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Some Estimates of the Bergman Kernel of Minimal Bounded Homogeneous Domains

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Abstract. We describe the Bergman kernel of any bounded homogeneous domain in a minimal realization relating to the Bergman kernels of the Siegel disks. Taking advantage of this expression, we obtain substantial estimates of the Bergman kernel of the homogeneous domain.
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1. Introduction

In this paper, we discuss the Bergman kernel $K_{\mathcal{U}}$ of a bounded homogeneous domain $\mathcal{U}$, which we assume to be minimal. One of our main results in the present work is the following estimate of $K_{\mathcal{U}}$, which will play a key role to characterize the boundedness of the Toeplitz operators in [15] (see also [16]).

Theorem A. Take any $\rho > 0$. Then, there exists $C_\rho > 0$ such that

$$C_\rho^{-1} \leq \left| \frac{K_{\mathcal{U}}(z,a)}{K_{\mathcal{U}}(a,a)} \right| \leq C_\rho$$

for all $z, a \in \mathcal{U}$ with $\beta_{\mathcal{U}}(z,a) \leq \rho$, where $\beta_{\mathcal{U}}$ means the Bergman distance on $\mathcal{U}$.

In the case that $\mathcal{U}$ is the Harish-Chandra realization of a bounded symmetric domain, the estimate is easily verified from properties of the Bergman kernel (see section 6). Besides, an estimate similar to Theorem A is shown for a homogeneous Siegel domain without difficulty (Proposition 2.2). However, for a general minimal bounded homogeneous domain, Theorem A does not seem to be trivial.

Our idea for the proof of Theorem A is to introduce certain equivariant holomorphic maps $\theta_{n_j} : \mathcal{U} \longrightarrow \mathcal{U}_{n_j}$ for $j = 1, \ldots, r\ (:= \operatorname{rank}\mathcal{U})$ from $\mathcal{U}$ into the Siegel disks $\mathcal{U}_{n_j} := \{W \in \mathrm{Sym}(n_j, \mathbb{C}) \mid I_{n_j} - W\overline{W} \text{ is positive definite}\}$ of rank n_j. Inspired by Xu [14], we obtain the following formula for the description of $K_{\mathcal{U}}$.

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Theorem B (Theorem 5.3). There exist integers $s_1, ..., s_r$ such that

$$K_{\mathcal{U}}(z,w) = \mathrm{Vol}(\mathcal{U})^{-1} \prod_{j=1}^{r} \left\{ \det \left(I_{n_j} - \theta_{n_j}(z)\overline{\theta_{n_j}(w)} \right) \right\}^{-s_j}$$

for $z, w \in \mathcal{U}$.

Recall that the Bergman kernel $K_{\mathcal{U}_m}$ of the Siegel disk $\mathcal{U}_m$ is given by

$$K_{\mathcal{U}_m}(Z,W) = \mathrm{Vol}\,(\mathcal{U}_m)^{-1} \det\left(I_m - Z\overline{W}\right)^{-(m+1)}.$$

Thus we obtain

$$K_{\mathcal{U}}(z,w) = C \prod_{j=1}^{r} K_{\mathcal{U}_{n_j}}(\theta_{n_j}(z), \theta_{n_j}(w))^{\frac{s_j}{n_j+1}},$$

which implies that the estimate in Theorem A for $\mathcal{U}$ is reduced to the ones for the symmetric domains $\mathcal{U}_{n_j}$.

Let us explain the organization of this paper. In section 2, we review properties of minimal domains, the Siegel upper half planes and homogeneous Siegel domains. In particular, we present in section 2.3 the matrix realization of any homogeneous Siegel domain introduced by the first author [7]. Based on this realization, we observe a relation between the Bergman distances on a homogeneous Siegel domain and the Siegel upper half planes (section 3), and introduce minor functions on a homogeneous cone in matrix realization, which coincide with the generalized power functions in Gindikin [5] (section 4). In section 5, we describe the Bergman kernel of minimal bounded homogeneous domains. The formula is expressed as a ratio of the Bergman kernels of the corresponding Siegel domain (Lemma 5.2), which together with the Cayley transform leads us to Theorem B. In section 6, we prove Theorem A, which yields another important estimate of $K_{\mathcal{U}}$ (Proposition 6.1).

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Notation. For an $N \times N$ matrix $A = (a_{ij}) \in \mathrm{Mat}(N, \mathbb{C})$ and $n = 1, \ldots, N$, we write $A^{[n]}$ for the $n \times n$ matrix $(a_{ij})_{1 \le i,j \le n}$. For real or complex domain D, we denote by $\mathrm{Cl}(D)$ the closure of D. The complexification of a real vector space V will be denoted by $V_{\mathbb{C}}$. For a holomorphic map ϕ defined on a neighborhood of $z \in \mathbb{C}^d$, we denote by $J(\phi, z)$ the complex Jacobi matrix of ϕ at z.

2. Preliminaries

2.1. Minimal domain.

First of all, we recall the definition and properties of minimal domains (see [8], [11]). Let D be a complex domain in $\mathbb{C}^d$ with finite volume and $t \in D$.

We say that D is a minimal domain with a center t if the following condition is satisfied: for every biholomorphism $\psi : D \longrightarrow D'$ with $\det J(\psi, t) = 1$, we have

$$\mathrm{Vol}\,(D') \geq \mathrm{Vol}\,(D).$$

We have the following convenient criterion for a domain to be minimal (see [8, Proposition 3.6], [11, Theorem 3.1]).

Proposition 2.1. *Let $D \subset \mathbb{C}^d$ be a bounded domain and $t \in D$. Then, D is a minimal domain with a center t if and only if*

$$K_D(z,t) = \frac{1}{\mathrm{Vol}\,(D)}$$

for any $z \in D$.

For example, a circular domain is minimal with a center 0, so that the Harish-Chandra realization for a bounded symmetric domain is also minimal, while there are many other minimal realizations for the symmetric domain. Recently in [8], a representative domain turns out to be a nice bounded realization of a bounded homogeneous domain, which is a generalization of the Harish-Chandra realization. The representative bounded homogeneous domain is always a minimal domain with a center 0 (see [8, Proposition 3.8]), though it is not circular unless it is symmetric. Therefore, in conclusion, every bounded homogeneous domain is biholomorphic to a minimal bounded homogeneous domain.

2.2. Siegel upper half plane.

Here we present basic facts used in this paper about the Siegel upper half plane $\mathcal{D}_n := \{Z \in \mathrm{Sym}(n,\mathbb{C}) \mid \mathrm{Im}\,Z \text{ is positive definite}\}$. It is well known that the real symplectic group $Sp(2n,\mathbb{R})$ acts on $\mathcal{D}_n$ transitively as linear fractional transforms:

$$\alpha \cdot Z = (PZ+Q)(RZ+S)^{-1} \qquad \left(\alpha = \begin{pmatrix} P & Q \\ R & S \end{pmatrix} \in Sp(2n,\mathbb{R}),\ Z \in \mathcal{D}_n\right).$$

Let H_n be the group of $n \times n$ lower triangular matrices with positive diagonals. We define

$$B_n := \left\{ \begin{pmatrix} T & 0 \\ 0 & {}^tT^{-1} \end{pmatrix} \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix} \,\middle|\, X \in \mathrm{Sym}(n,\mathbb{R}), T \in H_n \right\}.$$

Then B_n is a maximal connected split solvable Lie subgroup of $Sp(2n,\mathbb{R})$. The action of B_n on $\mathcal{D}_n$ is described as

$$\beta \cdot Z = T\,Z\,{}^tT + X \qquad \left(\beta = \begin{pmatrix} T & 0 \\ 0 & {}^tT^{-1} \end{pmatrix} \begin{pmatrix} I_n & X \\ 0 & I_n \end{pmatrix} \in B_n,\ Z \in \mathcal{D}_n\right), \tag{1}$$

so that the group B_n acts on $\mathcal{D}_n$ simply transitively.

Let $\mathcal{C}_n$ be the Cayley transform from $\mathcal{D}_n$ onto the Siegel disk $\mathcal{U}_n$ defined by

$$\mathcal{C}_n(Z) := (Z - iI_n)(Z + iI_n)^{-1} \tag{2}$$

for $Z \in \mathcal{D}_n$. It is easy to see that

$$I_n - \mathcal{C}_n(Z)\,\overline{\mathcal{C}_n(Z')} = \left(\frac{Z + iI_n}{2i}\right)^{-1} \left(\frac{Z - \overline{Z'}}{2i}\right) \overline{\left(\frac{Z' + iI_n}{2i}\right)^{-1}}. \tag{3}$$

2.3. Homogeneous Siegel domain.
Let Ω be a regular open convex cone in a real vector space V, W a complex vector space, and $F : W \times W \to V_{\mathbb{C}}$ a Hermitian map such that $F(u,u) \in \mathrm{Cl}(\Omega) \setminus \{0\}$ for $u \in W \setminus \{0\}$. Then the Siegel domain $D(\Omega, F) \subset V_{\mathbb{C}} \times W$ is defined by

$$D(\Omega, F) := \{\, (z,u) \in V_{\mathbb{C}} \times W \,|\, \mathrm{Im}\, z - F(u,u) \in \Omega \,\}.$$

For the degenerate case $F = 0$ with $W = \{0\}$, the Siegel domain becomes a tube domain $D(\Omega) = V + i\Omega \subset V_{\mathbb{C}}$. It is known that every bounded homogeneous domain is biholomorphic to some homogeneous Siegel domain ([13]). On the other hand, it is shown in [7] that every homogeneous Siegel domain is realized as a set of complex matrices with specific block decompositions in the following way.

Let $\nu_1, \dots, \nu_r$ be positive numbers, and $\{\mathcal{V}_{lk}\}_{1 \le k < l \le r}$ a system of real vector spaces $\mathcal{V}_{lk} \subset \mathrm{Mat}(\nu_l, \nu_k; \mathbb{R})$ satisfying

(V1) $A \in \mathcal{V}_{lk},\ B \in \mathcal{V}_{ki} \Longrightarrow AB \in \mathcal{V}_{li}$ for $1 \le i < k < l \le r$,
(V2) $A \in \mathcal{V}_{li},\ B \in \mathcal{V}_{ki} \Longrightarrow A\,{}^tB \in \mathcal{V}_{lk}$ for $1 \le i < k < l \le r$,
(V3) $A \in \mathcal{V}_{lk} \Longrightarrow A\,{}^tA \in \mathbb{R}I_{\nu_l}$ for $1 \le k < l \le r$.

We set $\nu := \nu_1 + \dots + \nu_r$. Let $\mathcal{V} \subset \mathrm{Sym}(\nu, \mathbb{R})$ be the space of real symmetric matrices X of the form

$$\begin{pmatrix} X_{11} & {}^tX_{21} & \dots & {}^tX_{r1} \\ X_{21} & X_{22} & & {}^tX_{r2} \\ \vdots & & \ddots & \\ X_{r1} & X_{r2} & & X_{rr} \end{pmatrix} \quad \begin{pmatrix} X_{kk} = x_{kk} I_{\nu_k},\ x_{kk} \in \mathbb{R} \quad (k = 1, \dots, r) \\ X_{lk} \in \mathcal{V}_{lk} \quad (1 \le k < l \le r) \end{pmatrix}.$$

We define $\Omega_{\mathcal{V}} := \{\, X \in \mathcal{V} \,|\, X \text{ is positive definite} \}$. Then $\Omega_{\mathcal{V}}$ is a regular open convex cone in the vector space $\mathcal{V}$. Let ν_0 be a positive integer, and $\{\mathcal{W}_k\}_{1 \le k \le r}$ a system of complex vector spaces $\mathcal{W}_k \subset \mathrm{Mat}\,(\nu_k, \nu_0; \mathbb{C})$ satisfying

(W1) $A \in \mathcal{V}_{lk},\ C \in \mathcal{W}_k \Longrightarrow AC \in \mathcal{W}_l$ for $1 \le k < l \le r$,
(W2) $C \in \mathcal{W}_l,\ C' \in \mathcal{W}_k \Longrightarrow C\,{}^t\overline{C'} \in (\mathcal{V}_{lk})_{\mathbb{C}}$ for $1 \le i < l \le r$,
(W3) $C \in \mathcal{W}_k \Longrightarrow C\,{}^t\overline{C} + \overline{C}\,{}^tC \in \mathbb{R}I_{\nu_k}$ for $1 \le k \le r$.

Let $\mathcal{W}$ be the space of complex matrices U of the form

$$U = \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_r \end{pmatrix} \in \mathrm{Mat}(\nu, \nu_0; \mathbb{C}) \quad \Big(U_k \in \mathcal{W}_k,\ \ k = 1, \dots, r\Big).$$

For $U, U' \in \mathcal{W}$, we define $F_{\mathcal{V},\mathcal{W}}(U, U') := (U\,{}^t\overline{U'} + \overline{U'}\,{}^tU)/4$. We see from (W1) – (W3) that $F_{\mathcal{V},\mathcal{W}}$ is a $\mathcal{V}_{\mathbb{C}}$-valued Hermitian form. Furthermore, it is easy to see that

$F_{\mathcal{V},\mathcal{W}}(U,U) \in \mathrm{Cl}(\Omega_{\mathcal{V}})\setminus\{0\}$ for $U \in \mathcal{W}\setminus\{0\}$. Then the Siegel domain $D(\Omega_{\mathcal{V}}, F_{\mathcal{V},\mathcal{W}})$ is defined by

$$D(\Omega_{\mathcal{V}}, F_{\mathcal{V},\mathcal{W}}) := \{(Z,U) \in \mathcal{V}_{\mathbb{C}} \times \mathcal{W} \,|\, \mathrm{Im}\, Z - F_{\mathcal{V},\mathcal{W}}(U,U) \in \Omega_{\mathcal{V}}\},$$

which we shall see to be homogeneous. First, let H be the set of $\nu \times \nu$ lower triangular matrices T of the form

$$\begin{pmatrix} T_{11} & & & \\ T_{21} & T_{22} & & \\ \vdots & & \ddots & \\ T_{r1} & T_{r2} & & T_{rr} \end{pmatrix} \qquad \begin{pmatrix} T_{kk} = t_{kk} I_{\nu_k},\ t_{kk} > 0 \quad (k = 1, \dots, r) \\ T_{lk} \in \mathcal{V}_{lk} \quad (1 \le k < l \le r) \end{pmatrix}.$$

Then H is a subgroup of the solvable group H_ν thanks to (V1). Moreover, H acts on the cone $\Omega_{\mathcal{V}}$ simply transitively by $\Omega_{\mathcal{V}} \ni X \mapsto TX{}^tT \in \Omega_{\mathcal{V}}$ $(T \in H)$. For $X \in \mathcal{V}, U \in \mathcal{W}$ and $T \in H$, we define an affine transform $b(X,U,T)$ on $\mathcal{V}_{\mathbb{C}} \times \mathcal{W}$ by

$$\begin{aligned} b(X,U,T)\cdot\zeta' \;:=\; & (T\,Z'\,{}^tT + X + 2i\,F_{\mathcal{V},\mathcal{W}}(TU',U) + iF_{\mathcal{V},\mathcal{W}}(U,U),\ TU'+U) \\ & (\zeta' = (Z',U') \in \mathcal{V}_{\mathbb{C}} \times \mathcal{W}). \end{aligned}$$

Then, each $b(X,U,T)$ preserves the domain $D(\Omega_{\mathcal{V}}, F_{\mathcal{V},\mathcal{W}})$. Let B be the set $\{b(X,U,T) \mid X \in \mathcal{V}, U \in \mathcal{W}, T \in H\}$, which forms a split solvable Lie group acting on $D(\Omega_{\mathcal{V}}, F_{\mathcal{V},\mathcal{W}})$ simply transitively. Therefore the Siegel domain $D(\Omega_{\mathcal{V}}, F_{\mathcal{V},\mathcal{W}})$ is homogeneous. Since every homogeneous Siegel domain can be obtained this way, we shall consider only the Siegel domains of this form. In particular, for the treatment of a Siegel domain of tube type, we set $\mathcal{W}_k = \{0\} \subset \mathrm{Mat}(\nu_k, \nu_0; \mathbb{C})$ $(k = 1, \dots, r)$. Thus we write F and D for $F_{\mathcal{V},\mathcal{W}}$ and $D(\Omega_{\mathcal{V}}, F_{\mathcal{V},\mathcal{W}})$ respectively in what follows for simplicity.

The Siegel domain D is embedded into the Siegel upper half plane $\mathcal{D}_N$ $(N := \nu_0 + \nu_1 + \cdots + \nu_r = \nu_0 + \nu)$ equivariantly with respect to the action of B. Namely, if we define an injective holomorphic map $\Phi : D \to \mathcal{D}_N$ and a group homomorphism $\phi : B \to B_N$ by

$$\Phi(\zeta) := \begin{pmatrix} iI_{\nu_0} & {}^tU \\ U & Z - \frac{i}{2}U\,{}^tU \end{pmatrix} \quad (\zeta = (Z,U) \in D), \tag{4}$$

and

$$\begin{aligned} &\phi(b(X,U,T)) \\ &:= \begin{pmatrix} I_{\nu_0} & & & \mathrm{Re}\,{}^tU \\ & I_\nu & \mathrm{Re}\,U & X + \frac{1}{2}\mathrm{Im}\,U\,{}^tU \\ & & I_{\nu_0} & \\ & & & I_\nu \end{pmatrix} \begin{pmatrix} I_{\nu_0} & & & \\ \mathrm{Im}\,U & T & & \\ & & I_{\nu_0} & -\mathrm{Im}\,{}^tU{}^tT^{-1} \\ & & & {}^tT^{-1} \end{pmatrix} \end{aligned}$$

respectively, we have by [7, p.601]

$$\phi(b)\cdot\Phi(\zeta) = \Phi(b\cdot\zeta) \tag{5}$$

for any $b \in B$ and $\zeta \in D$.

For the Siegel domain D, it is not difficult to obtain an estimate similar to Theorem A.

Proposition 2.2. *For any $\rho > 0$, there exists $A_\rho > 0$ such that*

$$A_\rho^{-1} \le \left| \frac{K_D(\zeta,\eta)}{K_D(\eta,\eta)} \right| \le A_\rho \tag{6}$$

for all $\zeta, \eta \in D$ with $\beta_D(\zeta,\eta) \le \rho$.

Proof. Put $p_0 := (iI_\nu, 0) \in D$. Since the function $f(\zeta') := \left| \frac{K_D(\zeta',p_0)}{K_D(p_0,p_0)} \right|$ is continuous and positive on the compact set $K_\rho := \{\zeta' \in D \mid \beta_D(\zeta',p_0) \le \rho\}$, we can take $A_\rho > 0$ for which

$$A_\rho^{-1} \le f(\zeta') \le A_\rho \qquad (\zeta' \in K_\rho). \tag{7}$$

On the other hand, since the group B acts on the domain D simply transitively, we can take $b \in B$ for which $b \cdot \eta = p_0$. Note that the Jacobian of b is constant because b is an affine transform. Thus we get

$$\left| \frac{K_D(\zeta,\eta)}{K_D(\eta,\eta)} \right| = \left| \frac{K_D(b\cdot\zeta, b\cdot\eta)\ \det J(b,\zeta)\,\overline{\det J(b,\eta)}}{K_D(b\cdot\eta, b\cdot\eta)\ \det J(b,\eta)\,\overline{\det J(b,\eta)}} \right| = \left| \frac{K_D(b\cdot\zeta, p_0)}{K_D(p_0,p_0)} \right|$$
$$= f(b\cdot\zeta).$$

By the invariance of the Bergman distance, we have $\beta_D(\zeta,\eta) = \beta_D(b\cdot\zeta, p_0)$, so that $\beta_D(\zeta,\eta) \le \rho$ implies $b\cdot\zeta \in K_\rho$. Therefore the estimate (6) follows from (7). ∎

3. Equivariant maps into the Siegel upper half planes

For $n = 1, \ldots, N$, let $\pi_n : \mathcal{D}_N \to \mathcal{D}_n$ be the surjective holomorphic map given by $\pi_n(Z) := Z^{[n]}$ $(Z \in \mathcal{D}_N)$. Let us observe the equivariance of π_n under actions of solvable groups. We define $\rho_n : B_N \longrightarrow B_n$ by

$$\rho_n\left(\begin{pmatrix} T & 0 \\ 0 & {}^tT^{-1} \end{pmatrix} \begin{pmatrix} I_N & X \\ 0 & I_N \end{pmatrix} \right) = \begin{pmatrix} T^{[n]} & 0 \\ 0 & {}^tT^{[n]-1} \end{pmatrix} \begin{pmatrix} I_n & X^{[n]} \\ 0 & I_n \end{pmatrix}.$$

Then ρ_n is a group homomorphism, and we see from (1) that

$$\pi_n(\beta \cdot Z) = \rho_n(\beta) \cdot \pi_n(Z) \tag{8}$$

for any $Z \in \mathcal{D}_N$ and $\beta \in B_N$. Now we define $\Phi_n := \pi_n \circ \Phi$ and $\phi_n := \rho_n \circ \phi$. From (5) and (8), we obtain the following proposition.

Proposition 3.1. *One has*

$$\phi_n(b) \cdot \Phi_n(\zeta) = \Phi_n(b\cdot\zeta) \tag{9}$$

for all $b \in B$ and $\zeta \in D$.

Using the group equivariance of Φ_n, we shall show that the map Φ_n is Lipschitz continuous with respect to the Bergman distances on D and $\mathcal{D}_n$.

Proposition 3.2. *There exists* $M_n > 0$ *such that*

$$\beta_{\mathcal{D}_n}(\Phi_n(\zeta), \Phi_n(\eta)) \leq M_n\, \beta_D(\zeta, \eta)$$

for any ζ, $\eta \in D$.

Proof. Let $ds^2_{\mathcal{D}_n}$ (resp. ds^2_D) be the Bergman metric on $\mathcal{D}_n$ (resp. D). It suffices to prove

$$ds^2_{\mathcal{D}_n}(\Phi_n(\zeta); J(\Phi_n, \zeta)X) \leq M_n\, ds^2_D(\zeta; X) \tag{10}$$

for all $\zeta \in D$ and $X \in \mathbb{C}^d$, where $d := \dim D$.

Since Hermitian forms $(ds^2_D)_{p_0}$ and $(\Phi^*_m ds^2_{\mathcal{D}_m})_{p_0}$ on $\mathbb{C}^d$ are positive definite and positive semi-definite respectively, we can take M_n for which

$$(\Phi^*_n ds^2_{\mathcal{D}_n})_{p_0} \leq M_n (ds^2_D)_{p_0}. \tag{11}$$

Using homogeneity, we will prove (10) holds for all $\zeta \in D$ and $X \in \mathbb{C}^d$. Let us take $b \in B$ such that $b \cdot \zeta = p_0$. Then, the right hand side of (10) is written as

$$ds^2_{\mathcal{D}_n}(\Phi_n(b \cdot p_0)\,; J(\Phi_n, b \cdot p_0)X). \tag{12}$$

Substituting $\zeta = p_0$ in (9), we have

$$\Phi_n(b \cdot p_0) = \phi_n(b) \cdot \Phi_n(p_0).$$

Furthermore, differentiating (9) at $\zeta = p_0$, we obtain

$$J(\Phi_n, b \cdot p_0)\, J(b, p_0) = J(\phi_n(b), \Phi_m(p_0))\;\; J(\Phi_n, p_0).$$

Therefore, (12) is equal to

$$ds^2_{\mathcal{D}_n}\left(\phi_n(b) \cdot \Phi_n(p_0)\,; J(\phi_n(b), \Phi_n(p_0))\, J(\Phi_n, p_0) J(b, p_0)^{-1} X\right),$$

which is equal to

$$ds^2_{\mathcal{D}_n}\left(\Phi_n(p_0)\,; J(\Phi_n, p_0) J(b, p_0)^{-1} X\right)$$

because $ds^2_{\mathcal{D}_n}$ is invariant under the holomorphic automorphism $\phi_n(b)$. By (11),

$$ds^2_{\mathcal{D}_n}\left(\Phi_n(p_0)\,; J(\Phi_n, p_0) J(b, p_0)^{-1} X\right) \leq M_n ds^2_D\left(p_0\,; J(b, p_0)^{-1} X\right)$$

and the right hand side is equal to

$$M_n ds^2_D\,(b \cdot p_0\,; X) = M_n ds^2_D\,(\zeta\,; X)\,,$$

because $b \in B$ is a biholomorphic map on D. Therefore, (10) is verified, whence Proposition 3.2 follows. ■

4. Minor functions

Definition 4.1. We set $\mu_1 := 1$ and $\mu_j := \nu_1+\nu_2+\cdots+\nu_{j-1}+1$ for $2 \le j \le r$. For $Z \in \Omega_{\mathcal{V}} + i\mathcal{V}\,(\subset \mathcal{V}_{\mathbb{C}})$, we define $Q_j(Z)\ \ (j = 1,\dots,r)$ by

$$Q_j(Z) := \frac{\det Z^{[\mu_j]}}{\det Z^{[\mu_j-1]}},$$

where we interpret $\det Z^{[0]} = 1$. We also define $Q^{\underline{s}}(Z)$ for $\underline{s} := (s_1, ..., s_r) \in \mathbb{R}^r$ by

$$Q^{\underline{s}}(Z) := Q_1(Z)^{s_1} \cdots Q_r(Z)^{s_r}.$$

The functions $Q_j(Z)$ and $Q^{\underline{s}}(Z)$ are denoted by $\chi_j(Z)$ and $Z^{\underline{s}}$ respectively in [5]. If D is a symmetric Siegel domain, then $Q^{\underline{s}}$ also coincides with the generalized power function $\Delta_{\underline{s}}$ in [4, P.122].

Example 4.2. Let $\mathcal{V}$ be the set of 4×4 symmetric matrices with real entries of the form

$$X = \begin{pmatrix} x_1 & 0 & x_4 & 0 \\ 0 & x_1 & 0 & x_5 \\ x_4 & 0 & x_2 & 0 \\ 0 & x_5 & 0 & x_3 \end{pmatrix}.$$

In this case, $\nu_1 = 2$, $\nu_2 = \nu_3 = 1$

$\mathcal{V}_{21} = \{\begin{pmatrix} x_4 & 0 \end{pmatrix} \mid x_4 \in \mathbb{R}\}, \mathcal{V}_{31} = \{\begin{pmatrix} 0 & x_5 \end{pmatrix} \mid x_5 \in \mathbb{R}\}$ and $\mathcal{V}_{32} = \{0\}$.

The cone $\Omega_{\mathcal{V}}$ is nothing but the Vinberg cone ([2], [12]). Let $\mathcal{W} = \{0\}$ and $F = 0$. Then D is the tube domain $\mathcal{V} + i\Omega_{\mathcal{V}}$ and we obtain

$$\begin{aligned} Q_1(X) &= x_1, \\ Q_2(X) &= \frac{x_1(x_1x_2 - x_4^2)}{x_1^2} = x_2 - \frac{x_4^2}{x_1}, \\ Q_3(X) &= \frac{\det X}{x_1(x_1x_2 - x_4^2)} = x_3 - \frac{x_5^2}{x_1}. \end{aligned}$$

These functions are considered in [2].

Take any $X \in \mathcal{V}$. There exists a unique lower triangular matrix $T \in H$ such that $T\,{}^tT = X$. Then we have

$$\det X^{[m]} = \det(T\,{}^tT)^{[m]} = (\det T^{[m]})^2 \tag{13}$$

for any $1 \le m \le \nu$. Since T is a lower triangular matrix, we can easily calculate the right hand side of (13). We have

$$\begin{aligned} \det X^{[\mu_k]} &= {t_{11}}^{2\nu_1} {t_{22}}^{2\nu_2} \cdots {t_{k-1,k-1}}^{2\nu_{k-1}} t_{kk}^2, \\ \det X^{[\mu_k-1]} &= {t_{11}}^{2\nu_1} {t_{22}}^{2\nu_2} \cdots {t_{k-1,k-1}}^{2\nu_{k-1}}. \end{aligned}$$

Therefore, we obtain $Q_k(X) = {t_{kk}}^2$. Hence we have

$$\det X^{[\mu_k]} = Q_1(X)^{\nu_1} Q_2(X)^{\nu_2} \cdots Q_{k-1}(X)^{\nu_{k-1}} Q_k(X). \tag{14}$$

Let $\det^{[m]}$ be the polynomial function on $\mathcal{V}_{\mathbb{C}}$ defined by $\det^{[m]} Z := \det Z^{[m]}$ $(Z \in \mathcal{V}_{\mathbb{C}})$.

Lemma 4.3. *Let $2 \le k \le r$. Then, one has*

$$Q_k = (\det{}^{[\mu_1]})^{c_{k1}} (\det{}^{[\mu_2]})^{c_{k2}} \cdots (\det{}^{[\mu_{k-1}]})^{c_{k,k-1}} \det{}^{[\mu_k]}, \tag{15}$$

where $c_{ki} = -\nu_k \prod_{i<p<k}(1-\nu_p)$ for $i = 1, \ldots, k-1$.

Proof. For $1 \le i < r$, we obtain

$$\frac{\det^{[\mu_{i+1}]}}{\det^{[\mu_i]}} = \frac{{Q_i}^{\nu_i} Q_{i+1}}{Q_i}$$

from (14). Therefore, we have

$$Q_{i+1} = {Q_i}^{1-\nu_i} \frac{\det^{[\mu_{i+1}]}}{\det^{[\mu_i]}}. \tag{16}$$

In particular,

$$Q_2 = {Q_1}^{1-\nu_1} \frac{\det^{[\mu_2]}}{\det^{[\mu_1]}} = (\det{}^{[\mu_1]})^{-\nu_1} \det{}^{[\mu_2]}.$$

Thus, the formula holds for $k = 2$ with $c_{21} = -\nu_1$. Assume that the statement holds for $l = j$. Substituting (15) to (16), we have

$$\begin{aligned} Q_{j+1} &= \det{}^{[\mu_{j+1}]} \left\{ \det{}^{[\mu_j]} \prod_{i<j} (\det{}^{[\mu_i]})^{c_{ji}} \right\}^{(1-\nu_j)} (\det{}^{[\mu_j]})^{-1} \\ &= \det{}^{[\mu_{j+1}]} (\det{}^{[\mu_j]})^{-\nu_j} \prod_{i<j} (\det{}^{[\mu_i]})^{(1-\nu_j)c_{ji}}. \end{aligned}$$

Therefore, if we put

$$\begin{aligned} c_{j+1,i} &= (1-\nu_j)c_{ji} = -\nu_j \prod_{i<p<j+1} (1-\nu_p) \quad \text{for } i = 1, \ldots, j, \\ c_{j+1,j} &= -\nu_j, \end{aligned}$$

we have (15) for $k = j+1$. ∎

Let $\underline{d} := (d_1, \ldots, d_r)$ and $\underline{b} := (b_1, \ldots, b_r)$ be r-tuples of respectively half integers and integers defined by

$$d_k := 1 + \frac{1}{2}\Big(\sum_{i<k} \dim \mathcal{V}_{ki} + \sum_{l>k} \dim \mathcal{V}_{lk}\Big), \quad b_k := \dim \mathcal{W}_k \qquad (k = 1, \ldots, r).$$

Then the Bergman kernel of the homogeneous Siegel domain D is given by

$$K_D(\zeta, \zeta') = C\, Q^{-(2\underline{d}+\underline{b})} \left(\frac{Z - \overline{Z'}}{2i} - F(U, U') \right)$$

for $\zeta := (Z, U), \zeta' := (Z', U') \in D$, where C is a constant depending only on the normalization of the Lebesgue measure on $\mathcal{V}$ (see [5, Proposition 5.1]). This formula together with Lemma 4.3 tells us the following.

Lemma 4.4. *Let c_{kj} $(1 \leq j < k \leq r)$ be the integer defined as in Lemma 4.3, and $c_{jj} = 1$ for $1 \leq j \leq r$. Setting*

$$s_j := \sum_{k \geq j} (2d_k + b_k) c_{kj} \qquad (j = 1, \ldots, r),$$

one has

$$K_D(\zeta, \zeta') = C \prod_{j=1}^{r} \left\{ \det{}^{[\mu_j]} \left(\frac{Z - \overline{Z'}}{2i} - F(U, U') \right) \right\}^{-s_j}.$$

5. Bergman kernel of the minimal bounded homogeneous domains

By the transformation formula of the Bergman kernel, we have the following general formula.

Lemma 5.1. *Let D_1 and D_2 be complex domains and α a biholomorphic map from D_1 onto D_2. Then, one has*

$$\frac{K_{D_2}(\alpha(\zeta_1), \alpha(\zeta_2))\, K_{D_2}(\alpha(\zeta_3), \alpha(\zeta_4))}{K_{D_2}(\alpha(\zeta_1), \alpha(\zeta_4))\, K_{D_2}(\alpha(\zeta_3), \alpha(\zeta_2))} = \frac{K_{D_1}(\zeta_1, \zeta_2)\, K_{D_1}(\zeta_3, \zeta_4)}{K_{D_1}(\zeta_1, \zeta_4)\, K_{D_1}(\zeta_3, \zeta_2)} \tag{17}$$

for $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in D_1$.

Let $\mathcal{U}$ be a minimal bounded homogeneous domain with a center t. By [13], $\mathcal{U}$ is biholomorphic to a homogeneous Siegel domain $D \subset \mathcal{V}_{\mathbb{C}} \times \mathcal{W}$. We set $p_0 := (iI_\nu, 0) \in D$ and take a biholomorphic map σ from D onto $\mathcal{U}$ such that $\sigma(p_0) = t$. From Lemma 5.1, we have the following lemma, which is crucial for the present work.

Lemma 5.2. *For any $\zeta, \zeta' \in D$, one has*

$$\mathrm{Vol}\,(\mathcal{U})\, K_{\mathcal{U}}(\sigma(\zeta), \sigma(\zeta')) = \frac{K_D(\zeta, \zeta') K_D(p_0, p_0)}{K_D(\zeta, p_0) K_D(p_0, \zeta')}.$$

Proof. By Lemma 5.1, we obtain

$$\frac{K_{\mathcal{U}}(\sigma(\zeta), \sigma(\zeta'))\, K_{\mathcal{U}}(t, t)}{K_{\mathcal{U}}\,(\sigma(\zeta), t)\; K_{\mathcal{U}}(t, \sigma(\zeta'))} = \frac{K_D(\zeta, \zeta')\, K_D(p_0, p_0)}{K_D(\zeta, p_0)\, K_D(p_0, \zeta')}. \tag{18}$$

By Proposition 2.1, we obtain

$$K_{\mathcal{U}}\,(\sigma(\zeta), t) = K_{\mathcal{U}}(t, \sigma(\zeta')) = K_{\mathcal{U}}(t, t) = \frac{1}{\mathrm{Vol}\,(\mathcal{U})}.$$

Therefore, the left hand side of (18) is equal to

$$\mathrm{Vol}\,(\mathcal{U})\, K_{\mathcal{U}}(\sigma(\zeta), \sigma(\zeta')),$$

and we complete the proof. ∎

For $1 \le n \le N$, let θ_n be the composition map $\mathcal{C}_n \circ \Phi_n \circ \sigma^{-1}$ from $\mathcal{U}$ into the Siegel disk $\mathcal{U}_n$. Using the maps θ_n, we can describe the Bergman kernel $K_{\mathcal{U}}$ as follows.

Theorem 5.3. *Putting $n_j = \nu_0 + \nu_1 + \cdots + \nu_{j-1} + 1$, one has*

$$\begin{aligned} K_{\mathcal{U}}(z, z') &= \frac{1}{\mathrm{Vol}\,(\mathcal{U})} \prod_{j=1}^{r} \left\{ \det\left(I_{n_j} - \theta_{n_j}(z)\overline{\theta_{n_j}(z')} \right) \right\}^{-s_j} \\ &= C \prod_{j=1}^{r} K_{\mathcal{U}_{n_j}}(\theta_{n_j}(z), \theta_{n_j}(z'))^{\frac{s_j}{n_j+1}} \end{aligned}$$

for $z, z' \in \mathcal{U}$, where s_j are integers defined by Lemma 4.4.

Proof. Let $\zeta = (Z, U)$ and $\zeta' = (Z', U')$ be elements $\sigma^{-1}(z) \in D$ and $\sigma^{-1}(z') \in D$ respectively. By Lemma 5.2, we have

$$\begin{aligned} & K_{\mathcal{U}}(z, z') = K_{\mathcal{U}}(\sigma(\zeta), \sigma(\zeta')) \\ &= \frac{1}{\mathrm{Vol}\,(\mathcal{U})} \frac{K_D(\zeta, \zeta') K_D(p_0, p_0)}{K_D(\zeta, p_0) K_D(p_0, \zeta')} \\ &= \frac{1}{\mathrm{Vol}\,(\mathcal{U})} \frac{Q^{-(2\underline{d}+\underline{b})}\left(\frac{Z - \overline{Z'}}{2i} - F(U, U')\right) Q^{-(2\underline{d}+\underline{b})}\left(\frac{iI_\nu - \overline{iI_\nu}}{2i} - F(0,0)\right)}{Q^{-(2\underline{d}+\underline{b})}\left(\frac{Z - \overline{iI_\nu}}{2i} - F(U, 0)\right) Q^{-(2\underline{d}+\underline{b})}\left(\frac{iI_\nu - \overline{Z'}}{2i} - F(0, U')\right)} \\ &= \frac{1}{\mathrm{Vol}\,(\mathcal{U})} \frac{Q^{-(2\underline{d}+\underline{b})}\left(\frac{Z - \overline{Z'}}{2i} - F(U, U')\right)}{Q^{-(2\underline{d}+\underline{b})}\left(\frac{Z - \overline{iI_\nu}}{2i}\right) Q^{-(2\underline{d}+\underline{b})}\left(\frac{iI_\nu - \overline{Z'}}{2i}\right)}. \end{aligned} \tag{19}$$

By Lemma 4.4, the right hand side of (19) is equal to

$$\frac{1}{\mathrm{Vol}\,(\mathcal{U})} \prod_{j=1}^{r} \left\{ \frac{\det^{[\mu_j]}\left(\frac{Z - \overline{Z'}}{2i} - F(U, U')\right)}{\det^{[\mu_j]}\left(\frac{Z - \overline{iI_\nu}}{2i}\right) \det^{[\mu_j]}\left(\frac{iI_\nu - \overline{Z'}}{2i}\right)} \right\}^{-s_j}.$$

We see from (4) that

$$\det\left(\frac{Z - \overline{Z'}}{2i} - F(U, U')\right) = \det\left(\frac{\Phi(\zeta) - \overline{\Phi(\zeta')}}{2i}\right).$$

Moreover, it is not difficult to check that

$$\det^{[m]}\left(\frac{Z - \overline{Z'}}{2i} - F(U, U')\right) = \det\left(\frac{\Phi_{m+\nu_0}(\zeta) - \overline{\Phi_{m+\nu_0}(\zeta')}}{2i}\right)$$

for $1 \leq m \leq \nu$. Since $n_j = \nu_0 + \mu_j$, we have

$$\frac{\det^{[\mu_j]}\left(\frac{Z-\overline{Z'}}{2i} - F(U,U')\right)}{\det^{[\mu_j]}\left(\frac{Z-\overline{iI_\nu}}{2i}\right)\det^{[\mu_j]}\left(\frac{iI_\nu-\overline{Z'}}{2i}\right)}$$
$$= \det\left\{\left(\frac{\Phi_{n_j}(\zeta)-\overline{\Phi_{n_j}(p_0)}}{2i}\right)^{-1}\left(\frac{\Phi_{n_j}(\zeta)-\overline{\Phi_{n_j}(\zeta')}}{2i}\right)\left(\frac{\Phi_{n_j}(p_0)-\overline{\Phi_{n_j}(\zeta')}}{2i}\right)^{-1}\right\}$$
$$= \det\left\{\left(\frac{\Phi_{n_j}(\zeta)+iI_{n_j}}{2i}\right)^{-1}\left(\frac{\Phi_{n_j}(\zeta)-\overline{\Phi_{n_j}(\zeta')}}{2i}\right)\overline{\left(\frac{\Phi_{n_j}(\zeta')+iI_{n_j}}{2i}\right)^{-1}}\right\}. \quad (20)$$

By (3), the last term of (20) is equal to

$$\det\left(I_{n_j} - \mathcal{C}_{n_j}(\Phi_{n_j}(\zeta))\,\overline{\mathcal{C}_{n_j}(\Phi_{n_j}(\zeta'))}\right).$$

Since $\theta_m = \mathcal{C}_m \circ \Phi_m \circ \sigma^{-1}$, we have

$$K_{\mathcal{U}}(z,z') = \frac{1}{\operatorname{Vol}(\mathcal{U})}\prod_{j=1}^{r}\left\{\det\left(I_{n_j} - \theta_{n_j}(z)\,\overline{\theta_{n_j}(z')}\right)\right\}^{-s_j}.$$

The second equality in the statement follows from the above and the formula ([6])

$$K_{\mathcal{U}_{n_j}}(w,w') = \frac{1}{\operatorname{Vol}(\mathcal{U}_{n_j})}\det\left(I_{n_j} - w\,\overline{w'}\right)^{-(n_j+1)} \quad (w,w' \in \mathcal{U}_{n_j}).$$ ■

6. Estimates of the Bergman kernel of minimal bounded homogeneous domains

In this section, we prove Theorem A. Let $\mathcal{U}$ be a minimal bounded homogeneous domain with a center t as in the previous section. For each $a \in \mathcal{U}$, we take φ_a an automorphism on $\mathcal{U}$ such that $\varphi_a(a) = t$. Since $K_{\mathcal{U}}(\cdot,t)$ is a constant function, we have

$$\begin{aligned}\frac{K_{\mathcal{U}}(z,a)}{K_{\mathcal{U}}(a,a)} &= \frac{K_{\mathcal{U}}(z,a)\,K_{\mathcal{U}}(a,t)}{K_{\mathcal{U}}(a,a)K_{\mathcal{U}}(z,t)}\\ &= \frac{K_{\mathcal{U}}(\varphi_a(z),t)\,K_{\mathcal{U}}(t,\varphi_a(t))}{K_{\mathcal{U}}(t,t)\,K_{\mathcal{U}}(\varphi_a(z),\varphi_a(t))}\\ &= \frac{K_{\mathcal{U}}(t,t)}{K_{\mathcal{U}}(\varphi_a(z),\varphi_a(t))}, \qquad (21)\end{aligned}$$

where the second equality follows from Lemma 5.1. For $\rho > 0$, let $B(t,\rho)$ denote the closed Bergman disk $\{z \in \mathcal{U} \mid \beta_{\mathcal{U}}(z,t) \leq \rho\}$, which is a compact subset of $\mathcal{U}$. For any $z, a \in \mathcal{U}$ with $\beta_{\mathcal{U}}(z,a) \leq \rho$, we have $(\varphi_a(z),\varphi_a(t)) \in B(t,\rho)\times\mathcal{U}$ because $\beta_{\mathcal{U}}(\varphi_a(z),t) = \beta_{\mathcal{U}}(\varphi_a(z),\varphi_a(a)) = \beta_{\mathcal{U}}(z,a) \leq \rho$. If $\mathcal{U}$ is a bounded symmetric domain, we know that

(P) $K_{\mathcal{U}}(z_1,z_2)$ extends to the compact set $B(t,\rho)\times \operatorname{Cl}(\mathcal{U})$ as a continuous and non-zero function (see [10, Theorem 2.10]). Therefore, we obtain Theorem A from (21). However, we don't know whether a nonsymmetric homogeneous domain has the property (P). Therefore, we take advantage of Theorem 5.3, which describes the Bergman kernel of the minimal homogeneous domain $\mathcal{U}$ in terms of the Bergman kernel of the Siegel disks $\mathcal{U}_{n_j}$. Moreover, we have an estimate of the Bergman distance in Proposition 3.2. Using these results, we will prove our main theorem.

Proof of Theorem A. Take any $z, a \in \mathcal{U}$ with $\beta_{\mathcal{U}}(z,a) \le \rho$. Then, there exist $\zeta, \eta \in D$ such that $\sigma(\zeta) = z$ and $\sigma(\eta) = a$. By Theorem 5.3, we have

$$\left|\frac{K_{\mathcal{U}}(z,a)}{K_{\mathcal{U}}(a,a)}\right| = \prod_{j=1}^{r} \left|\frac{K_{\mathcal{U}_{n_j}}(\theta_{n_j}(z), \theta_{n_j}(a))}{K_{\mathcal{U}_{n_j}}(\theta_{n_j}(a), \theta_{n_j}(a))}\right|^{\frac{s_j}{n_j+1}}. \tag{22}$$

On the other hand, since $\beta_D(\zeta,\eta) = \beta_{\mathcal{U}}(z,a) \le \rho$ and $\beta_{\mathcal{D}_{n_j}}(\Phi_{n_j}(\zeta), \Phi_{h_j}(\eta)) = \beta_{\mathcal{U}_{n_j}}(\theta_{n_j}(z), \theta_{n_j}(a))$, we obtain

$$\beta_{\mathcal{U}_{n_j}}(\theta_{n_j}(z), \theta_{n_j}(a)) \le M_{n_j}\rho \tag{23}$$

from Proposition 3.2. Since $\mathcal{U}_{n_j}$ is a bounded symmetric domain, there exists a positive constant C_j such that

$$C_j^{-1} \le \left|\frac{K_{\mathcal{U}_{n_j}}(w,w')}{K_{\mathcal{U}_{n_j}}(w',w')}\right| \le C_j$$

holds for any $w, w' \in \mathcal{U}_{n_j}$ with $\beta_{\mathcal{U}_j}(w,w') \le M_{n_j}\rho$. Therefore, thanks to (23), we have

$$C_j^{-1} \le \left|\frac{K_{\mathcal{U}_{n_j}}(\theta_{n_j}(z), \theta_{n_j}(a))}{K_{\mathcal{U}_{n_j}}(\theta_{n_j}(a), \theta_{n_j}(a))}\right| \le C_j.$$

In view of (22), we have

$$C_\rho^{-1} \le \left|\frac{K_{\mathcal{U}}(z,a)}{K_{\mathcal{U}}(a,a)}\right| \le C_\rho$$

with

$$C_\rho = \prod_{j=1}^{r} C_j^{\frac{s_j}{n_j+1}}.$$

∎

As an application of Theorem A, we obtain another important estimate of $K_{\mathcal{U}}$.

Proposition 6.1. *For any $\rho > 0$, there exists $M_\rho > 0$ such that $M_\rho^{-1} \le |K_{\mathcal{U}}(z,w)| \le M_\rho$ for any $z \in B(t,\rho)$ and $w \in \mathcal{U}$.*

Proof. Similarly to (21), we obtain

$$\left|\frac{K_{\mathcal{U}}(z,w)}{K_{\mathcal{U}}(t,t)}\right| = \left|\frac{K_{\mathcal{U}}(z,w)\, K_{\mathcal{U}}(t,t)}{K_{\mathcal{U}}(z,t)\, K_{\mathcal{U}}(t,w)}\right| = \left|\frac{K_{\mathcal{U}}(\varphi_w(t), \varphi_w(t))}{K_{\mathcal{U}}(\varphi_w(z), \varphi_w(t))}\right|. \tag{24}$$

Since $\beta_{\mathcal{U}}(\varphi_w(z), \varphi_w(t)) = \beta_{\mathcal{U}}(z,t) \leq \rho$, there exists $C_\rho > 0$ such that the right hand side of (24) is estimated by C_ρ and C_ρ^{-1} as in Theorem A. Note that the constant C_ρ is independent of z and w. Therefore, we obtain

$$C_\rho^{-1} K_{\mathcal{U}}(t,t) \leq |K_{\mathcal{U}}(z,w)| \leq C_\rho K_{\mathcal{U}}(t,t)$$

for any $z \in B(t,\rho)$ and $w \in \mathcal{U}$. ■

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