

Contact and 1-Quasiconformal Maps on Carnot Groups

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Communicated by G. Mauceri

Abstract. This paper answers to some questions that remained open for some time in the community of mathematicians working on quasiconformal mapping theory in subriemannian geometry. The first result presented here is the characterisation of the rigidity of Carnot groups in the class of C^2 contact maps, obtained by extending Tanaka theory from its classical domain of C^∞ contact vector fields to the pseudogroup of local C^2 contact mappings.

The second result is a Liouville type theorem proved for all Carnot groups other than \mathbb{R} or \mathbb{R}^2 . The proof rests upon recent results of Capogna and Cowling and classical results on prolonging the conformal Lie algebra. As an additional goal, this article aims to provide a partial exposition on Tanaka theory and to give an elementary proof of a key result due to Guillemin, Quillen and Sternberg concerning complex characteristics.

Mathematics Subject Classification 2000: 30C65, 58D05, 22E25, 53C17.

Key Words and Phrases: Contact map, conformal map, quasiconformal map, subriemannian geometry.

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1. Introduction

A nilpotent Lie algebra \mathfrak{n} is said to admit an s -step stratification if it decomposes as the direct sum of subspaces $\mathfrak{n} = \mathfrak{g}_{-s} \oplus \cdots \oplus \mathfrak{g}_{-1}$ which satisfy the bracket generating property $\mathfrak{g}_{j-1} = [\mathfrak{g}_{-1}, \mathfrak{g}_j]$, where $j = -1, \dots, -s + 1$, and \mathfrak{g}_{-s} is contained in the centre $\mathfrak{z}(\mathfrak{n})$.

Let N denote the connected, simply connected nilpotent Lie group with stratified Lie algebra \mathfrak{n} . The sub-bundle $\mathcal{H} \subseteq TN$ obtained by left translating \mathfrak{g}_{-1} is called the *horizontal bundle*. Furthermore, if \mathfrak{n} is equipped with an inner product $\langle \cdot, \cdot \rangle$ such that $\mathfrak{g}_i \perp \mathfrak{g}_j$ for every $i \neq j$, then N is called Carnot group.

The inner product on \mathfrak{n} defines an inner product on the tangent space $T_p N$ at every point $p \in N$ by left translation. In particular, for $X, Y \in T_p N$ we have

$$\langle X, Y \rangle_p = \langle (\tau_{p^{-1}})_*(X), (\tau_{p^{-1}})_*(Y) \rangle,$$

where $\tau_{p^{-1}}$ denotes left translation by $p^{-1} \in N$. A smooth curve γ is horizontal if its tangent vectors lie in the horizontal bundle, and the length of a horizontal curve is the integral of the lengths of its tangent vectors relative to the inner product described above. The Carnot-Carathéodory distance or subriemannian distance $d(p, q)$ between points p and q , is defined as the infimum of the lengths of all horizontal curves joining p and q . By a theorem of Chow [2], the bracket generating property implies that Carnot groups are horizontally path connected and so d is indeed a metric.

Local diffeomorphisms f of N whose differential f_* preserves the horizontal bundle are called *contact maps*. In our framework we shall always assume, unless otherwise stated, that contact diffeomorphisms are C^2 . The group N , or its Lie algebra \mathfrak{n} , is said to be *rigid* if the space of contact maps between open domains of N form a finite dimensional space. We shall call N *nonrigid* otherwise. The simplest examples of contact maps are left translations and dilations. A dilation by t of an element $X \in \mathfrak{n}$ is defined by $\delta_t(X) = \sum_{k=1}^s t^k X_{-k}$, and the corresponding map of N induced through the exponential defines dilation on N which we also denote by δ_t . We also note that left translations are isometries of the subriemannian metric and the dilations are homogeneous in the sense that $d(\delta_t(p), \delta_t(q)) = td(p, q)$.

The notion of rigidity makes sense with low regularity assumptions. For example in the theory of quasiconformal mappings on Carnot groups, the notion of contact map arises in the class of homeomorphisms with coordinate functions in the horizontal Sobolev space $HW_{loc}^{1,1}(N, \mathbb{R})$. However, the authors know no example in which the group is rigid in the class of C^∞ maps but nonrigid in $HW_{loc}^{1,1}(N, \mathbb{R})$. This situation provides significant motivation for studying rigidity and has been remarked upon by Korányi in his MathSciNet review of [14], by Heinonen in [9] and by Tyson in [26].

Our first main result is the following rigidity criterion for the pseudogroup of C^2 contact diffeomorphisms of N .

Theorem 1.1. *Let N be a real Carnot group with Lie algebra \mathfrak{n} , and denote by $\mathfrak{h}_0(\mathfrak{n})$ the subalgebra of the space of strata preserving derivations of \mathfrak{n} which annihilate all strata except \mathfrak{g}_{-1} , then the following conditions are equivalent.*

- (i) N is nonrigid;
- (ii) there exists $\varphi_{\mathbb{C}}$ in the complexification of $\mathfrak{h}_0(\mathfrak{n})$ of rank 1;
- (iii) there exists a vector $X_{\mathbb{C}}$ in the complexification of \mathfrak{g}_{-1} such that $\text{ad}X_{\mathbb{C}}$ has rank 0 or 1.

Some comments are in order. A suitably regular vector field U is said to be a *contact vector field* if its local flow is a 1-parameter group of contact maps. A contact vector field is characterised by the integrability condition $[U, \mathcal{H}] \subseteq \mathcal{H}$.

In [25] and [24], Tanaka developed an algebraic prolongation, which shows that the space \mathcal{CA}_p consisting of germs of C^∞ contact vector fields at a point p is finite dimensional if and only if the prolongation is finite dimensional [25, Theorem 6.1, page 38]. Furthermore, Tanaka showed that finiteness of his prolongation is equivalent to the finiteness of the usual prolongation of $\mathfrak{h}_0(\mathfrak{n})$ [25, Corollary 2, page 76]. Note here that the usual prolongation refers to the prolongation in the sense of Singer and Sternberg [22].

In [8] and [23] it is shown that infiniteness of the usual prolongation is equivalent to the complexified algebra containing rank 1 elements. Thus the equivalence of (i) and (ii), at the level of C^∞ contact vector fields, is a simple consequence of [25], [22] and [8]. However a standard integration argument extends the scope of this criterion to the pseudogroup of C^2 contact diffeomorphisms. In fact, we shall see that every C^2 contact map is induced, up to composition with some left translation, by an automorphism of the space \mathcal{CA}_e , where e is the identity of N . Consequently, if \mathcal{CA}_e forms a finite dimensional vector space then the automorphisms of \mathcal{CA}_e form a finite dimensional vector space thus implying finite dimensionality of local contact mappings.

As a further contribution we establish the equivalence of (ii) and (iii), which provides a rigidity test purely at the level of the Lie algebra \mathfrak{n} .

We remark that [22], [8], [7] and [23] all precede [25]. Furthermore Remark 2 on page 77 of [25] refers to [7], so we would argue that Tanaka was aware of (i) \Leftrightarrow (ii) in Theorem 1.1, at least at the infinitesimal level, although it is not explicitly stated in [25].

It is worthwhile to observe that if we substitute conditions (ii) and (iii) with

- (ii) $_{\mathbb{R}}$ there exists φ in $\mathfrak{h}_0(\mathfrak{n})$ of rank 0 or 1
- (iii) $_{\mathbb{R}}$ there exists a vector $X \in \mathfrak{g}_{-1}$ such that $\text{ad}X$ has rank 1

then one still has that (ii) $_{\mathbb{R}} \Leftrightarrow$ (iii) $_{\mathbb{R}}$ and they both imply (i). This was observed in [16] and [17].

A trivial case of nonrigidity occurs when \mathfrak{n} is degenerate in the following sense. We say that \mathfrak{n} is degenerate if \mathfrak{g}_{-1} contains nontrivial degenerate elements, that is elements $X \in \mathfrak{g}_{-1}$ such that $[X, \mathfrak{g}_{-1}] = \{0\}$. When \mathfrak{n} is degenerate, the

corresponding Carnot group is always the direct product with \mathbb{R}^n , where n is the dimension of the degenerate space. It follows that the set of local contact mappings contains the local diffeomorphisms of \mathbb{R}^n implying that the corresponding Carnot group is nonrigid.

The Tanaka prolongation theory can be used in order to characterise at the infinitesimal level any space of diffeomorphisms whose differential satisfies some kind of linear condition. A distinguished application is the generalisation of the Liouville theorem for 1-quasiconformal maps in Carnot groups. The classical result states that C^4 -conformal maps between domains of \mathbb{R}^3 are the restriction of the action of some element of the group $O(1, 4)$. The same result holds in \mathbb{R}^n when $n > 3$ (see, e.g., Nevanlinna [15]). A major advance in the theory was the passage from smoothness assumptions to metric assumptions (see Gehring [6] and Reshetnyak [21]): the conclusion of Liouville's theorem holds for 1-quasiconformal maps. When the ambient space is not Riemannian there are similar theorems. Capogna and Cowling proved in [1] that 1-quasiconformal maps defined on open subsets of a Carnot group are smooth and applied this to give some Liouville type results. In particular, they observe that 1-quasiconformal maps between open subsets of H-type groups whose Lie algebra has dimension larger than 2 form a finite dimensional space. This follows by combining the smoothness result in [1] and the work of Reimann [19], who established the corresponding infinitesimal result (see also [4, 3, 20]). Our contribution is the following statement.

Theorem 1.2. *The 1-quasiconformal maps between open domains of a Carnot group N , other than \mathbb{R} and \mathbb{R}^2 , form a finite dimensional space.*

The proof relies on Theorem 1.1 in [1] that states that 1-quasiconformal maps are C^∞ and in fact conformal in a familiar sense. This allows us to consider the problem at the infinitesimal level and to use the Tanaka criteria together with well known results on the usual prolongation of the conformal algebra $\mathfrak{co}(n)$.

The paper is organised as follows. The first section is dedicated to an overview of the Singer–Sternberg prolongation versus the Tanaka prolongation, and a detailed discussion on the result in [8] concerning the rank 1 condition. In the second section we prove Theorem 1.1, and in the third section we prove Theorem 1.2.

This work stems from discussions during the workshop on Geometry of ODE's and Vector Distributions held at the Stefan Banach International Mathematical Centre in January 2009. In particular we thank B. Kruglikov, A. Čap, B. Doubrov as well as M. Eastwood for pointing out the key results in [8] and [23]. We are particularly grateful to B. Kruglikov for detailed discussions on [8]. We also point out that in parallel to us, B. Doubrov and O. Radko (formerly O. Kuzmich) also have a manuscript in preparation [5], which uses (i) \Leftrightarrow (ii) in Theorem 1.1 to extend the results from [12] and discuss a number of illustrative examples including a proof that all metabelian Lie algebras with centre of dimension 2 are nonrigid.

2. Preliminaries

2.1. The Symmetric Algebra. Let $V = \text{Span}\{e_1, \dots, e_n\}$ be a vector space over the field $\mathbb{F} = \mathbb{R}$ or \mathbb{C} , and let $\{\varepsilon_1, \dots, \varepsilon_n\}$ denote the corresponding dual basis for V^* . For each integer $k \geq 0$ let $\otimes^k V$ and $\otimes^k V^*$ denote k -th tensor power of V and V^* respectively. Recall that the natural pairing $\langle \cdot, \cdot \rangle : \otimes^k V \times \otimes^k V^* \rightarrow \mathbb{F}$ is given by

$$\langle v_1 \otimes \dots \otimes v_k, \eta_1 \otimes \dots \otimes \eta_k \rangle = \prod_{i=1}^k \eta_i(v_i).$$

For each integer $k \geq 0$ let $S^k(V^*) \subset \otimes^k V^*$ denote the symmetric k -tensors. An element $K \in S^k(V^*)$ has the form

$$K = \sum_{j_1, \dots, j_k=1}^n K_{j_1, \dots, j_k} \varepsilon_{j_1} \otimes \dots \otimes \varepsilon_{j_k}$$

where the coefficients $K_{j_1, \dots, j_k} \in \mathbb{F}$ are symmetric with respect to permutation of the indices j_1, \dots, j_k . The symmetric product of elements of V^* is defined linearly by the symmetrisation operator, i.e.,

$$\varepsilon_{j_1} \odot \dots \odot \varepsilon_{j_k} = \text{Sym}(\varepsilon_{j_1} \otimes \dots \otimes \varepsilon_{j_k}) = \frac{1}{k!} \sum_{\sigma \in S_k} \varepsilon_{j_{\sigma(1)}} \otimes \dots \otimes \varepsilon_{j_{\sigma(k)}},$$

and the same definitions apply to elements of V . Note that $K \in S^k(V^*)$ if and only if $\text{Sym}(K) = K$, moreover

$$K = \sum_{1 \leq j_1 \leq \dots \leq j_k} k! K_{j_1, \dots, j_k} \varepsilon_{j_1} \odot \dots \odot \varepsilon_{j_k}.$$

The natural pairing restricted to $S^k(V) \otimes S^k(V^*) \rightarrow \mathbb{F}$ gives

$$\langle v_1 \odot \dots \odot v_k, \eta_1 \odot \dots \odot \eta_k \rangle = \frac{1}{k!} \sum_{\sigma} \prod_{i=1}^k \eta_{\sigma(i)}(v_i) \tag{1}$$

which is nondegenerate and thus identifies $S^k(V)^*$ with $S^k(V^*)$.

We obtain a symmetric multilinear map $V^k \rightarrow \mathbb{F}$, also denoted by K , by setting

$$\begin{aligned} K(v_1, \dots, v_k) &= \langle v_1 \otimes \dots \otimes v_k, K \rangle \\ &= \sum_{j_1, \dots, j_k=1}^n K_{j_1, \dots, j_k} \langle v_1 \otimes \dots \otimes v_k, \varepsilon_{j_1} \otimes \dots \otimes \varepsilon_{j_k} \rangle \\ &= \sum_{j_1, \dots, j_k=1}^n K_{j_1, \dots, j_k} \varepsilon_{j_1}(v_1) \varepsilon_{j_2}(v_2) \dots \varepsilon_{j_k}(v_k). \end{aligned} \tag{2}$$

Since a symmetric multilinear map $K : V^k \rightarrow \mathbb{F}$ always has the form given by (2), we conclude that the space of symmetric multilinear maps identifies with

$S^k(V^*)$. Furthermore, restricting K to the diagonal of V^k gives rise to a degree k homogeneous polynomial on V which we also denote by K and we write

$$K(v) = K(v, \dots, v) = \langle K, v^k \rangle.$$

Moreover, we can recover $K(v_1, \dots, v_k)$ as the coefficient of $\lambda_1 \lambda_2 \dots \lambda_k$ in the expression $K(\lambda_1 v_1 + \dots + \lambda_k v_k)$, and thus we have an identification of k -homogeneous polynomials on V with $S^k(V^*)$. Similarly k -homogeneous polynomials on V^* identify with $S^k(V)$ where $P \in S^k(V)$ is identified with the polynomial $P(\eta) = \langle P, \eta^k \rangle$. The passage from a polynomial to a symmetric multilinear map is called *polarisation*.

The set of symmetric multilinear maps $V^k \rightarrow V$ is isomorphic with $V \otimes S^k(V^*)$. An element $L \in V \otimes S^k(V^*)$ has the form

$$L = \sum_{i=1}^n \sum_{j_1, \dots, j_k=1}^n L_{j_1, \dots, j_k}^i e_i \otimes \varepsilon_{j_1} \otimes \dots \otimes \varepsilon_{j_k} \tag{3}$$

where L_{j_1, \dots, j_k}^i is symmetric in the lower indices and the corresponding multilinear map, also denoted by L , has the form

$$L(v_1, \dots, v_k) = \sum_{i=1}^n \left(\sum_{j_1, \dots, j_k=1}^n L_{j_1, \dots, j_k}^i \varepsilon_{j_1}(v_1) \dots \varepsilon_{j_k}(v_k) \right) e_i. \tag{4}$$

Since every symmetric multilinear map $V^k \rightarrow V$ has the form given by (4), we conclude that this space of maps identifies with $V \otimes S^k(V^*)$. Furthermore, interpreting $S^k(V^*)$ as the set of k -homogeneous polynomials on V we can view $V \otimes S^k(V^*)$ as the space of k -homogeneous polynomial maps $V \rightarrow V$. Obviously the same holds for $V^* \otimes S^k(V)$. Furthermore, we note that (3) can be written in the form

$$L = \sum_{j_2, \dots, j_k=1}^n \left(\sum_{i, j_1=1}^n L_{j_1, \dots, j_k}^i e_i \otimes \varepsilon_{j_1} \right) \otimes \varepsilon_{j_2} \otimes \dots \otimes \varepsilon_{j_k},$$

and the expression $A_{j_2, \dots, j_k} = \sum_{i, j_1=1}^n L_{j_1, \dots, j_k}^i e_i \otimes \varepsilon_{j_1}$ is an element of $\mathfrak{gl}(V)$ identifying with the matrix $(L_{j_1, \dots, j_k}^i)_{i, j_1} \in \mathfrak{gl}(\mathbb{F}^n)$. Since A_{j_2, \dots, j_k} is symmetric under permutation of the indices j_2, \dots, j_k , it follows that $V \otimes S^k(V^*)$ embeds into $\mathfrak{gl}(\mathbb{F}^n) \otimes S^{k-1}(V^*)$.

The symmetric algebra is given by

$$S(V^*) = \bigoplus_{k \geq 0} S^k(V^*) = \mathbb{F} \oplus V^* \oplus S^2(V^*) \oplus S^3(V^*) \oplus \dots, \tag{5}$$

and elements $T \in S(V^*)$ have the form $T = \sum_k T^k$, where $T^k \in S^k(V^*)$ and $T^k = 0$ for all but finitely many $k \in \mathbb{N}$.

2.2. Singer–Sternberg Prolongation. Let $\mathfrak{gl}(V)$ denote the Lie algebra of linear endomorphisms of V , and let $\mathfrak{g}^{(0)}$ be a Lie subalgebra of $\mathfrak{gl}(V)$. Then for each nonnegative integer k , the k -th *Singer–Sternberg prolongation* $\mathfrak{g}^{(k)}$ of $\mathfrak{g}^{(0)}$ is the space

$$\mathfrak{g}^{(k)} = \mathfrak{g}^{(0)} \otimes S^k(V^*) \cap V \otimes S^{k+1}(V^*).$$

Each $T \in \mathfrak{g}^{(k)}$ corresponds with a symmetric multilinear map $T : V^{k+1} \rightarrow V$, such that for each k -tuple $(v_1, \dots, v_k) \in V^k$, the endomorphism of V given by $v_{k+1} \rightarrow T(v_1, \dots, v_k, v_{k+1})$ is an element of $\mathfrak{g}^{(0)}$.

The Lie algebra $\mathfrak{g}^{(0)}$ is said to be of *finite type* if $\mathfrak{g}^{(k)} = \{0\}$ for some $k \geq 0$. Note that it follows that $\mathfrak{g}^{(k+\ell)} = \{0\}$ for all $\ell \geq 0$. In particular, we say that $\mathfrak{g}^{(0)}$ is of type k if this is the smallest positive integer for which $\mathfrak{g}^{(k)} = 0$. Naturally $\mathfrak{g}^{(0)}$ is said to be of *infinite type* otherwise.

The following examples are significant for our purposes, and they can be found in [10].

Example 2.1. Take $\mathfrak{g}^{(0)} = \mathfrak{co}(n)$, where

$$\mathfrak{co}(n) = \{A \in \mathfrak{gl}(n, \mathbb{R}) \mid A + A^{tr} = kI\}.$$

We show that $\mathfrak{g}^{(0)}$ is of type 2 if $n \geq 3$. The first prolongation $\mathfrak{g}^{(1)}$ is naturally isomorphic to the dual space \mathbb{R}^{n*} of \mathbb{R}^n . Indeed, let $T = (T_{jk}^i)$ be an element of $\mathfrak{g}^{(1)}$. Since the kernel of the homomorphism $A \in \mathfrak{co}(n) \rightarrow \text{trace}(A) \in \mathbb{R}$ is precisely $\mathfrak{o}(n) = \{A \in \mathfrak{gl}(n, \mathbb{R}) \mid A + A^{tr} = 0\}$, and since $\mathfrak{o}(n)$ is of type 1 [10], the linear mapping

$$T = (T_{jk}^i) \in \mathfrak{g}^{(1)} \rightarrow \xi = \left(\frac{1}{n} \sum_i T_{ik}^i \right) \in \mathbb{R}^{n*}$$

is injective (the kernel is the first prolongation of $\mathfrak{o}(n)$). To see that this mapping is also surjective, it is enough to observe that $\xi = (\xi_k)$ is the image of T with $T_{jk}^i = \delta_j^i \xi_k + \delta_k^i \xi_j - \delta_k^j \xi_i$. To prove that $\mathfrak{g}^{(2)} = 0$, set now $T = (T_{ijk}^h) \in \mathfrak{g}^{(2)}$. For each fixed k , T_{ijk}^h may be considered as the components of an element in $\mathfrak{g}^{(1)}$ and hence can be uniquely written

$$T_{ijk}^h = \delta_i^h \xi_{jk} + \delta_j^h \xi_{ik} - \delta_j^i \xi_{hk}.$$

Since T_{ijk}^h must be symmetric in all lower indices, we have $\sum_h T_{hjk}^h = \sum_h T_{hkj}^h$, from which follows $\xi_{jk} = \xi_{kj}$. From $\sum_h T_{hjk}^h = \sum_h T_{jkh}^h$, we obtain $(n - 2)\xi_{jk} = -\delta_{jk} \sum_h \xi_{hh}$, whence $(n - 2) \sum_h \xi_{hh} = -n \sum_h \xi_{hh}$ and thus $\sum_h \xi_{hh} = 0$. Therefore $(n - 2)\xi_{jk} = -\delta_{jk} \sum_h \xi_{hh} = 0$, which implies $\xi_{jk} = 0$ if $n \geq 3$.

Example 2.2. Suppose $\mathfrak{g}^{(0)}$ contains a linear endomorphism T of rank 1, then $T = v_0 \otimes \omega$ where $v_0 \in V$ and $\omega \in V^*$. For any positive integer k we can define a symmetric $(k + 1)$ -linear map $T_k : V^{k+1} \rightarrow V$ by setting

$$T_k(v_1, \dots, v_k, v_{k+1}) = \omega(v_1) \dots \omega(v_k) \omega(v_{k+1}) v_0$$

and the map

$$v_{k+1} \rightarrow T_k(v_1, \dots, v_k, v_{k+1})$$

is precisely $\omega(v_1) \dots \omega(v_k)T$ and thus a nontrivial element of $\mathfrak{g}^{(0)}$. Hence $\mathfrak{g}^{(k)} \neq \{0\}$ and $\mathfrak{g}^{(0)}$ is of infinite type.

A subalgebra $\mathfrak{g}^{(0)} \subset \mathfrak{gl}(V)$ is said to be *elliptic* if it does not contain any linear endomorphism of rank 1, hence finite type subalgebras are elliptic.

2.3. Complex characteristics. The goal here is to give an exposition on a key result of [8] which we state in an adapted form as Theorem 2.3 below. In fact in [8] they prove a more general result pertaining to finiteness of the prolongation of a principal symbol of a differential operators between complex vector bundles. They then apply this to the particular case where the differential operator represents the equation for the infinitesimal automorphisms of a flat G -structure. If $\mathfrak{g}^{(0)}$ is the Lie algebra of G then the $(k+1)$ -th order term in the formal Taylor expansion of such an infinitesimal automorphism is an element of $\mathfrak{g}^{(k)}$ and their finiteness criteria thus apply to the Singer–Sternberg prolongations. For the nonexpert it thus seems worth while to restate and prove their result in our specific setting using only Lie and symmetric algebra.

We remark that result of [8] is often called Spencer’s criteria even though [8] is the first published proof, furthermore the result was apparently known even earlier by Serre and Matsushima.

Throughout this section we consider V to be a vector space over \mathbb{C} . The reason lies in the fact that the proof of Theorem 2.3 relies on Hilbert’s Nullstellensatz.

Theorem 2.3 (Guillemin-Quillen-Sternberg [8]). *The subalgebra $\mathfrak{g}^{(0)} \subset \mathfrak{gl}(V)$ is of infinite type if and only if $\mathfrak{g}^{(0)}$ contains elements of rank 1.*

We begin by observing that

$$V \otimes S^{k+1}(V^*) \subset V \otimes V^* \otimes S^k(V^*) = \mathfrak{gl}(V) \otimes S^k(V^*)$$

which implies that

$$V \otimes S^{k+1}(V^*) = \mathfrak{gl}(V) \otimes S^k(V^*) \cap V \otimes S^{k+1}(V^*).$$

Let $F^{(0)} \subseteq \mathfrak{gl}(V)$ be a fixed complementary subspace, then it follows that

$$V \otimes S^{k+1}(V^*) = \mathfrak{g}^{(k)} \oplus F^{(k)}$$

where

$$F^{(k)} = F^{(0)} \otimes S^k(V^*) \cap V \otimes S^{k+1}(V^*). \quad (6)$$

Observe that if $\mathfrak{g}^{(0)}$ is of finite type, then for some $k \geq 1$ we have

$$V \otimes S^{k+1}(V^*) = F^{(k)},$$

and so every $P^* \in S^j(V^*)$, where $j \geq k$, it follows that $\mathfrak{g}^{(\ell)} \odot P^* \subseteq F^{(l+j)}$ for all ℓ . Conversely, if there exists a k such that every $P^* \in S^j(V^*)$ satisfies $\mathfrak{g}^{(\ell)} \odot P^* \subseteq F^{(l+j)}$ for all ℓ and $j \geq k$, then $\mathfrak{g}^{(0)}$ is of finite type.

We reformulate these observations in the convenient language of exact sequences and consider the problem in the dual. For every integer $k \geq -1$ the following sequence is exact

$$\mathfrak{g}^{(k)} \xrightarrow{\iota_k} V \otimes S^{k+1}(V^*) \xrightarrow{\pi_k} F^{(k)}.$$

Here we denoted by ι_k and π_k the canonical immersions and projections, and in the case $k = -1$ we put $\mathfrak{g}^{(-1)} = V$ and $F^{(-1)} = \{0\}$. The corresponding dual sequences take the form

$$\mathfrak{g}^{(k)*} \xleftarrow{\iota_k^*} V^* \otimes S^{k+1}(V) \xleftarrow{\pi_k^*} F^{(k)*}$$

where $\pi_k^*(F^{(k)*})$ is the set of all elements in $V^* \otimes S^{k+1}(V)$ which vanish on $\mathfrak{g}^{(k)}$, as one can see from standard algebra. The direct sum of the dual sequences gives the sequence

$$M^* \xleftarrow{\iota^*} V^* \otimes S(V) \xleftarrow{\pi^*} F^*,$$

where $M^* = \bigoplus_{k \geq -1} \mathfrak{g}^{(k)*}$, $\iota^*|_{V^* \otimes S^{k+1}(V)} = \iota_k^*$, $S(V) = \bigoplus_{k \geq 0} S^k(V)$, $\pi^*|_{F^{(k)*}} = \pi_k^*$ and $F^* = \bigoplus_{k \geq -1} F^{(k)*}$.

A nonzero element $\xi \in V^*$ is called a *characteristic* if there exists a nonzero $w \in V$ such that $w \otimes \xi \in \mathfrak{g}^{(0)}$. Equivalently, ξ is a characteristic if and only if the map $\sigma_\xi : V \rightarrow F^{(0)}$ defined by $\sigma_\xi(\cdot) = \pi_0(\cdot \otimes \xi)$ fails to be injective. In fact, note that $w \otimes \xi$ is a rank 1 element of $\mathfrak{g}^{(0)}$ and conversely every such element defines a characteristic. We also note that if $\xi \in V^*$ is not characteristic then σ_ξ is injective and so necessarily $\dim(V) \leq \dim F^{(0)}$ or equivalently $\dim \mathfrak{g}^{(0)} \leq n^2 - n$. Hence if $\dim \mathfrak{g}^{(0)} > n^2 - n$ then every element in V^* is characteristic and by Example 2.2 it follows that $\mathfrak{g}^{(0)}$ is of infinite type.

We say that an element $P \in S(V)$ *annihilates* M^* if

$$V^* \otimes S(V) \odot P \subset \pi^*(F^*).$$

We observe that such a P cannot have constant term and in fact must lie in $S^+(V) = \bigoplus_{\ell > 0} S^\ell(V)$. The set \mathcal{S} consisting of all annihilating P is an ideal in the ring $S(V)$. Moreover, the following two lemmas show that if \mathcal{C} denotes the set consisting of all characteristics, then the set of common zeros of \mathcal{S} is exactly $\mathcal{C} \cup \{0\}$.

Lemma 2.4. *If $P \in \mathcal{S}$, then $P(\xi) = 0$ for all $\xi \in \mathcal{C}$.*

Proof. Let P_ℓ be a homogeneous component of degree ℓ . Define a linear map $\alpha_{P_\ell}^* : V^* \rightarrow F^{(\ell-1)*}$ as follows: the annihilating assumption implies $\varepsilon_i \otimes P_\ell = \pi_{\ell-1}^*(B_i)$ for some $B_i \in F^{(\ell-1)*}$, so that we set $\alpha_{P_\ell}^*(\varepsilon_i) = B_i$. If $v^* = \sum_i a_i \varepsilon_i$ then

$$\pi_{\ell-1}^* \circ \alpha_{P_\ell}^*(v^*) = \pi_{\ell-1}^* \circ \sum_i a_i \alpha_{P_\ell}^*(\varepsilon_i) = \sum_i a_i \pi_{\ell-1}^*(B_i) = \sum_i a_i \varepsilon_i \otimes P_\ell = v^* \otimes P_\ell,$$

thus it follows that $\pi_{\ell-1}^* \circ \alpha_{P_\ell}^*$ is a right tensor multiplication by P_ℓ .

If ξ is a characteristic then $\pi_{\ell-1}(w \otimes \xi^\ell) = 0$ for some nonzero $w \in V$ and for all $\ell \geq 1$, whence

$$\begin{aligned} 0 &= \langle \alpha_{P_\ell} \circ \pi_{\ell-1}(w \otimes \xi^\ell), w^* \rangle \\ &= \langle w \otimes \xi^\ell, \pi_{\ell-1}^* \circ \alpha_{P_\ell}^*(w^*) \rangle \\ &= \langle w \otimes \xi^\ell, w^* \otimes P_\ell \rangle \\ &= \langle w, w^* \rangle \langle \xi^\ell, P_\ell \rangle, \end{aligned}$$

and so $\langle \xi^\ell, P_\ell \rangle = 0$. Hence if $P = \sum_{\ell \geq 1} P_\ell$ annihilates M^* , then $P(\xi) = 0$ for all characteristics ξ . \blacksquare

Now we assume ξ is not a characteristic so that $\sigma_\xi(\cdot) = \pi_0(\cdot \otimes \xi)$ defines a linear injection of V into $F^{(0)}$. Denote by $B_\xi : F^{(0)} \rightarrow V$ the obvious linear extension to $F^{(0)}$ of the left inverse of σ_ξ . For every $\eta \in V^*$ set $P_\xi(\eta) = \det(B_\xi \circ \sigma_\eta)$ where $\sigma_\eta(\cdot) = \pi_0(\cdot \otimes \eta)$. Then we have $P_\xi(\xi) = 1$ and $P_\xi \in S^n(V)$ where $n = \dim V$. Moreover, we prove the following.

Lemma 2.5. $P_\xi \in \mathcal{S}$

Proof. We begin by observing that the map $\eta \rightarrow B_\xi \circ \sigma_\eta$ is linear and so

$$B_\xi \circ \sigma_\eta = \sum_{\ell} \eta(e_\ell) B_\xi \circ \sigma_{e_\ell} = \sum_{\ell} \eta(e_\ell) Q_\xi^\ell.$$

For every $\eta \in V^*$ let $L_{\eta, \xi}$ denote the classical adjoint or adjugate of $B_\xi \circ \sigma_\eta$, that is $L_{\eta, \xi} \circ B_\xi \circ \sigma_\eta(v) = P_\xi(\eta)v$. If $A(j|i)$ denotes the matrix obtained by deleting the j -th row and i -th column of a matrix A , then we have the explicit formulae

$$[L_{\eta, \xi}]_{ij} = (-1)^{i+j} \det(B_\xi \circ \sigma_\eta(j|i)) = (-1)^{i+j} \det \sum_{\ell} \eta(e_\ell) Q_\xi^\ell(j|i).$$

Since $\eta(e_1)^{i_1} \dots \eta(e_n)^{i_n} = \langle e^I, \eta^{|I|} \rangle$ where $I = (i_1, \dots, i_n)$ and $|I| = \sum_{\ell} i_\ell$, it follows that $L_{\eta, \xi}$ has the form

$$L_{\eta, \xi} = \sum_{|I|=n-1} \langle e^I, \eta^{|I|} \rangle Y_I(\xi)$$

where the matrices $Y_I(\xi)$ do not depend on η .

If $v \in V$ and $\omega \in V^*$ are arbitrary, then it follows that

$$\begin{aligned} \langle v \otimes \eta^n, \omega \otimes P_\xi \rangle &= \langle P_\xi(\eta)v, \omega \rangle = \langle L_{\eta, \xi} \circ B_\xi \circ \sigma_\eta(v), \omega \rangle \\ &= \sum_{|I|=n-1} \langle e^I, \eta^{n-1} \rangle \langle Y_I(\xi) \circ B_\xi \circ \sigma_\eta(v), \omega \rangle \\ &= \sum_{|I|=n-1} \langle e^I, \eta^{n-1} \rangle \langle \sigma_\eta(v), B_\xi^* \circ Y_I(\xi)^*(\omega) \rangle \\ &= \sum_{|I|=n-1} \langle e^I, \eta^{n-1} \rangle \langle \pi_0(v \otimes \eta), B_\xi^* \circ Y_I(\xi)^*(\omega) \rangle \\ &= \sum_{|I|=n-1} \langle e^I, \eta^{n-1} \rangle \langle v \otimes \eta, \pi_0^*(B_\xi^* \circ Y_I(\xi)^*(\omega)) \rangle \\ &= \sum_{|I|=n-1} \langle v \otimes \eta^n, \pi_0^*(B_\xi^* \circ Y_I(\xi)^*(\omega)) \otimes e^I \rangle \\ &= \langle v \otimes \eta^n, \sum_{|I|=n-1} \pi_0^*(B_\xi^* \circ Y_I(\xi)^*(\omega)) \otimes e^I \rangle. \end{aligned}$$

It now follows by polarisation that

$$\omega \otimes P_\xi = \sum_{|I|=n-1} \pi_0^*(B_\xi^* \circ Y_I(\xi)^*(\omega)) \otimes e^I$$

and, since $B_\xi^* \circ Y_I(\xi)^*(\omega) \in F^{(0)*}$, we conclude from (6) that

$$\omega \otimes P_\xi \in \pi_0^*(F^{(0)*}) \otimes S^{n-1}(V) \cap V^* \otimes S^n(V) = \pi_{n-1}^*(F^{(n-1)*}).$$

Finally, observing that

$$V^* \otimes S(V) \odot P = V^* \otimes P \odot S(V),$$

it now follows that $P_\xi \in \mathcal{S}$, as desired. ■

We recall the Nullstellensatz. Let \mathbb{K} be an algebraically closed field and let $\mathbb{K}[x_1, \dots, x_n]$ be the ring of polynomials over \mathbb{K} in n indeterminates x_1, \dots, x_n . Given an ideal $J \subseteq \mathbb{K}[x_1, \dots, x_n]$ we define

$$V(J) = \{y \in \mathbb{K}^n \mid p(y) = 0 \text{ for all } P \in J\}.$$

Then we denote by $I(V(J))$ the ideal of all polynomials vanishing on $V(J)$ and we write

$$\sqrt{J} = \{r \in \mathbb{K}[x_1, \dots, x_n] \mid r^t \in J \text{ for some positive integer } t\}$$

for the radical of J . Hilbert's Nullstellensatz states that $I(V(J)) = \sqrt{J}$ for every ideal J .

We can now prove Theorem 2.3.

Proof. Let us suppose $\mathfrak{g}^{(0)}$ contains no elements of rank 1 which implies $\mathcal{S} = \emptyset$. Then we have $V(\mathcal{S}) = \{0\}$ because $V(\mathcal{S}) = \mathcal{S} \cup \{0\}$, and the Nullstellensatz implies that the radical of \mathcal{S} is $S^+(V)$. In particular, for t sufficiently large, $(e_j)^t \in \mathcal{S}$ for all $j = 1, \dots, m$. For k large enough, every element of $S^{k+1}(V)$ is the sum of terms of the form $Q_j \odot (e_j)^t$ where $Q_j \in S^{k+1-t}(V)$, whence $V^* \otimes S^{k+1}(V) \subseteq \pi^*(F^{(k)*})$. We then conclude that $\mathfrak{g}^{(k)*} = \{0\}$ and $\mathfrak{g}^{(k)} = \{0\}$ thus proving that if $\mathfrak{g}^{(0)}$ does not contain elements of rank one, then it is of finite type.

Conversely, Example 2.2 shows that if ξ is a characteristic and therefore $w \otimes \xi$ is a rank one element of $\mathfrak{g}^{(0)}$, we obtain $w \otimes \xi^{\ell+1} \in \mathfrak{g}^{(\ell)}$ for every $\ell > 0$, whence $\mathfrak{g}^{(0)}$ is of infinite type. ■

2.4. Complex vs Real. In this section we will use the symbol \lrcorner to denote contraction, that is for a multilinear map L defined on V^k , where V is a vector space, and $v_1, \dots, v_j \in V$ are given, we define $v_1 \lrcorner \dots \lrcorner v_j \lrcorner L$ to be the multilinear map defined on V^{k-j} given by

$$v_1 \lrcorner \dots \lrcorner v_j \lrcorner L(v_{j+1}, \dots, v_k) = L(v_1, \dots, v_k).$$

Let V be a real n -dimensional vector space and let $V_{\mathbb{C}}$ denote the complexification of V . In this section we prove that $\mathfrak{g}^{(0)} \subseteq \mathfrak{gl}(V)$ is of infinite type if and only if the complexified algebra $\mathfrak{g}_{\mathbb{C}}^{(0)} = \mathfrak{g}^{(0)} + i\mathfrak{g}^{(0)} \subseteq \mathfrak{gl}(V_{\mathbb{C}})$ is of infinite type. Obviously this result implies that $\mathfrak{g}^{(0)}$ is of finite type if and only if $\mathfrak{g}_{\mathbb{C}}^{(0)}$ is of finite type.

Assume T is a nonzero element of $\mathfrak{g}^{(k)}$. Given a point $w = (u_1 + iv_1, \dots, u_{k+1} + iv_{k+1}) \in V_{\mathbb{C}}^{k+1}$, set

$$T_{\mathbb{C}}(w) = \sum_{r \in I(w)} i^{\mu(r)} T(r)$$

where

$$I(w) = \{r = (r_1, \dots, r_{k+1}) \mid r_j \in \{u_j, v_j\}\}$$

and $\mu(r) =$ the number of v_j 's in r . It follows that $T_{\mathbb{C}}(w) \in V_{\mathbb{C}} \otimes S^{k+1}(V_{\mathbb{C}}^*)$, moreover

$$w_1 \lrcorner \dots \lrcorner w_k \lrcorner T_{\mathbb{C}}(w_{k+1}) = \sum_{r \in I(w)} i^{\mu(r)} r_1 \lrcorner \dots \lrcorner r_k \lrcorner T(r_{k+1}) \in \mathfrak{g}^{(0)} + i\mathfrak{g}^{(0)}$$

and we conclude that $T_{\mathbb{C}} \in \mathfrak{g}_{\mathbb{C}}^{(k)}$.

Assume T is a nonzero element of $\mathfrak{g}_{\mathbb{C}}^{(k)}$. Furthermore, let $\iota : V \hookrightarrow V_{\mathbb{C}}$ be the inclusion given by $\iota(u) = u + i0$. We note that $\text{Re}T$ and $\text{Im}T$ are not complex multilinear maps of $V_{\mathbb{C}}^{k+1}$, however $\text{Re}T \circ \iota$ and $\text{Im}T \circ \iota$ are real multilinear maps of V . For any k -tuple $w = (w_1, \dots, w_k) \in V_{\mathbb{C}}^k$ we also have

$$w_1 \lrcorner \dots \lrcorner w_k \lrcorner \text{Re}T = \text{Re}(w_1 \lrcorner \dots \lrcorner w_k \lrcorner T) \in \mathfrak{g}^{(0)},$$

and it follows that

$$u_1 \lrcorner \dots u_k \lrcorner \text{Re}T \circ \iota = \text{Re}(\iota(u_1) \lrcorner \dots \iota(u_k) \lrcorner T) \in \mathfrak{g}^{(0)},$$

with the same conclusions applying to $\text{Im}T$.

Furthermore, T is a complex linear combination of elements in the set

$$\{T(e_{j_1}, \dots, e_{j_{k+1}}) \mid e_i = \iota(e_i)\},$$

and so $T \circ \iota$ cannot be identically zero since T is not identically zero. Hence it follows that at least one of the real multilinear maps $\text{Re}T \circ \iota : V \rightarrow V$ or $\text{Im}T \circ \iota : V \rightarrow V$ defines a nonzero element of $\mathfrak{g}^{(k)}$.

2.5. Tanaka prolongation of stratified nilpotent Lie algebras. Consider a stratified nilpotent Lie algebra $\mathfrak{n} = \mathfrak{g}_{-s} \oplus \dots \oplus \mathfrak{g}_{-1}$. The Tanaka prolongation of \mathfrak{n} is the graded Lie algebra $\mathfrak{g}(\mathfrak{n})$ given by the direct sum $\mathfrak{g}(\mathfrak{n}) = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_k(\mathfrak{n})$, where $\mathfrak{g}_k(\mathfrak{n}) = \{0\}$ for $k < -s$, $\mathfrak{g}_k(\mathfrak{n}) = \mathfrak{g}_k$ for $-s \leq k \leq -1$, and for each $k \geq 0$, $\mathfrak{g}_k(\mathfrak{n})$ is inductively defined by

$$\mathfrak{g}_k(\mathfrak{n}) = \left\{ \varphi \in \bigoplus_{p < 0} \mathfrak{g}_{p+k}(\mathfrak{n}) \otimes \mathfrak{g}_p^* \mid \varphi([X, Y]) = [\varphi(X), Y] + [X, \varphi(Y)] \right\},$$

with $\mathfrak{g}_0(\mathfrak{n})$ consisting of the strata preserving derivations of \mathfrak{n} . If $\varphi \in \mathfrak{g}_k(\mathfrak{n})$, where $k \geq 0$, then the condition in the definition becomes the Jacobi identity upon setting $[\varphi, X] = \varphi(X)$ when $X \in \mathfrak{n}$. Furthermore, if $\varphi \in \mathfrak{g}_k(\mathfrak{n})$ and $\varphi' \in \mathfrak{g}_\ell(\mathfrak{n})$, where $k, \ell \geq 0$, then $[\varphi, \varphi'] \in \mathfrak{g}_{k+\ell}(\mathfrak{n})$ is defined inductively according to the Jacobi identity, that is

$$[\varphi, \varphi'](X) = [\varphi, [\varphi', X]] - [\varphi', [\varphi, X]].$$

In [25], Tanaka shows that $\mathfrak{g}(\mathfrak{n})$ determines the structure of the contact vector fields on the group N with Lie algebra \mathfrak{n} . In particular, there is an isomorphism between the set of contact vector fields and $\mathfrak{g}(\mathfrak{n})$ when the latter is finite dimensional. Also, $\mathfrak{g}(\mathfrak{n})$ is finite dimensional if and only if $\mathfrak{g}_k(\mathfrak{n}) = \{0\}$ for some $k \geq 0$ since $\mathfrak{g}_k(\mathfrak{n}) = \{0\}$ implies $\mathfrak{g}_{k+\ell}(\mathfrak{n}) = \{0\}$ for all $\ell > 0$. Hence the rigidity of a stratified nilpotent Lie group can be determined by studying the Tanaka prolongation of the Lie algebra.

For a given subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}_0(\mathfrak{n})$, the prolongation of the pair $(\mathfrak{n}, \mathfrak{g}_0)$ is defined as the graded subalgebra

$$\text{Prol}(\mathfrak{n}, \mathfrak{g}_0) = \mathfrak{n} \oplus \bigoplus_{k \geq 0} \mathfrak{g}_k \subset \mathfrak{g}(\mathfrak{n}),$$

where for each $k \geq 1$, \mathfrak{g}_k is inductively defined as the subspace of $\mathfrak{g}_k(\mathfrak{n})$ satisfying $[\mathfrak{g}_k, \mathfrak{g}_{-1}] \subseteq \mathfrak{g}_{k-1}$. The pair $(\mathfrak{n}, \mathfrak{g}_0)$ is said to be of finite type if $\mathfrak{g}_k = \{0\}$ for some k , otherwise it is of infinite type and $\mathfrak{g}(\mathfrak{n})$ is infinite dimensional.

The type of $(\mathfrak{n}, \mathfrak{g}_0)$ is determined by the subalgebra

$$\mathfrak{h} = \bigoplus_{k \geq -1} \mathfrak{h}_k \subset \text{Prol}(\mathfrak{n}, \mathfrak{g}_0)$$

where the subspaces $\mathfrak{h}_k \subset \mathfrak{g}_k$ are defined as follows: let

$$\hat{\mathfrak{n}} = \mathfrak{g}_{-s} \oplus \cdots \oplus \mathfrak{g}_{-2}$$

and for $k \geq -1$ define

$$\mathfrak{h}_k = \{u \in \mathfrak{g}_k \mid [u, \hat{\mathfrak{n}}] = \{0\}\}.$$

It follows that $[\mathfrak{h}_k, \mathfrak{g}_{-1}] \subset \mathfrak{h}_{k-1}$ for $k \geq 0$. The bracket generating property shows that \mathfrak{h}_0 identifies with a subalgebra $\mathfrak{h}^{(0)} \subseteq \mathfrak{gl}(\mathfrak{g}_{-1})$, moreover \mathfrak{h}_k identifies with $\mathfrak{h}^{(k)}$ and by [25, Corollary 2, page 76] $(\mathfrak{n}, \mathfrak{g}_0)$ is of infinite type if and only if $\mathfrak{h}^{(0)}$ is of infinite type in the usual sense. We extend the notation by writing $\mathfrak{h}_k(\mathfrak{n})$ and $\mathfrak{h}^{(k)}(\mathfrak{n})$ when $\mathfrak{g}_0 = \mathfrak{g}_0(\mathfrak{n})$, and summarise Tanaka’s result as follows.

Theorem 2.6. *[25, Corollary 2, page 76] The space of C^∞ contact vector fields on N is finite dimensional if and only if $\mathfrak{h}^{(0)}(\mathfrak{n})$ is of finite type in the usual sense.*

2.6. Tanaka prolongation of contact vector fields. In this section we focus on how the space of contact vector fields on N is related to $\mathfrak{g}(\mathfrak{n})$, the Tanaka prolongation algebra through $\mathfrak{g}_0(\mathfrak{n})$. We shall need this construction in Section 4, where we investigate the conformal vector fields which form a subalgebra of the Lie algebra of contact vector fields. In particular, we shall see that the Lie algebra of conformal vector fields is isomorphic to the Tanaka prolongation through a suitable subalgebra of $\mathfrak{g}_0(\mathfrak{n})$.

The argument we propose here follows [28, Section 2]. Given a contact vector field U on N , we shall define correspondingly a section of the bundle $N \times \mathfrak{g}(\mathfrak{n}) \rightarrow N$.

For each $i = 1, \dots, s$, let $\mathfrak{g}_{-i} = \text{span}\{X_{-i,k} \mid k = 1, \dots, d_i\}$ where s is the step of \mathfrak{n} and d_i is the dimension of \mathfrak{g}_{-i} . The Lie algebra of left invariant fields is then spanned by the vector fields $\tilde{X}_{-i,k} = (\tau_p)_*(X_{-i,k})$. Using this basis we write any two vector fields U and W as

$$U = \sum_{i,k} u_{-i,k} \tilde{X}_{-i,k} \quad W = \sum_{j,l} w_{-j,l} \tilde{X}_{-j,l}.$$

We define the \mathfrak{n} -valued function A_U as

$$A_U(p) = \sum_{i,k} u_{-i,k}(p) X_{-i,k}$$

and A_W accordingly. We further refine the notation by setting

$$A_U^{-i}(p) = \sum_k u_{-i,k}(p) X_{-i,k}$$

for each $i = 1, \dots, s$.

We then have the equation

$$A_{[U,W]}(p) = [A_U(p), A_W(p)] + U(A_W)(p) - W(A_U)(p), \tag{7}$$

where for example $U(A_W)(p) = \sum_{i,k} (Uw_{-i,k})(p)X_{-i,k}$.

If U is a contact vector field, and $W(p) = \tilde{X}_{-1}(p) = (\tau_p)_*(X_{-1})$ where $X_{-1} \in \mathfrak{g}_{-1}$, then $A_{\tilde{X}_{-1}}(p) = X_{-1}$ and (7) becomes

$$A_{[U, \tilde{X}_{-1}]}(p) = [A_U(p), X_{-1}] - \tilde{X}_{-1}(A_U)(p).$$

Since $A_{[U, \tilde{X}_{-1}]}(p) \in \mathfrak{g}_{-1}$, the previous equality implies that

$$\tilde{X}_{-1}(A_U^{-m})(p) = [A_U^{-m+1}(p), X_{-1}] \tag{8}$$

for all $m > 1$.

If \tilde{Y}_{-1} is another horizontal left invariant vector field, then (8) implies

$$\begin{aligned} \tilde{Y}_{-1}\tilde{X}_{-1}(A_U^{-m}) &= \tilde{Y}_{-1}[A_U^{-m+1}, X_{-1}] \\ &= \tilde{Y}_{-1}\left[\sum_k u_{-m+1,k}X_{-m+1,k}, X_{-1}\right] \\ &= \sum_k (\tilde{Y}_{-1}u_{-m+1,k})[X_{-m+1,k}, X_{-1}] \\ &= \left[\sum_k (\tilde{Y}_{-1}u_{-m+1,k})X_{-m+1,k}, X_{-1}\right] \\ &= [\tilde{Y}_{-1}(A_U^{-m+1}), X_{-1}] \\ &= [[A_U^{-m+2}, Y_{-1}], X_{-1}] \end{aligned} \tag{9}$$

for all $m > 2$, moreover

$$\begin{aligned} [\tilde{Y}_{-1}, \tilde{X}_{-1}](A_U^{-m})(p) &= [[A_U^{-m+2}(p), Y_{-1}], X_{-1}] - [[A_U^{-m+2}(p), X_{-1}], Y_{-1}] \\ &= [A_U^{-m+2}(p), [Y_{-1}, X_{-1}]] \end{aligned}$$

for all $m > 2$. By the bracket generating property we conclude that

$$\tilde{X}_{-r}(A_U^{-m})(p) = [A_U^{-m+r}(p), X_{-r}], \quad \forall X_{-r} \in \mathfrak{g}_{-r} \quad \text{s.t.} \quad r < m. \tag{10}$$

Define now $A_U^0 : N \rightarrow \bigoplus_{\ell < 0} \mathfrak{g}_\ell(\mathbf{n}) \otimes \mathfrak{g}_\ell^*$ by setting

$$A_U^0(p)(X_{-r}) = \tilde{X}_{-r}(A_U^{-r})(p), \quad X_{-r} \in \mathfrak{g}_{-r}. \tag{11}$$

For $X_{-r} \in \mathfrak{g}_{-r}$ and $Y_{-t} \in \mathfrak{g}_{-t}$, we have

$$\begin{aligned} A_U^0(p)([Y_{-t}, X_{-r}]) &= [\tilde{Y}_{-t}, \tilde{X}_{-r}](A_U^{-(r+t)})(p) \\ &= \tilde{Y}_{-t}\tilde{X}_{-r}(A_U^{-(r+t)})(p) - \tilde{X}_{-r}\tilde{Y}_{-t}(A_U^{-(r+t)})(p). \end{aligned} \tag{12}$$

Using (10) we have

$$\begin{aligned} \tilde{Y}_{-t}\tilde{X}_{-r}(A_U^{-(r+t)}) &= \tilde{Y}_{-t}[A_U^{-t}, X_{-r}] \\ &= \tilde{Y}_{-t}\left[\sum_k u_{-t,k}X_{-t,k}, X_{-r}\right] \\ &= \sum_k (\tilde{Y}_{-t}u_{-t,k})[X_{-t,k}, X_{-r}] \\ &= \left[\sum_k (\tilde{Y}_{-t}u_{-t,k})X_{-t,k}, X_{-r}\right] \\ &= [\tilde{Y}_{-t}(A_U^{-t}), X_{-r}], \end{aligned}$$

and so by (11) we have

$$\tilde{Y}_{-t}\tilde{X}_{-r}(A_U^{-(r+t)})(p) = [\tilde{Y}_{-t}(A_U^{-t})(p), X_{-r}] = [A_U^0(p)(Y_{-t}), X_{-r}]. \tag{13}$$

Similarly we also have $\tilde{X}_{-r}\tilde{Y}_{-t}(A_U^{-(r+t)})(p) = [A_U^0(p)(X_{-r}), Y_{-t}]$ which together with (13) and (12) gives

$$A_U^0(p)([Y_{-t}, X_{-r}]) = [A_U^0(p)(Y_{-t}), X_{-r}] + [Y_{-t}, A_U^0(p)(X_{-r})],$$

and we conclude that $A_U^0(p)$ is a strata preserving derivation of \mathfrak{n} . Furthermore, the previous equality is simply the Jacobi identity when we define

$$[A_U^0(p), X] := A_U^0(p)(X)$$

for any $X \in \mathfrak{n}$. In summary $A_U^0(p) \in \mathfrak{g}_0(\mathfrak{n})$ where

$$\mathfrak{g}_0(\mathfrak{n}) = \left\{ \varphi \in \bigoplus_{\ell < 0} \mathfrak{g}_\ell(\mathfrak{n}) \otimes \mathfrak{g}_\ell^* \mid \varphi([Y, X]) = [\varphi(Y), X] + [Y, \varphi(X)] \right\}.$$

Inductively on $n = 0, 1, 2, \dots$, we define $A_U^n : N \rightarrow \bigoplus_{\ell < 0} \mathfrak{g}_{\ell+n}(\mathfrak{n}) \otimes \mathfrak{g}_\ell^*$ by setting

$$A_U^n(p)(X_{-r}) = \tilde{X}_{-r}(A_U^{-r+n})(p) \quad \text{when } X_{-r} \in \mathfrak{g}_{-r}. \tag{14}$$

We remark that the right hand side of (14) is defined relative to the basis of $\mathfrak{g}_{-r+n}(\mathfrak{n})$ induced by the chosen basis of \mathfrak{n} and the corresponding dual basis. For example if $X_{-1} \in \mathfrak{g}_{-1}$ and $n = 1$ then $A_U^1(p)(X_{-1}) = \tilde{X}_{-1}(A_U^0)(p)$. Now if $\{dx_{-i,k}\}$ denotes the dual basis corresponding to $\{X_{-i,k}\}$ then

$$A_U^0(p) = \sum_{i,j,k,\ell} \lambda_{i,j,k,\ell}(p) X_{-i,k} \otimes dx_{-j,\ell}$$

and

$$\tilde{X}_{-1}(A_U^0)(p) = \sum_{i,j,k,\ell} (\tilde{X}_{-1}\lambda_{i,j,k,\ell})(p) X_{-i,k} \otimes dx_{-j,\ell}.$$

See Example 4.1 below for examples where $n > 1$.

A similar calculation to that which produced (13) shows that

$$\tilde{Y}_{-t}\tilde{X}_{-r}(A_U^{-(r+t)+n})(p) = [\tilde{Y}_{-t}(A_U^{-t+n})(p), X_{-r}] = [A_U^n(p)(Y_{-t}), X_{-r}]$$

and

$$\tilde{X}_{-r}\tilde{Y}_{-t}(A_U^{-(r+t)+n})(p) = [\tilde{X}_{-r}(A_U^{-r+n})(p), Y_{-t}] = [A_U^n(p)(X_{-r}), Y_{-t}],$$

and it follows that

$$\begin{aligned} A_U^n(p)([Y_{-t}, X_{-r}]) &= [\tilde{Y}_{-t}, \tilde{X}_{-r}](A_U^{-(r+t)+n})(p) \\ &= \tilde{Y}_{-t}\tilde{X}_{-r}(A_U^{-(r+t)+n})(p) - \tilde{X}_{-r}\tilde{Y}_{-t}(A_U^{-(r+t)+n})(p) \\ &= [A_U^n(p)(Y_{-t}), X_{-r}] + [Y_{-t}, A_U^n(p)(X_{-r})], \end{aligned}$$

which becomes the Jacobi identity when we set

$$[A_U^n(p), X_{-r}] := A_U^n(p)(X_{-r}). \tag{15}$$

In summary $A_U^n(p) \in \mathfrak{g}_n(\mathfrak{n})$ where

$$\mathfrak{g}_n(\mathfrak{n}) = \left\{ \varphi \in \bigoplus_{\ell < 0} \mathfrak{g}_{\ell+n}(\mathfrak{n}) \otimes \mathfrak{g}_\ell^* \mid \varphi([Y, X]) = [\varphi(Y), X] + [Y, \varphi(X)] \right\}.$$

For $A_W^m(p) \in \mathfrak{g}_m(\mathfrak{n})$ and $A_U^n(p) \in \mathfrak{g}_n(\mathfrak{n})$ we inductively define $[A_W^m(p), A_U^n(p)] \in \mathfrak{g}_{m+n}(\mathfrak{n})$ according to the Jacobi identity, that is for every $X \in \mathfrak{n}$ we set

$$\begin{aligned} [A_W^m(p), A_U^n(p)](X) &= [[A_W^m(p), A_U^n(p)], X] \\ &= [A_W^m(p), [A_U^n(p), X]] - [A_U^n(p), [A_W^m(p), X]]. \end{aligned}$$

Thus for every contact vector field U we obtain a section of the bundle $N \times \mathfrak{g}(\mathfrak{n}) \rightarrow N$, that is $\sum_{n \geq -s} A_U^n$.

3. Proof of Theorem 1.1

(ii) \Rightarrow (iii). Let $\varphi_{\mathbb{C}}$ be an element of rank 1 in the complexification of $\mathfrak{h}_0(\mathfrak{n})$. Corresponding to $\varphi_{\mathbb{C}}$ there exists a basis \mathcal{B} of $\mathfrak{n}_{\mathbb{C}}$, the complexification of \mathfrak{n} , and vectors $X_{\mathbb{C}}, Y_{\mathbb{C}} \in \mathcal{B}$ in the complexification of \mathfrak{g}_{-1} (not necessarily distinct), unique up to a scale, such that $\varphi_{\mathbb{C}}(Y_{\mathbb{C}}) = X_{\mathbb{C}}$ and $\varphi_{\mathbb{C}}(Z) = 0$ for every $Z_{\mathbb{C}} \in \mathcal{B} \setminus \{Y_{\mathbb{C}}\}$. For such a $Z_{\mathbb{C}}$ we have

$$0 = \varphi_{\mathbb{C}}[Y_{\mathbb{C}}, Z_{\mathbb{C}}] = [\varphi_{\mathbb{C}}Y_{\mathbb{C}}, Z_{\mathbb{C}}] + [Y_{\mathbb{C}}, \varphi_{\mathbb{C}}Z_{\mathbb{C}}] = [X_{\mathbb{C}}, Z_{\mathbb{C}}]$$

We conclude that $\text{rank ad } X_{\mathbb{C}} \leq 1$.

(iii) \Rightarrow (ii). If $X_{\mathbb{C}}$ is in the complexification of \mathfrak{g}_{-1} and $\text{ad } X_{\mathbb{C}}$ has rank zero, then \mathfrak{n} is degenerate. If we fix a basis of $\mathfrak{n}_{\mathbb{C}}$ containing $X_{\mathbb{C}}$, it suffices to take the derivation $\varphi_{\mathbb{C}}$ defined by $\varphi_{\mathbb{C}}(X_{\mathbb{C}}) = X_{\mathbb{C}}$ and zero for all other elements of the basis. Suppose now that $X_{\mathbb{C}}$ is such that $\text{rank ad } X_{\mathbb{C}} = 1$. Following [16], it is possible to complete $X_{\mathbb{C}}$ to a basis \mathcal{B} of $\mathfrak{n}_{\mathbb{C}}$, in such a way that there exists a unique $Y_{\mathbb{C}} \in \mathcal{B}$ for which $[X_{\mathbb{C}}, Y_{\mathbb{C}}] \neq 0$, moreover $Y_{\mathbb{C}}$ is in the complexification of \mathfrak{g}_{-1} . It follows directly that the endomorphism $\varphi_{\mathbb{C}} : \mathfrak{n}_{\mathbb{C}} \rightarrow \mathfrak{n}_{\mathbb{C}}$ defined by $\varphi_{\mathbb{C}}(Y_{\mathbb{C}}) = X_{\mathbb{C}}$ and zero in $\mathcal{B} \setminus Y_{\mathbb{C}}$ is an element of rank 1 in the complexification of $\mathfrak{h}_0(\mathfrak{n})$.

(ii) \Rightarrow (i). Let $\varphi_{\mathbb{C}}$ be a rank 1 element in the complexification of $\mathfrak{h}_0(\mathfrak{n})$. Then it identifies with an element of rank 1 in the complexification of $\mathfrak{h}^{(0)}(\mathfrak{n})$, say $\mathfrak{h}_{\mathbb{C}}^{(0)}(\mathfrak{n})$. By Theorem 2.3 it follows that $\mathfrak{h}_{\mathbb{C}}^{(0)}(\mathfrak{n})$ is of infinite type, and by the discussion in Theorem 2 $\mathfrak{h}^{(0)}(\mathfrak{n})$ is also of infinite type. Then Theorem 2.6 implies that N admits infinite dimensional families of C^∞ contact maps, thus N is nonrigid.

(i) \Rightarrow (ii). We show that if condition (ii) fails, then N is rigid in the class of C^2 maps. First, observe that if there is no rank 1 element in the complexification of $\mathfrak{h}_0(\mathfrak{n})$, then arguing as above we conclude that $\mathfrak{h}^{(0)}(\mathfrak{n})$ is of finite type, and so is $\mathfrak{g}(\mathfrak{n})$, the Tanaka prolongation of \mathfrak{n} . It follows that the set $CT(\mathcal{TU})$ consisting of

C^∞ contact vector fields on some open domain $\mathcal{U} \subseteq N$, forms a finite dimensional Lie algebra which is isomorphic to $\mathfrak{g}(\mathfrak{n})$.

If f is a C^2 contact map of \mathcal{U} and $U \in C\Gamma(T\mathcal{U})$, then f_*U is a C^1 contact vector field and thus satisfies the integrability condition $[f_*U, \mathcal{H}] \subseteq \mathcal{H}$. We approximate f_*U by the convolution $\psi_\epsilon \star f_*U$, where $\psi_\epsilon = \epsilon^{-Q}\psi \circ \delta_{1/\epsilon}$ and ψ is a smooth function with compact support contained in the unit ball $B_1(e)$. The standard properties of convolution imply that $\psi_\epsilon \star f_*U \in C\Gamma(T\mathcal{U})$ for all $\epsilon > 0$ and so $f_*U \in C\Gamma(T\mathcal{U})$.

By composing with suitable left translation if necessary, we can assume that \mathcal{U} contains the identity and that $f(e) = e$. If $\mathcal{T} : \mathfrak{g}(\mathfrak{n}) \rightarrow C\Gamma(T\mathcal{U})$ denotes the isomorphism given by Tanaka, then it follows that $\mathcal{T}^{-1} \circ f_* \circ \mathcal{T} \in \text{Aut } \mathfrak{g}(\mathfrak{n})$ which is a finite dimensional space. Thus a $C^2(\mathcal{U})$ contact map which fixes the identity is the unique solution to an initial value problem of the form

$$f_* = \mathcal{T} \circ \alpha \circ \mathcal{T}^{-1}, \quad f(e) = e$$

for some $\alpha \in \text{Aut } \mathfrak{g}(\mathfrak{n})$. Therefore we conclude that the C^2 contact maps on \mathcal{U} are parametrised by a finite dimensional space.

4. The Liouville theorem

In this section we prove Theorem 1.2.

Let $\mathcal{U}, \mathcal{V} \subset N$ be open domains, and let $f : \mathcal{U} \rightarrow \mathcal{V}$ be a homeomorphism. For $p \in \mathcal{U}$ and for small $t \in \mathbb{R}$ we define the distortion as

$$H_f(p, t) = \frac{\max\{d(f(p), f(q)) \mid d(p, q) = t\}}{\min\{d(f(p), f(q)) \mid d(p, q) = t\}}.$$

We say that f is K -quasiconformal if there exists a constant $K \geq 1$ such that

$$\limsup_{t \rightarrow 0} H_f(p, t) \leq K$$

for all $p \in \mathcal{U}$.

Quasiconformal maps are Pansu differentiable, see [18], which implies they satisfy the contact conditions almost everywhere. We recall that the Pansu differential $Df(p)$ is the automorphism of N defined as

$$Df(p)q = \lim_{t \rightarrow 0} \delta_t^{-1} \circ \tau_{f(p)}^{-1} \circ f \circ \tau_p \circ \delta_t(q),$$

where $p, q \in N$. The Lie derivative of $Df(p)$, say $df(p)$, is a grading-preserving automorphism of \mathfrak{n} . Suitably regular contact maps are Pansu differentiable. In particular C^1 contact maps are Pansu differentiable everywhere, see [27], and more generally almost everywhere Pansu differentiable when locally Lipschitz, see [13].

In [1], the authors prove that 1-quasiconformal maps are C^∞ and characterised by the conditions of being locally quasiconformal and $Df(p)$ coinciding with a similarity, i.e. a product of a dilation and an isometry of the subriemannian metric. This implies in particular that $df(p)$ is a similarity when restricted to \mathfrak{g}_{-1} [1, Lemma 5.2].

The exponential coordinate $x_{-i,k}(p)$ of p is defined so that

$$p = \exp\left(\sum_{-i,k} x_{-i,k}(p) X_{-i,k}\right)$$

and for a mapping f we write $f_{-i,k}(p) = x_{-i,k}(f(p))$. The dual one forms which correspond with the left invariant vector fields $\tilde{X}_{-i,k}$ are then given by $\theta_{-i,k}(p) = (\tau_p^{-1})^* dx_{-i,k}$. The tangent mapping f_* then has the block matrix form $M_f = (M_{i,j})$ where $M_{i,j} = (\theta_{-i,k}(f_* \tilde{X}_{-j,\ell}))$. Furthermore, if f is Pansu differentiable at p , then $M_{i,j}(p) = 0$ when $j < i$, moreover $M_{1,1} = (\tilde{X}_{-1,\ell} f_{-1,k})$ since $\theta_{-1,k}(p) = dx_{-1,k}$.

Since f is Pansu differentiable at p , [1, Lemma 3.4] implies that the restriction of $df(p)$ to \mathfrak{g}_{-1} in exponential coordinates is $M_{1,1}(p)$. Thus for all $X, Y \in \mathfrak{g}_{-1}$ we have

$$\langle M_{1,1}(p)(X), M_{1,1}(p)(Y) \rangle = \lambda(p) \langle X, Y \rangle \tag{16}$$

and so for every $p \in \mathcal{U}$, $M_{1,1}(p)$ is given by an element in the conformal group

$$CO(n) = \{M \in GL(n, \mathbb{R}) \mid M^{tr} M = \lambda I\},$$

where $n = \dim \mathfrak{g}_{-1}$. In view of this fact we shall also refer to 1-quasiconformal maps as conformal maps.

Let f^t be a flow of conformal maps induced by a C^∞ contact vector field $U = \sum_{i,k} u_{-i,k} \tilde{X}_{-i,k}$, and let \mathcal{U} be an open set such that f^t is defined on \mathcal{U} for all t sufficiently small. By definition

$$u_{-i,k}(p) = \theta_{-i,k}(p) \left(\frac{d}{dt} f^t(p) \Big|_{t=0} \right)$$

and

$$u_{-1,k}(p) = dx_{-1,k} \left(\frac{d}{dt} f^t(p) \Big|_{t=0} \right) = \frac{d}{dt} x_{-1,k}(f^t(p)) \Big|_{t=0} = \frac{d}{dt} f_{-1,k}^t(p) \Big|_{t=0}.$$

Now if $M_{1,1}^t = (\tilde{X}_{-1,\ell} f_{-1,k}^t)$, then by Schwarz's theorem, we have

$$\frac{d}{dt} M_{1,1}^t(p) \Big|_{t=0} = (\tilde{X}_{-1,\ell} \frac{d}{dt} f_{-1,k}^t(p) \Big|_{t=0}) = (\tilde{X}_{-1,\ell} u_{-1,k}^t(p)),$$

and we conclude that if $X \in \mathfrak{g}_{-1}$ then

$$\frac{d}{dt} M_{1,1}^t(p) \Big|_{t=0} X = \tilde{X} A_U^{-1}(p) = A_U^0(p)(X).$$

From (16) we have

$$\frac{d}{dt} M_{1,1}^t(p)^{tr} M_{1,1}^t(p) \Big|_{t=0} = \frac{d}{dt} \lambda_t(p) \Big|_{t=0} I = \lambda'(p) I$$

and so if $\bar{A}_U^0(p)$ denotes the restriction of $A_U^0(p)$ of \mathfrak{g}_{-1} then

$$\bar{A}_U^0(p)^{tr} + \bar{A}_U^0(p) = \lambda'(p) I,$$

and we conclude that $\bar{A}_U^0(p)$ coincides with an element of

$$\mathfrak{co}(n) = \{A \in \mathfrak{gl}(n, \mathbb{R}) \mid A^{tr} + A = \lambda' I \text{ for some } \lambda' \in \mathbb{R}\},$$

where $n = \dim \mathfrak{g}_{-1}$.

It follows from [25, Lemma 6.3] and [28, §2] that the Lie algebra of conformal vector fields is determined by the Tanaka prolongation on \mathfrak{n} through $\mathfrak{g}_0 = \{\varphi \in \mathfrak{g}_0(\mathfrak{n}) : \varphi \in \mathfrak{co}(n)\}$. We illustrate with the case of the three dimensional Heisenberg group.

Example 4.1. It is well known [11] that the $2n + 1$ -dimensional Heisenberg group is nonrigid with respect to its contact structure, whereas the conformal maps coincide with the action of some element in the Lie group $SU(1, n)$. We retrieve the result for the conformal maps at the infinitesimal level by computing the Tanaka prolongation of the conformal vector fields. The notation is that of Section 2.

Assume now that \mathfrak{n} is the three dimensional Heisenberg algebra, and N is the the corresponding Heisenberg group. We fix a basis $\{X_1, X_2, Y\}$ of \mathfrak{n} such that $[X_1, X_2] = Y$ is the only nonzero bracket and define a scalar product for which $\mathfrak{g}_{-1} = \text{span}\{X_1, X_2\}$ and $\mathfrak{g}_{-2} = \text{span}\{Y\}$ are orthogonal. Choose exponential coordinates on the group $(x_1, x_2, y) = \exp y Y \exp x_2 X_2 \exp x_1 X_1$. We identify the Lie algebra with the tangent space $T_e N$ to N at the identity e , and for X in \mathfrak{n} we write \tilde{X} for the left-invariant vector field that agrees with X at e . The left invariant vector fields corresponding to the basis vectors are

$$\begin{aligned} \tilde{X}_1 &= \frac{\partial}{\partial x_1} & \tilde{X}_2 &= \frac{\partial}{\partial x_2} + x_1 \frac{\partial}{\partial y} \\ \tilde{Y} &= \frac{\partial}{\partial y}. \end{aligned}$$

Let $U = f_1 \tilde{X}_1 + f_2 \tilde{X}_2 + g \tilde{Y}$ be a conformal vector field. The contact conditions are

$$\tilde{X}_1 g = -f_2 \quad \text{and} \quad \tilde{X}_2 g = f_1$$

and the conformality conditions are

$$\tilde{X}_1 f_1 = \tilde{X}_2 f_2 \quad \text{and} \quad \tilde{X}_2 f_1 = -\tilde{X}_1 f_2.$$

Thus contact and conformality imply

$$\tilde{X}_1^2 g = \tilde{X}_2^2 g \quad \text{and} \quad \tilde{X}_1 \tilde{X}_2 g = -\tilde{X}_2 \tilde{X}_1 g. \tag{17}$$

We gather some implications of (17) which will be required in the computations. First we note that the second equation in (17) implies

$$\tilde{Y} g = 2\tilde{X}_1 \tilde{X}_2 g, \tag{18}$$

and the bracket conditions $[\tilde{X}_1, [\tilde{X}_1, \tilde{X}_2]] = [\tilde{X}_2, [\tilde{X}_1, \tilde{X}_2]] = 0$ imply

$$\tilde{X}_2 \tilde{X}_1^2 g = -3\tilde{X}_1^2 \tilde{X}_2 g \quad \tilde{X}_1 \tilde{X}_2^2 g = -3\tilde{X}_2^2 \tilde{X}_1 g. \tag{19}$$

Let $\{dx_1, dx_2, dy\}$ denote the basis of \mathfrak{n}^* , dual to the basis we fixed for \mathfrak{n} . For every $p \in \mathcal{U}$ we have

$$\begin{aligned} A_U^{-1}(p) &= \tilde{X}_2 g(p) X_1 - \tilde{X}_1 g(p) X_2 \\ A_U^{-2}(p) &= g(p) Y, \end{aligned}$$

and for each integer $k \geq 0$ we have

$$\begin{aligned} A_U^k(p) &= A_U^k(p)(X_1) \otimes dx_1 + A_U^k(p)(X_2) \otimes dx_2 + A_U^k(p)(Y) \otimes dy \\ &= \tilde{X}_1(A_U^{k-1})(p) \otimes dx_1 + \tilde{X}_2(A_U^{k-1})(p) \otimes dx_2 + \tilde{Y}(A_U^{k-2})(p) \otimes dy. \end{aligned}$$

A direct calculation shows that

$$A_U^0(p) = (\tilde{Y}g)(p)A_1 + (\tilde{X}_1\tilde{X}_1g)(p)A_2$$

where

$$\begin{aligned} A_1 &= \frac{1}{2}X_1 \otimes dx_1 + \frac{1}{2}X_2 \otimes dx_2 + Y \otimes dy \\ A_2 &= X_1 \otimes dx_2 - X_2 \otimes dx_1. \end{aligned}$$

Setting $\mathfrak{g}_0 = \text{span}\{A_1, A_2\}$, we can compute the Lie brackets of $\mathfrak{n} \oplus \mathfrak{g}_0$. The higher order prolongation spaces can then be computed in a purely symbolic way using the Jacobi identity and the bracket generating property. The advantage in taking this symbolic approach is that no more information is required from the contact conditions or the conformality conditions. In order to compute $A_U^1(p)$ we use (17) and (19). We obtain

$$A_U^1(p) = (\tilde{Y}\tilde{X}_1g)(p)B_1 + (\tilde{Y}\tilde{X}_2g)(p)B_2 \tag{20}$$

where

$$\begin{aligned} B_1 &= A_1 \otimes dx_1 - \frac{3}{2}A_2 \otimes dx_2 - \tilde{X}_2 \otimes dy \\ B_2 &= A_1 \otimes dx_2 + \frac{3}{2}A_2 \otimes dx_1 + \tilde{X}_1 \otimes dy. \end{aligned}$$

In particular, we have used the identities $\tilde{X}_1^3g = \frac{3}{2}\tilde{Y}\tilde{X}_2g$ and $\tilde{X}_2\tilde{X}_1^2g = -\frac{3}{2}Y\tilde{X}_1g$, which are easily derived from (17), (18) and (19).

At the next step of prolongation we see that

$$A_U^2(p) = (\tilde{Y}\tilde{X}_1^2g)(p)C_1 + (\tilde{Y}^2g)(p)C_2$$

where

$$\begin{aligned} C_1 &= B_1 \otimes dx_1 + B_2 \otimes dx_2 + A_2 \otimes dy \\ C_2 &= \frac{1}{2}B_2 \otimes dx_1 - \frac{1}{2}B_1 \otimes dx_2 + A_1 \otimes dy. \end{aligned}$$

We now see that $\tilde{Y}\tilde{X}_1^2g(p) = 0$ by observing that

$$\begin{aligned}\tilde{Y}\tilde{X}_1^2g &= \tilde{X}_1\tilde{X}_2\tilde{X}_1^2g - \tilde{X}_2\tilde{X}_1\tilde{X}_1^2g \\ &= \tilde{X}_1\tilde{X}_2\tilde{X}_1^2g - \tilde{X}_2\tilde{X}_1\tilde{X}_2^2g \quad \text{by (17)} \\ &= -3\tilde{X}_1\tilde{X}_1^2\tilde{X}_2g + 3\tilde{X}_2\tilde{X}_2^2\tilde{X}_1g \quad \text{by (19)} \\ &= -3\tilde{X}_1^2\tilde{X}_1\tilde{X}_2g - 3\tilde{X}_2^2\tilde{X}_1\tilde{X}_2g \quad \text{by (17)} \\ &= -\frac{3}{2}\tilde{Y}(\tilde{X}_1^2g + \tilde{X}_2^2g) \quad \text{by (18)} \\ &= -3\tilde{Y}\tilde{X}_1^2g \quad \text{by (17)}.\end{aligned}$$

Finally, we observe using (19) that $\tilde{Y}\tilde{X}_1^2g = 0$ implies

$$\tilde{X}_1\tilde{Y}^2g = \tilde{X}_2\tilde{Y}^2g = Y^3g = 0,$$

since

$$\tilde{X}_1\tilde{Y}^2g = -\frac{2}{3}\tilde{X}_2\tilde{Y}\tilde{X}_1^2g, \quad \tilde{X}_2\tilde{Y}^2g = \frac{2}{3}\tilde{X}_1\tilde{Y}\tilde{X}_1^2g \quad \text{and} \quad \tilde{Y}^3g = \frac{4}{3}\tilde{X}_2^2\tilde{Y}\tilde{X}_1^2g.$$

It now follows that

$$\text{Prol}(\mathfrak{n}, \mathfrak{g}_0) = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 \quad (21)$$

where

$$\mathfrak{g}_0 = \text{span}\{A_1, A_2\}, \quad \mathfrak{g}_1 = \text{span}\{B_1, B_2\} \quad \text{and} \quad \mathfrak{g}_2 = \text{span}\{C_2\}.$$

Setting $H_1 = 2A_1$, $H_2 = A_2$, $\bar{X}_1 = 2B_1$, $\bar{X}_2 = 2B_2$ and $\bar{Y} = -4C_2$, one recognises in (21) the root space decomposition of the simple Lie algebra $\mathfrak{su}(1, 2)$, and direct calculation according to the bracket definition in Section 2 gives the correct bracket relations (see Table 1).

We now come to the proof of Theorem 1.2.

Proof. By the discussion above, the Lie algebra of conformal vector fields is $\text{Prol}(\mathfrak{n}, \mathfrak{g}_0)$. Theorem 2.6 implies that the character of $\text{Prol}(\mathfrak{n}, \mathfrak{g}_0)$ is determined by the usual prolongation of $\mathfrak{h}_0(\mathfrak{n}) \cap \mathfrak{g}_0$. This space identifies with a subalgebra $\mathfrak{h}^{(0)}$ of $\mathfrak{co}(n)$.

If $n \geq 3$, it follows from Example 2.1 that the usual prolongation of $\mathfrak{co}(n)$ is zero at the second step and so in particular $\mathfrak{h}^{(2)} = \{0\}$. Thus $\text{Prol}(\mathfrak{n}, \mathfrak{g}_0)$ has finite dimension if $\dim \mathfrak{g}_{-1} \geq 3$.

Now set $n = 2$. Since \mathfrak{n} is not abelian, there exist X_1, X_2 in \mathfrak{g}_{-1} such that $[X_1, X_2] = Y \neq 0$. Then it follows that

$$\mathfrak{co}(n) \cap \mathfrak{h}^{(0)}(\mathfrak{n}) \subseteq \left\{ \begin{bmatrix} 0 & b \\ -b & 0 \end{bmatrix} \mid b \in \mathbb{R} \right\},$$

with equality if \mathfrak{n} is the three dimensional Heisenberg algebra. It follows easily that the space on the right hand side of the inclusion above is of type 1 in the sense

described in Section 2, which proves Theorem 1.2 for the case $n = 2$. Notice that the same conclusion can of course be reached as a consequence of Example 4.1.

Finally, let f be a conformal map on some open domain \mathcal{U} . Composing with left translation if necessary, we may assume that $f(e) = e$. Arguing as we did at the end of Section 3, we conclude that f is induced by an automorphism of $\text{Prol}(\mathfrak{n}, \mathfrak{g}_0)$, so that the proof of Theorem 1.2 is complete. ■

| | Y | X_1 | X_2 | H_1 | H_2 | \bar{X}_1 | \bar{X}_2 | \bar{Y} |
|-------------|---------|--------------|-------------|-------------|--------------|--------------|-------------|--------------|
| Y | 0 | 0 | 0 | $-2Y$ | 0 | $2X_2$ | $-2X_1$ | $2H_1$ |
| X_1 | 0 | 0 | Y | $-X_1$ | X_2 | $-H_1$ | $-3H_2$ | \bar{X}_2 |
| X_2 | 0 | $-Y$ | 0 | $-X_2$ | $-X_1$ | $3H_2$ | $-H_1$ | $-\bar{X}_1$ |
| H_1 | $2Y$ | X_1 | X_2 | 0 | 0 | $-\bar{X}_1$ | \bar{X}_2 | $-2\bar{Y}$ |
| H_2 | 0 | $-X_2$ | X_1 | 0 | 0 | $-\bar{X}_2$ | \bar{X}_1 | 0 |
| \bar{X}_1 | $-2X_2$ | H_1 | $-3H_2$ | \bar{X}_1 | \bar{X}_2 | 0 | $-2\bar{Y}$ | 0 |
| \bar{X}_2 | $2X_1$ | $3H_2$ | H_1 | \bar{X}_2 | $-\bar{X}_1$ | $2\bar{Y}$ | 0 | 0 |
| \bar{Y} | $-2H_1$ | $-\bar{X}_2$ | \bar{X}_1 | $2\bar{Y}$ | 0 | 0 | 0 | 0 |

Table 1: Bracket table of $\mathfrak{su}(1, 2)$.

References

- [1] Capogna, L., and M. Cowling, *Conformality and q -harmonicity in carnot groups*, Duke Math. J. **135** (2006), 455–479.
- [2] Chow, Wei-Liang, „Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung“, Math. Ann. **117** (1939), 98–105.
- [3] Cowling, M., F. De Mari, A. Korányi, and H. M. Reimann. *Contact and conformal maps in parabolic geometry, I*, Geom. Dedicata **111** (2005), 65–86.
- [4] Cowling, M., and H. M. Reimann. *Quasiconformal mappings on Carnot groups: three examples*, In: “Harmonic analysis at Mount Holyoke, (South Hadley, MA, 2001), Vol. **320** of Contemp. Math., Amer. Math. Soc., Providence, RI, 2003, 111–118.
- [5] Doubrov, B., and O. Radko. *Graded nilpotent lie algebras of infinite type*, arXiv:1001.0379.

- [6] Gehring, F. W., *Rings and quasiconformal mappings in space*, Trans. Amer. Math. Soc. **103** (1962), 353–393.
- [7] Guillemin, V., D. Quillen, and S. Sternberg, *The classification of the complex primitive infinite pseudogroups*, Proc. Nat. Acad. Sci. U.S.A. **55** (1966), 687–690. y
- [8] —, *The classification of the irreducible complex algebras of infinite type*, J. Analyse Math. **18** (1967), 107–112.
- [9] Heinonen, J., *Calculus on Carnot groups*, in: “Fall School in Analysis (Jyväskylä, 1994,” Vol. **68** of Report Univ. Jyväskylä, 1995, 1–31.
- [10] Kobayashi, Sh., “Transformation groups in differential geometry,” Classics in Mathematics. Springer-Verlag, Berlin, 1995, Reprint of the 1972 edition.
- [11] Korányi, A., and H. M. Reimann. *Quasiconformal mappings on the Heisenberg group*, Invent. Math. **80** (1985), 309–338.
- [12] Kuzmich, O., *Graded nilpotent Lie algebras in low dimensions*, Lobachevskii J. Math. **3** (electronic), (1999), 147–184.
- [13] Magnani, V., *Contact equations, Lipschitz extensions and isoperimetric inequalities*, Calc. Var. Partial Differential Equations, to appear.
- [14] Margulis, G. A., and G. D. Mostow, *The differential of a quasi-conformal mapping of a Carnot-Carathéodory space*, Geom. Funct. Anal. **5** (1995), 402–433.
- [15] Nevanlinna, R., *On differentiable mappings*, in: “Analytic Functions,” Princeton Math. Ser. **24** Princeton Univ. Press, 1960, 3–9.
- [16] Ottazzi, A. *A sufficient condition for nonrigidity of Carnot groups*, Math. Z. **259** (2008), 617–629.
- [17] Ottazzi, A., and B. Warhurst, *Algebraic prolongation and rigidity of Carnot groups*, Monatsh. Math. **162** (2009), 179–195.
- [18] Pansu, P., Métriques de Carnot-Carathéodory et quasiisométries des espaces symétriques de rang un. Ann. of Math. **129** (1989), 1–60.
- [19] Reimann, H. M., *Rigidity of H-type groups*, Math. Z. **237** (2001), 697–725.
- [20] Reimann, H. M., and F. Ricci, *The complexified Heisenberg group*. In: “Proceedings on Analysis and Geometry,” (Russian) (Novosibirsk Akademgorodok, 1999), Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000, 465–480.
- [21] Reshetnyak, Y. G. *Liouville’s conformal mapping theorem under minimal regularity hypotheses*, Sib. Math. J. **8** (1967), 631–634.

- [22] Singer, I. M., and Shlomo Sternberg *The infinite groups of Lie and Cartan. I. The transitive groups*, J. Analyse Math. **15** (1965), 1–114.
- [23] Spencer, D. C, *Overdetermined systems of linear partial differential equations*, Bull. Amer. Math. Soc. **75** (1969), 179–239.
- [24] Noboru Tanaka. *On generalized graded Lie algebras and geometric structures. I.* J. Math. Soc. Japan, 19:215–254, 1967.
- [25] Tanaka. N., *On differential systems, graded Lie algebras and pseudogroups*, J. Math. Kyoto Univ. **10** (1970), 1–82.
- [26] Tyson, J. T., *RNC workshop quasiconformal maps on metric spaces: questions and conjectures*, in: “Future trends in geometric function theory,” Vol. **92** of Rep. Univ. Jyväskylä Dep. Math. Stat., Univ. Jyväskylä, 2003, 249–262.
- [27] B. Warhurst. *Contact and Pansu differentiable maps on Carnot groups*, Bull. Aust. Math. Soc. **77** (2008), 495–507.
- [28] Yamaguchi, K., *Differential systems associated with simple graded Lie algebras*, in: “Progress in differential geometry,” Vol. **22** of Adv. Stud. Pure Math., Math. Soc. Japan, Tokyo, 1993, 413–494.

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Received September 2, 2010
and in final form March 15, 2011