Journal of Lie Theory
Volume **21** (2011) 457–468

Automorphism Groups of Some Stable Lie Algebras

Jongwoo Lee, Xueqing Chen, Seul Hee Choi, and Ki-Bong Nam

Communicated by A. Fialowski

Abstract. A degree stable Lie algebra is defined in the paper [14]. The Lie algebra automorphism group $Aut_{Lie}(S^+(2))$ of the Lie algebra $S^+(2)$ is found in the paper [14]. The Lie algebra automorphism group of the Lie algebra $W(1,0,2)$ is also found in this paper [2]. We find the algebra automorphism groups of the Lie algebras $W(1^2,1,1)$ and $W(1^2,2,0)$ in this work. We show that the Cartan subalgebras of $W(1^2,1,1)$ and $W(1^2,2,0)$ are one dimensional.
Mathematics Subject Classification 2000: 17A36.
Key Words and Phrases: Simple, Witt algebra, degreeing Lie algebra, Cartan subalgebra.

1. Introduction

The automorphism groups of some self-centralizing Lie algebras are studied in the papers [9], [10], [12]. Rudakov found the continuous automorphisms of the topological Cartan type Lie algebras in the paper [16]. The automorphism group $Aut_{Lie}(S^+(2))$ of the Lie algebra $S^+(2) = S(0,0,2)$ is found in the paper [14]. In this work, we find the automorphism groups $Aut_{Lie}(W(1^2,1,1)$ and Aut_{Lie} $(W(1^2,2,0))$ of the Lie algebras $W(1^2,1,1)$ and $W(1^2,2,0)$ which contain $S^+(2) = S(0,0,2)$ (see [13]). We show that there is no automorphism θ of $W(1^2,1,1)$ such that $\theta(\partial_1) = c_1\partial_1 + c_2\partial_2$ where c_i are non-zero scalars for $i=1,2$. We also show that $Tor(W(1^2,1,1))$ and $Tor(W(1^2,2,0))$ are ones. We show that the Cartan subalgebras of $W(1^2,1,1)$ and $W(1^2,2,0)$ are one dimensional.

2. Preliminaries

Let $\mathbb{F}$ be the field of characteristic zero (not necessarily algebraically closed). Throughout the paper, $\mathbb{N}$ and $\mathbb{Z}$ denote the non-negative integers and the integers, respectively. Let $\mathbb{F}^\bullet$ be the multiplicative group of non-zero elements of $\mathbb{F}$. Let L be a Lie algebra over $\mathbb{F}$ with a basis $S = \{s_u | u \in I\}$ where I is an index set. The Lie algebra L is degreeing if for any $s \in S$ we define the Lie degree $deg_{Lie}(s) \in \mathbb{Z}$ of s. Thus for any l of L, we may define $deg_{Lie}(l)$ as the highest Lie degree of non-zero basis terms of l. An element l of L is degree stable if for any

ISSN 0949–5932 / $2.50

$l_1 \in L$ $deg_{Lie}([l, l_1]) \leq deg_{Lie}(l_1)$ holds. For a degreeing Lie algebra L, the degree stabilizer $St_{Lie}(L)$ of the Lie algebra L is the vector subspace of L spanned by all the elements which are degree stable. For any $\theta \in Aut_{Lie}(L)$ we have the following diagram:

$$\begin{array}{ccc} St_{Lie}(L) & \longrightarrow & St_{Lie}(L) \\ \downarrow \iota & & \downarrow \iota \\ L & \longrightarrow & L \end{array}$$

Figure 1

where $Aut_{Lie}(L)$ is the automorphism group of the Lie algebra L and ι is an embedding from $St_{Lie}(L)$ to L as vector spaces. It is an interesting to note that the equality

$$St_{Lie}(L) = \theta(St_{Lie}(L)) \tag{1}$$

sometimes holds and sometimes does not hold for any $\theta \in Aut_{Lie}(L)$. A Lie algebra L is degree-stabilizing if $St_{Lie}(L)$ is auto-invariant, i.e., the equality (1) holds. Kaplansky generalizes the Witt algebra as follows:
Let $\mathbf{V}$ be a vector space over $\mathbb{F}$ and G a total additive group of functionals on $\mathbf{V}$. Let A be the vector space direct sum of copies of $\mathbf{V}$, one for each element of A. An element of A is $\sum_{x \in \mathbf{V}, \alpha \in G} c_{x,\alpha}(x, \alpha)$ where $c_{x,\alpha} \in \mathbb{F}$. If we define the multiplication as $[(x, \alpha), (y, \beta)] = \alpha(y)(x, \alpha + \beta) - \beta(x)(y, \alpha + \beta)$, then we have a Lie algebra (see [7]). Kaplansky shows that if $dim(\mathbf{V}) \neq 1$, then the Lie algebra is simple in the paper [7]. Kawamoto defines an infinite dimensional generalized Witt Lie algebra which is simple in his paper [8]. Đoković and K. Zhao also define a class of infinite dimensional generalized Witt Lie algebras which are simple in the papers [4], [5], [17]. The other generalized Witt algebra are defined on a stable algebra in the formal power series ring $\mathbb{F}[[x_1, \cdots, x_n]]$ or on the localization of the stable algebra (see [3], [6], [12]). One of those kinds of algebras is defined as follows: for fixed positive integers $t_{11} > \cdots > t_{1p}, \cdots, t_{n1} > \cdots > t_{np}$, we define the $\mathbb{F}$-algebra $\mathbb{F}[n^{p+\cdots+q}, m, s]$ which is spanned by

$$\begin{aligned} &\{e^{a_{11}x_1^{t_{11}}} \cdots e^{a_{1p}x_1^{t_{1p}}} \cdots e^{a_{n1}x_n^{t_{n1}}} \cdots e^{a_{nq}x_n^{t_{nq}}} x_1^{i_1} \cdots x_m^{i_m} x_{m+1}^{i_{m+1}} \cdots x_{m+s}^{i_{m+s}} | \\ &a_{11}, \cdots, a_{np}, i_1, \cdots, i_m \in \mathbb{Z}, i_{m+1}, \cdots, i_{m+s} \in \mathbb{N}\} \end{aligned} \tag{2}$$

such that the algebra $\mathbb{F}[n^{p+\cdots+q}, m, s] := \mathbb{F}[n^*, m, s]$ contains the polynomial ring $\mathbb{F}[x_1, x_2, \cdots, x_{m+s}]$ where e^{x_r} is the exponential function for $r \in \{1, \cdots, n\}$ etc. (see [1], [6], [10], [11]). For $n, m, s \in \mathbb{N}$, the Lie admissible algebra

$$NW(n^{p+\cdots+q}, m, s) := NW(n^*, m, s)$$

has the standard basis

$$\begin{aligned} B_{W(n,m,s)} = \{&e^{a_{11}x_1^{t_{11}}} \cdots e^{a_{1p}x_1^{t_{1p}}} \cdots e^{a_{n1}x_n^{t_{n1}}} \cdots e^{a_{nq}x_n^{t_{nq}}} x_1^{i_1} \cdots x_m^{i_m} x_{m+1}^{i_{m+1}} \cdots x_{m+s}^{i_{m+s}} \partial_u | \\ &a_{11}, \cdots, a_{np}, i_1, \cdots, i_m \in \mathbb{Z}, i_{m+1}, \cdots, i_{m+s} \in \mathbb{N}, \\ &1 \leq u \leq m + s, n \leq max\{m, s\}\} \end{aligned} \tag{3}$$

with the obvious addition such that the multiplication $*$ is defined as follows:

$$f\partial_u * g\partial_v = f\partial_u(g)\partial_v$$

for $f, g \in NW(n^*, m, s)$ where ∂_u is the partial derivative on $\mathbb{F}[n^*, m, s]$ with respect to x_u, $1 \le u \le m+s$. The antisymmetrized algebra of $NW(n^*, m, s)$ is the Witt type Lie algebra $W(n^*, m, s)$. The Lie algebra $W(n^*, m, s)$ is $\mathbb{Z}^{p+\cdots+q}$-graded as follows:

$$W(n^*, m, s) = \bigoplus_{a_{11},\cdots,a_{nq}} W_{a_{11},\cdots,a_{nq}} \tag{4}$$

where $W_{a_{11},\cdots,a_{nq}}$ is the vector subspace of $W(n^*, m, s)$ spanned by

$$\{e^{a_{11}x_1^{t_{11}}} \cdots e^{a_{1p}x_1^{t_{1p}}} \cdots e^{a_{n1}x_n^{t_{n1}}} \cdots e^{a_{nq}x_n^{t_{nq}}} x_{i_1}^{i_1} \cdots x_{m+s}^{i_{m+s}} \partial_u |$$
$$i_1, \cdots, i_m \in \mathbb{Z}, i_{m+1}, \cdots, i_{m+s} \in \mathbb{N}, 1 \le u \le m+s, n \le max\{m, s\}\}$$

(see [16]). For each basis element

$$e^{a_{11}x_1^{t_{11}}} \cdots e^{a_{1p}x_1^{t_{1p}}} \cdots e^{a_{n1}x_n^{t_{n1}}} \cdots e^{a_{nq}x_n^{t_{nq}}} x_1^{i_1} \cdots x_m^{i_m} x_{m+1}^{i_{m+1}} \cdots x_{m+s}^{i_{m+s}} \partial_u$$

of $W(n^*, m, s)$, we define the Lie degree of the basis element as follow:

$$deg_{Lie}(e^{a_{11}x_1^{t_{11}}} \cdots e^{a_{1p}x_1^{t_{1p}}} \cdots e^{a_{n1}x_n^{t_{n1}}} \cdots e^{a_{nq}x_n^{t_{nq}}} x_1^{i_1} \cdots x_m^{i_m} x_{m+1}^{i_{m+1}} \cdots x_{m+s}^{i_{m+s}} \partial_u)$$
$$= |i_1| + \cdots + |i_m| + i_{m+1} + \cdots + i_{m+s}$$

(see [16]). For any l of $W(n^*, m, s)$, we can define the Lie degree $deg_{Lie}(l)$ as the highest degree of non-zero terms of l. The Witt algebra $W(0, 0, 1)$ and the centerless Virasoro algebra $W(0, 1, 0)$ are self-centralizing (see [15]). Furthermore they are degree-stabilizing (see [7]). Let A be a subset of a Lie algebra L. The centralizer $Cl_L(A)$ is the set $\{l \in L | [l, l_1] = 0$ for any $l_1 \in A\}$. For any l in the Lie algebra L, l_1 is ad-diagonal with respect to l, if $[l, l_1] = cl$ holds where $c \in \mathbb{F}$. For a Lie algebra L, an element l in L is ad-diagonal of the set A in L, if for any $x \in A$, $[l, x] = c_x x$ holds where $c_x \in \mathbb{F}$. For a given basis B of a Lie algebra L, the toral $tor_L(B) = tor(B)$ of B is n, if there are n ad-diagonal elements $\{l_1, \cdots l_n\}$ with respect to B such that the set $\{l_1, \cdots, l_n\}$ is the linearly independent maximal subset of L. For a Lie algebra L, $Tor(L)$ is defined as follows:

$$Tor(L) = max\{tor(B) | B \text{ is a basis of } L\}.$$

A Lie algebra L is n-toral, if $Tor(L) = n$. The Lie algebras $W(0, 1, 0)$ and $W(0, 0, 1)$ are 1-toral and self-centralizing (see [9]). For an algebra A, two bases B_1 and B_2 of A are equivalent denoted by $B_1 \sim B_2$, if for any element b_1 of B_1, there is an element b_2 of B_2 such that $b_1 = cb_2$ holds for some non-zero scalar c.

3. Automorphism group of $W(1^2, 1, 1)$

Note 1. It is well know that the non-associative algebra $NW(n^*, m, s)$ and the Lie (or its antisymmetrized) algebra $W(n^*, m, s)$ are simple (see [3], [11], [12]). Thus every non-zero endomorphism of $NW(n^*, m, s)$ or $W(n^*, m, s)$ is injective.

Note that the standard basis of $W(1^2, 1, 1)$ is
$\{e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_u | a, b, i \in \mathbb{Z}, j \in \mathbb{N}, 1 \leq u \leq 2\}$.
Generally, it is not easy to prove that $St_{Lie}(L)$ is a Lie subalgebra of L or not, i.e., it depends on the Lie algebra. For any basis elements $e^{a_1x^{t_1}}e^{b_1x^{t_2}}x^{i_1}y^{j_1}\partial_u$ and $e^{a_2x}e^{b_2x^{t_2}}x^{i_2}y^{j_2}\partial_v$ of $W(1^2, 1, 1)$, let us define the natural order $>_{Lie}$ as follows:

$$c_1e^{a_1x^{t_1}}e^{b_1x^{t_2}}x^{i_1}y^{j_1}\partial_u >_{Lie} c_2e^{a_2x^{t_1}}e^{b_2x^{t_2}}x^{i_2}y^{j_2}\partial_v, \tag{5}$$

if and only if $a_1 > a_2$, or $a_1 = a_2$ and $b_1 > b_2$, or $a_1 = a_2$, $b_1 = b_2$, and $i_1 > i_2$, or $\cdots$, and $a_1 = a_2$, $b_1 = b_2$, $i_1 = i_2$ $j_1 = j_2$, and $u < v$ for any non-zero scalars c_1 and c_2. Thus we can define the natural order on $W(1^2, 1, 1)$. In (5), note that a coefficient of a basis element does not affect the order $>_{Lie}$ of $W(1^2, 1, 1)$. Thus we may define $deg_{Lie}(l)$ of any element $l \in W(1^2, 1, 1)$ as the highest Lie degree of non-zero basis terms of l. Note that $W(1^2, 1, 1)$ is simple (see [12]). From now on, let us assume that $t_1 > t_2$.

Lemma 3.1. *$St_{Lie}(W(1^2, 1, 1))$ is a Lie subalgebra of the Lie algebra $W(1^2, 1, 1)$ spanned by $\{x\partial_2, y\partial_2, \partial_2\}$.*

Proof. It is obvious that the Lie subalgebra $< \{x\partial_2, y\partial_2, , \partial_2\} >$ of $W(1^2, 1, 1)$ spanned by $\{x\partial_2, y\partial_2, \partial_2\}$ is in $St_{Lie}(W(1^2, 1, 1))$. It is trivial to prove that every element which is not in $< \{x\partial_2, y\partial_2, \partial_1, \partial_2\} >$ cannot be degree stable. This implies that $St_{Lie}(W(1^2, 1, 1)) =< \{x\partial_2, y\partial_2, \partial_2\} >$. Therefore we have proven the lemma. ■

To find the automorphism group $Aut_{Lie}(W(1^2, 1, 1))$ of the Lie algebra $W(1^2, 1, 1)$, we will find the stable Lie subalgebra of $W(1^2, 1, 1)$ and an auto-invariant set of $W(1^2, 1, 1)$.

Lemma 3.2. *For any $\theta \in Aut_{Lie}(W(1^2, 1, 1))$, the element $\theta(y\partial_2)$ is in the stabilizer $St_{Lie}(W(1^2, 1, 1))$ of the Lie algebra $W(1^2, 1, 1)$.*

Proof. For any $\theta \in Aut_{Lie}(W(1^2, 1, 1))$ and a basis element $e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_u$ of the algebra $W(1^2, 1, 1)$, we have that

$$\theta([y\partial_2, e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_u] = (j - \delta_{2,u})\theta(e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_u) \tag{6}$$

where $\delta_{2,u}$ is the Kronecker delta. By (6) and the fact that $W(1^2, 1, 1)$ is simple, for any $l \in W(1^2, 1, 1)$, we have that

$$deg_{Lie}([\theta(y\partial_2), \theta(l)]) \leq deg_{Lie}(\theta(l)).$$

This implies that $\theta(y\partial_2) \in St_{Lie}(W(1^2, 1, 1))$ and so $\theta(y\partial_2)$ can be written as follows:

$$\theta(y\partial_2) = d_1x\partial_2 + d_2y\partial_2 + d_3\partial_2 \tag{7}$$

where $d_1, d_2, d_3 \in \mathbb{F}$. ■

Lemma 3.3. *There is no automorphism θ of $W(1^2, 1, 1)$ such that*

$$\theta(y\partial_2) = d_1 x\partial_2 + d_2 y\partial_2 + d_3\partial_2 \tag{8}$$

where $d_1, d_2 \in \mathbb{F}^\bullet$ and $d_3 \in \mathbb{F}$.

Proof. Let θ be the automorphism of $W(1^2, 1, 1)$ such that it holds the conditions of the lemma for the element in the lemma. $\theta(x^u\partial_1)$ can be written as follow:

$$\begin{aligned}\theta(x^u\partial_1) = c(a_{u1}, b_{u1}, i_{u1}, j_{u1}, 1)e^{a_{u1}x^{t_1}}e^{b_{u1}x^{t_2}}x^{i_{u1}}y^{j_{u1}}\partial_1 + \\ c(a_{u1}, b_{u1}, i_{u1}, j_{u1}, 1)e^{a_{u1}x^{t_1}}e^{b_{u1}x^{t_2}}x^{i_{u1}}y^{j_{u1}}\partial_2 + \#_1\end{aligned} \tag{9}$$

where either $e^{a_1x^{t_1}}e^{b_1x^{t_2}}x^{i_{u1}}y^{j_{u1}}\partial_1$ or $e^{a_1x^{t_1}}e^{b_1x^{t_2}}x^{i_{u1}}y^{j_{u1}}\partial_2$ is the maximal term of the element $\theta(x^u\partial_1)$ depending on their coefficients and $\#_1$ is the sum of the remaining terms of $\theta(\partial_1)$ with appropriate coefficients using the order $>_{Lie}$ and $u \in \mathbb{N}$. Furthermore, by Lemma 3 of [2], we can assume that $b_{u1} \neq 0$. If $j_{u1} \neq 0$, then $x^u\partial_1$ cannot centralize $y\partial_2$. We have that

$$\begin{aligned}\theta(x^u\partial_1) = c(a_{u1}, b_{u1}, i_{u1}, 0, 1)e^{a_{u1}x^{t_1}}e^{b_{u1}x^{t_2}}x^{i_{u1}}\partial_1 + \\ c(a_{u1}, b_{u1}, i_{u1}, 0, 1)e^{a_{u1}x^{t_1}}e^{b_{u1}x^{t_2}}x^{i_{u1}}\partial_2 + \#_1.\end{aligned} \tag{10}$$

Since $\theta(x\partial_1)$ is an ad-diagonal element with respect to $\{\theta(x^v\partial_1)|v \in \mathbb{N}\}$, every maximal term of $\theta(x^v\partial_1)$ is in the (a_{u1}, b_{u1})-homogeneous component $W_{a_{u1},b_{u1}}$. Since $\theta(y\partial_2)$ centralizes $\theta(x^u\partial_1)$ and $d_1, d_2 \neq 0$, if $c(a_{u1}, b_{u1}, i_{u1}, j_{u1}, 1) \neq 0$, then $c(a_{u1}, b_{u1}, i_{u1}, j_{u1}, 2) \neq 0$ and vice versa. Since $\theta(x\partial_1)$ is an ad-diagonal element with respect to $\{\theta(x^v\partial_1)|v \in \mathbb{N}\}$, $\theta(x^v\partial_1)$ and $\theta(\partial_1)$ have terms in the same homogeneous components. This implies that all terms of the elements $\theta(x^u\partial_1)$, $u \in \mathbb{N}$, have the same maximal terms with appropriate coefficients. Let us prove the lemma by induction on the number $H(\theta(x\partial_1))$ of homogeneous components of $\theta(x\partial_1)$ such that the homogeneous components have a non-zero term of $\theta(x\partial_1)$. Let us assume that $H(\theta(x\partial_1))$ is one. Since $\theta(x\partial_1)$ is an ad-diagonal element with respect to $\{\theta(x^v\partial_1)|v \in \mathbb{N}\}$, it has a term in the $(0,0)$-homogeneous component $W_{0,0}$. By assumption, there is no room of $\theta(x\partial_1)$ to have a term of $W_{0,0}$. This contradiction shows that we can assume that $H(\theta(x\partial_1)) \geq 2$. This implies that $\theta(x\partial_1)$ has a non-zero term of $W_{0,0}$. There is an element $\theta(x^u\partial_1)$ which also has a non-zero term of $W_{0,0}$ such that the degree of the maximal term of $\theta(x^u\partial_1)$ is greater than zero where $u \neq 1$. $\theta(x\partial_1)$ and $\theta(x^u\partial_1)$ have the same maximal terms of $W_{0,0}$ with appropriate scalars. Thus every non-zero term of $\theta(x\partial_1)$ which is not in $W_{0,0}$ is a non-zero term of $\theta(x^u\partial_1)$ with appropriate coefficients and vice versa. Since $H(\theta(x\partial_1)) = H(\theta(x\partial_1)) \geq 2$, there are $c \in \mathbb{F}$ and $u \in \mathbb{N}$ such that

$$[\theta(x\partial_1) - c\theta(x^u\partial_1), \theta(x^u\partial_1)] \neq (u-1)\theta(x^u\partial_1). \tag{11}$$

This contradiction shows that we can assume that $x^r\partial_1$ is the maximal term of $\theta(x\partial_1)$ for an integer $r > 1$. This gives a similar contradiction as (11). This implies that $\theta(x\partial_1) \in W_{0,0}$. This implies that $\theta(y\partial_2)$ cannot centralize $\theta(x^u\partial_1)$. This contradiction shows that there is no automorphism θ of $W(1^2, 0, 2)$ which holds (3.7). Therefore we have proven the lemma. ■

Lemma 3.4. *There is no automorphism θ of $W(1^2, 1, 1)$ such that*

$$\theta(y\partial_2) = d_1 x\partial_2 + d_2\partial_2 = (d_1 x + d_2)\partial_2 \tag{12}$$

holds where $d_1 \in \mathbb{F}^\bullet$ and $d_2 \in \mathbb{F}$.

Proof. Let θ be the automorphism of $W(1^2, 0, 2)$ such that it holds (12). By Lemma 3.3, we are able to prove that $\theta(y\partial_2)$ cannot centralize an element $\theta(x^u\partial_1)$, $u > 1$. This contradiction shows that there is no automorphism θ of $W(1^2, 0, 2)$ which holds (12). Therefore we have proven the lemma. ■

Lemma 3.5. *For any automorphism θ of $W(1^2, 1, 1)$ and any basis element $y^k\partial_2$ of $W(1^2, 1, 1)$,*

$$\theta(y^k\partial_2) = d^{1-k}(y + d_1)^k\partial_2 \tag{13}$$

holds where $d_1 \in \mathbb{F}$ and $d \in \mathbb{F}^\bullet$.

Proof. Let θ be the automorphism of $W(1^2, 1, 1)$. By Lemmas 3.1, 3.2, 3.3, 3.4, we have that $\theta(y\partial_2) = (y + d_1)\partial_2$ holds for $d_1 \in \mathbb{F}$. This implies that $\theta(\partial_2) = d\partial_2$ holds for $d \in \mathbb{F}^\bullet$. By induction on $k \in \mathbb{N}$ of $y^k\partial_2$, we are able to prove that $\theta(y^k\partial_2) = d^{1-k}(y + d_1)^k\partial_2$ holds. Therefore we have proven the lemma. ■

Lemma 3.6. *For any automorphism θ of $W(1^2, 1, 1)$ and any basis element $x^i\partial_1$ of $W(1^2, 1, 1)$,*

$$\theta(x^i\partial_1) = c^{1-i}x^i\partial_1 \tag{14}$$

holds where $c \in \mathbb{F}^\bullet$.

Proof. Let θ be the automorphism of $W(1^2, 1, 1)$. By Lemma 3.5, we have that $\theta(y^k\partial_2) = d^{1-k}(y + d_1)^k\partial_2$ holds for $d_1 \in \mathbb{F}$ and $d \in \mathbb{F}^\bullet$. So we are able to prove that $\theta(\partial_1) = c\partial_1$ holds for $c \in \mathbb{F}^\bullet$. Since the Lie subalgebra $W(0, 1, 0)$ of $W(1^2, 1, 1)$ spanned by $\{x^u\partial_1 | u \in \mathbb{Z}\}$ is a self-centralizing Lie algebra, we have two cases, Case I: $\theta(x\partial_1) = -(x+c_1)\partial_1$ and Case II: $\theta(x\partial_1) = (x+c_1)\partial_1$ for $c_1 \in \mathbb{F}$.

Case I. Let us assume that $\theta(x\partial_1) = -(x+c_1)\partial_1$ holds. By $\theta([\partial_1, x\partial_1]) = \theta(\partial_1)$, we have that $-[\theta(\partial), (x + c_1)\partial_1] = \theta(\partial_1)$, we have that $\theta(\partial_1) = \alpha_0(x + c_1)^2\partial_1$ for $\alpha_0 \in \mathbb{F}^\bullet$. This implies that $\theta(x^2\partial_1) = \alpha_2\partial_1$ for $\alpha_2 \in \mathbb{F}^\bullet$. By $\theta([x^{-1}\partial_1, x^2\partial_1]) = 3\theta(\partial_1)$, we have that $c_1 = 0$ and $\theta(x^{-1}\partial_1) = \alpha_1 x^3\partial_1$ $\alpha_1 \in \mathbb{F}^\bullet$. By induction on i of $x^{-i}\partial_1$, we have that $\theta(x^{-i}\partial_1) = \alpha_{-i}x^{i+2}\partial_1$ for $\alpha_{-i} \in \mathbb{F}^\bullet$. By $\theta([x^{-t_1+1}\partial_1, e^{x^{t_1}}\partial_1]) = \theta(e^{x^{t_1}}\partial_1)$, we are able to prove that

$$[\alpha_{-t_1+1}x^{t_1+1}\partial_1, \theta(e^{x^{t_1}}\partial_1)] = \theta(e^{x^{t_1}}\partial_1) \tag{15}$$

holds. Since $\theta(e^{x^{t_1}}\partial_1) \notin W_{0,0}$ and $t_1 + 1$ is positive, there is no element $\theta(e^{x^{t_1}}\partial_1)$ of the algebra which holds the equality (15). This contradiction shows that there is no automorphism which holds $\theta(x\partial_1) = -(x + c_1)\partial_1$.

Case II. Let us assume that $\theta(x\partial_1) = (x+c_1)\partial_1$ holds. By induction on $i \in \mathbb{N}$, we are able to prove that $\theta(x^i\partial_1) = c^{1-i}(x+c_1)^i\partial_1$ holds. By $\theta([x^{-2}\partial_1, x^3\partial_1]) = 5\theta(\partial_1)$, we have that $[\theta(x^{-2}\partial_1), c^{-2}(x + c_1)^3\partial_1] = 5c\partial_1$. This implies that $c_1 = 0$ and $\theta(x^{-2}\partial_1) = c^3(x^{-2}\partial_1$. By induction on i of $x^i\partial_1$, we can prove that $\theta(x^i\partial_1) = c^{1-i}x^i\partial_1$ easily. Therefore we have proven the lemma. ■

Note 2. For any basis elements $e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_1$ and $e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_1$ of $W(1^2, 1, 1)$, $c_{11}, c_{12}, d_{11}, d_{12} \in \mathbb{F}^\bullet$, and $c_{13} \in \mathbb{F}$, if we define a linear map $\theta_{c_{11},c_{12},c_{13},d_{11},d_{12},1}$ from $W(1^2, 1, 1)$ to itself as follows:

$$\theta_{c_{11},c_{12},c_{13},d_{11},d_{12},1}(e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_1) = c_{11}^{1-i}c_{12}^{-j}d_{11}^a d_{12}^b e^{ax^{t_1}}e^{bx^{t_2}}x^i(y + c_{13})^j\partial_1,$$
$$\theta_{c_{11},c_{12},c_{13},d_{11},d_{12},1}(e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_2) = c_{11}^{-i}c_{12}^{1-j}d_{11}^a d_{12}^b e^{ax^{t_1}}e^{bx^{t_2}}x^i(y + c_{13})^j\partial_2, \quad (16)$$

then $\theta_{c_{11},c_{12},c_{13},d_{11},d_{12},1}$ can be linearly extended to a Lie automorphism of $W(1^2, 1, 1)$ such that $c_{11}^{t_1} = c_{11}^{t_2} = 1$.

Note 3. For any basis elements $e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_1$ and $e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_1$ of $W(1^2, 1, 1)$, $c_{21}, c_{22}, d_{21}, d_{22} \in \mathbb{F}^\bullet$, and $c_{23} \in \mathbb{F}$, if we define a linear map $\theta_{c_{21},c_{22},c_{23},d_{21},d_{22},2}$ from $W(1^2, 1, 1)$ to itself as follows:

$$\theta_{c_{21},c_{22},c_{23},d_{21},d_{22},2}(e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_1) = c_{21}^{1-i}c_{22}^{-j}d_{21}^a d_{22}^b e^{-ax^{t_1}}e^{bx^{t_2}}x^i(y + c_{23})^j\partial_1,$$
$$\theta_{c_{21},c_{22},c_{23},d_{21},d_{22},2}(e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_2) = c_{21}^{-i}c_{22}^{1-j}d_{21}^a d_{22}^b e^{-ax^{t_1}}e^{bx^{t_2}}x^i(y + c_{23})^j\partial_2, \quad (17)$$

then $\theta_{c_{21},c_{22},c_{23},d_{21},d_{22},2}$ can be linearly extended to a Lie automorphism of $W(1^2, 1, 1)$ such that $c_{21}^{t_1} = -1$ and $c_{21}^{t_2} = 1$.

Note 4. For any basis elements $e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_1$ and $e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_1$ of $W(1^2, 1, 1)$, $c_{31}, c_{32}, d_{31}, d_{32} \in \mathbb{F}^\bullet$, and $c_{33} \in \mathbb{F}$, if we define a linear map $\theta_{c_{31},c_{32},c_{33},d_{31},d_{32},3}$ from $W(1^2, 1, 1)$ to itself as follows:

$$\theta_{c_{31},c_{32},c_{33},d_{31},d_{32},3}(e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_1) = c_{31}^{1-i}c_{32}^{-j}d_{31}^a d_{32}^b e^{ax^{t_1}}e^{-bx^{t_2}}x^i(y + c_{33})^j\partial_1,$$
$$\theta_{c_{31},c_{32},c_{33},d_{31},d_{32},3}(e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_2) = c_{31}^{-i}c_{32}^{1-j}d_{31}^a d_{32}^b e^{ax^{t_1}}e^{-bx^{t_2}}x^i(y + c_{33})^j\partial_2, \quad (18)$$

then $\theta_{c_{31},c_{32},c_{33},d_{31},d_{32},3}$ can be linearly extended to a Lie automorphism of $W(1^2, 1, 1)$ such that $c_{31}^{t_1} = 1$ and $c_{31}^{t_2} = -1$. □

Note 5. For any basis elements $e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_1$ and $e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_1$ of $W(1^2, 1, 1)$, $c_{41}, c_{42}, d_{41}, d_{42} \in \mathbb{F}^\bullet$, and $c_{43} \in \mathbb{F}$, if we define a linear map $\theta_{c_{41},c_{42},c_{43},d_{41},d_{42},4}$ from $W(1^2, 1, 1)$ to itself as follows:

$$\theta_{c_{41},c_{42},c_{43},d_{41},d_{42},4}(e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_1) = c_{41}^{1-i}c_{42}^{-j}d_{41}^a d_{42}^b e^{-ax^{t_1}}e^{-bx^{t_2}}x^i(y + c_{43})^j\partial_1,$$
$$\theta_{c_{41},c_{42},c_{43},d_{41},d_{42},4}(e^{ax^{t_1}}e^{bx^{t_2}}x^iy^j\partial_2) = c_{41}^{-i}c_{42}^{1-j}d_{41}^a d_{42}^b e^{-ax^{t_1}}e^{-bx^{t_2}}x^i(y + c_{43})^j\partial_2, \quad (19)$$

then $\theta_{c_{41},c_{42},c_{43},d_{41},d_{42},4}$ can be linearly extended to a Lie automorphism of $W(1^2, 1, 1)$ such that $c_{41}^{t_1} = c_{41}^{t_2} = -1$.

Lemma 3.7. *For any automorphism θ of $W(1^2, 1, 1)$, θ is one of the automorphisms $\theta_{c_{11},c_{12},c_{13},d_{11},d_{12},1}$, $\theta_{c_{21},c_{22},c_{23},d_{21},d_{22},2}$, $\theta_{c_{31},c_{32},c_{33},d_{31},d_{32},3}$, and $\theta_{c_{41},c_{42},c_{43},d_{41},d_{42},4}$ as shown in Notes 2-5 with appropriate constant conditions.*

Proof. Let θ be the automorphism of $W(1^2,1,1)$ in the theorem. By Lemma 3.5 and Lemma 3.6, we can assume that (13) and (14) hold with the same constants. Thus by induction on i,j of $x^iy^j\partial_u$, $1 \le u \le 2$, we are able to prove that $\theta(W(0,0,2)) = W(0,0,2)$ holds, i.e., $W(0,0,2)$ is θ-invariant or auto-invariant. Since $y^u\partial_2$ centralizes $e^{x^{t_1}}\partial_1$ and $y\partial_2 \in St_{Lie}(W(1^2,1,1))$, we have that

$$\theta(e^{x^{t_1}}\partial_1) = c_{a,b,i,0,1}e^{ax^{t_1}}e^{bx^{t_2}}x^i\partial_1 + \#_1 \tag{20}$$

holds where $e^{ax^{t_1}}e^{bx^{t_2}}x^i\partial_1$ is the maximal term of $\theta(e^{x^{t_1}}\partial_1)$ and $\#_1$ does not have a term with ∂_2. We have three cases, Case I: $a,b \neq 0$, Case II: $a=0$ and $b \neq 0$, and Case III: $a \neq 0$ and $b = 0$.

Case I. Let us assume that $a,b \neq 0$. We have that $\theta(e^{-x^{t_1}}\partial_1)$ has a similar form as (20). By

$$\theta([e^{x^{t_1}}\partial_1, e^{x^{t_1}}\partial_1]) \in W(0,0,2), \tag{21}$$

we have that the maximal term of $\theta(e^{-x^{t_1}}\partial_1)$ is in W_{a_1,b_1} or in $W_{-a_1,-b_1}$. Let us assume that the maximal term of $\theta(e^{-x^{t_1}}\partial_1)$ is in W_{a_1,b_1}. Thus by (22), $\theta(e^{x^{t_1}}\partial_1)$ and $\theta(e^{-x^{t_1}}\partial_1)$ have terms in the same homogeneous components. Furthermore we can assume that $H(\theta(e^{x^{t_1}}\partial_1) = \theta(e^{-x^{t_1}}\partial_1) \geq 2$. This implies that there is non-zero constant c such that

$$[\theta(e^{x^{t_1}}\partial_1), \theta(e^{x^{t_1}}\partial_1 - ce^{-x^{t_1}}\partial_1)] \neq -2ct_1\theta(x^{t_1-1}\partial_1). \tag{22}$$

Thus we can assume that the maximal term of $\theta(e^{-x^{t_1}}\partial_1)$ is in $W_{-a_1,-b_1}$. If $H(\theta(e^{x^{t_1}}\partial_1) \neq 1$, then we can derive a contradiction because of the minimal term of $\theta([e^{t_1}\partial_1, e^{-x^{t_1}}])$. If $H(\theta(e^{x^{t_1}}\partial_1) = 1$, then we have that $\theta([e^{t_1}\partial_1, e^{-x^{t_1}}]) \neq -2t_1\theta(x^{t_1-1}\partial_1)$. This gives a contradiction. Thus $a,b \neq 0$ does not hold.

Case II. Let us assume that $a = 0$ and $b \neq 0$. This implies that $\theta(e^{x^{t_1}}\partial_1) = c'e^{x^{bt_2}}\partial_1 + \#_2$ holds. By $\theta([\partial_1, e^{x^{t_1}}\partial_1]) = t_1\theta(e^{x^{t_1}}x^{t_1-1}\partial_1)$, we have that

$$[c\partial_1, c'e^{x^{bt_2}}\partial_1 + \#_2] = cc'bt_2e^{x^{t_1}}x^{i+t_1-1}\partial_1 + \#_3$$

holds. This implies that

$$\theta(e^{x^{t_1}}x^{t_1-1}\partial_1) = cc'b\frac{t_2}{t_1}e^{x^{t_1}}x^{i+t_1-1}\partial_1 + \frac{\#_3}{t_1}.$$

This implies that

$$\begin{aligned}\theta([x\partial_1, e^{x^{t_1}}x^{t_1-1}\partial_1]) &= t_1\theta(e^{x^{t_1}}x^{2t_1-1}\partial_1) + (t_1-2)\theta(e^{x^{t_1}}x^{t_1-1}\partial_1)\\ &= [(x+c_1)\partial_1, cc'b\frac{t_2}{t_1}e^{x^{t_1}}x^{i+t_1-1}\partial_1 + \frac{\#_3}{t_1}]\end{aligned}$$

holds. This implies that $\theta(e^{x^{t_1}}x^{2t_1-1}\partial_1) = cc'b^2\frac{t_2^2}{t_1^2}e^{bx^{t_1}}x^{i+2t_1-1}\partial_1 + \#_4$. Note that

$$\theta([x^{t_1}\partial_1, e^{x^{t_1}}\partial_1]) = t_1\theta(e^{x^{t_1}}x^{2t_1-1}\partial_1) - t_1 + \theta(e^{x^{t_1}}x^{t_1-1}\partial_1).$$

This implies that

$$[c^{1-t_1}(x+c_1)^{t_1}\partial_1, c_1e^{bx^{t_2}}x^i\partial_1] = cc'b^2\frac{t_2^2}{t_1^2}e^{bx^{t_1}}x^{i+2t_2-1}\partial_1 + \#_5 \tag{23}$$

holds. Since $i+t_1+t_2-1 \neq i+2t_2-1$, the equality (23) does not hold. So we have a contradiction. Thus there is no automorphism of $W(1^2,1,1)$ which holds $a=0$ and $b\neq 0$.

Case III. Let us assume that $a\neq 0$ and $b=0$. This implies that $\theta(e^{x^{t_1}}\partial_1) = c'e^{x^{at_1}}\partial_1 + \#_6$. Let us assume that $\#_6 \neq 0$. Let us assume that $H(e^{x^{t_1}}\partial_1)\geq 1$. This implies that $H(e^{x^{-t_1}}\partial_1) = H(e^{x^{t_1}}\partial_1)$ holds. This implies that there is a non-zero scalar c such that

$$\theta([e^{-x^{-t_1}}\partial_1, e^{x^{-t_1}}\partial_1 - ce^{x^{-t_1}}\partial_1]) \neq 2t_1\theta(e^{x^{-t_1}}\partial_1). \tag{24}$$

This contradiction shows that $H(e^{x^{-t_1}}\partial_1) = 1$. Similarly we can prove that $\theta(e^{x^{t_1}}\partial_1) = de^{ax^{t_1}}\partial_1$ and $\theta(e^{-x^{t_1}}\partial_1) = d_1e^{-ax^{t_1}}\partial_1$. This implies that $\theta(x^i\partial_1) = c^{1-i}x^i\partial_1$. By induction on i,k of $x^iy^k\partial_u$, $1\leq u\leq 2$, we are able to prove that

$$\theta(x^iy^k\partial_u) = c^{\delta_{1u}-i}d^{\delta_{2u}-k}x^i(y+c)^k\partial_u \tag{25}$$

where δ_{1u} and δ_{2u} are Kronecker deltas. Since

$$\{x^py^j\partial_u, e^{x^{t_1}}\partial_1, e^{-x^{t_1}}\partial_1 | a,i\in\mathbb{Z}, u,i\in\mathbb{N}, 1\leq u\leq 2\}$$

is a generator of the Lie subalgebra $W(1^1,1,1)$ of $W(1^2,1,1)$, we have that a is either 1 or -1. So we have two subcases, Subcase I: $a=1$ and Subcase II: $a=-1$.

Subcase I. Let us assume that $\theta(e^{x^{t_1}}\partial_1) = d_1e^{x^{t_1}}\partial_1$ holds for $d_1\in\mathbb{F}^\bullet$.
By $\theta([e^{-x^{t_1}}\partial_1, e^{x^{t_1}}\partial_1]) = 2t_1\theta(x^{t_1-1}\partial_1)$,
we have that $[\theta(e^{-x^{t_1}}\partial_1), d_1e^{x^{t_1}}\partial_1] = 2c^{2-t_1}t_1(x+c_1)^{t_1-1}\partial_1$
holds. Therefore $c_1=0$ and $\theta(e^{-x^{t_1}}\partial_1) = \frac{c^{2-t_1}}{d_1}e^{-x^{t_1}}\partial_1$ hold. By $\theta([x\partial_1, e^{x^{t_1}}\partial_1]) = t_1\theta(e^{x^{t_1}}x^{t_1}\partial_1) - \theta(e^{x^{t_1}}\partial_1)$, we have that $\theta(e^{x^{t_1}}x^{t_1}\partial_1) = de^{x^{t_1}}x^{t_1}\partial_1$ holds. By Lemma 3.6 and $\theta([x^{-t_1+1}\partial_1, e^{x^{-t_1}}\partial_1]) = t_1\theta(e^{x^{t_1}}\partial_1) - (-t_1+1)\theta(e^{x^{t_1}}x^{-t_1}\partial_1)$, we have that $[c^{t_1}x^{-t_1+1}\partial_1, d_1e^{x^{-t_1}}\partial_1] = t_1d_1e^{x^{t_1}}\partial_1 + (t_1-1)\theta(e^{x^{t_1}}x^{-t_1}\partial_1)$, This implies that $c=1$. Similarly we can prove that A: $\theta(e^{x^{t_2}}\partial_1) = d_2e^{x^{t_2}}\partial_1$ and B: $\theta(e^{x^{t_2}}\partial_1) = d_2e^{-x^{t_2}}\partial_1$ where $d_2\in\mathbb{F}^\bullet$.

A. Let us assume that $\theta(e^{x^{t_2}}\partial_1) = d_2e^{x^{t_2}}\partial_1$ holds. Since $x^{-t_2+1}\partial_1$ is an ad-diagonal element with respect to $e^{x^{t_2}}\partial_1$, we can prove that $c^{t_2}=1$ holds. By induction on b of $e^{x^{bt_2}}\partial_1$, we can prove that $\theta(e^{x^{bt_2}}\partial_1) = d_2^be^{x^{bt_2}}\partial_1$. So we have that

$$\theta(e^{x^{at_1}}e^{x^{bt_2}}x^i\partial_1) = c^{1-i}d_1^ad_2^be^{x^{at_1}}e^{x^{bt_2}}x^i\partial_1 \tag{26}$$

holds. So by (25) and (26), we can prove that θ can be linearly extended to the automorphism $\theta_{c_{11},c_{12},c_{13},d_{11},d_{12},1}$ as shown Note 2 with appropriate constants.

B. Let us assume that $\theta(e^{x^{t_2}}\partial_1) = d_2 e^{-x^{t_2}}\partial_1$. Similarly we can prove that $c^{t_2} = -1$. Similarly to A, we are able to prove that θ can be linearly extended to the automorphism $\theta_{c_{31},c_{32},c_{33},d_{31},d_{32},3}$ as shown Note 4 with appropriate constants.

Subcase II. Let us assume that $\theta(e^{x^{t_1}}\partial_1) = d_1 e^{-x^{t_1}}\partial_1$ holds for $d_1 \in \mathbb{F}^\bullet$. Similarly to Subcase I, we have that C: $\theta(e^{x^{t_2}}\partial_1) = d_2 e^{x^{t_2}}\partial_1$ and D: $\theta(e^{x^{t_2}}\partial_1) = d_2 e^{-x^{t_2}}\partial_1$ where $d_2 \in \mathbb{F}^\bullet$.

C. If we assume that $\theta(e^{x^{t_2}}\partial_1) = d_2 e^{x^{t_2}}\partial_1$, then similarly to A, θ can be linearly extended to the automorphism $\theta_{c_{21},c_{22},c_{23},d_{21},d_{22},2}$ as shown Note 3 with appropriate constants.

D. If we assume that $\theta(e^{x^{t_2}}\partial_1) = d_2 e^{-x^{t_2}}\partial_1$, then similarly to A, θ can be linearly extended to the automorphism $\theta_{c_{41},c_{42},c_{43},d_{41},d_{42},4}$ as shown Note 5 with appropriate constants.

This implies that θ can be linearly extended to one of the the automorphisms $\theta_{c_{11},c_{12},c_{13},d_{11},d_{12},1}$, $\theta_{c_{21},c_{22},c_{23},d_{21},d_{22},2}$, $\theta_{c_{31},c_{32},c_{33},d_{31},d_{32},3}$, and $\theta_{c_{41},c_{42},c_{43},d_{41},d_{42},4}$ as shown in Notes 2-5. Therefore we have proven the lemma. ∎

Theorem 3.8. *The automorphism group of the algebra $W(1^2,1,1)$ is generated by the automorphisms $\theta_{c_{11},c_{12},c_{13},d_{11},d_{12},1}$, $\theta_{c_{21},c_{22},c_{23},d_{21},d_{22},2}$, $\theta_{c_{31},c_{32},c_{33},d_{31},d_{32},3}$, and $\theta_{c_{41},c_{42},c_{43},d_{41},d_{42},4}$ as shown in Notes 2-5 with appropriate constant conditions.*

Proof. Let θ be an automorphism of $W(1^2,1,1)$. By Lemma 3.7, θ is one of the automorphisms $\theta_{c_{11},c_{12},c_{13},d_{11},d_{12},1}$, $\theta_{c_{21},c_{22},c_{23},d_{21},d_{22},2}$, $\theta_{c_{31},c_{32},c_{33},d_{31},d_{32},3}$, and $\theta_{c_{41},c_{42},c_{43},d_{41},d_{42},4}$ as shown in Notes 2-5 with appropriate constant conditions. Thus the automorphism group $Aut(W(1^2,1,1))$ of the algebra $W(1^2,1,1)$ is generated by the automorphisms $\theta_{c_{11},c_{12},c_{13},d_{11},d_{12},1}$, $\theta_{c_{21},c_{22},c_{23},d_{21},d_{22},2}$, $\theta_{c_{31},c_{32},c_{33},d_{31},d_{32},3}$, and $\theta_{c_{41},c_{42},c_{43},d_{41},d_{42},4}$ as shown in Notes 2-5 with appropriate constant conditions. Therefore we have proven the theorem. ∎

Remark 3.9. Thanks to Theorem 3.8, we have that the automorphism group of $W(1^2,2,0)$ is generated by the automorphisms $\theta_{c_{11},c_{12},0,d_{11},d_{12},1}$, $\theta_{c_{21},c_{22},0,d_{21},d_{22},2}$, $\theta_{c_{31},c_{32},0,d_{31},d_{32},3}$, and $\theta_{c_{41},c_{42},0,d_{41},d_{42},4}$ which are defined on the algebra $W(1^2,2,0)$ as similar Notes 2-5 of $W(1^2,2,0)$.

4. Cartan Subalgebra

A Cartan subalgebra $\mathfrak{C}$ of the algebra $W(1^2,1,1)$ (resp. $W(1^2,2,0)$) is spanned by $y\partial_2 + c\partial_2$ (resp. $y\partial_2$) where $c \in \mathbb{F}$. Note that $\mathfrak{C}$ is one dimensional and $Tor(W(1^2,1,1))$ (resp. $Tor(W(1^2,2,0))$) is one. The root space decomposition of $W(1^2,1,1)$ (resp. $W(1^2,2,0)$) with respect to $\mathfrak{C}$ is the following:

$$W(1^2,1,1) = \bigoplus_{j\in\mathbb{N}\cup\{-1\}} W_j \qquad (\text{ resp. } W(1^2,2,0) = \bigoplus_{j\in\mathbb{Z}} W_j) \tag{27}$$

where W_j is a vector subspace of $W(1^2,1,1)$ (resp. $W(1,2,0)$) spanned by $\{e^{a_1x^{t_1}}e^{a_2x^{t_2}}x^{i_1}(y+c)^j\partial_1, e^{a_1x^{t_1}}e^{a_2x^{t_2}}x^{i_2}(y+c)^{j+1}\partial_2 | a_1,a_2,i_1 \in \mathbb{Z}, i_2 \in \mathbb{N}\}$ and $[y\partial_2 + c\partial_2, W_j] \subset W_j$ (resp. $\{e^{a_1x^{t_1}}e^{a_2x^{t_2}}x^{i_1}y^j\partial_1,\ e^{a_1x^{t_1}}e^{a_2x^{t_2}}x^{i_2}e^{a_2x^{t_2}}y^{j+1}\partial_2 | a_1,a_2,i_1,i_2 \in \mathbb{Z}\}$ and $[y\partial_2, W_j] \subset W_j$). We have the following proposition.

Proposition 4.1. *For any automorphism θ of the algebra $W(1^2,1,1)$ (resp. $W(1,2,0)$), $\theta(y\partial_2) = y\partial_2 + c\partial_2$ (resp. $y\partial_2$) where $c \in \mathbb{F}$.*

Proof. Since any Cartan subalgebra $\mathfrak{C}$ of $W(1^2,1,1)$ (resp. $W(1^2,2,0)$) is one dimensional, the Cartan subalgebra $\mathfrak{C}$ is auto-invariant. Thus the proof of the proposition is obvious. ∎

Remark 4.2. Thanks to Proposition 4.1, we are also able to find the automorphism groups of the algebras $W(1^2,1,1)$ and $W(1^2,2,0)$.

References

[1] Chang, H. J., *Über Wittsche Lie-Ringe* (German), Abh. Math. Sem. Hansischen Univ. **14**, (1941). 151–184.

[2] Choi, Seul Hee, and Ki-Bong Nam, *Grade Stable Lie Algebras I*, Rocky Mountain Journal of Mathematics **40** (2010), 813–824.

[3] Choi, Seul Hee, Jongwoo Lee, and Ki-Bong Nam, *Derivations of a restricted Weyl-type algebra containing the polynomial ring*, Comm. Algebra **36** (2008), 3435–3446.

[4] Đoković, D. Ž., and K. Zhao, *Derivations, isomorphisms, and second cohomology of generalized Witt algebras* Trans. Amer. Math. Soc. **350** (1998), 643–664.

[5] —, *Generalized Cartan type W Lie algebras in characteristic zero*, J. Algebra **195** (1997), 170–210.

[6] Kac, V. G., *Description of Filtered Lie Algebra with which Graded Lie algebras of Cartan type are Associated*, Izv. Akad. Nauk SSSR, Ser. Mat. Tom **38** (1974), 832–834.

[7] Kaplansky, I., *Seminar on simple Lie algebras*, Bull. Amer. Math. Soc. **60** (1954), 470–471.

[8] Kawamoto, N., *Generalizations of Witt algebras over a field of characteristic zero*, Hiroshima Math. J. **16** (1986), 417–426.

[9] —, *On G-graded automorphisms of generalized Witt algebras*, Contemp. Math. Amer. Math. Soc. **184** (1995), 225–230.

[10] Kawamoto, Naoki, Atsushi Mitsukawa, Ki-Bong Nam, and Moon-Ok Wang, *The automorphisms of generalized Witt type Lie algebras*, Journal of Lie Theory, **13** (2003), 571–576.

[11] Lee, Jongwoo, and Ki-Bong Nam, *Non-associative algebras containing the matrix ring*, Linear Algebra Appl. **429** (2008), 72–78.

[12] Nam, Ki-Bong, *Generalized W and H Type Lie Algebras*, Algebra Colloquium **6** (1999), 329–340.

[13] —, *Generalized S-type Lie Algebras*, Rocky Mountain Journal of Mathematics **37** (2007), 1291–1230.

[14] Nam, Ki-Bong, and Seul Hee Choi, *Degree Stable Lie Algebras I*, Algebra Colloquium **13** (2006), 487–494.

[15] Nam, Ki-Bong, and Jonathan Pakianathan, *On generalized Witt algebras in one variable*, Turkish Journal of Mathematics (2010), to appear.

[16] Rudakov, A. N., *Groups of automorphisms of infinite-dimensional simple Lie algebras*, Math. USSR-Izvestija **3** (1969), 707–722.

[17] Zhao, K., *Isomorphisms between generalized Cartan type W Lie algebras in characteristic* 0, Canad. J. Math. **50** (1998), 210–224.

Jongwoo Lee
Graduate School of Railroad
Seoul National Univ. of Technology
172 Kongneung-dong Nowongu
Seoul, Korea
saganlee@snut.ac.kr

Xueqing Chen
Dept. of Math and Computer Sciences
Univ. of Wisconsin-Whitewater
Whitewater, WI 53190, USA
chenx@uww.edu

Seul Hee Choi
Dept. of Mathematics
Univ. of Jeonju
Jeonju 560-759, Korea
chois@jj.ac.kr

Ki-Bong Nam
Dept. of Math and Computer Sciences
Univ. of Wisconsin-Whitewater
Whitewater, WI 53190, USA
namk@uww.edu

Received November 11, 2009
and in final form January 20, 2011