 , this implies that $|I_j| = \min\{|I_k| : k = 1, \dots, r\}$. In other words, λ_j is the length of the minimal row of the tableau T_π . By definition of T_π (see section 2.3.1), n is the last entry of the λ_j -th column of T_π , hence it lies in the last row of T_π . By consequent, n lies in the last column of the transposed tableau $T^* = (T_\pi)^*$.

Note that the last column of T^* has length λ_r . Let $(T^*)'$ be the subtableau of T^* of entries $1, \dots, n - 1$. By Lemma 7.2, there is a locally trivial fiber bundle $\mathcal{K}^{T^*} \rightarrow \mathbb{P}^{\lambda_r-1}$ of fiber isomorphic to $\mathcal{K}^{(T^*)'}$.

Let $\lambda'_r = \lambda_r - 1$ and for $k \neq r$, let $\lambda'_k = \lambda_k$. Similarly as in section 4.2, we consider $u' \in \text{End}(\mathbb{C}^{n-1})$ of Jordan form $\lambda(u') = (\lambda'_1, \dots, \lambda'_r)$. Set $I'_j = I_j \setminus \{n\}$,

and $I'_k = I_k$ for $k \neq j$. Let $\pi' = (I'_1, \dots, I'_r)$. As in section 4.2, we see that $\pi' \in \Pi_{u'}^1$. Moreover, the tableau $T_{\pi'}$ is the subtableau of T_π of entries $1, \dots, n-1$. It follows that $(T^*)'$ coincides with the transposed tableau $(T_{\pi'})^*$. Then, by induction hypothesis, the component $\mathcal{K}^{(T^*)'}$ is an iterated fiber bundle of base $(\mathbb{P}^1, \dots, \mathbb{P}^{\lambda_1-1}, \dots, \mathbb{P}^1, \dots, \mathbb{P}^{\lambda_r-2})$. Therefore, \mathcal{K}^{T^*} is an iterated fiber bundle of the desired base.

(2) Suppose there are $j, k \in \{1, \dots, r\}$, $j \neq k$ such that $1 \in I_j$, $n \in I_k$.

We set in this case $n_1 = \max I_j$. As $n \notin I_j$, we have $n_1 < n$. Again according to the definition of the set Π_u^1 , the sequence π has no crossing, which implies that for all l , we have either $I_l \subset \{1, \dots, n_1\}$ or $I_l \subset \{n_1 + 1, \dots, n\}$. We consider the two subsequences $\pi_1 = (I_l : l = 1, \dots, r, I_l \subset \{1, \dots, n_1\})$ and $\pi_2 = (\phi(I_l) : l = 1, \dots, r, I_l \subset \{n_1 + 1, \dots, n\})$, where

$$\phi : \{n_1 + 1, \dots, n\} \rightarrow \{1, \dots, n - n_1\}$$

is defined by $\phi(i) = i - n_1$. Let $u_1 \in \text{End}(\mathbb{C}^{n_1})$ and $u_2 \in \text{End}(\mathbb{C}^{n-n_1})$ be nilpotent of Jordan forms $\lambda(u_1) = (\lambda_l : l = 1, \dots, r, I_l \subset \{1, \dots, n_1\})$ and $\lambda(u_2) = (\lambda_l : l = 1, \dots, r, I_l \subset \{n_1 + 1, \dots, n\})$, respectively. Then, we clearly have $\pi_1 \in \Pi_{u_1}^1$ and $\pi_2 \in \Pi_{u_2}^1$.

Let $T_1 = T_{\pi_1}$ and $T_2 = T_{\pi_2}$ be the standard tableaux associated to π_1 and π_2 , and let $(T_1)^*$ and $(T_2)^*$ be their respective transposes. From the definition of the tableaux $T_\pi, T_{\pi_1}, T_{\pi_2}$, it is easy to see that the transpose $T^* = (T_\pi)^*$ is in fact the sum $T^* = (T_1)^* + (T_2)^*$. By Proposition 7.1, it follows that the component \mathcal{K}^{T^*} is isomorphic to the product $\mathcal{K}^{(T_1)^*} \times \mathcal{K}^{(T_2)^*}$. Moreover, by induction hypothesis, $\mathcal{K}^{(T_1)^*}$ is an iterated fiber bundle of base $(\mathbb{P}^1, \dots, \mathbb{P}^{\lambda_l-1})_l$ where l runs over $\{l = 1, \dots, r : I_l \subset \{1, \dots, n_1\}\}$. Similarly, $\mathcal{K}^{(T_2)^*}$ is an iterated fiber bundle of base $(\mathbb{P}^1, \dots, \mathbb{P}^{\lambda_l-1})_l$ for l running over $\{l = 1, \dots, r : I_l \subset \{n_1 + 1, \dots, n\}\}$. We infer that \mathcal{K}^{T^*} is an iterated fiber bundle of the desired base.

The proof of Theorem 2.7 is then complete.

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Index of the notation

- §1 $V, n, u, \mathcal{B}, \mathcal{B}_u,$
 2.1.1 $\lambda(u), Y(u), \lambda^*(u),$
 2.1.2 $Y_i^T, Y(u|_{V_i}), \mathcal{B}_u^T, \mathcal{K}^T,$
 2.1.3 $\Lambda_u, \Lambda_u^*, \mathcal{U}_\pi^{\text{BC}}, \mathcal{K}_\pi^{\text{BC}}, \mathcal{K}_\pi^{\text{R}},$
 2.1.4 $T_\pi (\pi \in \Lambda_u), T_\pi^* (\pi \in \Lambda_u^*), u^*, T^*,$
 2.2 $\pi \geq \rho,$
 2.3 $Z_u,$
 2.3.1 $\Pi_u, \pi : \{1, \dots, n\} \rightarrow \{\emptyset, 1, \dots, n\}, \mathcal{Z}_\pi, T_\pi (\pi \in \Pi_u), \Pi_u^0,$
 2.3.2 $I_j < I_k, \Pi_u^1,$
 6.2 $\tau_0, \sigma(\tau_0), \mathbf{S}_u, F_\sigma, D, \tau_{\underline{d}}, \sigma_{\underline{d}}, F_{\underline{d}},$
 6.3 $\Omega_{\underline{d}}, \phi_{i,j},$
 7.1 $Y_1 + Y_2, T_1 + T_2.$

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