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The Commutator Subalgebra and Schur Multiplier of a Pair of Nilpotent Lie Algebras

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Abstract. Let (L, N) be a pair of finite dimensional nilpotent Lie algebras, in which N is an ideal in L. In the present article, we prove that if the factor Lie algebras L/N and $N/Z(L, N)$ are of dimensions m and n, respectively, then the commutator subalgebra $[L, N]$ is of dimension at most $\frac{1}{2}n(n + 2m - 1)$, and also determine when $\dim([L, N]) = \frac{1}{2}n(n + 2m - 1)$. In addition, we introduce the notion of the Schur multiplier $\mathcal{M}(L, N)$ of an arbitrary pair (L, N) of Lie algebras, and show that if N admits a complement K in L with $\dim(N) = n$ and $\dim(K) = m$, then the dimension of $\mathcal{M}(L, N)$ is bounded above by $\frac{1}{2}n(n + 2m - 1)$. In this case, we characterize the pairs (L, N) for which $\dim(\mathcal{M}(L, N))$ is either $\frac{1}{2}n(n + 2m - 1)$ or $\frac{1}{2}n(n + 2m - 1) - 1$.
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1. Introduction and preliminary

All Lie algebras are considered over a fixed field Λ *and* $[\ ,\ ]$ *denotes the Lie bracket.*

Let L, N be two Lie algebras. By an action of L on N we mean a Λ-bilinear map $L \times N \to N, (l, n) \mapsto {}^l n$ satisfying

$$^{[l,l']}n = {}^l({}^{l'}n) - {}^{l'}({}^l n) \ \text{ and } \ {}^l[n, n'] = [{}^l n, n'] + [n, {}^l n'],$$

for all $l, l' \in L$ and $n, n' \in N$. Evidently, if L is a subalgebra of some Lie algebra P and N is an ideal in P, then the Lie multiplication in P induces an action of L on N. In fact, $l \in L$ acts on $n \in N$ by ${}^l n = [l, n]$.

Given the action of L on N, we define the L-*commutator subalgebra* of N to be the subalgebra $[L, N]$ generated by elements of the form ${}^l n$ with $l \in L, n \in N$, and the L-*central* of N to be the central subalgebra $Z(L, N) = \{n \in N \mid {}^l n = 0, \text{ for all } l \in L\}$. In particular, if N is an ideal in L then $[L, N]$ and $Z(L, N)$ denote the usual commutator subalgebra and the centralizer of L in N, respectively. In this case, we define $Z_2(L, N)$ to be the pre-image in N of $Z(L/Z(L, N), N/Z(L, N))$.

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Let (L,N) be a pair of Lie algebras, where N is an ideal in L. Then we define the *Schur multiplier* of the pair (L,N) to be the abelian Lie algebra $\mathcal{M}(L,N)$ appearing in the following natural exact sequence of Lie algebras

$$H_3(L) \longrightarrow H_3(L/N) \longrightarrow \mathcal{M}(L,N) \longrightarrow \mathcal{M}(L) \longrightarrow \mathcal{M}(L/N) \\ \longrightarrow L/[L,N] \longrightarrow L/L^2 \longrightarrow L/(L^2+N) \longrightarrow 0,$$

where $\mathcal{M}(-)$ and $H_3(-)$ denote the Schur multiplier and the third homology of a Lie algebra, respectively. This is analogous to the definition of the Schur multiplier of a pair of groups given by Ellis [4] (see also [7,8]). In [5], it is proved that $\mathcal{M}(L,N) \cong \ker(N \wedge L \longrightarrow L)$, in which $N \wedge L$ denotes the non-abelian exterior product of Lie algebras. Also using the above sequence, one may easily observe that if the ideal N possesses a complement in L, then $\mathcal{M}(L) \cong \mathcal{M}(L,N) \oplus \mathcal{M}(L/N)$. In this case, for any free presentation $0 \longrightarrow R \longrightarrow F \longrightarrow L \longrightarrow 0$ of L, $\mathcal{M}(L,N)$ is isomorphic to the factor Lie algebra $(R \cap [S,F])/[R,F]$, where S is an ideal in F such that $S/R \cong N$. In particular, if $N = L$, then the Schur multiplier of (L,N) will be $\mathcal{M}(L) = (R \cap F^2)/[R,F]$ (see [3,6,10,11]).

A pair (L,N) of finite dimensional nilpotent Lie algebras is said to be *Heisenberg* provided $[L,N]$ and $Z(L,N)$ are the same subalgebras of dimension one. In the special case $N = L$, the Lie algebra L must be of odd dimension.

Let L be a Lie algebra with central factor Lie algebra of dimension n. Then Moneyhum [9] proved that $\frac{1}{2}n(n-1)$ is an upper bound for the dimension of the derived subalgebra L^2. She also showed that the dimension of the Schur multiplier of a Lie algebra of dimension n is bounded above by $\frac{1}{2}n(n-1)$. Using these results, Batten et al. [2] obtained the following two theorems.

Theorem. *Assume that L is a finite dimensional nilpotent Lie algebra such that $\dim(L/Z(L)) = n$. If $\dim(L^2) = \frac{1}{2}n(n-1)$, then $L/Z(L)$ is either abelian or $H(1)$, where $H(1)$ denotes the Heisenberg algebra of dimension 3.*

Theorem. *Let L be an n-dimensional nilpotent Lie algebra. Then*

(i) $\dim(\mathcal{M}(L)) = \frac{1}{2}n(n-1)$ *if and only if L is abelian.*

(ii) $\dim(\mathcal{M}(L)) = \frac{1}{2}n(n-1) - 1$ *if and only if $L = H(1)$.*

Now in this article, we extend the above results to a pair of finite dimensional nilpotent Lie algebras, as follows.

Theorem A. *Let (L,N) be a pair of finite dimensional nilpotent Lie algebras with L/N and $N/Z(L,N)$ of dimensions m and n, respectively. Then*

(i) $\dim([L,N]) \leq \frac{1}{2}n(n+2m-1)$.

(ii) *If $\dim([L,N]) = \frac{1}{2}n(n+2m-1)$, then either $N/Z(L,N)$ is an abelian Lie algebra or the pair $(L/Z(L,N), N/Z(L,N))$ is Heisenberg.*

Theorem B. *Let (L,N) be a pair of finite dimensional nilpotent Lie algebras and K be the complement of N in L. Assume N and K are of dimensions n and m, respectively. Then the following statements hold:*

(i) $\dim(\mathcal{M}(L,N))+\dim([L,N])\le \frac{1}{2}n(n+2m-1)$.
(ii) *If* L *is abelian, then* $\dim(\mathcal{M}(L,N))=\frac{1}{2}n(n+2m-1)$.
(iii) *If* $\dim(\mathcal{M}(L,N))=\frac{1}{2}n(n+2m-1)$, *then* N *is abelian.*
(iv) *If* $\dim(L/(L^2+N))=k$ *and* $\dim(N/(L^2\cap N))=d$, *then*

$$\frac{1}{2}d(d+2k-1)\le \dim(\mathcal{M}(L,N))+\dim([L,N]).$$

(v) *If* $\dim(\mathcal{M}(L,N))=\frac{1}{2}n(n+2m-1)-1$, *then either* N *is central or the pair* (L,N) *is Heisenberg.*

2. Proof of theorems

Let (L,N) be a pair of finite dimensional nilpotent Lie algebras with $\dim(L/N)=m$ and $\dim(N/Z(L,N))=n$. It is readily verified that for any vector $z\in Z_2(L,N)-Z(L,N)$, $[L,z]\subseteq[L,N]\cap Z(L,N)$ and the adjoint map $\mathrm{ad}z : L\longrightarrow [L,z]$ is an epimorphism such that $\ker(\mathrm{ad}z)=C_L(z)$ contains the ideal $[L,N]+Z(L,N)$. We consider two non-negative integers $a(z)$ and $b(z)$ such that

$$a(z)=\dim([L,z]) \quad \text{and} \quad b(z)=\dim(\frac{L/[L,z]}{Z(L/[L,z],N/[L,z])}).$$

Since $Z(L,N)\subset\langle z,Z(L,N)\rangle\subseteq C_L(z)$, $a(z)=\dim(L/C_L(z))<\dim(L/Z(L,N))$ $=m+n$. Also, $z+[L,z]\in Z(L/[L,z],N/[L,z])-(Z(L,N)/[L,z])$ yields that

$$b(z)<\dim(\frac{L/[L,z]}{Z(L,N)/[L,z]})=m+n.$$

The following lemmas shorten the proof of Theorem A.

Lemma 2.1. *Using the above assumptions and notations, we have*
(i) $\dim([L,N])\le\frac{1}{2}n(n+2m-1)$.
(ii) *Suppose for some non-negative integer* s *we have*

$$\dim([L,N])=\frac{1}{2}n(n+2m-1)-s;$$

then $\dim([L/Z(L,N),N/Z(L,N)])\le s+1$.

Proof. (i) Let $\{\overline{x}_1,\dots,\overline{x}_n\}$ be a basis of $N/Z(L,N)$. Extend this set to a basis of $L/Z(L,N)$, say $\{\overline{x}_1,\dots,\overline{x}_n,\overline{x}_{n+1},\dots,\overline{x}_{n+m}\}$. Then $[L,N]$ is spanned by $\{[x_i,x_j]\mid 1\le i\le n \text{ and } i<j\le m+n\}$.

(ii) Choose a vector $z\in Z_2(L,N)-Z(L,N)$. Then

$$\dim(\frac{N/[L,z]}{Z(L/[L,z],N/[L,z])})=b(z)-m\le n-1,$$

and hence part (i) indicates that $\dim([L/[L,z],N/[L,z]])\le\frac{1}{2}(n-1)(n+2m-2)$. Therefore $\frac{1}{2}n(n+2m-1)-s=\dim([L,N])\le\frac{1}{2}(n-1)(n+2m-2)+a(z)$, and then $a(z)\ge n+m-s-1$. Since $[L,N]+Z(L,N)$ is an ideal in L contained in $C_L(z)$, it follows that

$$\dim(\frac{L}{[L,N]+Z(L,N)})\ge\dim(\frac{L}{C_L(z)})=a(z)\ge n+m-s-1.$$

Alternatively, $(L/Z(L,N))/(([L,N]+Z(L,N))/Z(L,N)) \cong L/([L,N]+Z(L,N))$. Hence, the last inequality yields

$$\dim([\frac{L}{Z(L,N)}, \frac{N}{Z(L,N)}]) = \dim(\frac{L}{Z(L,N)}) - \dim(\frac{L}{[L,N]+Z(L,N)}) \leq s+1.$$

This completes the proof. ■

Lemma 2.2. *Using the above assumptions and notations, if*

$$\dim([\frac{L}{Z(L,N)}, \frac{N}{Z(L,N)}]) = s+1-k$$

for some $0 \leq k \leq s+1$, *then for all* $z \in Z_2(L,N) - Z(L,N)$, $a(z) \leq m+n-1-s+k$ *.In particular, if* $k=0$ *then* $a(z) = m+n-s-1$ *and* $C_L(z) = [L,N]+Z(L,N)$.

Proof. ¿From the proof of Lemma 2.1 and the hypothesis, we have

$$n+m-s-1 \leq a(z) = \dim(\frac{L}{C_L(z)}) \leq \dim(\frac{L}{[L,N]+Z(L,N)})$$

$$= \dim(\frac{L/Z(L,N)}{[L/Z(L,N), N/Z(L,N)]}) = n+m-1-s+k,$$

for all $z \in Z_2(L,N) - Z(L,N)$. Now, assuming $k=0$, the above inequalities imply that $a(z) = m+n-s-1$, and so $C_L(z) = [L,N]+Z(L,N)$, as required. ■

Now, we are ready to prove Theorem A.

Proof. [Theorem A] (i) It has proved in Lemma 2.1(i).

(ii) By applying Lemma 2.1(ii) in the case $s=0$, we have

$$\dim([L/Z(L,N), N/Z(L,N)]) \leq 1.$$

If $\dim([L/Z(L,N), N/Z(L,N)]) = 0$, then $N/Z(L,N)$ is central in $L/Z(L,N)$. So, suppose that $\dim([L/Z(L,N), N/Z(L,N)]) = 1$. Since $L/Z(L,N)$ is nilpotent, by [12, Proposition 7], $[L/Z(L,N), N/Z(L,N)] \cap Z(L/Z(L,N)) \neq 0$ and hence

$$[L/Z(L,N), N/Z(L,N)] \subseteq Z_2(L,N)/Z(L,N).$$

We claim that $dim(Z_2(L,N)/Z(L,N)) = 1$. Assume, to the contrary, that there exist vectors $x, y \in Z_2(L,N) - Z(L,N)$ such that $x + Z(L,N)$ and $y + Z(L,N)$ are linearly independent in $Z_2(L,N)/Z(L,N)$. By Lemma 2.2, $C_L(x) = [L,N] + Z(L,N) = C_L(y)$ and thus $y \in C_L(x)$. But $\dim(C_L(x)/Z(L,N)) = \dim(L/Z(L,N)) - \dim(L/C_L(x)) = n+m-(n+m-1) = 1$. This is a contradiction to the linear independence of $x + Z(L,N)$ and $y + Z(L,N)$. Therefore, $\dim(Z_2(L,N)/Z(L,N)) = 1$ and the pair $(L/Z(L,N), N/Z(L,N))$ is Heisenberg. ■

Now we obtain the following corollary which is of interest in its own account.

Corollary 2.3. *Let (L,N) be a pair of finite dimensional nilpotent Lie algebras with* $\dim(L/N)=m$, $\dim(N/Z(L,N))=n$, *and* $\dim([L,N])=\frac{1}{2}n(n+2m-1)-s$ *for some $s\geq 0$. If there is a $z\in Z_2(L,N)-Z(L,N)$ such that $a(z)=m+n-1-s$, then $b(z)=m+n-1$ and the factor Lie algebra $\dfrac{N/[L,z]}{Z(L/[L,z],N/[L,z])}$ is abelian or the pair $\left(\dfrac{L/[L,z]}{Z(L/[L,z],N/[L,z])},\dfrac{N/[L,z]}{Z(L/[L,z],N/[L,z])}\right)$ is Heisenberg.*

Proof. By Lemma 2.1(i),

$$\dim([L/[L,z],N/[L,z]])\leq \tfrac{1}{2}(b(z)(b(z)-1)-m(m-1)).$$

Consequently,

$$\begin{aligned}\frac{1}{2}n(n+2m-1)-s=\dim([L,N])&=\dim([\frac{L}{[L,z]},\frac{N}{[L,z]}])+\dim([L,z])\\&\leq\frac{1}{2}(b(z)(b(z)-1)-m(m-1))+m+n-1-s,\end{aligned}$$

whence $b(z)=m+n-1$. Hence $\dim(\dfrac{N/[L,z]}{Z(L/[L,z],N/[L,z])})=n-1$ and $\dim([\dfrac{L}{[L,z]},\dfrac{N}{[L,z]}])=\frac{1}{2}(n-1)(n+2m-2)$. Therefore, Theorem A(ii) gives the result. ∎

To prove Theorem B, we need the following definition and propositions.

Definition 2.4. A Lie homomorphism $\sigma:N^*\longrightarrow L$ together with an action of L on N^* is called a *cover* (or *covering pair*) of the pair (L,N) of Lie algebras if the following conditions hold:

(i) $\sigma(N^*)=N$;
(ii) $\sigma({}^{l}n)={}^{l}\sigma(n)$, for all $l\in L$, $n\in N^*$;
(iii) ${}^{\sigma(n_1)}n={}^{n_1}n$, for all $n,n_1\in N^*$;
(iv) $\ker\sigma\subseteq Z(L,N^*)\cap[L,N^*]$;
(v) $\ker\sigma\cong\mathcal{M}(L,N)$.

One may readily observe that a cover $\sigma:N^*\longrightarrow L$ of the pair (L,L) together with action ${}^{l}n={}^{\sigma(n_1)}n$, where $l=\sigma(n_1)$ for some $n_1\in N^*$, gives the usual notion of a covering Lie algebra N^* of L. The following result gives the existence of cover of a given pair (L,N), in which N has a complement in L. In special case, when L is finite dimensional and $N=L$ then the result of Batten and Stitzinger [1] is obtained.

Proposition 2.5. *Let (L,N) be a pair of Lie algebras such that N has a complement in L. Then (L,N) admits at least one cover.*

Proof. Let $0\longrightarrow R\longrightarrow F\longrightarrow L\longrightarrow 0$ be a free presentation of L and S an ideal in F such that $N\cong S/R$. Let $T/[R,F]$ be a complement of $\mathcal{M}(L,N)$ in $R/[R,F]$, for some suitable ideal T in F. Consider the mapping $\sigma:S/T\longrightarrow F/R$

given by $\sigma(s+T)=s+R$ together with the action ${}^{f+R}(s+T)=[f,s]+T$, for all $f\in F$ and $s\in S$. Then for each $s_1,s_2\in S$, $f\in F$, $r\in R$, we have

$$\sigma({}^{f+R}(s_1+T))=\sigma([f,s_1]+T)=[f,s_1]+R={}^{f+R}\sigma(s_1+T),$$
$${}^{\sigma(s_1+T)}(s_2+T)={}^{s_1+R}(s_2+T)=[s_1,s_2]+T={}^{s_1+T}(s_2+T),$$
$${}^{f+R}(r+T)=[f,r]+T=T.$$

Also $\sigma(S/T)=S/R$ and $\ker\sigma\cong\mathcal{M}(L,N)$. Moreover,

$$\frac{R}{T}\subseteq\frac{[F,S]+T}{T}=\langle\,{}^{f}s{+}T\mid f\in F,s\in S\rangle=\langle\,{}^{f+R}(s{+}T)\mid f\in F,s\in S\rangle=[\frac{F}{R},\frac{S}{T}].$$

Therefore $\sigma:S/T\longrightarrow F/R$ is a cover of (L,N). ■

Proposition 2.6. *Let (L,N) be a pair of finite dimensional Lie algebras such that N has a complement in L. Then*

$$\dim(\mathcal{M}(\frac{L}{L^2},\frac{N+L^2}{L^2}))\leq\dim(\mathcal{M}(L,N))+\dim([L,N]).$$

Proof. Let $0\to R\to F\to L\to 0$ be a free presentation of L and S an ideal in F such that $N\cong S/R$. Then

$$\mathcal{M}(\frac{L}{L^2},\frac{N+L^2}{L^2})\cong\frac{(F^2+R)\cap[F^2+S,F]}{[F^2+R,F]}=\frac{[F^2+S,F]}{[F^2+R,F]}=\frac{[S,F]+[F^2,F]}{[R,F]+[F^2,F]}.$$

So,

$$\begin{aligned}\dim(\mathcal{M}(L,N))+\dim([L,N])&=\dim(\frac{R\cap[S,F]}{[R,F]})+\dim(\frac{[S,F]}{R\cap[S,F]})\\&=\dim(\frac{[S,F]}{[R,F]})\geq\dim(\mathcal{M}(\frac{L}{L^2},\frac{N+L^2}{L^2})),\end{aligned}$$

as required. ■

Now we are able to prove Theorem B.

Proof. [Theorem B] Let homomorphism $\sigma:N^*\longrightarrow L$ together with an action of L on N^* be a cover of (L,N). We define a homomorphism $\psi:K\longrightarrow Der(N^*)$ given by $\psi(k)=\psi_k$, where $\psi_k:N^*\longrightarrow N^*$ is a derivation given by $\psi_k(x)={}^{k}x$, in which ${}^{k}x$ is induced by the action of L on N^*. Set H to be the semidirect sum of N^* by K. Then it is easily seen that the subalgebras $[L,N^*]$ and $Z(L,N^*)$ are identical with the commutator subalgebra $[H,N^*]$ and the centralizer of N^* in H, $Z(H,N^*)$, respectively. If $\delta:H\longrightarrow L$ is the mapping defined by $\delta(x+k)=\sigma(x)+k$, for all $x\in N^*$ and $k\in K$, then it can be shown that δ is an epimorphism with $\ker\delta=\ker\sigma$. Also, the factor Lie algebras $H/Z(L,N^*)$ and $N^*/Z(L,N^*)$ are isomorphic to L and N, respectively.

(i) Since $\dim(H/N^*)=m$ and $\dim(N^*/Z(L,N^*))\leq\dim(N^*/\ker\sigma)=n$, Lemma 2.1(i) shows that $\dim([H,N^*])\leq\frac{1}{2}n(n+2m-1)$. Now, the isomorphisms $[L,N]\cong[H,N^*]/\ker\sigma$ and $\ker\sigma\cong\mathcal{M}(L,N)$ follows the result.

(ii) Since the exact sequence $0 \longrightarrow N \longrightarrow L \longrightarrow K \longrightarrow 0$ splits, $\mathcal{M}(L) \cong \mathcal{M}(L,N) \oplus \mathcal{M}(K)$. By [9; Lemma 2.3], $\dim(\mathcal{M}(L)) = \frac{1}{2}(m+n)(m+n-1)$ and $\dim(\mathcal{M}(K)) = \frac{1}{2}m(m-1)$. So, $\dim(\mathcal{M}(L,N)) = \frac{1}{2}n(n+2m-1)$, as required.

(iii) If $\ker\sigma$ is a proper subalgebra of $Z(L,N^*)$, then the above discussion indicates that $\dim(H/N^*) = m$ and $\dim(N^*/Z(H,N^*)) = \dim((N^*/Z(L,N^*))) \leq n-1$. So, using Lemma 2.1(i) and part (i), $\dim(\mathcal{M}(L,N)) = \frac{1}{2}n(n+2m-1) \leq \dim([H,N^*]) \leq \frac{1}{2}(n-1)(n+2m-2)$. Hence we must have $m+n \leq 1$ and L is abelian.

Now, suppose that $\ker\sigma$ is equal to $Z(L,N^*)$, then $\dim([H,N^*]) = \frac{1}{2}n(n+2m-1)$. So, by Theorem A(ii), either $N^*/Z(H,N^*)$ is abelian or the pair $(N^*/Z(H,N^*), H/Z(H,N^*))$ is Heisenberg. If (L,N) is Heisenberg then

$$\dim([L,N]) + \dim(\mathcal{M}(L,N)) = \frac{1}{2}n(n+2m-1)+1,$$

which contradicts to part (i). Therefore in both cases N is abelian.

(iv) Proposition 2.6 and part (iii) yield the result.

(v) Suppose that $\ker\sigma$ is a proper subalgebra of $Z(L,N^*)$. As part (iii), we conclude that $\frac{1}{2}n(n+2m-1) \leq \dim([H,N^*]) \leq \frac{1}{2}(n-1)(n+2m-2)$, and so $m+n \leq 2$. Hence the possible values for the pair (m,n) must be one of the following cases:

$$(m,n) = (1,0),(0,1),(1,1),(2,0),(0,2).$$

It is readily seen that the values $(1,0),(0,1),(0,2)$ are impossible. Moreover, if $(m,n) = (2,0)$ or $(1,1)$ then $\dim(L) = 2$ and $\dim(\mathcal{M}(L)) = 0$, which is a contradiction to [2; Theorem 3]. Therefore, we must have $\ker\sigma = Z(L,N^*)$. By the assumption and Lemma 2.1(i), $\frac{1}{2}n(n+2m-1)-1 \leq \dim([L,N^*]) = \dim([H,N^*]) \leq \frac{1}{2}n(n+2m-1)$. Now, if $\dim([H,N^*]) = \frac{1}{2}n(n+2m-1)-1$ then either N is abelian or the pair (L,N) is Heisenberg. If $\dim([H,N^*]) = \frac{1}{2}n(n+2m-1)$ then $[H,N^*] = \mathcal{M}(L,N) = Z(H,N^*)$ and so, again N is abelian. This proves the theorem. ∎

The following examples show that both outcomes obtained in the assertion (v) of Theorem B occur.

Examples: (i) Let $L = H(m) \oplus A$, where $H(m)$ denotes the Heisenberg algebra of dimension $2m+1$ and A is a 1-dimensional Lie algebra. Then it is easy to verify that A is a central ideal in L, $\dim(\mathcal{M}(L)) = 2m^2+m-1$ (by [1; Example 3]) and $\dim(\mathcal{M}(L,A)) = 2m$.

(ii) Let $\{f,g,z\}$ be a basis for $H(1)$ with $[f,g] = z$. Then $H(1)$ is a semidirect sum of $\langle f,z\rangle$ by $\langle g\rangle$, the pair $(H(1),\langle f,z\rangle)$ is Heisenberg and $\dim(\mathcal{M}(L,A)) = 2$.

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