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Invariant Strong KT Geometry on Four-Dimensional Solvable Lie Groups

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Abstract. A strong KT (SKT) manifold consists of a Hermitian structure whose torsion three-form is closed. We classify the invariant SKT structures on four-dimensional solvable Lie groups. The classification includes solutions on groups that do not admit compact four-dimensional quotients. It also shows that there are solvable groups in dimension four that admit invariant complex structures but have no invariant SKT structure.
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1. Introduction

On any Hermitian manifold (M, g, J) there is a unique Hermitian connection [10], called the Bismut connection, which has torsion a three-form. Explicitly the Bismut connection is given by

$$\nabla^{\mathrm{B}} = \nabla^{\mathrm{LC}} + \tfrac{1}{2}T^{\mathrm{B}}, \qquad c^{\mathrm{B}} = \left(T^{\mathrm{B}}\right)^{\flat} = -Jd\omega, \tag{1.1}$$

where $\omega = g(J\cdot, \cdot)$ is the fundamental two-form and $Jd\omega = -d\omega(J\cdot, J\cdot, J\cdot)$. If the torsion three-form c^{B} is closed, we have a *strong Kähler manifold with torsion*, or briefly an SKT *manifold*. The study of SKT structures has received notable attention over recent years, see [7] for a survey and for an approach through generalized geometry, see [3]. This has been motivated partly by the quest for canonical choices of metric compatible with a given complex structure and partly by the relevance of such geometries to super-symmetric theories from physics [8, 13, 14, 15, 23].

Kähler manifolds are precisely the SKT manifolds with torsion three-form identically zero. However, most SKT manifolds are non-Kähler. For example compact semisimple Lie groups cannot be Kähler since they have second Betti number equal to zero, but any even-dimensional compact Lie group can be endowed

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with the structure of an SKT manifold, see Appendix 6. The SKT geometry of nilpotent Lie groups was studied by Fino, Parton & Salamon [6], who provided a full classification in dimension 6 for left-invariant structures.

In this paper we classify left-invariant SKT structures on four-dimensional solvable Lie groups, showing that there are a number of new examples; see Table 4.1, only the first two entries belong to the nilpotent classification. The greater variety and complexity of this case is already seen from the classification results for complex structures: Salamon [19] classified the 6-dimensional nilpotent Lie groups with left-invariant integrable complex structure, whereas in the solvable case there is a classification only in dimension four [1, 18, 21].

In dimension four, a Hermitian manifold (M, g, J) is an SKT manifold precisely when the associated Lee one-form $\theta = Jd^*\omega$ is co-closed. When M is compact, Gauduchon [9] showed that, up to homothety, there is a unique such metric in each conformal class of Hermitian metrics. The situation for non-compact manifolds is less clear. Our classification includes non-compact SKT manifolds that admit no compact quotient, and also shows that there are invariant complex structures that admit no compatible invariant SKT metric.

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2. Solvable Lie algebras

Since we are interested in invariant structures on a simply-connected Lie group G, it is sufficient to study the corresponding structures on the Lie algebra $\mathfrak{g}$. To $\mathfrak{g}$ one associates two series of ideals: the *lower central series*, which is given by $\mathfrak{g}_1 = \mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{g}_k = [\mathfrak{g}, \mathfrak{g}_{k-1}]$ and the *derived series* defined by $\mathfrak{g}^1 = \mathfrak{g}'$, $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}^{k-1}]$. The Lie algebra is *nilpotent* (resp. *solvable*) if its lower (resp. derived) series terminates after finitely many steps.

One has that $\mathfrak{g}^j \subseteq \mathfrak{g}_j$, so that nilpotent algebras are solvable. On the other hand, consider a solvable Lie algebra $\mathfrak{g}$. Lie's Theorem applied to the adjoint representation of the complexification $\mathfrak{g}_{\mathbb{C}}$, gives a complex basis for $\mathfrak{g}_{\mathbb{C}}$ with respect to which each ad_X is upper triangular. One then has the well-known:

Lemma 2.1. *A finite-dimensional Lie algebra $\mathfrak{g}$ is solvable if and only if its derived algebra $\mathfrak{g}'$ is nilpotent.* ∎

Remark 2.2. For $\mathfrak{g}$ solvable of dimension four, $\mathfrak{g}'$ has dimension at most three and so is one of a known list. Lemma 2.1 then implies that $\mathfrak{g}'$ is either Abelian or the Heisenberg algebra $\mathfrak{h}_3$, which has basis elements E_1, E_2, E_3 with only one non-trivial Lie bracket $[E_1, E_2] = E_3$.

Identifying $\mathfrak{g}$ with left-invariant vector fields on G, and $\mathfrak{g}^*$ with left-invariant one-forms one has the relation

$$da(X, Y) = -a([X, Y])$$

for all $X, Y \in \mathfrak{g}$ and $a \in \mathfrak{g}^*$. We may describe for example $\mathfrak{h}_3$ by letting e_1, e_2, e_3 be the dual basis in $\mathfrak{g}^*$ to E_1, E_2, E_3 and computing $de_1 = 0$, $de_2 = 0$, $de_3 = e_2 \wedge e_1$. We will use the compact notation $\mathfrak{h}_3 = (0, 0, 21)$ to encode these relations.

Let $\Lambda^*\mathfrak{g}^*$ be the exterior algebra on $\mathfrak{g}^*$ and write $\mathcal{I}(A)$ for the ideal in $\Lambda^*\mathfrak{g}^*$ generated by a subset A. We interpret the condition for $\mathfrak{g}$ to be solvable dually via the elementary:

Lemma 2.3. *A finite-dimensional Lie algebra $\mathfrak{g}$ is solvable if and only if there are maximal subspaces $\{0\} = W_0 < W_1 < \dots < W_r = \mathfrak{g}^*$ such that*

$$dW_i \subseteq \mathcal{I}(W_{i-1}) \tag{2.1}$$

for each i. ■

Concretely $W_1 = \ker(d\colon \mathfrak{g}^* \to \Lambda^2\mathfrak{g}^*)$ (cf. [19]) and W_i is defined inductively to be the maximal subspace satisfying (2.1). We will sometimes find it useful to choose a filtration $\{0\} = V_0 < V_1 < \dots < V_n = \mathfrak{g}^*$ with

$$\dim_{\mathbb{R}} V_i = i \quad \text{and} \quad dV_i \subseteq \mathcal{I}(V_{i-1}) \qquad \text{for each } i. \tag{2.2}$$

One way to construct such filtrations is to refine the spaces W_i, however in some cases other choices may be possible and useful.

3. The SKT structural equations

A left-invariant almost Hermitian structure on G is determined by an inner product g on the Lie algebra $\mathfrak{g}$ and a linear endomorphism J of $\mathfrak{g}$ such that $J^2 = -1$ and $g(JX, JY) = g(X, Y)$ for all $X, Y \in \mathfrak{g}$. The SKT condition consists of the requirement that J be integrable and that $dJd\omega = 0$ where $\omega(X, Y) = g(JX, Y)$. In the differential algebra, integrability of J may be expressed as the condition that $d\Lambda^{1,0} \subseteq \Lambda^{2,0} + \Lambda^{1,1}$. If $\mathfrak{g}$ is four-dimensional and solvable, we now show that there is one of two choices of possible good bases $\{a, Ja, b, Jb\}$ for $\mathfrak{g}^*$. We will later determine the SKT condition in each case.

Lemma 3.1. *Let $\mathfrak{g}$ be a solvable Lie algebra of dimension four. If (g, J) is an integrable Hermitian structure on $\mathfrak{g}$ then there is an orthonormal set $\{a, b\}$ in $\mathfrak{g}^*$ such that $\{a, Ja, b, Jb\}$ is a basis for $\mathfrak{g}^*$ and either*

Complex case: *$\mathfrak{g}$ has structural equations*

$$\begin{aligned} da = 0, \quad d(Ja) = x_1 aJa, \quad db = y_1 aJa + y_2 ab + y_3 aJb + z_1 bJa + z_2 JaJb,\\ d(Jb) = u_1 aJa + u_2 ab + u_3 aJb + v_1 bJa + v_2 JaJb + w_1 bJb, \end{aligned} \tag{3.1}$$

or

Real case: *$\mathfrak{g}$ has structural equations*

$$\begin{aligned} da = 0, \quad d(Ja) = x_1 aJa + x_2(ab + JaJb) + x_3(aJb + bJa) + y_2 bJb,\\ db = z_1 aJa + z_2 ab + z_3 aJb,\\ d(Jb) = u_1 aJa + u_2 ab + u_3 aJb + v_1 bJa + v_2 bJb + w_1 JaJb. \end{aligned} \tag{3.2}$$

In the complex case, $\{a, Ja, b, Jb\}$ may be chosen orthonormal and $\omega = aJa + bJb$, omitting $\wedge$ signs. In the real case, $\omega = aJa + bJb + t(ab + JaJb)$ for some $t \in (-1, 1)$.

Proof. Let V_i be a refined filtration of $\mathfrak{g}^*$ as in (2.2). As $\dim_{\mathbb{R}} V_2 = 2$ we have two possibilities for the complex subspace $V_2 \cap JV_2$, either it is non-trivial so $V_2 = JV_2$ or it is zero. If the filtration V_i can be chosen with $V_2 = JV_2$ we will say we are in the complex case, otherwise we are in the real case.

For the complex case, $JV_2 = V_2$ and $V_1 \subseteq V_2 \cap \ker d$, so we may take an orthonormal basis $\{a, Ja\}$ of V_2 with $a \in V_1$. We have $da = 0$ and solvability implies $d(Ja) \in \mathcal{I}(a) \cap \Lambda^2 = \mathbb{R}aJa \oplus a \wedge V_2^{\perp}$. As J is integrable, we must have $d(Ja) \in \Lambda^{1,1}$ too, so $d(Ja) = x_1 aJa$.

In the real case, choose $a \in V_1$ and $b \in V_2 \cap V_1^{\perp}$ of unit length. Then $da = 0$ and the form of $d(Ja)$ follows from the condition $d(Ja) \in \Lambda^{1,1}$. The form of ω follows from $t = g(b, Ja)$ which has absolute value less than 1 by the Cauchy-Schwarz inequality. ■

The above equations are necessary but far from sufficient. For integrability it remains to impose $d(b - iJb)^{0,2} = 0$, and to obtain a Lie algebra the Jacobi identity must be satisfied. The latter is equivalent to the condition $d^2 = 0$. Both of these conditions are straightforward to compute. We list the results below. In each case the first line comes from the integrability condition on J, in the last line we provide the SKT condition and the remaining equations are from $d^2 = 0$.

Lemma 3.2. *The structural equations of Lemma 3.1 give an SKT structure on a solvable Lie algebra if and only if the following quantities vanish:*

Complex case:

$$\begin{gathered}
y_2 - z_2 - u_3 + v_1, \quad y_3 - z_1 + u_2 - v_2, \\
x_1 z_1 - y_3 v_1 - z_2 u_2, \quad (x_1 - y_2 + u_3) z_2 - y_3(z_1 + v_2), \\
y_2 w_1, \quad y_3 w_1, \quad z_1 w_1, \quad z_2 w_1, \\
(x_1 + y_2 - u_3) v_1 - (z_1 + v_2) u_2 + u_1 w_1, \\
x_1 v_2 + y_1 w_1 - y_3 v_1 - z_2 u_2, \\
(x_1 + y_2 + u_3)(y_2 + u_3) + (z_1 - v_2)^2 - u_1 w_1.
\end{gathered} \tag{3.3}$$

Real case:

$$\begin{gathered}
z_2 - u_3 + v_1, \quad z_3 + u_2 - w_1, \\
x_2 u_2 - x_3(z_2 - v_1) - y_2 u_1, \quad (-x_1 + z_2 + u_3) y_2 + x_2^2 + x_3(x_3 - v_2), \\
x_2 u_3 - x_3(w_1 + z_3) + y_2 z_1, \quad (x_1 + z_2 - u_3) v_1 - (x_3 - v_2) u_1 - u_2 w_1, \\
x_2 v_2 - y_2 w_1, \quad x_3 z_1 + z_3 v_1, \quad y_2 z_1 + z_3 v_2, \quad x_2 z_1 + z_3 w_1, \quad x_2 v_1 - x_3 w_1, \\
x_2 w_1 + x_3 v_1 - y_2 u_1 + z_2 v_2, \quad x_1 w_1 - x_2 u_1 + z_1 v_2 - z_3 v_1, \\
\big\{(x_1 + z_2 + u_3)(-y_2 + z_2 + u_3) + x_2(x_2 - z_1 + t v_2) \\
\qquad + \big(x_3 - u_1 + t(u_2 - w_1)\big)(x_3 + v_2) + w_1^2\big\}.
\end{gathered} \tag{3.4}$$

In some cases the SKT structure reduces to Kähler. This occurs if and only if the following additional conditions hold:

Complex case:

$$y_1 = 0 = u_1, \quad u_3 = -y_2, \quad v_2 = z_1 \tag{3.5}$$

Real case:

$$\begin{gathered} x_2 - z_1 = t(x_1 + u_3), \quad x_3 - u_1 = -tu_2, \quad y_2 - z_2 - u_3 = tx_2, \\ w_1 = t(x_3 + v_2). \end{gathered} \tag{3.6}$$

4. The SKT classification

We are now ready to describe the simply-connected four-dimensional solvable real Lie groups admitting invariant SKT structures. The notation for and distinguishing characteristics of all the solvable real Lie algebras in dimensions up to four are summarised in Appendix 7 following the classification in [1].

Theorem 4.1. *Let G be a simply-connected four-dimensional solvable real Lie group. Then G admits a left-invariant* SKT *structure if and only if its Lie algebra $\mathfrak{g}$ is listed in Table 4.1. Furthermore the left-invariant* SKT *structures on G may be explicitly determined and the dimension and number of connected components of the moduli space up to homotheties are as in Table 4.1.*

The table also indicates which groups admit invariant Kähler metrics, and gives the dimensions of the Lie algebra cohomology.

$\mathfrak{g}'$	$\mathfrak{g}$	dim	π_0	Kähler	$(b_1 \dots b_4)$
$\{0\}$	$\mathbb{R}^4$	0	1	✓	$(4,6,4,1)$
$\mathbb{R}$	$\mathbb{R} \times \mathfrak{h}_3$	0	1	×	$(3,4,3,1)$
	$\mathbb{R} \times \mathfrak{r}_{3,0}$	1	1	✓	$(3,3,1,0)$
$\mathbb{R}^2$	$\mathbb{R} \times \mathfrak{r}'_{3,0}$	1	1	✓	$(2,2,2,1)$
	$\mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}}$	2	1	✓	$(2,1,0,0)$
$\mathbb{R}^3$	$\mathfrak{r}'_{4,\lambda,0}$ $(\lambda > 0)$	1	2	✓	$(1,1,1,0)$
	$\mathfrak{r}_{4,-1/2,-1/2}$	1	1	×	$(1,0,1,1)$
	$\mathfrak{r}'_{4,2\lambda,-\lambda}$ $(\lambda > 0)$	1	2	×	$(1,0,1,1)$
$\mathfrak{h}_3$	$\mathfrak{d}_4$	2	1	×	$(1,0,1,1)$
	$\mathfrak{d}_{4,2}$	2	1	✓	$(1,1,1,0)$
	$\mathfrak{d}'_{4,0}$	2	1	×	$(1,0,1,1)$
	$\mathfrak{d}_{4,1/2}$	1	1	✓	$(1,0,0,0)$
	$\mathfrak{d}'_{4,\lambda}$ $(\lambda > 0)$	1	1	✓	$(1,0,0,0)$

Table 4.1: The four-dimensional solvable Lie algebras that admit an SKT structure. Of these, only $\mathbb{R}^4$ fails to admit an SKT structure that is not Kähler. In the table, dim and π_0 are the dimension and number of components of the SKT moduli space modulo homotheties, b_k denotes $\dim H^k(\mathfrak{g})$.

The proof will occupy the rest of this section. Following Remark 2.2 we analyse the possible solutions to the equations of §3 case by case after the type of $\mathfrak{g}'$. We use the Lie algebra structure of $\mathfrak{g}$ combined with the SKT geometry to determine a canonical choice of basis $\{a, Ja, b, Jb\}$ with $\{a, b\}$ orthonormal, refining the approach of §3. When talking of the SKT moduli space, we consider only left-invariant structures on the given G. These are determined by (g, J) on $\mathfrak{g}$. Two SKT pairs (g_1, J_1) and (g_2, J_2) on $\mathfrak{g}$ are considered equivalent if there is a Lie algebra automorphism ϕ with $\phi^* g_2 = g_1$ and $\phi \circ J_1 = J_2 \circ \phi$. Equivalent structures have canonical bases with the same structure constants and any remaining parameters in the structure equations are parameters for the SKT moduli space.

4.1. Trivial derived algebra.

For $\mathfrak{g}' = \{0\}$, $\mathfrak{g} \cong \mathbb{R}^4$ is Abelian, $d \equiv 0$ so all structure constants are zero and each almost Hermitian structure is Kähler. All these Kähler structures are equivalent.

4.2. One-dimensional derived algebra.

For $\mathfrak{g}' = \mathbb{R}$, we have $\dim W_1 = 3$. It follows that we can choose a, Ja, $b \in W_1$ and are thus in the case $V_2 = JV_2$. The structural equations for $\mathfrak{g}$ in this case are

$$\begin{gathered} da = 0 = d(Ja) = db, \\ d(Jb) = u_1 aJa + u_2(ab + JaJb) + u_3(aJb + bJa) + w_1 bJb, \end{gathered}$$

where the coefficients satisfy $0 = u_2^2 + u_3^2 - u_1 w_1$ and $d(Jb) \neq 0$. Rotating a, Ja in V_2, we may ensure that $u_2 = 0$ and $u_3 \geqslant 0$, so $u_1 w_1 = u_3^2$. Replacing b by $-b$, we obtain $w_1 \geqslant 0$.

If $w_1 = 0$ then $u_3 = 0$ and we may take $u_1 > 0$, after an appropriate choice of b. Thus we have the algebra given by

$$da = 0 = d(Ja) = db, \quad d(Jb) = u_1 aJa. \tag{4.1}$$

Any other orthonormal Hermitian basis $\{a', Ja', b', Jb'\}$ with $a', Ja' \in V_2$, $b' \in W_1$ and $u_1' > 0$ has $b' = b$, $a' = \cos\theta\, a + \sin\theta\, Ja$ and $d(Jb') = u_1'\, a'Ja' = u_1\, aJa$. The parameter $u_1 > 0$ thus describes inequivalent SKT solutions. Scaling of the metric by a homothety, $g \mapsto \lambda^2 g$, $\lambda > 0$, is realised by $a \mapsto \lambda a$, $b \mapsto \lambda b$ and gives $u_1 \mapsto u_1/\lambda$. Thus the resulting SKT metrics are all homothetic to each other. These SKT structures are not Kähler. Moreover we see that $\mathfrak{g}$ is nilpotent and so isomorphic to $\mathbb{R} \times \mathfrak{h}_3$.

If $w_1 > 0$ then $\mathfrak{g}$ is not nilpotent and so isomorphic to $\mathbb{R} \times \mathfrak{r}_{3,0}$. As $u_1 w_1 = u_3^2 \geqslant 0$ we have the structural equations

$$da = 0 = d(Ja) = db, \quad d(Jb) = u_1 aJa + u_3(aJb + bJa) + w_1 bJb,$$

with $u_3 = \sqrt{u_1 w_1}$, $u_1 \geqslant 0$. This is Kähler only if $u_1 = 0$. The non-Kähler solutions have $u_1, u_3, w_1 > 0$ and $u_2 = 0$, which fixes the choice of basis $\{a, Ja, b, Jb\}$. Up to homothety the only parameter is u_1. The moduli space is thus connected.

4.3. Two-dimensional derived algebra.

For $\mathfrak{g}' = \mathbb{R}^2$, we have $\dim W_1 = 2$, and we shall distinguish between the cases $W_1 = JW_1$ and $W_1 \cap JW_1 = \{0\}$ where $W_1 = \ker d$ is complex or real.

Complex kernel We have $W_1 = JW_1$ and taking $V_2 = W_1$ thus have the structural equations

$$\begin{aligned} da &= 0 = d(Ja), \\ db &= y_1 aJa + y_3 aJb + z_2 JaJb, \\ d(Jb) &= u_1 aJa - y_3 ab + z_2 bJa \end{aligned}$$

with no restrictions on the coefficients other than that db and $d(Jb)$ are linearly independent. Rotating a, Ja we may put $z_2 = 0$, $y_3 > 0$. Rotating b, Jb we can then get $u_1 \geqslant 0$, $y_1 = 0$, reducing the structure to

$$da = 0 = d(Ja), \quad db = y_3 aJb, \quad d(Jb) = u_1 aJa - y_3 ab.$$

The solution is Kähler if and only if $u_1 = 0$. For $u_1 > 0$ the Hermitian basis is unique. The SKT moduli space is connected of dimension 1 modulo homotheties. The Lie algebra $\mathfrak{g}$ is isomorphic to $\mathbb{R} \times \mathfrak{r}'_{3,0}$.

Real kernel Here $W_1 \cap JW_1 = \{0\}$ and we again take $V_2 = W_1$ putting us in the real case and giving the structural equations

$$\begin{aligned} da &= 0 = db, \\ d(Ja) &= x_1 aJa + x_3(aJb + bJa) + y_2 bJb, \\ d(Jb) &= u_1 aJa + u_3(aJb + bJa) + v_2 bJb, \end{aligned}$$

where the last two lines are linearly independent and the coefficients satisfy

$$\begin{gathered} (x_1 - u_3)y_2 = (-v_2 + x_3)x_3, \quad u_1(v_2 - x_3) = u_3(u_3 - x_1), \\ u_3 x_3 = u_1 y_2, \quad (u_1 - x_3)(v_2 + x_3) = (u_3 + x_1)(u_3 - y_2). \end{gathered} \tag{4.2}$$

Lemma 4.2. *We have* $\mathfrak{z}(\mathfrak{g}) = \{0\}$ *and* $\mathfrak{u}(\mathfrak{g}) \cong \mathfrak{r}_{3,-1}$*, so* $\mathfrak{g} \cong \mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}}$.

Proof. We compute the centre via $\mathfrak{z}(\mathfrak{g}) = \{\, X \in \mathfrak{g} : X \lrcorner d\alpha = 0 \text{ for all } \alpha \in \mathfrak{g}^* \,\}$. Writing $X = pA + qB + p'JA + q'JB$, where $\{A, B, JA, JB\}$ is the dual basis to $\{a, b, Ja, Jb\}$, one finds that $X \in \mathfrak{z}(\mathfrak{g})$ implies $(p, q, 0)^T$ and $(0, p, q)^T$ lie in the one-dimensional null space of the rank two matrix

$$Q = \begin{pmatrix} x_1 & x_3 & y_2 \\ u_1 & u_3 & v_2 \end{pmatrix}.$$

We conclude that $p = 0 = q$. The same calculation applies to p' and q', so $X = 0$ and $\mathfrak{z}(\mathfrak{g}) = \{0\}$.

Writing $\mathbf{a} = \left(\begin{smallmatrix} x_1 \\ x_3 \end{smallmatrix}\right)$, $\mathbf{b} = \left(\begin{smallmatrix} x_3 \\ y_2 \end{smallmatrix}\right)$, $\mathbf{c} = \left(\begin{smallmatrix} u_1 \\ u_3 \end{smallmatrix}\right)$, $\mathbf{d} = \left(\begin{smallmatrix} u_3 \\ v_2 \end{smallmatrix}\right)$, equations (4.2) may be interpreted geometrically as saying that $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{a} - \mathbf{d}$ are mutually parallel and that $\mathbf{b} - \mathbf{c}$ is parallel to $\mathbf{a} + \mathbf{d}$. Imposing the constraint $\operatorname{rank} Q = 2$, then leads to the fact that $\mathbf{a}$ and $\mathbf{d}$ are linearly independent.

The map $\chi = \operatorname{Tr}\operatorname{ad}\colon \mathfrak{g} \to \mathbb{R}$ is given by $\chi(A) = -(x_1 + u_3)$, $\chi(B) = -(x_3 + v_2)$, $\chi(JA) = 0 = \chi(JB)$. This is zero only if $\mathbf{a} = -\mathbf{d}$, which by the above remark, is not possible. Thus $\mathfrak{g}$ is not unimodular. Choosing $a \in \operatorname{Im}\chi^* \leqslant \ker d$, we have $0 = a(B) \propto \chi(B)$ and so $v_2 = -x_3$.

Write $\mathbf{a}-\mathbf{d} = 2k\mathbf{v}$ with $\mathbf{v} = \binom{c}{s}$, $c^2+s^2 = 1$. Then (4.2) implies $\mathbf{b}, \mathbf{c} \in \langle \mathbf{v} \rangle$. However $\mathbf{a}+\mathbf{d} \notin \langle \mathbf{v} \rangle$ but is parallel to $\mathbf{b}-\mathbf{c}$, so we find $\mathbf{b} = \mathbf{c} = h\mathbf{v}$, for some $h \in \mathbb{R}$. This gives $x_3 = ks = hc$, so we may write $k = \ell c$, $h = \ell s$ for some non-zero $\ell \in \mathbb{R}$. Changing the sign of $\mathbf{v}$ we may force $\ell > 0$. We get

$$Q = \ell \begin{pmatrix} c^2+1 & cs & s^2 \\ cs & s^2 & -cs \end{pmatrix}.$$

The last two columns specify the exterior derivative d on $\mathfrak{u}(\mathfrak{g})^* \cong \mathfrak{g}^*/\operatorname{Im}\chi^*$. One sees that $\mathfrak{u}(\mathfrak{g}) \cong \mathfrak{r}_{3,-1}$ as B acts with eigenvalues $\pm\ell s$. ∎

To summarise, we get a unique choice of basis $\{a, Ja, b, Jb\}$ with $\{a, b\}$ orthonormal by taking $a \in \operatorname{Im}\chi^*$, $b \in \ker d \cap (\operatorname{Im}\chi^*)^\perp$ with $x_1 > 0$ and $v_2 > 0$.

We may describe the isomorphism of $\mathfrak{g}$ with $\mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}}$ explicitly by introducing half-angles. Writing $c = \sigma^2 - \tau^2$, $s = 2\sigma\tau$, $\sigma^2 + \tau^2 = 1$, $\sigma > 0$ and using the orthogonal transformation $a' = \sigma a + \tau b$, $b' = -\tau a + \sigma b$, gives the structural equations

$$d(Ja') = 2\ell\sigma\, a'Ja', \qquad d(Jb') = -2\ell\tau\, b'Jb'.$$

We have $\ell, \sigma > 0$ and, replacing b' by $-b'$ if necessary, we may ensure that $\tau < 0$. The SKT moduli space is thus parameterised by $\sigma/\tau \in (-1, 0)$, $\ell > 0$ and the parameter $t = g(b', Ja') \in (-1, 1)$ in the metric. Up to homotheties it is connected of dimension 2. The solutions are Kähler precisely when $t = 0$.

Remark 4.3. If one considers the complex structure on $\mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}}$ with $de = 0$, $d(Je) = eJe$, $df = 0$, $d(Jf) = fJf$ one sees that a metric with $\omega = eJe + fJf + t(eJf + fJe)$ is SKT (indeed Kähler) only if $t = 0$. Thus for a given complex structure the SKT condition depends on the choice of metric. This is in contrast to the study of SKT structures on six-dimensional nilmanifolds [6].

4.4. Three-dimensional Abelian derived algebra. For $\mathfrak{g}' = \mathbb{R}^3$, we have $\dim W_1 = 1$, and moreover the assumption that $\mathfrak{g}'$ is Abelian implies that $d(Ja)$, db, $d(Jb) \in \mathcal{I}(a)$. So it is legitimate to assume that $V_2 = JV_2$. The structural equations are thus

$$da = 0, \quad d(Ja) = x_1 aJa,$$
$$db = y_1 aJa + y_2 ab + y_3 aJb, \quad d(Jb) = u_1 aJa - y_3 ab + y_2 aJb.$$

with coefficients satisfying the equation

$$0 = y_2(2y_2 + x_1)$$

and non-degeneracy conditions $x_1 \neq 0$, $y_2^2 + y_3^2 \neq 0$. One may choose a, b so that $x_1 > 0$, $y_1 \geqslant 0$ and $u_1 = 0$. This choice is unique if $y_1 > 0$, for $y_1 = 0$, b is an arbitrary unit vector in $V_2^\perp$. The solutions are then Kähler only if y_1 and y_2 are zero.

If $y_2 = 0$, then $y_3 \neq 0$ and $\mathfrak{g} \cong \mathfrak{r}'_{4,|x_1/y_3|,0}$. Thus on a given $\mathfrak{r}'_{4,\lambda,0}$, $\lambda > 0$, the SKT moduli up to homothety has dimension 1, parameter y_3, with two connected components determined by the sign of y_3, and contains the Kähler solutions as $y_1 = 0$.

For $y_2 \neq 0$, we have $x_1 = -2y_2$. There are two cases. For $y_3 = 0$, we have $\mathfrak{g} \cong \mathfrak{r}_{4,-1/2,-1/2}$ and there is a one-dimensional connected family of solutions up to homothety. For $y_3 \neq 0$, the Lie algebra $\mathfrak{g}$ is $\mathfrak{r}'_{4,2\lambda,-\lambda}$ with $\lambda = |y_2/y_3|$. Again the moduli is of dimension 1 up to homothety and has two connected components.

4.5. Three-dimensional non-Abelian derived algebra.

For $\mathfrak{g}' = \mathfrak{h}_3$, as above we have $\dim W_1 = 1$. Let d' denote the exterior derivative on $\mathfrak{g}'$. We distinguish between the complex and real cases $V_2 = JV_2$ and $V_2 \cap JV_2 = \{0\}$.

Complex case We have $a \in W_1 = V_1$, and $Ja \in V_2 = JV_2$. Moreover it is possible to take $b \in V_2^\perp$ with $d'b = 0$. The condition $\mathfrak{g}' \cong \mathfrak{h}_3$ then forces $d'(Jb) \in \langle bJa \rangle$, giving the structural equations

$$\begin{gathered} da = 0, \quad d(Ja) = x_1 aJa, \\ db = y_1 aJa + y_2 ab + y_3 aJb, \quad d(Jb) = u_1 aJa + u_2 ab + u_3 aJb + v_1 bJa, \end{gathered}$$

with x_1, $y_2^2 + y_3^2$ and v_1 non-zero. Adjusting the choice of a, we may take $x_1 > 0$. The SKT equations are now the vanishing of

$$\begin{gathered} y_2 - u_3 + v_1, \quad y_3 + u_2, \quad y_3 v_1, \\ v_1(x_1 + y_2 - u_3), \quad (y_2 + u_3)(y_2 + u_3 + x_1). \end{gathered}$$

We deduce that $y_3 = 0 = u_2$, $v_1 = x_1$ and $u_3 = y_2 + x_1$, leaving the condition $(2y_2 + x_1)(y_2 + x_1) = 0$.

If $y_2 = -x_1$, then the structural equations are

$$\begin{gathered} da = 0, \quad d(Ja) = x_1 aJa, \\ db = y_1 aJa - x_1 ab, \quad d(Jb) = u_1 aJa + x_1 bJa \end{gathered}$$

subject only to $x_1 > 0$. We see that $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}')$ is isomorphic to $\mathfrak{r}_{3,-1}$, so $\mathfrak{g}$ itself is isomorphic to $\mathfrak{d}_4$. The only ambiguity in the basis is $b \mapsto -b$, corresponding to $(y_1, u_1) \mapsto (-y_1, -u_1)$. The SKT moduli modulo homotheties is connected and has dimension 2. There are no Kähler solutions.

For $x_1 = -2y_2$, we have the structural equations

$$\begin{gathered} da = 0, \quad d(Ja) = x_1 aJa, \\ db = y_1 aJa - \tfrac12 x_1 ab, \quad d(Jb) = u_1 aJa + \tfrac12 x_1 aJb + x_1 bJa, \end{gathered}$$

again with $x_1 > 0$. The quotient $\mathfrak{g}/\mathfrak{z}(\mathfrak{g}')$ is isomorphic to $\mathfrak{r}_{3,-1/2}$, and $\mathfrak{g}$ is thus isomorphic to $\mathfrak{d}_{4,2}$. The solutions are Kähler only for $y_1 = 0 = u_1$. There is the same $b \mapsto -b$ ambiguity as above. Again the SKT moduli space up to homotheties is connected of dimension 2.

Real case First note that $\dim W_2 = 3$, so we may choose b to be a unit vector in $W_2 \cap \langle a, Ja\rangle^\perp$. This gives $t = g(b, Ja) = 0$. Now $d'b = 0$, where d' is the differential on $\mathfrak{g}'$, as above. As $\mathfrak{h}'_3 = \mathbb{R}$, we have that $d'(Ja)$ and $d'(Jb)$ are linearly dependent, but not both zero. In fact, if $d'(Ja) = 0$, we may take $V_2 = \langle a, Ja\rangle$ and reduce to the complex case described above, so we assume instead $d'(Ja) \neq 0$.

Write $(x_2, x_3, y_2) = m\mathbf{p}$, $(w_1, v_1, v_2) = n\mathbf{p}$ for some unit vector $\mathbf{p} = (p, q, r)$, $m \neq 0$. The structural equations of $\mathfrak{h}_3$, imply $b \wedge d'x = 0$ is zero for all $x \in \mathfrak{g}'$, giving $p = 0$ and $x_2 = 0 = w_1$. Now $q^2 + r^2 = 1$ and one may normalise so that $r \geqslant 0$. Then

$$d'(Ja) = m\, bJc, \quad d'(Jb) = n\, bJc,$$

where

$$c = qa + rb.$$

¿From this one sees $d'(nJa - mJb) = 0$ and so $(nJa - mJb) \wedge d'x = 0$ is zero too. We conclude that $qJa + rJb$ and $nJa - mJb$ are parallel and write $n = kq$, $m = -kr$, for some $k \neq 0$.

The structural equations are now

$$\begin{gathered} da = 0, \quad d(Ja) = x_1 aJa - kqr(aJb + bJa) - kr^2\, bJb, \\ db = z_1 aJa + z_2 ab + z_3 aJb, \\ d(Jb) = u_1 aJa + u_2 ab + u_3 aJb + kq^2\, bJa + kqr\, bJb, \end{gathered}$$

with $q^2 + r^2 = 1$, $r > 0$, the forms $d(Ja)$, db, $d(Jb)$ non-zero, and subject to

$$\begin{gathered} u_3 = z_2 + kq, \quad u_2 = -z_3, \quad rz_1 = qz_3, \\ kq^3 - qz_2 - ru_1 = 0, \quad 2kq^2 + x_1 - z_2 - u_3 = 0, \\ q(q(x_1 + z_2 - u_3) - 2ru_1) = 0, \quad (x_1 + z_2 + u_3)(z_2 + u_3 + kr^2) = 0. \end{gathered} \tag{4.3}$$

Substituting the first three equations into the remaining four, one sees that the first equation on the last line follows from the two on the middle line. There are thus two cases corresponding to the two factors of the last equation.

The first case is $z_2 = -x_1 - u_3$, which reduces to $x_1 = -kq^2 = -u_3$, $z_2 = 0$, $u_1 = kq^3/r$, giving the structural equations

$$da = 0, \quad d(Ja) = -k\, cJc, \quad db = z_3 r^{-1}\, aJc, \quad d(Jb) = -z_3\, ab + kqr^{-1}\, cJc,$$

with $z_3 \neq 0$. Now $\tilde{\mathfrak{g}}^* = \mathfrak{g}/\mathfrak{z}(\mathfrak{g}')^* \cong \langle a, b, c\rangle$, with $c' = c/r$, has structural equations $\tilde{d}a = 0$, $\tilde{d}b = z_3 ac'$, $\tilde{d}c' = -z_3 ab$ and so is isomorphic to $\mathfrak{r}'_{3,0}$. This gives $\mathfrak{g} \cong \mathfrak{d}'_{4,0}$.

In this case the solutions are never Kähler. The SKT moduli up to homotheties has dimension 2 and is connected. To see this note that a is specified up to sign, which may be fixed by requiring $k > 0$. If $q \neq 0$, replacing b by $\pm b$, we may then ensure $z_3 > 0$. This uniquely specifies b, and the remaining parameter is given by q. For $q = 0$, we may rotate in the b, Jb plan, but this does not change the solution.

The final case is $z_2 = -u_3 - kr^2$. Here one finds $x_1 = -k(1+q^2)$, $z_2 = -k/2$, $u_1 = -kq(2q^2 + 1)/2r$ giving

$$\begin{gathered} da = 0, \quad d(Ja) = -k(aJa + cJc), \quad db = -\tfrac{1}{2}k\, ab + z_3 r^{-1}\, aJc, \\ d(Jb) = \tfrac{1}{2}kr^{-1}\, a(qJa - rJb) - z_3\, ab + kqr^{-1}\, cJc. \end{gathered} \tag{4.4}$$

This time computing the structural equations for $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{z}(\mathfrak{g}')$ gives $\tilde{d}a = 0$, $\tilde{d}b = -\frac{1}{2}kab + z_3ac'$, $\tilde{d}c' = -z_3ab - \frac{1}{2}kac'$. If $z_3 \neq 0$, we have $\tilde{\mathfrak{g}} \cong \mathfrak{r}'_{3,\lambda}$ with $\lambda = |k/2z_3|$ giving $\mathfrak{g} \cong \mathfrak{d}'_{4,\lambda}$. The analysis for the choices of a, b is as above. For $z_3 = 0$, we have $\tilde{\mathfrak{g}} \cong \mathfrak{r}_{3,1}$ and $\mathfrak{g} \cong \mathfrak{d}_{4,1/2}$. The basis analysis is similar to above: $k > 0$ fixes a; for $q \neq 0$, b is fixed by $q > 0$; for $q = 0$ we may rotate in the b, Jb plane without changing the solution.

The solutions are Kähler precisely when $q = 0$. The SKT moduli up to homotheties has dimension 1 and is connected both for $\mathfrak{g} = \mathfrak{d}'_{4,\lambda}$ and for $\mathfrak{g} = \mathfrak{d}_{4,1/2}$.

This completes the proof of Theorem 4.1.

5. Consequences and concluding remarks

Let us first emphasise Remark 4.3 that for four-dimensional solvable groups the SKT condition depends explicitly on both the metric and the complex structure, in contrast to the situation [6] for six-dimensional nilpotent groups.

Corollary 5.1. *There are four-dimensional solvable complex Lie groups whose family of compatible invariant Hermitian metrics contains both* SKT *and non-*SKT *structures.* ∎

An alternative approach to our classification of invariant SKT structures in Theorem 4.1 would be to start with results for complex structures on four-dimensional solvable Lie groups (Ovando [17, 18], Snow [21]) and then to impose the SKT condition. We have used this approach to cross check our results, but also found that the lists given in [18] for Kähler forms and algebras with complex structures have some errors and omissions. Some of these are corrected in [1], but we wish to emphasise that the proof given in §4 is independent of those calculations. In contrast to the compact case we see:

Corollary 5.2. *The four-dimensional solvable Lie algebras* $\mathfrak{g}$ *that admit invariant complex structures but no compatible invariant* SKT *metric are:* $\mathbb{R} \times \mathfrak{r}_{3,1}$, $\mathbb{R} \times \mathfrak{r}'_{3,\lambda>0}$, $\mathfrak{aff}_{\mathbb{C}}$, $\mathfrak{r}_{4,1}$, $\mathfrak{r}_{4,\mu,\lambda}$, $(\mu = \lambda \neq -\frac{1}{2}$ *or* $\mu \leqslant \lambda = 1)$, $\mathfrak{r}'_{4,\mu,\lambda}$ $(\lambda \neq 0, -\mu/2)$, $\mathfrak{d}_{4,\lambda}$ $(\lambda \neq \frac{1}{2}, 2)$, $\mathfrak{h}_4$. ∎

Here the given constraints on the parameters are in addition to the defining constraints for the algebras.

On the other hand if G admits a discrete co-compact subgroup Γ then $M = \Gamma\backslash G$ is a compact manifold (a solvmanifold). By Gauduchon's Theorem [9] any complex structure on M admits an SKT metric (indeed one in any compatible conformal class). If G has an invariant complex structure one may then construct a compatible invariant SKT structure on G via pull-back from M (cf. [5]). A necessary condition for Γ to exist is that G be unimodular, which is equivalent to $b_4(\mathfrak{g}) = 1$, but in general this is not sufficient. The correct classification of complex solvmanifolds in dimension four has recently been provided by Hasegawa [12]. In our notation, one obtains

1. tori from $\mathfrak{g} = \mathbb{R}^4$,

2. primary Kodaira surfaces from $\mathfrak{g} = \mathbb{R} \times \mathfrak{h}_3$,
3. hyperelliptic surfaces from $\mathfrak{g} = \mathbb{R} \times \mathfrak{r}'_{3,0}$,
4. Inoue surfaces of type S^0 from $\mathfrak{g} = \mathfrak{r}_{4,-\frac{1}{2},-\frac{1}{2}}$ and from $\mathfrak{g} = \mathfrak{r}'_{4,2\lambda,-\lambda}$,
5. Inoue surfaces of type $S^{\pm}$ from $\mathfrak{g} = \mathfrak{d}_4$ and
6. secondary Kodaira surfaces from $\mathfrak{g} = \mathfrak{d}'_{4,0}$.

Comparing this list with our classification we conclude:

Corollary 5.3. *Each unimodular solvable four-dimensional Lie group G with invariant* SKT *structure admits a compact quotient by a lattice.* ■

Recall that an HKT structure is given by three complex structures I, J, $K = IJ = -JI$ with common Hermitian metric such that $Id\omega_I = Jd\omega_J = Kd\omega_K$. If (g, I) is already SKT then (g, J) and (g, K) are necessarily SKT and the HKT structure is strong. However the list of HKT structures on solvable Lie groups is known in dimension four from [2].

Corollary 5.4. *The only four-dimensional solvable Lie algebra that is strong* HKT *is $\mathbb{R}^4$, which is hyperKähler. The algebra $\mathfrak{d}_{4,1/2}$ admits both* HKT *and* SKT *structures; these structures are distinct. The remaining* HKT *algebras $\mathfrak{aff}_{\mathbb{C}}$ and $\mathfrak{r}_{4,1,1}$ do not admit invariant* SKT *structures.* ■

In the case of $\mathfrak{d}_{4,1/2}$ one may use (4.4) to check that the HKT and SKT metrics are inequivalent.

Finally, let us make the following observation which follows from case-by-case study of the algebras found in our SKT classification Theorem 4.1. The *symmetry rank* of an SKT manifold (M, g, J) is the dimension of the maximal Abelian group of isometries that preserve J, cf. [11, 4].

Corollary 5.5. *Each invariant* SKT *structure on a four-dimensional solvable Lie group G has symmetry rank at least 2.* ■

This motivates a future study of SKT structures on Abelian principal bundles over Riemann surfaces.

APPENDIX

6. SKT structures on compact Lie groups

The existence of SKT structures on compact even-dimensional Lie groups, is briefly indicated in the introduction to [6], and attributed to [22]. However, the result is not explicit in the latter reference and neither specifies the complex structures. We therefore give a proof for reference.

Proposition 6.1. *Any even-dimensional compact Lie group G admits a left-invariant* SKT *structure. Moreover each left-invariant complex structure on G admits a compatible invariant* SKT *metric.*

Proof. Let $\mathfrak{t}^{\mathbb{C}}$ be a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$. By [20], left-invariant complex structures J on G are in one-to-one correspondence with pairs $(J_{\mathfrak{t}}, P)$, where $J_{\mathfrak{t}}$ is any complex structure on $\mathfrak{t}$ and $P \subseteq \Delta$ is a system of positive roots: one defines

$$\mathfrak{g}^{1,0} = \mathfrak{t}^{1,0} \oplus \bigoplus_{\alpha \in P} \mathfrak{g}^{\mathbb{C}}_{\alpha}. \tag{6.1}$$

Extend the negative of the Killing form on $[\mathfrak{g}, \mathfrak{g}]$ to a J-compatible positive definite inner product g on $\mathfrak{g}$. The associated Levi-Civita connection on G has $\nabla^{\mathrm{LC}}_X Y = \frac{1}{2}[X, Y]$, for $X, Y \in \mathfrak{g}$. Consider now the left-invariant connection given by

$$\nabla_X Y = 0, \quad \text{for } X, Y \in \mathfrak{g}. \tag{6.2}$$

This connection preserves the metric g and the complex structure J and has torsion $T^{\nabla}(X, Y) = -[X, Y]$, so $(T^{\nabla})^{\flat}(X, Y, Z) = -g([X, Y], Z)$, which is a closed three-form. Thus (G, g, J) is an SKT manifold. ∎

7. Low-dimensional solvable Lie algebras

The four-dimensional solvable real Lie algebras are classified in [1]. In this section we summarise the classification and provide the notation for §4.

The map $\chi\colon \mathfrak{g} \to \mathbb{R}$, $\chi(x) = \mathrm{Tr}(\mathrm{ad}(x))$, is a Lie algebra homomorphism. Its kernel $\mathfrak{u}(\mathfrak{g})$, the *unimodular kernel of* $\mathfrak{g}$, is an ideal in $\mathfrak{g}$ containing the derived algebra $\mathfrak{g}'$. The Lie algebra $\mathfrak{g}$ is said to be *unimodular* if $\chi \equiv 0$. Note that if G admits a co-compact discrete subgroup then $\mathfrak{g}$ is necessarily unimodular [16].

Our notation for the three-dimensional solvable Lie algebras will be as given in Table 7.1. Note that $\mathfrak{r}_{3,0} \cong \mathbb{R} \times \mathfrak{aff}_{\mathbb{R}}$.

$\mathfrak{aff}_{\mathbb{R}}$	$(0, 21)$	
$\mathfrak{h}_3$	$(0, 0, 21)$	
$\mathfrak{r}_3$	$(0, 21+31, 31)$	
$\mathfrak{r}_{3,\lambda}$	$(0, 21, \lambda 31)$	$\lvert\lambda\rvert \leqslant 1$
$\mathfrak{r}'_{3,\lambda}$	$(0, \lambda 21 + 31, -21 + \lambda 31)$	$\lambda \geqslant 0$

Table 7.1: Non-Abelian solvable Lie algebras of dimension at most three that are not of product type.

The four dimensional solvable Lie algebras are classified as follows.

Theorem 7.1 ([1]). *Let $\mathfrak{g}$ be a four dimensional solvable real Lie algebra. Then $\mathfrak{g}$ is isomorphic to one and only one of the following Lie algebras: $\mathbb{R}^4$, $\mathfrak{aff}_{\mathbb{R}} \times \mathfrak{aff}_{\mathbb{R}}$, $\mathbb{R} \times \mathfrak{h}_3$, $\mathbb{R} \times \mathfrak{r}_3$, $\mathbb{R} \times \mathfrak{r}_{3,\lambda}$ $(\lvert\lambda\rvert \leqslant 1)$, $\mathbb{R} \times \mathfrak{r}'_{3,\lambda}$ $(\lambda \geqslant 0)$, or one of the algebras in Table 7.2.*

Among these the unimodular algebras are: $\mathbb{R}^4$, $\mathbb{R} \times \mathfrak{h}_3$, $\mathbb{R} \times \mathfrak{r}_{3,-1}$, $\mathbb{R} \times \mathfrak{r}'_{3,0}$, $\mathfrak{n}_4$, $\mathfrak{r}_{4,-1/2}$, $\mathfrak{r}_{4,\mu,-1-\mu}$ $(-1 < \mu \leqslant -\frac{1}{2})$, $\mathfrak{r}'_{4,\mu,-\mu/2}$, $\mathfrak{d}_4$, $\mathfrak{d}'_{4,0}$.

In the Table 7.3 the four-dimensional solvable real Lie algebras are sorted by their derived algebra $\mathfrak{g}'$. In some cases it is easy to recognise which algebra is at hand using the following observations:

$\mathfrak{n}_4$	$(0,0,21,31)$	
$\mathfrak{aff}_{\mathbb{C}}$	$(0,0,31-42,41+32)$	
$\mathfrak{r}_4$	$(0,21+31,31+41,41)$	
$\mathfrak{r}_{4,\lambda}$	$(0,21,\lambda 31+41,\lambda 41)$	
$\mathfrak{r}_{4,\mu,\lambda}$	$(0,21,\mu 31,\lambda 41)$	$\mu,\lambda \in \mathscr{R}_4$
$\mathfrak{r}'_{4,\mu,\lambda}$	$(0,\mu 21,\lambda 31+41,-31+\lambda 41)$	$\mu > 0$
$\mathfrak{d}_4$	$(0,21,-31,32)$	
$\mathfrak{d}_{4,\lambda}$	$(0,\lambda 21,(1-\lambda)31,41+32)$	$\lambda \geqslant \frac{1}{2}$
$\mathfrak{d}'_{4,\lambda}$	$(0,\lambda 21+31,-21+\lambda 31,2\lambda.41+32)$	$\lambda \geqslant 0$
$\mathfrak{h}_4$	$(0,21+31,31,2.41+32)$	

Table 7.2: Four-dimensional solvable Lie algebras not of product type. The set $\mathscr{R}_4$ consists of the $(\mu,\lambda) \in [-1,1]^2$ with $\lambda \geqslant \mu$ and $\mu,\lambda \neq 0$ and satisfying $\lambda < 0$ if $\mu = -1$.

$\mathfrak{g}'$	$\mathfrak{z}(\mathfrak{g})$	$\mathfrak{g}$
$\{0\}$		$\mathbb{R}^4$
$\mathbb{R}$		$\mathbb{R}\times\mathfrak{h}_3,\ \mathbb{R}\times\mathfrak{r}_{3,0}$
$\mathbb{R}^2$	$\{0\}$	$\mathfrak{aff}_{\mathbb{R}}\times\mathfrak{aff}_{\mathbb{R}},\ \mathfrak{aff}_{\mathbb{C}},\ \mathfrak{d}_{4,1}$
	$\mathbb{R}$	$\mathbb{R}\times\mathfrak{r}_3,\ \mathbb{R}\times\mathfrak{r}_{3,\lambda\neq 0},\ \mathbb{R}\times\mathfrak{r}'_{3,\lambda},\ \mathfrak{r}_{4,0},\ \mathfrak{n}_4$
$\mathbb{R}^3$		$\mathfrak{r}_4,\ \mathfrak{r}_{4,\lambda\neq 0},\ \mathfrak{r}_{4,\mu,\lambda},\ \mathfrak{r}'_{4,\mu,\lambda}$
$\mathfrak{h}_3$		$\mathfrak{d}_4,\ \mathfrak{d}_{4,\lambda\neq 1},\ \mathfrak{d}'_{4,\lambda},\ \mathfrak{h}_4$

Table 7.3: The four-dimensional solvable Lie algebras sorted by $\mathfrak{g}'$ and, where necessary, $\mathfrak{z}(\mathfrak{g})$. The conditions on the parameters are in addition to those from Tables 7.1 and 7.2.

$\mathfrak{g}' = \mathbb{R}$: $\mathbb{R}\times\mathfrak{h}_3$ is nilpotent, $\mathbb{R}\times\mathfrak{r}_{3,0}$ is not.

$\mathfrak{g}' = \mathbb{R}^2$, $\mathfrak{z}(\mathfrak{g}) = \{0\}$: $\mathfrak{aff}_{\mathbb{R}}\times\mathfrak{aff}_{\mathbb{R}}$ and $\mathfrak{d}_{4,1}$ are completely solvable, $\mathfrak{aff}_{\mathbb{C}}$ is not. Moreover these algebras have different unimodular kernels:

$$\mathfrak{u}(\mathfrak{aff}_{\mathbb{R}}\times\mathfrak{aff}_{\mathbb{R}}) \cong \mathfrak{r}_{3,-1},\quad \mathfrak{u}(\mathfrak{d}_{4,1}) \cong \mathfrak{h}_3,\quad \mathfrak{u}(\mathfrak{aff}_{\mathbb{C}}) \cong \mathfrak{r}'_{3,0}.$$

$\mathfrak{g}' = \mathfrak{h}_3$: the algebras are distinguished by $\tilde{\mathfrak{g}} = \mathfrak{g}/\mathfrak{z}(\mathfrak{g}')$ as follows:

$$\tilde{\mathfrak{d}}_4 \cong \mathfrak{r}_{3,-1},\quad \tilde{\mathfrak{d}}_{4,\lambda\neq 1} \cong \mathfrak{r}_{3,(1-\lambda)/\lambda},\quad \tilde{\mathfrak{d}}'_{4,\lambda} \cong \mathfrak{r}'_{3,\lambda},\quad \tilde{\mathfrak{h}}_4 \cong \mathfrak{r}_3.$$

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