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The Smoothness of Orbital Measures on Exceptional Lie Groups and Algebras

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Abstract. Suppose that G is a compact, connected, simple, exceptional Lie group with Lie algebra $\mathfrak{g}$. We determine the sharp minimal exponent k_0, which depends on G or $\mathfrak{g}$, such that the convolution of any k_0 continuous, G-invariant measures is absolutely continuous with respect to Haar measure. The exponent k_0 is also the minimal integer such that any k_0-fold product of conjugacy classes in G or k_0-fold sum of adjoint orbits in $\mathfrak{g}$ has non-empty interior. Unlike in the classical case, the answer can be less than the rank of G or $\mathfrak{g}$.

We also establish a dichotomy for orbital measures μ, supported on non-trivial conjugacy classes or adjoint orbits of minimal non-zero dimension: for each k, either $\mu^k \in L^2$ or μ^k is singular with respect to Haar measure.
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1. Introduction

Let G be a compact, connected, simple Lie group. In [12], Ragozin proved the surprising fact that the convolution of $\dim(G)$ continuous G-invariant measures is absolutely continuous with respect to Haar measure on G. His work implies that a product of $\dim(G)$ non-trivial conjugacy classes in G has positive measure and even non-empty interior. He was unable to decide if $\dim(G)$ was minimal with these properties, and speculated that it was not.

In [4] and [5], the minimum number of convolution powers with this absolute continuity property and the minimal integer k such that every product (or sum) of k non-trivial conjugacy classes in G (respectively, adjoint orbits in $\mathfrak{g}$) has non-empty interior was determined for each of the classical Lie groups and algebras. The answer depends on the Lie type, but in all cases was between r and $2r$, where $r = \operatorname{rank} \mathfrak{g}$. Ragozin's result was also improved in [8] for the five exceptional, compact, simple Lie groups with the exponent reduced to n for the groups of type

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E_n (where $n = 6, 7, 8$), to 6 for a group of type F_4, and to 3 for a group of type G_2.[1]

In this paper, we complete Ragozin's project by determining the sharp exponent for the exceptional Lie groups and algebras. We prove that $\mu_1 * \cdots * \mu_k$ is absolutely continuous with respect to Haar measure for all G-invariant, continuous measures μ_j on an exceptional Lie group G or Lie algebra $\mathfrak{g}$ if and only if $k \geq k_0$, where k_0 depends on the type of the exceptional Lie group or algebra and is specified below:

$$k_0 = \begin{cases} 3 & \text{if } G \text{ or } \mathfrak{g} \text{ is of Lie type } E_6, E_7, \text{ or } E_8 \\ 3 & \text{if } G \text{ is the Lie group of type } G_2 \\ 4 & \text{if } G \text{ is the Lie group of type } F_4 \\ 2 & \text{if } \mathfrak{g} \text{ is the Lie algebra of type } F_4 \text{ or } G_2. \end{cases} \tag{1.1}$$

Standard arguments show that k_0 is also the minimal integer such that every k_0-fold product (sum) of non-trivial conjugacy classes in G (adjoint orbits in $\mathfrak{g}$) has non-empty interior.

The approach taken in [8] was to use estimates on the rate of decay of characters on the group, together with the Peter–Weyl theorem, to deduce $L^2(G)$ results for convolutions of continuous, orbital measures on G, an important class of G-invariant measures. In this paper we take, instead, the direct approach of studying the l^2 norm of the Fourier transform of powers of orbital measures on G, using the combinatorial method developed in [7]. We prove that $\mu^{k_0} \in L^2(G)$ for all continuous orbital measures μ on G, and from this fact one can deduce that k_0 is sufficient for Ragozin's absolute continuity problem in the group case. For the Lie algebra problem, we apply a transference argument.

Easy geometric arguments are used to show that the non-trivial conjugacy classes (or adjoint orbits) of minimum dimension have the property that their $(k_0 - 1)$-fold product (respectively, sum) has Haar measure zero. Consequently, orbital measures μ, supported on the minimal non-trivial conjugacy classes or adjoint orbits satisfy a dichotomy: for each positive integer k, either $\mu^k \in L^2$ or μ^k is singular with respect to Haar measure on G or $\mathfrak{g}$. This same dichotomy is known to hold for the classical Lie groups and algebras (see [4]–[7]). Sums of orbits, products of conjugacy classes and convolutions of measures supported on these manifolds were also investigated by Ricci and Stein in [13], [14], by Dooley and Wildberger in [2], [3], and by Wildberger in [15].

2. Background Results

2.1. Notation. Let G be a compact, connected, simple Lie group with Lie algebra $\mathfrak{g}$, let $\mathbb{T}$ be a maximal torus of G and $\mathfrak{t}$ be its Lie algebra, which we also call a torus. We denote by Φ the set of roots of the complexification of $\mathfrak{g}$ with respect to the complexified torus and write Φ^+ for the positive roots. The group G acts on its Lie algebra by the adjoint action Ad_G, and acts on itself by the

[1] The dimensions of these groups are listed in Table 5 for comparison with Ragozin's result.

operation of conjugation, which we denote in the same way. The meaning will be clear from the context.

A finite, complex Borel measure μ on G (or on $\mathfrak{g}$) is called *G-invariant* if $\mu(E)=\mu(\mathrm{Ad}_G(g)E)$ for all $g\in G$ and Borel sets $E\subseteq G$ (respectively, $E\subseteq\mathfrak{g}$). The G-invariant measures on the group are often also called *central* since they commute with all other measures under convolution.

Given $X\in\mathfrak{g}$, the *orbital measure* μ_X is the Borel measure on $\mathfrak{g}$ defined by the rule

$$\int_{\mathfrak{g}} f\,d\mu_X=\int_G f(\mathrm{Ad}_G(g)X)\,dm_G(g)$$

for any continuous, compactly supported function f on $\mathfrak{g}$. Here m_G denotes the Haar measure on G. The probability measure μ_X is G-invariant and supported on the compact adjoint orbit $O_X\subseteq\mathfrak{g}$, the image of X under the Ad_G action. Given $x\in G$, the orbital measure μ_x on G is defined similarly, and is the G-invariant probability measure supported on the conjugacy class C_x in G containing x. A measure is said to be continuous if the measure of every singleton is zero. All orbital measures μ_X or μ_x, with $X\in\mathfrak{g}$ and $x\in G$, are continuous except if $X=0$ or x belongs to the center of G. Of course, x is in the center of the group G if and only if C_x is a singleton, and we call these the trivial conjugacy classes.

Every adjoint orbit and conjugacy class has zero Haar measure, being a proper submanifold, consequently, all orbital measures are singular with respect to Haar measure. We also recall that every orbit and conjugacy class contains a torus element.

Roots are defined not only on the torus of the Lie algebra, but also on torus elements in the group by the formula $\alpha(x)=\alpha(X)\mod 2\pi$, where $X\in\mathfrak{t}$ is any element with $\exp X=x\in\mathbb{T}$. We say that the root $\alpha\in\Phi$ *annihilates* the element $X\in\mathfrak{t}$ or $x\in\mathbb{T}$ if $\alpha(X)=0$ or $\alpha(x)=0\mod 2\pi$. The set of annihilating roots is a root subsystem of Φ and thus has a Lie type. By the *type* of x we mean the Lie type of its set of annihilating roots. The elements x and $\mathrm{Ad}_G(g)x$ have the same type, so we may also speak of the type of an adjoint orbit or conjugacy class.

The following geometric fact will be useful later in finding k_0. By $(k)O_X$ we mean the k-fold sum of orbit O_X, and by C_x^k we mean the k-fold product of the conjugacy class C_x.

Lemma 2.1. *If $X\in\mathfrak{t}$ or $x\in\mathbb{T}$ and the number non-annihilating roots of X (or x) is less than $\dim(\mathfrak{g})/k$, then $(k)O_X$ (or C_x^k) has measure zero and μ_X^k (or μ_x^k, respectively) is singular with respect to Haar measure.*

Proof. It is known that the dimension of the orbit, O_X, is equal to the number of non-annihilating roots of $X\in\mathfrak{t}$ [11, VI.4], so the hypothesis guarantees that $(k)O_X$ has Haar measure zero. Since the k-fold convolution product μ_X^k is supported on $(k)O_X$, it is clearly a singular measure. The argument for μ_x^k and C_x^k are similar. ∎

To prove the sufficiency of the value of k_0, it is actually enough to show that $\mu_x^{k_0}\in L^2(G)$ for all x not in the center of G. The reason for this is that the

$L^2(\mathfrak{g})$ results for orbital measures on the Lie algebra $\mathfrak{g}$ will then follow from an application of a transference principle established in [7], as we explain below.

Lemma 2.2. *Suppose $X \in \mathfrak{t}$ and $\mu_x^k \in L^2(G)$ whenever $x \in \mathbb{T}$ has the same Lie type as X. Then $\mu_X^k \in L^2(\mathfrak{g})$.*

Proof. Fix a neighbourhood $U \subseteq \mathfrak{g}$ on which the exponential map is a diffeomorphism. For almost all $\lambda > 0$, the elements $X, \lambda X \in \mathfrak{t}$ and $\exp(\lambda X) \in \mathbb{T}$ have exactly the same set of annihilating roots. Choose such a λ, sufficiently small that $(k)O_{\lambda X} \subseteq U$. If $\mu_{\exp \lambda X}^k \in L^2(G)$, then $\mu_{\lambda X}^k \in L^2(\mathfrak{g})$, according to the transference principle [7, Cor. 7.3]. But the Fourier transform of μ_X and $\mu_{\lambda X}$ are dilates, hence $\mu_X^k \in L^2(\mathfrak{g})$ if and only if $\mu_{\lambda X}^k \in L^2(\mathfrak{g})$. ∎

2.2. Combinatorial Criterion. To prove that $\mu_x^k \in L^2(G)$ for suitable exponents k, we rely heavily on a combinatorial criterion established in [7], which we briefly review. We suppose that $\{\alpha_1, \dots, \alpha_n\}$ is a base for a root system Φ of rank n and $\{\lambda_1, \dots, \lambda_n\}$ is the set of fundamental dominant weights, that is, the dual basis vectors which satisfy $(\alpha_i, \lambda_j) = \delta_{ij}$. (In our application, Φ will be the root system of one of the exceptional groups.) Define

$$S_j = \{\alpha \in \Phi^+ : (\alpha, \lambda_j) \neq 0\}.$$

Given a set of l integers, $i_1, \dots, i_l$, satisfying

$$n \geq i_1 > i_2 > \cdots > i_l \geq 1,$$

and a root subsystem Ψ of Φ, inductively define

$$X_j = S_{i_j} \setminus \bigcup_{k=1}^{j-1} S_{i_k} = \left\{ \alpha \in \Phi^+ \setminus \bigcup_{k=1}^{j-1} X_k : (\alpha, \lambda_{i_j}) \neq 0 \right\} \text{ for } j = 1, \dots, l,$$

$$B_j = B_j(\Psi) = \left\{ \alpha \in \Psi^+ \setminus \bigcup_{k=1}^{j-1} B_k : (\alpha, \lambda_{i_j}) \neq 0 \right\} \text{ for } j = 1, \dots, l$$

and put

$$G_j = X_j \setminus B_j.$$

We call G_j and B_j the 'good' and 'bad' roots, respectively, arising at step j relative to the given set of indices. The expressions $|X_j|$, $|B_j|$ and $|G_j|$ denote the cardinalities of these sets.

Let $\kappa(i_1, \dots, i_l, \Psi)$ be the minimum integer k such that

$$\sum_{j=1}^{l} ((k-1)|X_j| - k|B_j|) = \sum_{j=1}^{l} ((k-1)|G_j| - |B_j|) > \frac{l}{2}.$$

The combinatorial criterion for μ_x^k to belong to $L^2(G)$ is the content of the next theorem, whose proof relies upon the Weyl character and degree formulas and the Peter–Weyl theorem.

Table 1: The positive roots for the exceptional groups

Type	Positive Roots Φ^+	#
E_8	$e_i \pm e_j : 1 \le i < j \le 8$; $\frac{1}{2}(e_8 + \sum_{k=1}^{7} s_k e_k) : s_k = \pm 1, \prod_{k=1}^{7} s_k = 1$	120
E_7	$e_7 - e_8$; $e_i \pm e_j : 1 \le i < j \le 6$; $\frac{1}{2}(e_8 - e_7 + \sum_{k=1}^{6} s_k e_k) : s_k = \pm 1, \prod_{k=1}^{6} s_k = 1$	63
E_6	$e_i \pm e_j : 1 \le i < j \le 5$; $\frac{1}{2}(e_8 - e_7 - e_6 + \sum_{k=1}^{5} s_k e_k) : s_k = \pm 1, \prod_{k=1}^{5} s_k = 1$	36
F_4	$e_i \pm e_j : 1 \le i < j \le 4; e_l : 1 \le l \le 4$; $\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)$	24
G_2	$e_i - e_j : 1 \le i < j \le 3$; $2e_i - e_j - e_k : i \ne j \ne k \in \{1, 2, 3\}$	6

Theorem 2.3 ([7, Thm 6.1]). *Suppose that G is a compact, connected, simple Lie group of rank n, and $\Phi(x)$ is the set of annihilating roots of $x \in \mathbb{T}$. Let*

$$\kappa_0(x) = \max\{\kappa(i_1, \ldots, i_l, \Psi)\}$$

where the maximum is taken over all $l \in \{1, \ldots, n\}$, all sets of indices that satisfy $n \ge i_1 > i_2 > \cdots > i_l \ge 1$, and all root subsystems Ψ that are conjugate under the Weyl group to $\Phi(x)$. Then $\mu_x^{\kappa_0(x)} \in L^2(G)$.

2.3. Roots of the Exceptional Groups. For the convenience of the reader, we list the positive roots for the exceptional groups in Table 1, and the total number of positive roots. Throughout, e_i denotes one of the standard basis vectors. For further background on roots and root systems we refer the reader to [9].

3. The exceptional Lie groups and algebras E_6, E_7 and E_8

We continue to use the notation and terminology described in the previous section. The goal of this section is to prove the following theorem.

Theorem 3.1. *Suppose that G is a compact, connected, simple, exceptional Lie group of type E_6, E_7 or E_8, with Lie algebra $\mathfrak{g}$. If μ is any continuous orbital measure on G or $\mathfrak{g}$, then $\mu^3 \in L^2$. Moreover, there exists $X \in \mathfrak{g}$ such that $O_X + O_X$ has Haar measure zero and $\mu_X * \mu_X$ is singular with respect to Haar measure on $\mathfrak{g}$. Similarly, there exists $x \in G$ such that $m_G(C_x^2) = 0$ and $\mu_x * \mu_x$ is singular with respect to m_G.*

The difficult part of this argument is proving that $\mu^3 \in L^2$. Since every $X \in \mathfrak{t}$ has the same type as some $x \in \mathbb{T}$, Lemma 2.2 implies that it is sufficient to prove that $\mu_x^3 \in L^2(G)$ whenever μ_x is a continuous orbital measure on G.

According to Theorem 2.3, it suffices to prove that $\kappa_0(x) \leq 3$ for every non-central $x \in \mathbb{T}$. Furthermore, since the set of annihilating roots of a non-central torus element is a proper root subsystem [1], and any proper root subsystem is contained in a maximal subsystem, it suffices to verify that $\kappa(i_1, \ldots, i_l, \Psi) \leq 3$ whenever Ψ is a maximal root subsystem contained in the root system of type E_n. We will do this for E_6, E_7, and E_8 separately.

We begin by recording some facts about the structure of the root system E_8. The roots may be divided into two classes: the *regular roots*, those of the form $\pm e_i \pm e_j$; and the others, which we call the *peculiar roots*. In E_8 there are 56 regular positive roots and 64 peculiar positive roots. The regular roots form a root subsystem of type D_8.

In fact, the regular roots form a root subsystem in all the exceptional groups of type E_n (where $n = 6, 7, 8$). This is a consequence of the fact that if the sum or difference of two regular roots is a root, then it is a regular root. More generally, the following easy observation will be very useful for us.

Lemma 3.2. *Let Ψ be a root subsystem. Then the regular roots in Ψ form a root subsystem.*

Proof. An intersection of root subsystems is a root subsystem. ■

The same is not true of the peculiar roots. Indeed, two peculiar roots are either orthogonal or one of their sum or difference is a regular root.

We introduce the following notation for the positive peculiar roots. By $P_{j_1,j_2,\ldots,j_l}$ we mean the positive peculiar root with a minus sign in positions $j_1, \ldots, j_l$, that is,

$$P_{j_1,j_2,\ldots,j_l} = \tfrac{1}{2}\left(e_8 - \sum_{k=1}^{l} e_{j_k} + \sum_{k \neq j_1,\ldots,j_l,8} e_k\right).$$

We write P_0 for the peculiar root with all plus signs and P_q^- for the positive peculiar root with plus signs only in positions q and 8.

Here are some useful identities. Different letters denote different indices in $1, \ldots, 7$.

$$P_0 - P_{ij} = e_i + e_j; \qquad P_0 + P_q^- = e_8 + e_q; \qquad P_{ij} - P_{ijkl} = e_k + e_l;$$
$$P_{ij} + P_j^- = e_8 - e_i; \qquad P_{ij} - P_{ik} = e_k - e_j; \qquad P_q^- - P_m^- = e_q - e_m;$$
$$P_{ijkl} + P_{imnq} = e_8 - e_i; \qquad P_{ijkl} - P_{ijkm} = e_m - e_l;$$
$$P_{ij} + P_{klmn} = e_8 + e_q; \qquad P_{ijkl} - P_{ijklmn} = e_m + e_n.$$

3.1. Exceptional group of type E_8. It is known (see [9, p. 65]) that a base for E_8 is given by the eight vectors

$$\alpha_1 = \tfrac{1}{2}(e_1 + e_8 - \sum_{i=2}^{7} e_i),$$
$$\alpha_2 = e_1 + e_2,$$
$$\alpha_k = e_{k-1} - e_{k-2} \quad \text{where } k = 3, \ldots, 8.$$

Table 2: The possibilities for the sets S_j for the case of E_8

	Positive regular roots	#	Positive peculiar roots	#
S_1	$e_j \pm e_8 : j = 1, \ldots, 7$	14	all	64
S_2	$e_i + e_j,\ e_j \pm e_8 :$ $i, j = 1, \ldots, 7$	35	all but $P_q^- :$ $q \neq 8$	57
S_3	$e_i + e_j,\ e_1 - e_i,\ e_l \pm e_8 :$ $i, j = 2, \ldots, 7,\ l = 1, \ldots, 7$	35	all but P_1^-	63
S_4	$e_k \pm e_8,\ e_i + e_j,\ e_l \pm e_j :$ $k = 1, \ldots, 7,$ $i, j = 3, \ldots, 7,\ l = 1, 2$	44	all but P_1^-, P_2^-	62
S_5	$e_k \pm e_8,\ e_i + e_j,\ e_l \pm e_j :$ $k = 1, \ldots, 7,$ $i, j = 4, \ldots, 7,\ l = 1, 2, 3$	44	all but P_{4567}, $P_i^- :$ $i = 1, 2, 3$	60
S_6	$e_k \pm e_8,\ e_i + e_j,\ e_l \pm e_j :$ $k = 1, \ldots, 7,$ $i, j = 5, 6, 7,\ l = 1, \ldots, 4$	41	all but P_j^-, $P_{j567} :$ $j = 1, \ldots, 4$	56
S_7	$e_k \pm e_8,\ e_6 + e_7,\ e_i \pm e_j :$ $k = 1, \ldots, 7,$ $j = 6, 7,\ i = 1, \ldots, 5$	35	all but P_{67}, P_j^-, $P_{ij67} :$ $i, j = 1, \ldots, 5$	48
S_8	$e_8 + e_7,\ e_i \pm e_j :$ $j = 7, 8,\ i = 1, \ldots, 6$	25	P_0, P_7^-, P_{ij}, $P_{ijkl} :$ $i, j, k, l = 1, \ldots, 6$	32

It is a routine exercise to show that the fundamental dominant weights for E_8 corresponding to the base above are the eight vectors

$$\lambda_1 = 2e_8$$

$$\lambda_2 = \tfrac{5}{2}e_8 + \tfrac{1}{2}\sum_{i=1}^{7} e_i,$$

$$\lambda_3 = \tfrac{7}{2}e_8 + \tfrac{1}{2}\left(\sum_{i=2}^{7} e_i - e_1\right),$$

$$\lambda_k = \sum_{i=k-1}^{7} e_i + (9-k)e_8 \quad \text{where } k = 4, \ldots, 8.$$

From these descriptions, it is easy to determine the sets $S_1, \ldots, S_8$. In Table 2, we list the positive regular and peculiar roots in each of the sets S_j. The numbers of such roots are also given.

The following lemmas also have analogues for E_6 and E_7.

Lemma 3.3. *Let Ψ be a root subsystem of E_8. Suppose that there are a pair of distinct indices $i, j \in \{1, \ldots, 8\}$ such that both roots $e_i \pm e_j$ belong to $\Phi \setminus \Psi$.*

Then Ψ *contains at most* 32 *positive peculiar roots.*

Proof. First, suppose that $e_i \pm e_j \in \Phi \setminus \Psi$ for some $i,j \in \{1,\dots,7\}$. By symmetry, we may assume that $i=1$ and $j=2$. Because Ψ is a root subsystem, if both P_0 and P_{12} belong to Ψ, then $P_0 - P_{12} = e_1 + e_2$ must also belong to Ψ. Since this is not the case, at most one of P_0 or P_{12} belongs to Ψ. Similarly, each of the following 32 pairs,

$$P_0, P_{12};\ P_1^-, P_2^-;\ P_{12ij}, P_{ij};\ P_{1j}, P_{2j};\ P_{1ijk}, P_{2ijk};\ P_{12ijkl}, P_{ijkl}$$

with $i,j,k,l \in \{3,\dots,7\}$ (with different letters denoting different indices) has the property that one of their sum or difference is equal to either e_1+e_2 or e_1-e_2. If both peculiar roots of one of these 32 pairs belongs to Ψ, then so must their sum or difference, hence Ψ contains at most one of each pair.

Otherwise, the missing pair must be $e_j \pm e_8$ where, without loss of generality, $j=1$. A similar argument applies with the 32 pairs

$$P_0, P_1^-;\ P_{1j}, P_j^-;\ P_{iq}, P_{klmn};\ P_{1ijk}, P_{1lmn} \quad \text{with } i,j,k,l,m,n \in \{2,\dots,7\},$$

and the proof is finished. ■

Lemma 3.4. *Suppose that* $i,j \in \{1,\dots,6\}$ *and* Ψ *is a root subsystem of* E_8.

(a) If both $e_i \pm e_j \in \Phi \setminus \Psi$, *then there are at most* 16 *peculiar roots in* $\Psi \cap S_8$.

(b) If only one of $e_i \pm e_j$ *belongs to* Ψ, *then there are at most* 24 *peculiar roots in* $\Psi \cap S_8$.

Proof. Without loss of generality, $i,j = 1,2$. Consider the sixteen pairs of peculiar roots, all of which belong to S_8,

$$P_0, P_{12};\ P_{12ij}, P_{ij};\ P_{1j}, P_{2j};\ P_{1ijk}, P_{2ijk};\ P_7^-, P_{3456}$$

with $i,j,k \in \{3,\dots,6\}$. Eight of these pairs have their sum or difference equal to e_1+e_2 and the other eight give e_1-e_2. Now argue as above. ■

Lemma 3.5. *Suppose that* Ψ *is a root subsystem of* E_8 *and there is a pair* $i,j \in \{1,\dots,5\}$ *with both* $e_i \pm e_j \in \Phi \setminus \Psi$. *Then* $\Psi \cap S_7$ *contains at most* 24 *peculiar roots.*

Proof. The peculiar roots of S_7 may be partitioned into 24 distinct pairs, each of which has sum or difference equal to one of $e_i \pm e_j$. ■

The expression $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z.

Lemma 3.6. *A root subsystem of type* A_m *in* D_n, *where* $m \geq 4$, *contains at most* $\lfloor \frac{1}{4}(m+1)^2 \rfloor$ *positive roots of the form* $e_i + e_j$.

Proof. A root subsystem of type A_m in D_n, where $m \geq 4$, must have the structure

$$\{s_i e_i - s_j e_j : i,j \in I, i \neq j\},$$

where I is an $m+1$ element subset of $\{1,\ldots,8\}$ and s_i is a choice of ± 1. The root e_i+e_j belongs to A_m if and only if $s_i s_j=-1$, and the maximum number of such roots is $\frac{1}{4}(m+1)^2$ if k is odd, or $\frac{1}{4}m(m+2)$ if m is even. ■

3.2. Proof of Theorem 3.1 for E_8. Identify the torus of the Lie algebra of E_8 with $\mathbb{R}^8$ and consider the element

$$X=(0,\ldots,0,\pi,\pi)\in\mathfrak{t}.$$

The set of annihilating roots of X is of Lie type E_7 (exactly as described in section 2) and the set of annihilating roots of $x=\exp X$ is of type $E_7\times A_1$ (namely, the E_7 described previously, together with $\pm(e_7+e_8)$). Thus X and x have 114 and 112 non-annihilating roots respectively. As $\frac{1}{2}\dim(E_8)=124$, Lemma 2.1 implies that μ_X^2 and μ_x^2 are singular measures in $\mathfrak{g}$ and G respectively, and O_X+O_X and C_x^2 have Haar measure zero.

We now turn to proving that $\mu_x^3\in L^2(G)$ for all continuous orbital measures μ_x on G. The maximal proper root subsystems of E_8 may be deduced from the Borel–Siebenthal theorem and are listed in [10, p. 136]. They are of type $E_7\times A_1$, D_8, $E_6\times A_2$, A_8, and $A_4\times A_4$, and have 64, 56, 39, 36 and 20 positive roots respectively. It clearly suffices to check that $\kappa(i_1,\ldots,i_l,\Psi)\le 3$ when Ψ is one of these types.

Since each set S_j contains at least 57 elements, if $|\Psi^+|\le 36$, then each S_j contains at least 21 good roots. Of course, $\sum_{j=1}^{l}|B_j|\le|\Psi^+|$, thus

$$\sum_{j=1}^{l}\left((k-1)\,|G_j|-|B_j|\right)\ge(k-1)21-36>\tfrac{8}{2}\quad\text{if } k=3.$$

This shows that $\kappa(i_1,\ldots,i_l,\Psi)\le 3$ if $|\Psi^+|\le 36$. Thus we only need to study Ψ of type $E_7\times A_1$, D_8 or $E_6\times A_2$.

3.2.1. Case 1: Ψ is of type $E_7\times A_1$.

This is the most difficult case. Our strategy will be to first consider the root subsystem of Ψ consisting of the regular roots. This is a root subsystem contained in one of type D_8, the root subsystem of the regular roots of E_8, as well as being contained in one of type $E_7\times A_1$. Hence the set of regular roots of Ψ must be one of the following types: A_7, A_6, a subsystem of $D_6\times A_1\times A_1$, a subsytem of $A_5\times A_1\times A_1$, or $A_{j_1}\times\cdots\times A_{j_l}$ where $j_i\le 4$ and $j_1+\cdots+j_l\le 8$.

All of these root subsystems, with the exception of A_7, have the property that there is a pair of regular roots $e_i\pm e_j$, both of which belong to $\Phi\setminus\Psi$. By Lemma 3.3, Ψ contains at most 32 positive peculiar roots. As a root subsystem of type $E_7\times A_1$ contains 64 positive roots, this implies that Ψ contains at least 32 positive regular roots. This observation eliminates all the root subsystems on the list except A_7 and $D_6\times A_1\times A_1$. Thus we may assume that the regular roots in Ψ are either type A_7, with 28 regular and 36 peculiar positive roots, or type $D_6\times A_1\times A_1$, with 32 regular and 32 peculiar positive roots. We further remark that a root subsystem of type $D_6\times A_1\times A_1$ in D_8 has the form

$$\{e_i\pm e_j: i,j\in I\}\cup\{e_i\pm e_j: i,j\in J\}\tag{3.1}$$

where I and J are disjoint subsets of $\{1,\ldots,8\}$ of cardinalities 6 and 2 respectively.

Choose a set of indices $1 \leq i_l < \cdots < i_1 \leq 8$. First, suppose that one of the indices i_j is 4 or 5. Then $\left|S_{i_j}\right| \geq 99$ and therefore $|G_j| \geq 99 - 64 = 35$. Moreover, $\sum_{j=1}^{l} |B_j| \leq |\Psi^+| = 64$, hence

$$\sum_{j=1}^{l} (2|G_j| - |B_j|) \geq 75 - 64 > \tfrac{8}{2},$$

so in this situation $\kappa(i_1,\ldots,i_l,\Psi) \leq 3$.

In particular, note that one of the indices must be 4 or 5 if $l = 7$ or 8, hence we may assume that $l \leq 6$. In this case, one of the indices must be one of $3, 4$ or 5 and then, as $|S_3| = 98$,

$$\sum_{j=1}^{l} (2|G_j| - |B_j|) \geq 68 - 64 > \tfrac{6}{2}.$$

Again we see that $\kappa(i_1,\ldots,i_l,\Psi) \leq 3$.

Thus we may assume that $l \leq 5$. We first consider the case when one of the indices is 1. As S_1 contains 64 peculiar positive roots, S_1 must contain 32 good peculiar roots if the set of regular roots is of type $D_6 \times A_1 \times A_1$ and 28 good peculiar roots if the set of regular roots is of type A_7. When the set of regular roots is of type $D_6 \times A_1 \times A_1$ there are either 4 or 12 good regular roots in S_1, depending on whether the index 8 belongs to I or J. When the set of regular roots is of type A_7, exactly one of each pair $e_j \pm e_8$ with $j = 1,\ldots,7$ belongs to Ψ^+. Thus S_1 contains at least 7 good regular positive roots. In either situation, there are at least 35 good roots in S_1 and this shows $\kappa(i_1,\ldots,1,\Psi) \leq 3$.

Since $|S_6| = 97$, the set S_6 contains at least 33 good roots and this is enough to deduce that $\kappa(i_1,\ldots,i_l,\Psi) \leq 3$ if $l \leq 3$ and one of the indices, i_j say, is 6. Also, $|S_2 \cup S_6| = 107$, so $\sum_{j=1}^{l} |G_j| \geq 43$ if two of the indices are 2 and 6, and this gives $\kappa \leq 3$ for any $l \leq 5$. If $l = 4$ and one of the indices is 6, then at least one other must be chosen from $\{1,2,3,4,5\}$. Combined with the previous observations, this shows that $\kappa \leq 3$ if any $i_j = 6$.

The set S_2 contains at least $57 - 32$, that is, 25, good peculiar roots if the set of regular roots is of type $D_6 \times A_1 \times A_1$ and at least 21 good peculiar roots if the set of regular roots is of type A_7. In the former case, S_2 contains either 14 or 21 good regular roots, depending on whether the index 8 belongs to I or J, for a total of at least 39 good roots. According to Lemma 3.6, at most 16 of the 28 roots $e_i + e_j$ belong to Ψ if Ψ is type A_7. Thus, in the latter case, S_2 contains at least 12 good regular roots for a total of 33 good roots. As with index 6, it follows that $\kappa \leq 3$ if one of the indices is 2.

It only remains to consider the situation when $l = 1$ or 2 and the indices are 7 or 8.

(a) Assume that the set of regular roots in Ψ is of type $D_6 \times A_1 \times A_1$. If only one of $6, 7, 8$ belongs to I (as defined in (3.1)), say $6 \in I$ without loss of generality, then the regular roots $e_i \pm e_j$ for $j \in \{7,8\}$ and $i \in \{1,\ldots,5\}$, $e_6 \pm e_8$

and e_7+e_6 are all good roots in S_7. A simple cardinality argument shows there are at least 16 good peculiar roots in S_7, for a total of at least 39 good roots. Otherwise, two or more of the indices $6,7,8$ belong to I and then one of the indices $j_0\in\{1,2,3,4,5\}$ is not in I. Consequently neither of the roots $e_i\pm e_{j_0}$, for any (fixed) choice of $i\in\{1,\dots,5\}\cap I$, belong to the $D_6\times A_1\times A_1$ root subsystem, and therefore neither belong to Ψ. Lemma 3.5 implies S_7 contains at least 24 good peculiar roots. Since the regular roots in S_7 of the form $e_i\pm e_j$, with $i\in I$ and $j\in J$, are good roots, one can verify that S_7 contains at least 12 good regular roots, for a total of 36 good roots. In either case, we may conclude that $\kappa(7,\Psi)\leq 3$ and $\kappa(8,7,\Psi)\leq 3$.

Last, we suppose that $l=1$, $i_1=8$. If both $7,8\in J$, then the roots $e_i\pm e_j$, for $j\in\{7,8\}$ and $i\in\{1,\dots,6\}$, are all good regular roots in S_8. Hence $|G_1|\geq 24$ and $|B_1|\leq 57-24=33$ and this obviously implies $\kappa(8,\Psi)\leq 3$. Otherwise, there must be a pair $e_i\pm e_j$, with $i,j\in\{1,\dots,6\}$, in $\Phi\setminus\Psi$. Lemma 3.4(a) implies there are at least 16 good peculiar roots in S_8. An easy argument shows that there are also at least 8 good regular roots, so again we see $|G_1|\geq 24$.

(b) Otherwise the set of regular roots in Ψ is of type A_7. Since A_7 contains only one of each pair, $e_i\pm e_j$, the set S_7 contains at least 17 good regular roots, the set $S_7\cup S_8$ contains at least 18 and the set S_8 at least 12. A simple cardinality argument shows that S_7, and therefore also $S_7\cup S_8$, contains at least 12 good peculiar roots. By Lemma 3.4(b), S_8 has at least 8 good peculiar roots. Thus S_7 has at least 29 good and at most 54 bad roots, S_8 has at least 20 good and at most 37 bad roots, and $S_7\cup S_8$ has at least 30 good and at most 54 bad roots. Thus if $i_1=7$ or 8 and $l=1$, or $i_1,i_2=8,7$ and $l=2$, we have $\sum_{j=1}^{l}(2|G_j|-|B_j|)>\frac{1}{2}l$.

This completes the argument that $\kappa(i_1,\dots,i_l,\Psi)\leq 3$ for all indices $i_1,\dots,i_l$ when Ψ is of type $E_7\times A_1$.

3.2.2. Case 2: Ψ is of type D_8.

As with Case 1, we begin by considering the root subsystem consisting of the regular roots in Ψ. Lemma 3.3 implies that if some pair $e_i\pm e_j$ were contained in $\Phi\setminus\Psi$, then Ψ would have at least 24 regular positive roots. This means that the root subsystem consisting of the regular roots in Ψ is one of the following types: D_8, A_7, D_7, $D_6\times A_1\times A_1$, $D_6\times A_1$, D_6, $D_5\times D_3$ and $D_4\times D_4$. Of course, a root system of type D_m, which consists only of regular roots, must have the form $\{e_i\pm e_j : i,j\in I\}$ where the index set $I\subseteq\{1,\dots,8\}$ is of cardinality m.

We remark that if the indices $1\leq i_l<\cdots<i_1\leq 8$ are chosen so that $|S_{i_1}\cup\cdots\cup S_{i_l}|\geq 87$, then

$$\sum_{j=1}^{l}(2|G_j|-|B_j|)\geq 62-56>\tfrac{8}{2}$$

and thus $\kappa\leq 3$. More generally, if $\sum_{j=1}^{l}|G_j|\geq 31$, we may also conclude $\kappa\leq 3$. Since $|S_j|\geq 92$ if $j=2,\dots,6$ and S_1 contains at least 32 good peculiar roots, we only need to further investigate the cases where $l=1,\ 2$ and $i_j=7,\ 8$.

(a) Assume the set of regular roots are either type D_7 or D_8. Then Ψ^+ has either zero or 14 peculiar roots. In either case, S_7 and $S_7\cup S_8$ contain at least 34 good peculiar roots, which is enough to ensure $\kappa \leq 3$ if $l = 2$ or $l = 1$ and $i_1 = 7$. The set S_8 contains 32 good peculiar roots in the D_8 case and 18 otherwise. But in the latter case, there are also at least four good regular roots. Thus S_8 contains at least 22 good roots and at most 35 bad roots, which shows $\kappa(8, \Psi) \leq 3$.

(b) Assume the set of regular roots is of type A_7. Then Ψ^+ contains 28 regular and 28 peculiar roots. As one of each pair, $e_i \pm e_j$, is a good root, the set S_7 and $S_7 \cup S_8$ therefore contains at least 20 good peculiar roots and at least 17 good regular roots, for a total of 37 good roots. Similarly, S_8 contains at least 12 good regular roots and by Lemma 3.4(b), at least 8 good peculiar roots. Thus S_8 contains at least 20 good roots and at most 37 bad roots. In all these cases we may conclude $\kappa \leq 3$.

(c) Assume the set of regular roots are types $D_6 \times A_1 \times A_1$, $D_6 \times A_1$ or D_6. Then Ψ^+ has at most 26 peculiar roots, assuring that S_7 contains at least 22 good peculiar roots. By considering which of $6, 7, 8$ belong to the set of six indices generating the D_6, one can check that there are at least 12 good regular roots in S_7, for a total of 34 good roots.

If D_6 is not based on the indices $\{1, \ldots, 6\}$, then Lemma 3.4 (a) implies S_8 has at least 16 good peculiar roots and at least 8 good regular roots, for a total of 24 good and 33 bad roots. Otherwise, all the regular roots in S_8 are good. Either situation yields $\kappa \leq 3$.

(d) If the set of regular roots is of type $D_5 \times D_3$ or $D_4 \times D_4$ the reasoning is similar.

This completes the argument that $\kappa(i_1, \ldots, i_l, \Psi) \leq 3$ when Ψ is type D_8.

3.2.3. Case 3: Ψ is of type $E_6 \times A_2$.

Since $|S_j| \geq 78$ for all $j \neq 8$, each of these sets contains at least 39 good roots. As $2 \times 39 - 39 > \frac{8}{2}$, it only remains to consider the case where $l = 1$, $i_1 = 8$.

As usual, consider the set of regular roots in Ψ and suppose first that they form a root system of type $A_5 \times A_1 \times A_1$, $A_5 \times A_1$ or A_5, with the A_5 constructed on the indices $\{1, \ldots, 6\}$, in all cases. Then all the roots $e_i \pm e_j$, $i = 7, 8$ and $j = 1, \ldots, 6$ are good and this is sufficient to see that $\kappa \leq 3$.

Otherwise, an application of Lemma 3.4(a) ensures that the set S_8 contains at least 16 good peculiar roots. One can verify there are at most 17 positive regular roots in Ψ if the set of regular roots is of type $A_{j_1} \times \cdots \times A_{j_t}$. Otherwise the set of regular roots is a subset of a root system of type $D_5 \times A_2$, and thus has at most 23 positive roots. In addition to the 16 good peculiar roots in S_8, by considering which of the five indices is the index set for the D_5 one can easily verify that S_8 also contains at least 12 good regular roots.

This completes the argument that $\kappa \leq 3$ when the maximal root subsystem is $E_6 \times A_2$ and concludes the proof of the theorem for E_8.

3.3. Exceptional groups of type E_6 and E_7. The arguments for E_6 and E_7 are similar. We sketch the main ideas.

3.4. Proof of Theorem 3.1 for E_6. The fundamental dominant weights are listed in [8] and the corresponding sets S_j are described in Table 3, with the number of regular and peculiar positive roots given in brackets.

Table 3: The possibilities for the sets S_j for the case of E_6

	Positive Regular roots	#	Positive Peculiar roots	#
S_1	none	0	all	16
S_2	$e_i + e_j$: $i, j = 1, \ldots, 5$	10	all but P_q^- : $q = 1, \ldots, 5$	11
S_3	$e_i + e_j,\ e_1 - e_i$: $i, j = 2, \ldots, 5$	10	all but P_1^-	15
S_4	$e_i + e_j,\ e_k \pm e_j$: $i, j = 3, 4, 5,\ k = 1, 2$	15	all but P_1^-, P_2^-	14
S_5	$e_4 + e_5,\ e_k \pm e_j$: $j = 4, 5,\ k = 1, 2, 3$	13	all but P_{4567}, P_i^- : $i = 1, 2, 3$	12
S_6	$e_i \pm e_5$: $i = 1, \ldots, 4$	8	P_{67}, P_5^-, P_{ij67} : $i, j = 1, \ldots, 4$	8

In the same fashion as the earlier lemmas, one can prove the following results.

Lemma 3.7. *Assume Ψ is a root subsystem of E_6.*

(a) If there is some $i, j \in \{1, \ldots, 5\}$ such that $e_i \pm e_j \in \Phi \setminus \Psi$, then Ψ contains at most 8 positive peculiar roots.

(b) If there is some $i, j \in \{1, \ldots, 4\}$ such that $e_i \pm e_j \in \Phi \setminus \Psi$, then $\Psi \cap S_6$ contains at most 4 peculiar roots and $\Psi \cap S_2$ contains at most 7 peculiar roots.

(c) If there is some $i, j \in \{1, \ldots, 3\}$ such that $e_i \pm e_j \in \Phi \setminus \Psi$, then $\Psi \cap S_5$ contains at most 6 peculiar roots.

(d) If Ψ does not contain one of $e_i \pm e_j$, for some $i, j \in \{1, \ldots, 4\}$, $i \neq j$, then there are most 6 peculiar roots in $\Psi \cap S_6$.

The maximal root subsystems of E_6 are of types $A_2 \times A_2 \times A_2$, $A_5 \times A_1$ and D_5 with 9, 16 and 20 positive roots, respectively [10, p. 136]. The argument that $\kappa(i_l, \ldots, i_1, \Psi) \leq 3$ is very easy for $A_2 \times A_2 \times A_2$ and is left to the reader.

Suppose that Ψ is of type $A_5 \times A_1$. Since the set of regular roots of E_6 is of type D_5, one can use Lemma 3.7(a) to deduce that the set of regular roots in Ψ is either of type A_4 or $D_3 \times D_2$. The fact that an A_4 root subsystem contains only one of each pair $e_i \pm e_j$, and at most six roots of the form $e_i + e_j$, makes it easy to check that $\kappa \leq 3$ in this case. When the set of regular roots is of type $D_3 \times D_2$, Lemma 3.7(b) can be used to count good and bad roots and verify that $\kappa \leq 3$.

Now suppose that Ψ is of type D_5. This is the case if, for example, $\Psi = \Phi(X)$ or $\Psi = \Phi(x)$, where $X = (0, 0, 0, 0, 0, -1, -1, 1) \in \mathfrak{t}$ and $x = \exp X \in \mathbb{T}$, under the natural identification of the torus of E_6 with the suitable 6 dimensional subspace of $\mathbb{R}^8$. The elements X and x have 32 non-annihilating roots. As the dimension of E_6 is 78, Lemma 2.1 implies μ_X^2 and μ_x^2 are singular measures, and

$O_X + O_X$ and C_x^2 have Haar measure zero.

Our usual arguments show that the set of regular roots must be of type D_5, D_4 or A_4. The D_5 case is trivial since all the peculiar roots are good. When the set of regular roots is of type D_4, then a pair $e_i \pm e_j \in \Phi \setminus \Psi$ for some $i, j = 1, \ldots, 4$, so Lemma 3.7(b) applies. Part (c) is also helpful in counting good and bad roots. The argument that $\kappa \leq 3$ is entirely routine, but somewhat tedious because of the number of sets $S_j \cup S_k$, which must be considered.

The arguments are the most delicate when the set of regular roots is of type A_4, say $\{s_i e_i - s_j e_j : i \neq j = 1, \ldots, 5\}$, for a suitable choice of signs $s_i = \pm 1$. There is no loss of generality in assuming that $s_i = -1$ for an even number of i, and thus with a suitable Weyl conjugation, ω, consisting of an even number of sign changes, we may suppose that $\omega(\Psi) = \Psi'$ is of type D_5 and has a standard A_4 (by which we mean all $s_i = +1$) as its set of regular roots. This new set Ψ' has 10 regular and 10 peculiar positive roots.

If the peculiar root $P_{67} \in \Psi'$, then no peculiar root of the form P_{ij67} belongs to Ψ' since $P_{67} - P_{ij67} = e_i + e_j$ does not belong to Ψ'. It follows that Ψ' contains at most six peculiar roots, namely, the roots P_{67} and P_q^- for $q = 1, \ldots, 5$, which is a contradiction. Similarly, if $P_q^- \in \Psi'$ for some $q \in \{1, \ldots, 5\}$, then since $P_q^- - (e_q - e_l) = P_l^-$, all P_l^- must belong to Ψ'. Since $P_{ij67} - P_{ijkl67} = e_k + e_l$, no P_{ij67} belongs to Ψ'. Again this gives an insufficient number of peculiar roots.

So it must be that the 10 peculiar roots in Ψ' are precisely the set $P^{(2)} = \{P_{ij67} : i, j = 1, \ldots, 5, i \neq j\}$. Hence Ψ is $P^{(2)} \cup \{e_i - e_j : i, j = 1, \ldots, 5, i \neq j\}$ or is the conjugate of this set under an element of the Weyl group, with the Weyl group element being either two or four sign changes. In the first case the counting is straightforward. Otherwise, it is perhaps easiest to apply the Weyl conjugation to the sets S_j and assume that Ψ is in standard form. The arguments are elementary, but require some consideration of which signs are the ones that are changed. The details are straightforward and are left to the reader.

3.5. Proof of Theorem 3.1 for E_7. The fundamental dominant weights are listed in [8], and the corresponding sets S_j are listed in Table 4 together with the numbers of regular and peculiar roots.

As with E_8, one can prove the following elementary facts.

Lemma 3.8. *Assume that Ψ is a root subsystem of E_7.*

(a) If there is some $i, j \in \{1, \ldots, 6\}$ such that $e_i \pm e_j \in \Phi \setminus \Psi$, or if $e_7 - e_8 \in \Phi \setminus \Psi$, then Ψ contains at most 16 positive peculiar roots.

(b) If there is some $i, j \in \{1, \ldots, 5\}$ such that $e_i \pm e_j \in \Phi \setminus \Psi$, then $\Psi \cap S_7$ contains at most 8 peculiar roots.

(c) If there is some $i, j \in \{1, \ldots, 4\}$ such that $e_i \pm e_j \in \Phi \setminus \Psi$, then $\Psi \cap S_6$ contains at most 12 peculiar roots.

(d) If there is some $i, j \in \{1, \ldots, 3\}$ such that $e_i \pm e_j \in \Phi \setminus \Psi$, then $\Psi \cap S_5$ contains at most 14 peculiar roots.

(e) If Ψ^+ contains at most one of each pair $e_i \pm e_j$ and $e_k \pm e_l$, for some $i, j, k, l \in \{1, \ldots, 5\}$, then there are most 10 peculiar roots in $\Psi \cap S_7$.

The maximal root subsystems are of types $A_5 \times A_2$, A_7, $A_1 \times$ D_6 and E_6

Table 4: The possibilities for the sets S_j for the case of E_7

	Positive Regular roots	#	Positive Peculiar roots	#
S_1	$e_7 - e_8$	1	all	32
S_2	$e_7 - e_8,\ e_i + e_j$: $i, j = 1, \ldots, 6$	16	all but P_q^- : $q = 1, \ldots, 6$	26
S_3	$e_7 - e_8,\ e_i + e_j,\ e_1 - e_j$: $i, j = 2, \ldots, 6$	16	all but P_1^-	31
S_4	$e_7 - e_8,\ e_i + e_j,\ e_k \pm e_j$: $i, j = 3, \ldots, 6,\ k = 1, 2$	23	all but $P_1^-,\ P_2^-$	30
S_5	$e_7 - e_8, e_i + e_j, e_k \pm e_j$: $i, j = 4, 5, 6,\ k = 1, 2, 3$	22	all but $P_{4567},\ P_j^-$: $j = 1, 2, 3$	28
S_6	$e_7 - e_8,\ e_5 + e_6,\ e_k \pm e_j$: $j = 5, 6,\ k = 1, 2, 3, 4$	18	all but $P_j^-,\ P_{j567}$: $j = 1, 2, 3, 4$	24
S_7	$e_7 - e_8,\ e_i \pm e_6$: $i = 1, \ldots, 5$	11	$P_6^-,\ P_{j7},\ P_{ijk7}$: $i, j, k = 1, \ldots, 5$	16

with 18, 28, 31 and 36 positive roots, respectively [10, p. 136]. The arguments are fairly straightforward for the case of $A_5 \times A_2$.

We first sketch the key ideas when the root subsystem is of type A_7. The set of regular roots of E_7 is of type $D_6 \times A_1$, so our usual arguments, using part (a) of the lemma, allow us to deduce that the subset of regular roots must be either type A_5, $A_5 \times A_1$ or $D_3 \times D_3$. In the first two cases there are either 15 regular and 13 peculiar positive roots, or 16 and 12, respectively. The fact that A_5 contains at most 9 roots of the form $e_i + e_j$ is useful for the analysis of S_2. Lemma 3.8(e) is helpful for the set S_7. The desired calculations for the other sets, S_j, follow easily from cardinality arguments. If the set of regular roots is type $D_3 \times D_3$, then for some $i, j \in \{1, \ldots, 5\}$, the pair of regular roots $e_i \pm e_j$ belongs to $\Phi \setminus \Psi$. Part (b) of the lemma can be applied and we argue in the customary fashion.

Next, suppose that the root subsystem is of type $A_1\times\ D_6$. Then the set of regular roots must be of type $A_1\times\ D_6$, D_6, $D_5 \times A_1$, D_5, $D_4 \times A_1 \times A_1 \times A_1$, $A_5 \times A_1$ or A_5. No special tricks are needed here. The first two cases are easy as then Ψ has at most one peculiar root. For $A_5 \times A_1$ or A_5, use part (e) of the lemma. For $D_5 \times A_1$ or D_5, either (b) of the lemma applies or the root system of type D_5 must be built on indices $\{1, \ldots, 5\}$, in which case S_7 contains at least 10 good regular roots. The analysis for the other sets, S_j, is routine. In the case when the regular roots form a root system of type $D_4 \times A_1 \times A_1 \times A_1$ use (b) again.

The case when the root subsystem Ψ is of type E_6 is the difficult (and sharp) case. The set of annihilating roots of

$$X = (0, \ldots, 0, \tfrac{1}{2}, \tfrac{1}{2}, 1) \in \mathfrak{t}$$

is of type E_6. Since $\dim(E_7) = 133$, the measures μ_X^2 and $\mu_{\exp x}^2$ are singular, and

$O_X + O_X$ and $C^2_{\exp X}$ have measure zero.

All the root subsystems of Ψ that are contained in one of type $D_6 \times A_1$, with the exception of D_5, A_5, or $A_5 \times A_1$, have cardinality at most 12 and omit a pair, $e_i \pm e_j$, where $i, j \in \{1, \ldots, 6\}$. Together with Lemma 3.8(a), this implies that Ψ could have only 28 positive roots, which is a contradiction. If the set of regular roots is of type A_5, then $e_7 - e_8$ is omitted and the same reasoning applies. Thus the set of regular roots is either of type D_5 or $A_5 \times A_1$.

Parts (c) and (d) of the lemma are useful in handling the case when the set of regular roots is of type D_5. In addition to the sets S_j, a number of the pairs $S_j \cup S_k$ need to be considered, but no new ideas are required.

Last, suppose that the set of regular roots in Ψ form a subsystem of type $A_5 \times A_1$. This means Ψ^+ has 16 regular and 20 peculiar positive roots. The regular roots must have the form

$$\{s_i e_i - s_j e_j : i, j = 1, \ldots, 6, i \neq j\} \cup \{\pm(e_7 - e_8)\}.$$

First, suppose that an odd number of $s_i = -1$. Letting ω be the Weyl conjugation with an even number of sign changes and permuting the indices in $\{1, \ldots, 6\}$, as necessary, we can assume that only $s_1 = -1$. The 32 positive peculiar roots of E_7 may be paired as follows:

P_{1237}, P_6^-	P_{1247}, P_5^-	P_{1257}, P_4^-	P_{1267}, P_3^-	P_{1347}, P_2^-	P_{4567}, P_1^-
P_{17}, P_{1457}	P_{27}, P_{2347}	P_{37}, P_{3457}	P_{47}, P_{2457}	P_{57}, P_{2357}	P_{67}, P_{2367}
P_{1357}, P_{3567}	P_{1367}, P_{3467}	P_{1467}, P_{2467}	P_{1567}, P_{4567}.		

The last four pairs have the property that their difference is $e_j - e_1$, none of which belong to Ψ. All the other pairs have the property that their difference is one of $e_j + e_k$ where $j, k \in \{2, \ldots, 6\}$, and none of these belong to Ψ. Consequently, Ψ has only one of each pair, for a maximum of 16 peculiar roots. But Ψ contains 20 peculiar positive roots, so this is impossible.

Hence there must be an even number of $s_i = -1$ and after applying a Weyl conjugation ω, with an even number of sign changes, we may suppose that all $s_i = +1$ in $\omega(\Psi)$. Note that if for one index i, the root $P_{i7} \in \omega(\Psi)$, then since $P_{j7} = P_{i7} + e_i - e_j$, all $P_{j7} \in \omega(\Psi)$. Also, $P_{i7} - P_{ijk7} = e_j + e_k \notin \omega(\Psi)$, so none of the peculiar roots P_{ijk7} would belong to $\omega(\Psi)$. This implies the only positive peculiar roots in $\omega(\Psi)$ are the 12 peculiar roots P_{i7} or P_i^-, $i = 1, \ldots, 6$. But $\omega(\Psi)$ has 20 positive peculiar roots. The arguments are similar if $\omega(\Psi)$ contains one P_i^-. Thus $\omega(\Psi)$ must consist of the 20 positive peculiar roots P_{ijk7} (where $i, j, k = 1, \ldots, 6$) and their negatives, the roots $e_i - e_j$ (where $i, j = 1, \ldots, 6$), and $\pm(e_7 - e_8)$.

If the Weyl conjugation ω is the identity it is easy to do the counting with the sets S_j. Otherwise, we may suppose that ω is two sign changes and instead do the counting of $\omega(\Psi)$ in $\omega(S_j)$. This completes the proof for the exceptional Lie group and algebra of type E_7.

4. The Exceptional Lie groups and algebras F_4 and G_2

Theorem 4.1. *(a) If G is the compact Lie group of type F_4 (or of type G_2) then $\mu_x^4 \in L^2(G)$ (or $\mu_x^3 \in L^2(G)$ respectively) for all continuous orbital measures on G. There exists $x \in G$ such that the measure of C_x^3 (respectively, C_x^2) is zero and μ_x^3 (or μ_x^2) is singular with respect to Haar measure on G.*

(b) If $\mathfrak{g}$ is the compact Lie algebra of type F_4 or G_2, then $\mu_X^2 \in L^2(\mathfrak{g})$ for all continuous orbital measures on $\mathfrak{g}$.

Proof. We use a similar strategy to that used for the exceptional groups E_n. In particular, we continue to use the notation S_j and $\kappa(i_1,\ldots,i_l,\Psi)$, and speak of the good and bad roots.

4.1.1. The case of F_4

Note that F_4 has 12 long positive roots, $e_i \pm e_j$, where $i,j = 1,\ldots,4$ and $i<j$, and 12 short positive roots, these being the four roots e_i, (where $i = 1,\ldots,4$), and the 8 peculiar positive roots. The sets S_j are easily seen to be as follows:

$$\begin{aligned}
S_1 &= \{e_i, e_1+e_2, e_i \pm e_j : i=1,2, j=3,4\} \cup \{\tfrac{1}{2}(e_1+e_2 \pm e_3 \pm e_4)\},\\
S_2 &= \{e_1, e_2, e_3, \text{all } e_i \pm e_j \text{ except } e_2 - e_3\}\\
&\quad \cup \{\tfrac{1}{2}(e_1+e_2 \pm e_3 \pm e_4), \tfrac{1}{2}(e_1 - e_2 + e_3 \pm e_4)\},\\
S_3 &= \{e_1, e_i, e_1 \pm e_j, e_i + e_j : i,j = 2,3,4\}\\
&\quad \cup \{\tfrac{1}{2}(e_1+e_2 \pm e_3 \pm e_4), \tfrac{1}{2}(e_1 - e_2 + e_3 \pm e_4), \tfrac{1}{2}(e_1 - e_2 - e_3 + e_4)\},\\
S_4 &= \{e_1, e_1 \pm e_j : j = 2,3,4\} \cup \{\tfrac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}.
\end{aligned}$$

These sets have cardinalities 15, 20, 20 and 15 respectively.

The maximal root subsystems of F_4 are of types B_4, $C_3 \times A_1$ and $A_2 \times A_2$, with 16, 10 and 6 positive roots, respectively. Since the cardinality of each set S_j is at least 15, a trivial counting argument shows that $\kappa(i_1,\ldots,i_l,\Psi) \le 4$ if Ψ is either of type $C_3 \times A_1$ or $A_2 \times A_2$.

We identify the torus of the Lie algebra of F_4 with $\mathbb{R}^4$ and consider the group element $x = \exp(\pi,\pi,\pi,\pi) \in \mathbb{T}$. Its set of annihilating roots,

$$\Phi(x) = \{\pm e_i \pm e_j, \tfrac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4)\},$$

is of type B_4, and the usual appeal to Lemma 2.1 shows that μ_x^3 is singular and $m_G(C_x^2) = 0$.

A root subsystem of type B_4 has 12 long and 4 short positive roots. The 12 long positive roots must be the roots, $e_i \pm e_j$ where $i,j = 1,\ldots,4$ and $i<j$. The four short roots are mutually orthogonal, and so must be either the four short regular roots, e_i, where $i = 1,\ldots,4$, or four mutually orthogonal, peculiar positive roots. In the first case, it is simple to count the good and bad roots. In the second case, we remark that the four peculiar roots must have the same parity in their number of minus signs and this fact makes the counting straightforward to verify that $\kappa \le 4$.

Now consider the Lie algebra of type F_4. A rank four root subsystem cannot be the set of annihilating roots of a non-zero element in the Lie algebra, hence the root subsystems we need to consider are those of type B_3, C_3, $B_2 \times A_1$, A_3 and $A_1 \times A_1 \times A_1$. Since $|S_j| \geq 15$ for all j, the latter three root subsystems, with at most six positive roots, trivially satisfy $\kappa \leq 2$. This leaves B_3 and C_3 to study.

In a root subsystem, Ψ, of type B_3, there are three mutually orthogonal, positive short roots and six positive long roots. Consequently, the short roots are either three of $e_1, \ldots, e_4$, or three peculiar roots from one of the two sets of four mutually orthogonal, peculiar roots. (These are the two sets of four peculiar roots with the same parity of minus signs.) Since sums or differences of annihilating roots are also annihilating roots, in the first case we see that Ψ is a standard B_3, on three of the four indices, and it is easy to check that taking $\kappa = 2$ works. In the second case, by taking sums or differences of the three annihilating peculiar roots, one can show that the six long roots form a root subsystem of type A_3 on the indices $\{1, 2, 3, 4\}$. For example, if the three peculiar roots are $\frac{1}{2}(e_1 + e_2 \pm (e_3 + e_4))$ and $\frac{1}{2}(e_1 - e_2 + e_3 - e_4)$, then taking sums and differences we see that

$$\{e_1 + e_2, e_1 + e_3, e_1 - e_4, e_3 + e_4, e_2 + e_4, e_2 - e_3\}$$

is the set of long roots. More generally, Ψ is simply a change in signs from this particular case. In particular, the long roots have the property that for each pair, i, j, one of $e_i \pm e_j$ is a good root and the other is bad. With this observation the counting is easy.

A root subsystem of type C_3 has six short and three long positive roots. The short roots come in orthogonal pairs, which are not orthogonal to any other short root. Thus the short roots must consist of e_i, e_j and two pairs of peculiar roots, one pair from each of the subsets of four that are mutually orthogonal. Being mutually orthogonal, the three long roots must consist of the pair $e_i \pm e_j$ and one of $e_k \pm e_l$. We assume that the third root is $e_k + e_l$, without loss of generality. Each pair of short roots is orthogonal to one of the long roots. The pair, e_i, e_j, is orthogonal to $e_k + e_l$. The two pairs of peculiar roots must be orthogonal to $e_i \pm e_j$ and not orthogonal to the root $e_k + e_l$, hence they must be the roots $\frac{1}{2}(e_i \pm e_j \pm (e_k + e_l))$. By considering the possibilities for i, j, k, l chosen from $\{1, 2, 3, 4\}$, one can verify that $\kappa \leq 2$.

This completes the argument for F_4.

4.1.2. The case of G_2.

It was already noted in [8] that $\mu_x^3 \in L^2(G)$ for any continuous, orbital measure on the Lie group of type G_2. The element $x = \exp(2\pi, -\frac{4}{3}\pi, -\frac{2}{3}\pi) \in \mathbb{T}$ is of type A_2, with the three long roots being its annihilators. Thus μ_x^2 is singular with respect to Haar measure and $m_G(C_x^2) = 0$.

In the Lie algebra of type G_2, the set of annihilating roots of a non-zero element cannot have rank 2. Thus an element has either no annihilating roots or one positive annihilating root. Since the sets S_1, S_2 (listed in [8]) each have five elements, it follows trivially that $\kappa_0 = 2$. ∎

5. Concluding remarks

Ragozin [12] proved that if $n = \dim(G)$ and $\mu_1, \ldots, \mu_n$ are continuous, G-invariant measures on G, then $\mu_1 * \cdots * \mu_n$ is absolutely continuous with respect to m_G, and if $x_1, \ldots, x_n \in \mathbb{T}$ are not in the center of G, then $C_{x_1} \cdots C_{x_n}$ has non-empty interior in G. We improve these results, as well, for the exceptional Lie groups and algebras.

Corollary 5.1. *Let G be a compact, connected, simple, exceptional Lie group and let $\mathfrak{g}$ be its Lie algebra. Let k_0 be as given in (1.1). Then the following hold.*

(i) *$\mu^k \in L^2$ for all continuous orbital measures μ on G or $\mathfrak{g}$ if and only if $k \geq k_0$. Furthermore, there is a continuous orbital measure μ such that μ^{k_0-1} is singular with respect to Haar measure.*

(ii) *The convolution products $\mu_1 * \cdots * \mu_k$ belong to L^2 for all continuous orbital measures μ_j if and only if $k \geq k_0$.*

(iii) *The set $O_1 + \cdots + O_k$ (or $C_1 \cdots C_k$) has non-empty interior for all non-trivial adjoint orbits $O_j \subseteq \mathfrak{g}$ (or non-trivial conjugacy classes $C_j \subseteq G$ respectively) if and only if $k \geq k_0$.*

(iv) *The measures $\mu_1 * \cdots * \mu_k$ are absolutely continuous with respect to Haar measure for all G-invariant, continuous measures μ_j on G or $\mathfrak{g}$ if and only if $k \geq k_0$.*

Proof. Part (i) is established in Theorems 3.1 and 4.1 of this paper and part (ii) follows as a direct consequence of Hölder's inequality.

For part (iii), we note that k_0-fold sums (or products) of non-trivial adjoint orbits (respectively, conjugacy classes) support probability measures that are absolutely continuous with respect to Haar measure, and consequently must have positive measure. It is known that for these sets having positive measure is equivalent to having non-empty interior (see [12]). Our theorems show the necessity of the choice of k_0.

Finally, part (iv), the sharp answer to Ragozin's absolute continuity problem, follows from part (iii) by the same reasoning as used in [12]. ∎

In Table 5, we record information about the non-trivial adjoint orbits and conjugacy classes that are minimal in dimension.

Ragozin conjectured that the sharp answer to the absolute continuity problem would be $\lceil \dim(G)/\min_x(\dim(C_x)) \rceil$, where x varies over the non-central elements of G; by $\lceil s \rceil$ we mean the least integer greater or equal to s. As Lemma 2.1 shows, this is the least possible answer; this integer is always too small for the classical Lie groups and algebras. In contrast, our results imply that Ragozin's conjecture is correct for all the exceptional Lie groups and algebras.

Note also that in each case it was the adjoint orbits or conjugacy classes of minimal dimension, and their orbital measures, that were used to demonstrate the sharpness of the choice of k_0. Thus the following L^2-singular dichotomy holds, as in the classical case.

Table 5: Minimal dimension conjugacy classes and orbits

Lie Type	G_2	F_4	E_6	E_7	E_8
Dimension	14	52	78	133	248
Type of minimal conjugacy class	A_2	B_4	D_5	E_6	$E_7 \times A_1$
Type of minimal orbit	A_1	B_3, C_3	D_5	E_6	E_7
Dimension of minimal conjugacy class	6	16	32	62	112
Dimension of minimal orbit	10	30	32	62	114

Corollary 5.2. *Suppose that $x \in G$ generates a non-trivial conjugacy class of minimal dimension. The orbital measure μ_x satisfies the dichotomy that either μ_x^k is singular with respect to Haar measure on G, or $\mu_x^k \in L^2(G)$. A similar statement holds for μ_X when X generates a non-trivial, adjoint orbit of minimal dimension in $\mathfrak{g}$.*

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