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The Image of the Lepowsky Homomorphism for SO$(n,1)$ and SU$(n,1)$

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Abstract. Let G_o be a classical rank one semisimple Lie group and let K_o denote a maximal compact subgroup of G_o. Let $U(\mathfrak{g})$ be the complex universal enveloping algebra of G_o and let $U(\mathfrak{g})^K$ denote the centralizer of K_o in $U(\mathfrak{g})$. Also let $P: U(\mathfrak{g}) \longrightarrow U(\mathfrak{k}) \otimes U(\mathfrak{a})$ be the projection map corresponding to the direct sum $U(\mathfrak{g}) = \big(U(\mathfrak{k}) \otimes U(\mathfrak{a})\big) \oplus U(\mathfrak{g})\mathfrak{n}$ associated to an Iwasawa decomposition of G_o adapted to K_o. In this paper we give a characterization of the image of $U(\mathfrak{g})^K$ under the injective antihomorphism $P: U(\mathfrak{g})^K \longrightarrow U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ when G_o is locally isomorphic to SO$(n,1)$ and SU$(n,1)$.
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1. Introduction

Let G_o be a connected, noncompact, real semisimple Lie group with finite center, and let K_o denote a maximal compact subgroup of G_o. We denote with $\mathfrak{g}_o$ and $\mathfrak{k}_o$ the Lie algebras of G_o and K_o, and $\mathfrak{k} \subset \mathfrak{g}$ will denote the respective complexified Lie algebras. Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and let $U(\mathfrak{g})^K$ denote the centralizer of K_o in $U(\mathfrak{g})$.

Let $P: U(\mathfrak{g}) \longrightarrow U(\mathfrak{k}) \otimes U(\mathfrak{a})$ be the projection map corresponding to the direct sum $U(\mathfrak{g}) = \big(U(\mathfrak{k}) \otimes U(\mathfrak{a})\big) \oplus U(\mathfrak{g})\mathfrak{n}$, associated to an Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ adapted to $\mathfrak{k}$. Let $G_o = K_oA_oN_o$ be the corresponding Iwasawa decomposition for G_o.

If $U(\mathfrak{k})^M$ denotes the centralizer of M_o in $U(\mathfrak{k})$, M_o being the centralizer of A_o in K_o, then it is known (see [11]) that one has the exact sequence

$$0 \longrightarrow U(\mathfrak{g})^K \xrightarrow{P} U(\mathfrak{k})^M \otimes U(\mathfrak{a}),$$

and that P becomes an antihomomorphism of algebras if $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ is given the tensor product algebra structure. However, the image of P is not yet well understood, we refer the reader to [10], [12] and [4] for further information.

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In order to determine the actual image $P(U(\mathfrak{g})^K)$, Tirao introduced in [12] a subalgebra B of $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ defined by a set of linear equations derived from certain embeddings between Verma modules, and proved that $P(U(\mathfrak{g})^K) = B^{W_\rho}$ when G_0 is locally isomorphic to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$. Here, W is the Weyl group of the pair $(\mathfrak{g},\mathfrak{a})$, ρ is half the sum of the positive roots of $\mathfrak{g}$, and B^{W_ρ} is the subalgebra of all elements in B that are invariant under the tensor product action of W on $U(\mathfrak{k})^M$ and the translated action of W on $U(\mathfrak{a})$. Recently, in [4], we extended this result to the symplectic group $\mathrm{Sp}(n,1)$. In fact we obtained the following stronger result.

Theorem 1.1. *If G_o is locally isomorphic to $Sp(n,1)$ then $P(U(\mathfrak{g})^K) = B$.*

We announced in [4] that the above result also holds for $\mathrm{SO}(2n,1)$, $\mathrm{SU}(n,1)$ and F_4. We did not know at that time whether $B = B^{W_\rho}$ for $\mathrm{SO}(2n+1,1)$ and we had not yet completed the proof for F_4. Now we know that the following theorem holds,

Theorem 1.2. *If G_o be locally isomorphic to $SO(n,1)$ or $SU(n,1)$ it follows that $P(U(\mathfrak{g})^K) = B$, and moreover $B = B^{W_\rho}$.*

This paper is devoted to proving this theorem. As we mentioned above, it was proved in [12] that $P(U(\mathfrak{g})^K) = B^{W_\rho}$ when G_0 is locally isomorphic to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$. Thus the main contribution of this paper is that $B = B^{W_\rho}$ for $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$. Additionally, we give a new and simpler proof of the fact that $P(U(\mathfrak{g})^K) = B$. We are still working to complete the details for F_4.

The projection P was originally introduced by Kostant long time ago in order to contribute to the understanding of the structure and representation theory of $U(\mathfrak{g})^K$. The need for the study of the algebra $U(\mathfrak{g})^K$ arises from the fundamental work of Harish-Chandra relating the infinite-dimensional representation theory of G_o to the finite-dimensional representation theory of $U(\mathfrak{g})^K$. Since then, there were a number of results on the structure of $U(\mathfrak{g})^K$, see notably [7]. However, the study of $U(\mathfrak{g})^K$ is acknowledged to be very difficult and the infinite-dimensional representation theory of G_o has been approached by different means.

On the other hand, the algebra B turns out to be an isomorphic copy of $U(\mathfrak{g})^K$ strictly[1] contained in $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ that is defined by a set of linear equations. The fact that we were able to prove that $B = B^{W_\rho}$ keeps alive the hope that it could help to understand the structure of $U(\mathfrak{g})^K$.

2. The algebra B and the image of $U(\mathfrak{g})^K$

Assume that G_o is a connected, noncompact real semisimple Lie group, with finite center and split rank one. Let $G_o = K_oA_oN_o$ be the an Iwasawa decomposition of G_o, let $\mathfrak{k}_o$, $\mathfrak{a}_o$ and $\mathfrak{n}_o$ be the corresponding Lie algebras and let $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ be their complexifications.

Let $\mathfrak{t}_o$ be a Cartan subalgebra of the Lie algebra $\mathfrak{m}_o$ of M_o. Set $\mathfrak{h}_o = \mathfrak{t}_o \oplus \mathfrak{a}_o$ and let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be the corresponding complexification. Then $\mathfrak{h}_o$ and $\mathfrak{h}$ are Cartan subalgebras of $\mathfrak{g}_o$ and $\mathfrak{g}$, respectively. Choose a Borel subalgebra $\mathfrak{t} \oplus \mathfrak{m}^+$ of the complexification $\mathfrak{m}$ of $\mathfrak{m}_o$ and take $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{m}^+ \oplus \mathfrak{n}$ as a Borel subalgebra of $\mathfrak{g}$.

[1]See for example Theorem 2.2

Let Δ and Δ^+ be, respectively, the corresponding sets of roots and positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. As usual ρ is half the sum of the positive roots, θ the Cartan involution and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$ corresponding to (G_o, K_o). Also, if $\alpha \in \Delta$, X_α will be a nonzero root vector associated to α and $E_\alpha = X_{-\alpha} + \theta X_{-\alpha}$.

If $\langle\, ,\, \rangle$ denotes the Killing form of $\mathfrak{g}$, for each $\alpha \in \Delta$ let $H_\alpha \in \mathfrak{h}$ be the unique element such that $\phi(H_\alpha) = 2\langle \phi, \alpha\rangle / \langle \alpha, \alpha\rangle$ for all $\phi \in \mathfrak{h}^*$, and let $\mathfrak{h}_\mathbb{R}$ be the real span of $\{H_\alpha : \alpha \in \Delta\}$. Also set $H_\alpha = Y_\alpha + Z_\alpha$ where $Y_\alpha \in \mathfrak{t}$ and $Z_\alpha \in \mathfrak{a}$, and let $P_+ = \{\alpha \in \Delta^+ : Z_\alpha \neq 0\}$. For each $\alpha \in P_+$ we can consider the elements in $U(\mathfrak{k}) \otimes U(\mathfrak{a})$ as polynomials in Z_α with coefficients in $U(\mathfrak{k})$. Then, let B be the algebra of all $b \in U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ that satisfy

$$E_\alpha^n b(n - Y_\alpha - 1) \equiv b(-n - Y_\alpha - 1) E_\alpha^n \mod \big(U(\mathfrak{k})\mathfrak{m}^+\big) \tag{1}$$

for all simple roots $\alpha \in P_+$ and all $n \in \mathbb{N}$. We know, from Theorem 5 and Corollary 6 of [12], that $P(U(\mathfrak{g})^K) \subset B$ for all rank one gropus, and moreover, that $P(U(\mathfrak{g})^K) = B^{W_\rho}$ for $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$.

Since in this paper we shall be concerned with G_o locally isomorphic to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$, we recall that in this case there is only one simple root in P_+ if G_o is locally isomorphic to $\mathrm{SO}(n,1)$ for $n > 3$, and there are two simple roots in P_+ if G_o is locally isomorphic to $\mathrm{SO}(3,1)$ or $\mathrm{SU}(n,1)$ for $n \geq 2$.

Let G be the adjoint group of $\mathfrak{g}$ and let K be the connected Lie subgroup of G with Lie algebra $ad_\mathfrak{g}(\mathfrak{k})$. Also let $M = \mathrm{Centr}_K(\mathfrak{a})$, $M' = \mathrm{Norm}_K(\mathfrak{a})$ and $W = M'/M$. Let Γ denote the set of all equivalence classes of irreducible holomorphic finite dimensional K-modules V_γ such that $V_\gamma^M \neq 0$. Any $\gamma \in \Gamma$ can be realized as a submodule of all harmonic polynomial functions on $\mathfrak{p}$, homogeneous of degree d, for a uniquely determined $d = d(\gamma)$ (see [9]). If V is any K-module and $\gamma \in \hat{K}$ then V_γ will denote the isotypic component of V corresponding to γ. Let $U(\mathfrak{k})_d^M = \bigoplus U(\mathfrak{k})_\gamma^M$, where the sum extends over all $\gamma \in \Gamma$ such that $d(\gamma) \leq d$. Then $U(\mathfrak{k})^M = \bigcup_{d \geq 0} U(\mathfrak{k})_d^M$ is an ascending filtration of $U(\mathfrak{k})^M$. If $b \in U(\mathfrak{k})^M$ define $d(b) = \min\{d \in \mathbb{N}_o : b \in U(\mathfrak{k})_d^M\}$ and call it the *Kostant degree* of b. Since we shall be mainly concerned with representations $\gamma \in \Gamma$ that occur as subrepresentations of $U(\mathfrak{k})$ we set,

$$\Gamma_1 = \{\gamma \in \Gamma : \gamma \text{ is a subrepresentation of } U(\mathfrak{k})\}. \tag{2}$$

If $0 \neq b \in U(\mathfrak{k}) \otimes U(\mathfrak{a})$ we write $b = b_m \otimes Z_\alpha^m + \cdots + b_0$ in a unique way with $b_j \in U(\mathfrak{k})$ for $0 \leq j \leq m$, and $b_m \neq 0$, for any simple root $\alpha \in P_+$. We shall refer to m as the *degree* of b and to $\widetilde{b} = b_m \otimes Z_\alpha^m$ as the *leading term* of b. Let $\big(U(\mathfrak{k})^M \otimes U(\mathfrak{a})\big)^W$ denote the ring of W-invariants in $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ under the tensor product of the action of W on $U(\mathfrak{k})^M$ and the action of W on $U(\mathfrak{a})$. The following result was proved in Proposition 2.6 of [4] for any connected, noncompact, real semisimple Lie group G_o, with finite center and split rank one.

Proposition 2.1. *If $\widetilde{b} = b_m \otimes Z_\alpha^m \in \big(U(\mathfrak{k})^M \otimes U(\mathfrak{a})\big)^W$ and $d(b_m) \leq m$, then there exits $u \in U(\mathfrak{g})^K$ such that $\widetilde{b}$ is the leading term of $b = P(u)$.*

From this result it follows that Theorem 1.2 is a consequence of the following theorem. We shall prove this statment in Proposition 2.3 below.

Theorem 2.2. *If $b = b_m \otimes Z^m + \cdots + b_0 \in B$ and $b_m \neq 0$, then $d(b_m) \leq m$ and its leading term $\widetilde{b} = b_m \otimes Z_\alpha^m \in \big(U(\mathfrak{k})^M \otimes U(\mathfrak{a})\big)^W$.*

Proposition 2.3. *Theorem 2.2 implies Theorem 1.2.*

Proof. We mentioned above that $P(U(\mathfrak{g})^K) = B^{W_\rho} \subset B$ for $\mathrm{SO}(n,1)$ and $\mathrm{SU}(n,1)$. Let us prove by induction on the degree m of $b \in B$, that $B \subset P(U(\mathfrak{g})^K)$. If $m = 0$ we have $b = b_0 \in U(\mathfrak{k})^M$ and Theorem 2.2 implies that $d(b_0) = 0$. If $\gamma \in \Gamma_1$ and $d(\gamma) = 0$ then γ can be realized by constant polynomial functions on $\mathfrak{p}$ and these fuctions are K-invariant. Thus $b_0 \in U(\mathfrak{k})^K$ and therefore $b = b_0 = P(b_0) \in P(U(\mathfrak{g})^K)$.

If $b \in B$ and $m > 0$, from Theorem 2.2 and Proposition 2.1 we know that there exists $v \in U(\mathfrak{g})^K$ such that $\widetilde{P(v)} = \widetilde{b}$. Then $b - P(v)$ lies in B and the degree of $b - P(v)$ is strictly less than m. Hence, by the induction hypothesis, there exists $u \in U(\mathfrak{g})^K$ such that $P(u) = b - P(v)$ and $b = P(u+v) \in P(U(\mathfrak{g})^K)$. This completes the induction argument. Therefore we obtain that $B \subset P(U(\mathfrak{g})^K) = B^{W_\rho} \subset B$. ■

The rest of the paper will be devoted to proving Theorem 2.2 when G_0 is locally isomorphic to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$.

3. The equations defining B

To simplify the notation, for a given simple root $\alpha \in P_+$ set $E = E_\alpha$, $Y = Y_\alpha$ and $Z = Z_\alpha$. It follows from Lemma 29 of [12] that $[E, Y] = cE$, where $c = 1$ if G_o is locally isomorphic to $\mathrm{SO}(n,1)$, and $c = \frac{3}{2}$ if G_o is locally isomorphic to $\mathrm{SU}(n,1)$.

We identify $U(\mathfrak{k}) \otimes U(\mathfrak{a})$ with the polynomial ring in one variable $U(\mathfrak{k})[x]$, replacing Z by the indeterminate x. To study the equation (1) we shall change the unknown $b(x) \in U(\mathfrak{k})[x]$ by $c(x) \in U(\mathfrak{k})[x]$ defined by

$$c(x) = b(x + H - 1), \tag{3}$$

where $H = 0$ if $c = 1$, and when $c = \frac{3}{2}$, H is an appropriate vector in $\mathfrak{t}$ to be chosen later, depending on the simple root $\alpha \in P_+$ and such that $[H, E] = \frac{1}{2}E$ (see (10)). If $\widetilde{Y} = Y + H$, we have $[E, \widetilde{Y}] = E$. This is the main reason for introducing H, because it allow us to treat (1) in a unified way in both cases, $c = 1, \frac{3}{2}$.

Then $b(x) \in U(\mathfrak{k})[x]$ satisfies (1) if and only if $c(x) \in U(\mathfrak{k})[x]$ satisfies

$$E^n c(n - \widetilde{Y}) \equiv c(-n - \widetilde{Y}) E^n \tag{4}$$

for all $n \in \mathbb{N}$. Observe that (4) is an equation in the noncommutative ring $U(\mathfrak{k})$.

Now, if p is a polynomial in one indeterminate x with coefficients in a ring let $p^{(n)}$ denote the n-th discrete derivative of p. That is, $p^{(1)}(x) = p(x+\frac{1}{2}) - p(x-\frac{1}{2})$ and in general $p^{(n)}(x) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} p(x + \frac{n}{2} - j)$. If $p = p_m x^m + \cdots + p_0$, then

$$p^{(n)}(x) = \begin{cases} 0, & \text{if } n > m \\ m! p_m, & \text{if } n = m. \end{cases} \tag{5}$$

Also, if $X \in \mathfrak{k}$ we shall denote with $\dot{X}$ the derivation of $U(\mathfrak{k})$ induced by $\operatorname{ad}(X)$. Moreover if D is a derivation of $U(\mathfrak{k})$ we shall denote with the same symbol the unique derivation of $U(\mathfrak{k})[x]$ which extends D and such that $Dx = 0$. Thus for $b \in U(\mathfrak{k})[x]$ and $b = b_m x^m + \cdots + b_0$, we have $Db = (Db_m)x^m + \cdots + (Db_0)$. Observe that these derivations commute with the operation of taking the discrete derivative in $U(\mathfrak{k})[x]$.

Next theorem gives a triangularized version of the system (1) that defines the algebra B. The meaning of this will be clarified after the statement of the theorem. Its proof is contained in [2] where the system (4) is studied in a more abstract setting, in particular, an LU-decomposition of its coefficient matrix is obtained.

Theorem 3.1. *Let $c \in U(\mathfrak{k})[x]$. Then the following systems of equations are equivalent:*

(i) $E^n c(n - \widetilde{Y}) \equiv c(-n - \widetilde{Y})E^n$, $(n \in \mathbb{N}_0)$;

(ii) $\dot{E}^{n+1}\big(c^{(n)}\big)(\frac{n}{2} + 1 - \widetilde{Y}) + \dot{E}^n\big(c^{(n+1)}\big)(\frac{n}{2} - \frac{1}{2} - \widetilde{Y})E \equiv 0$, $(n \in \mathbb{N}_0)$.

Moreover, if $c \in U(\mathfrak{k})[x]$ is a solution of one of the above systems, then for all $\ell, n \in \mathbb{N}_0$ we have

(iii) $(-1)^n \dot{E}^\ell\big(c^{(n)}\big)(-\frac{n}{2} + \ell - \widetilde{Y})E^n - (-1)^\ell \dot{E}^n\big(c^{(\ell)}\big)(-\frac{\ell}{2} + n - \widetilde{Y})E^\ell \equiv 0$.

Observe that if $c \in U(\mathfrak{k})[x]$ is of degree m and $c = c_m x^m + \cdots + c_0$, then all the equations of the system (ii) corresponding to $n > m$ are trivial because $c^{(n)} = 0$. Moreover, the equation corresponding to $n = m$ reduces to $\dot{E}^{m+1}(c_m) \equiv 0$ and the equation associated to $n = j$, for $j < m$, only involves the coefficients $c_m, \ldots, c_j$. In other words the system (ii) is a triangular system of $m+1$ linear equations in the $m+1$ unknowns $c_m, \ldots, c_0$.

Since we are going to use equations (iii) of Theorem 3.1, it is convenient to consider a basis of $\mathbb{C}[x]$ that behaves well under the discrete derivative. Then let $\{\varphi_n\}_{n\geq 0}$ be the basis of $\mathbb{C}[x]$ defined by,

$$\varphi_0 = 1, \tag{i}$$

$$\varphi_n^{(1)} = \varphi_{n-1} \qquad \text{if } n \geq 1, \tag{ii}$$

$$\varphi_n(0) = 0 \qquad \text{if } n \geq 1. \tag{iii}$$

The existence and uniqueness of the family $\{\varphi_n\}_{n\geq 0}$ follows inductively from conditions (i), (ii) and (iii) above. Moreover it is easy to see that,

$$\varphi_n(x) = \tfrac{1}{n!}x(x + \tfrac{n}{2} - 1)(x + \tfrac{n}{2} - 2)\cdots(x - \tfrac{n}{2} + 1), \quad n \geq 1.$$

To simplify the notation from now on we shall write $u \equiv v$ instead of $u \equiv v$ mod $\big(U(\mathfrak{k})\mathfrak{m}^+\big)$, for any $u, v \in U(\mathfrak{k})$.

Lemma 3.2. *Let $u \in U(\mathfrak{k})$ and $X \in \mathfrak{k} - \mathfrak{m}^+$ be such that $\dot{X}(\mathfrak{m}^+) \subset \mathfrak{m}^+$. Then, if $n \in \mathbb{N}$ and $uX^n \equiv 0$ we have $u \equiv 0$.*

Proof. Choose a basis $\{Z_1, \ldots, Z_q\}$ of $\mathfrak{m}^+$ and complete it to a basis of $\mathfrak{k}$ by adding vectors $X_1, \ldots, X_p$ with $X_p = X$. Then by Poincaré-Birkhoff-Witt theorem the ordered monomials $X^I = X_1^{i_1} \cdots X_p^{i_p}$, $I = \{i_1, \ldots, i_p\}$, and $Z^J = Z_1^{j_1} \cdots Z_q^{j_q}$, $J = \{j_1, \ldots, j_q\}$, form a basis $\{X^I Z^J\}$ of $U(\mathfrak{k})$.

It is enough to prove the lemma for $n=1$. If $u=\sum a_{I,J}X^IZ^J$ we have

$$uX=\sum a_{I,J}X^IXZ^J-\sum a_{I,J}X^I\dot{X}(Z^J).$$

Then, since $\dot{X}(Z^J)\equiv 0$ it follows that $uX\equiv\sum a_{I,J}X^IXZ^J$. Therefore $uX\equiv 0$ implies that $a_{I,J}=0$ if $J=0$. Hence the lemma follows. ∎

The following result was proved in Theorem 3.11 of [3].

Theorem 3.3. *Let G_o be locally isomorphic to $SO(n,1)$ for $n\geq 3$, or to $SU(n,1)$, $n\geq 2$. Then, $\sum_{j\geq 0}\dot{E}^j\big(U(\mathfrak{k})^M\big)$ is a direct sum and we have*

$$\Big(\sum_{j\geq 0}\dot{E}^j\big(U(\mathfrak{k})^M\big)\Big)\cap U(\mathfrak{k})\mathfrak{m}^+=0.$$

4. Representations in Γ

It is well known that (K,M) is a Gelfand pair when G_o is locally isomorphic to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$. In particular $\dim(V_\gamma^M)=1$ for all $\gamma\in\Gamma$. In these cases we have an alternative and convenient description of the Kostant degree of $\gamma\in\Gamma$. In fact, given a simple root $\alpha\in P_+$ set $E=X_{-\alpha}+\theta X_{-\alpha}$ for any $X_{-\alpha}\neq 0$. Then if $\gamma\in\Gamma$ define

$$q(\gamma)=\max\{q\in\mathbb{N}:E^q(V_\gamma^M)\neq 0\}. \tag{6}$$

The following propositions establish the relation between $q(\gamma)$ and $d(\gamma)$ for any $\gamma\in\Gamma$ as well as other facts about the representations in Γ. Some of these results where first established in [6], others were proved in [3] for G_o locally isomorphic to $\mathrm{SO}(n,1)$ or $\mathrm{SU}(n,1)$, and in [5] they were generalized to any real rank one semisimple Lie group.

Proposition 4.1. *Let G_o be locally isomorphic to $SO(n,1)$ for $n\geq 3$. Then there exists a Borel subalgebra $\mathfrak{b}_\mathfrak{k}=\mathfrak{h}_\mathfrak{k}\oplus\mathfrak{k}^+$ of $\mathfrak{k}$ such that $\mathfrak{m}^+\subset\mathfrak{k}^+$ and $E\in\mathfrak{k}^+$. For any such a Borel subalgebra there exists a fundamental weight ξ_o with the following properties:*

(i) If $\gamma\in\hat{K}$ and ξ_γ denotes its highest weight then $\gamma\in\Gamma$ if and only if $\xi_\gamma=k\xi_o$ when $n\geq 4$ and $\xi_\gamma=2k\xi_o$ if $n=3$, for some $k\in\mathbb{N}_o$.
(ii) If rank $(G_o)=$ rank (K_o) (that is, n is even) we have, $\gamma\in\Gamma_1$ if and only if $\xi_\gamma=k\xi_o$ with k even.
(iii) If $\gamma\in\Gamma$ we have $E^{q(\gamma)}(V_\gamma^M)=V_\gamma^{\mathfrak{k}^+}$, $\xi_\gamma=q(\gamma)\xi_o$ if $n\geq 4$, and $\xi_\gamma=2q(\gamma)\xi_o$ if $n=3$. Moreover $d(\gamma)=q(\gamma)$.

As we indicated before if G_0 is locally isomorphic to $\mathrm{SU}(n,1)$ there are two simple roots $\alpha=\alpha_1,\alpha_n$ in P_+ (see Section 6 for more details). Hence, in this case we set $E_1=X_{-\alpha_1}+\theta X_{-\alpha_1}$ and $E_2=X_{-\alpha_n}+\theta X_{-\alpha_n}$. The following proposition summarizes some results about the representations $\gamma\in\Gamma$ for the group $\mathrm{SU}(n,1)$.

Proposition 4.2. *Let G_o be locally isomorphic to $SU(n,1)$ for $n\geq 2$. Then for $E=E_1$ (respectively $E=E_2$) there exists a Borel subalgebra $\mathfrak{b}_\mathfrak{k}=\mathfrak{h}_\mathfrak{k}\oplus\mathfrak{k}^+$ of $\mathfrak{k}$ such that $\mathfrak{m}^+\subset\mathfrak{k}^+$ and $E_1\in\mathfrak{k}^+$ (respectively $E_2\in\mathfrak{k}^+$). Moreover:*

(i) The Cartan complement $\mathfrak{p}=\mathfrak{p}_1\oplus\mathfrak{p}_2$*, where* $\mathfrak{p}_1$ *and* $\mathfrak{p}_2$ *are irreducible* $\mathfrak{k}$*-modules and* $\mathfrak{p}_1=\mathfrak{p}_2^*$*.*
(ii) If ξ_1 *and* ξ_2 *are the highest weights of* $\mathfrak{p}_1$ *and* $\mathfrak{p}_2$ *respectively, and* ξ_γ *denotes the highest weight of any* $\gamma\in\hat{K}$*, then* $\gamma\in\Gamma$ *if and only if* $\xi_\gamma=k_1\xi_1+k_2\xi_2$ *with* $k_1,k_2\in\mathbb{N}_o$*, and* $d(\gamma)=k_1+k_2$*.*
(iii) We have $\gamma\in\Gamma_1$ *if and only if* $\xi_\gamma=k(\xi_1+\xi_2)$ *for* $k\in\mathbb{N}_o$*.*
(iv) Let $\gamma\in\Gamma_1$*,* $E=E_1$ *(respectively* $E=E_2$*) and let* $q(\gamma)$ *be as in (6). Then* $E^{q(\gamma)}(V_\gamma^M)=V_\gamma^{\mathfrak{k}+}$*,* $\xi_\gamma=q(\gamma)(\xi_1+\xi_2)$ *and* $d(\gamma)=2q(\gamma)$*.*
(v) If we set $X=[E_1,E_2]$ *then* $X\neq 0$*,* $X\in\mathfrak{m}^+$ *if* $n\geq 3$ *and* $X\in\mathfrak{t}+\mathfrak{z}(\mathfrak{k})$ *if* $n=2$*. Moreover* $[X,E_1]=[X,E_2]=0$ *if* $n\geq 3$*. For* $\gamma\in\Gamma_1$ *let* $0\neq b\in V_\gamma^M$*, then* $E_2^kE_1^\ell(b)=E_1^\ell E_2^k(b)$ *for all* $\ell,k\geq 0$ *and* $E_2^{q(\gamma)}E_1^{q(\gamma)}(b)\neq 0$*.*

For the construction of the Borel subalgebra $\mathfrak{b}_\mathfrak{k}$ of Propositions 4.1 and 4.2 we refer the reader to Section 3 of [5] and for the other statements of the above propositions we refer the reader to Proposition 4.4, Theorem 4.5 and Theorem 5.3 of [5].

The Weyl group $W=M'/M$ preserves the one dimensional space V_γ^M for any $\gamma\in\Gamma$ and since $W=\{1,w_0\}$, it follows that w_0 is either the identity or minus the identity on V_γ^M. It is well known that if rank $(G_o)=$ rank (K_o) (that is, G_o is locally isomorphic to $\mathrm{SO}(2p,1)$ or $\mathrm{SU}(n,1)$) the element w_0 acts as the identity on $\mathfrak{k}$ and therefore it acts as the identity on V_γ^M for all $\gamma\in\Gamma_1$. On the other hand, if rank $(G_o)=$ rank $(K_o)+1$ we have $\Gamma=\Gamma_1$ and the following proposition describes the action of w_0 on V_γ^M.

Proposition 4.3. *Let* G_o *be locally isomorphic to* $SO(2p+1,1)$ *with* $p\geq 1$ *and let* $\gamma\in\Gamma$ *with* $\xi_\gamma=k\xi_o$*. Then* w_0 *is the identity on* V_γ^M *if and only if* k *even.*

Proof. Since $\Gamma=\Gamma_1$ we may assume that $V_\gamma^M\subset U(\mathfrak{k})^M$ and let $v_0\in V_\gamma^M$ be a non zero element. Since $U(\mathfrak{k})^M\simeq U(\mathfrak{k})^K\otimes U(\mathfrak{m})^M$ (see [7] and [13]) there exist unique $x_i\in U(\mathfrak{k})^K$ and $y_i\in U(\mathfrak{m})^M$ for $i=1,\dots,r$, such that

$$v_0=\sum_{i=1}^r x_iy_i,$$

where $\{x_i\}$ is a linearly independent set in $U(\mathfrak{k})^K$. Then, $w_0v_0=\pm v_0$ if and only if $w_0y_i=\pm y_i$ for all $i=1,\dots,r$. On the other hand,

$$y_i\equiv t_i \mod \big(U(\mathfrak{m})\mathfrak{m}^+\big)$$

where $t_i\in U(\mathfrak{t})$ is the image of y_i by the Harish-Chandra isomorphism $U(\mathfrak{m})^M\to U(\mathfrak{t})^{W(\mathfrak{m},\mathfrak{t})_{\rho_\mathfrak{m}}}$. Here $W(\mathfrak{m},\mathfrak{t})_{\rho_\mathfrak{m}}$ denotes de action of the Weyl group $W(\mathfrak{m},\mathfrak{t})$ on $\mathfrak{t}$ translated by $\rho_\mathfrak{m}$.

If $\{T_1,T_2,\dots,T_p\}$ is an orthonormal basis of $\mathfrak{t}$ with respect to the Killing form, the elements $q_k=\sum_{i=1}^p T_i^{2k}$ for $k=1,\dots,p-1$, and $q_p=T_1T_2\dots T_p$ are the generators of $S(\mathfrak{t})^{W(\mathfrak{m},\mathfrak{t})}$ (see [1]). Note that q_k has even degree in T_i for all $i=1,\dots,p$ and all $k=1,\dots,p-1$, but q_p has degree one in T_i for all $i=1,\dots,p$. Let $\widetilde{q}_k\in U(\mathfrak{t})^{W(\mathfrak{m},\mathfrak{t})_{\rho_\mathfrak{m}}}$ be the translated element by $\rho_\mathfrak{m}$ corresponding to q_k, for example $\widetilde{q}_p=(T_1+\rho_\mathfrak{m}(T_1))\dots(T_p+\rho_\mathfrak{m}(T_p))$. We know that $t_i = Q_i(\widetilde{q}_1, \dots, \widetilde{q}_p) = Q'_i(T_1, \dots, T_p)$, where Q_i and Q'_i are polynomials in $\mathbb{C}[x_1, \dots, x_p]$ for all $i = 1, \dots, p$.

It is not difficult to see that the basis $\{T_1, T_2, \dots, T_p\}$ and a representative of w_0 can be chosen so that,

(i) $w_0 T_i = T_i$ for all $i = 1, \dots, p-1$ and $w_0 T_p = -T_p$;

(ii) $w_0 \mathfrak{m}^+ = \mathfrak{m}^+$;

(iii) $\dot{E}(T_1) = -E$ and $\dot{E}(T_i) = 0$ for $i = 2, \dots, p$.

Property (ii) implies that $w_0 y_i = \pm y_i$ if and only if $w_0 t_i = \pm t_i$, and property (i) implies that $w_0 t_i = t_i$ (respectively $w_0 t_i = -t_i$) if and only if Q_i is an even (respectively odd) polynomial in $\widetilde{q}_p$.

Now assume that $w_0 t_i = t_i$ for all $i = 1, \dots, r$. Then Q'_i has even degree in all the variables $T_1, \dots, T_p$. On the other hand, property (iii) implies that

$$\dot{E}^s(T_1^j) = \Big(\sum_{\ell=1}^{s} (-1)^\ell \tbinom{s}{\ell} (T_1 + \ell - s)^j\Big) E^s = \begin{cases} 0, & \text{if } s > j \\ j! E^s, & \text{if } s = j, \end{cases} \tag{7}$$

hence for $s \in \mathbb{N}_0$ and $1 \le i \le r$, there exists a polynomial $\widetilde{Q}'_i \in \mathbb{C}[x_1, \dots, x_p]$ (that depends on s) such that $\dot{E}^s(Q'_i(T_1, \dots, T_p)) = \widetilde{Q}'_i(T_1, \dots, T_p) E^s$. Now, since $v_0 \in V_\gamma^M$ and $\xi_\gamma = k\xi_o$, from Proposition 4.1 we know that $\dot{E}^k(v_0) \neq 0$ and $\dot{E}^{k+1}(v_0) = 0$. Then,

$$\begin{aligned} 0 &= \dot{E}^{k+1}(v_0) \\ &= \sum_{i=1}^{r} x_i \dot{E}^{k+1}(y_i) \\ &\equiv \sum_{i=1}^{r} x_i \dot{E}^{k+1}(Q'_i(T_1, \dots, T_p)) \\ &= \sum_{i=1}^{r} x_i \widetilde{Q}'_i(T_1, \dots, T_p) E^{k+1}. \end{aligned}$$

Hence, in view of Lemma 3.2 this implies that

$$\sum_{i=1}^{r} x_i \widetilde{Q}'_i(T_1, \dots, T_p) \equiv 0.$$

Now, since $\{x_i\}$ is a linearly independent set in $U(\mathfrak{k})^K$ and $\widetilde{Q}'_i(T_1, \dots, T_p) \in U(\mathfrak{t})$, we obtain that $\widetilde{Q}'_i(T_1, \dots, T_p) = 0$ for $i = 1, \dots, r$ (see Proposition 13 of [13]). This implies that $\dot{E}^{k+1}(Q'_i(T_1, \dots, T_p)) = 0$ for $i = 1, \dots, r$. On the other hand, since $\dot{E}^k(v_0) \neq 0$ there exists some $1 \le j \le r$ such that $\dot{E}^k(Q'_j(T_1, \dots, T_p)) \neq 0$. These two results about $Q'_j(T_1, \dots, T_p)$, together with (7), imply that k is equal to the degree of Q'_i in the variable T_1 which we know is even.

Finally, if we assume that $w_0 t_i = -t_i$ for all $i = 1, \dots, r$, we obtain that Q'_i has odd degree in all the variables $T_1, \dots, T_p$. Then the same argument as above shows that k is odd. This completes the proof of the proposition. ∎

5. The case SO(n,1)

In this section we shall prove Theorem 1.2 when G_o is locally isomorphic to $\mathrm{SO}(n,1)$ with $n \geq 3$.

5.1. Preliminary results. As we pointed out before, there is only one simple root $\alpha_1 \in P_+$ if $n \geq 4$ and there are two α_1, α_2 if $n = 3$. In all cases we set $\alpha = \alpha_1$, $E = E_\alpha$, $Y = Y_\alpha$ and $Z = Z_\alpha$. Also as in (3), to any $b(x) \in U(\mathfrak{k})[x]$ we associate $c(x) \in U(\mathfrak{k})[x]$ defined by $c(x) = b(x-1)$. If $b(x) \in U(\mathfrak{k})[x]$, $b(x) \neq 0$, we shall find it convenient to write, in a unique way, $b = \sum_{j=0}^m b_j x^j$, $b_j \in U(\mathfrak{k})$, $b_m \neq 0$, and the corresponding $c = \sum_{j=0}^m c_j \varphi_j$ with $c_j \in U(\mathfrak{k})$. Then the following result establishes the relation between the coefficients b_j and c_j. Since its proof is straightforward we ommit it.

Lemma 5.1. *Let $b = \sum_{j=0}^m b_j x^j \in U(\mathfrak{k})[x]$ and set $c(x) = b(x-1)$. Then, if $c = \sum_{j=0}^m c_j \varphi_j$ with $c_j \in U(\mathfrak{k})$ we have*

$$c_i = \sum_{j=i}^{m} t_{ij} b_j \qquad 0 \leq i \leq m, \tag{8}$$

where t_{ij} are rational numbers and $t_{ii} = i!$. In other words, the vectors $(b_0, \dots, b_m)^t$ and $(c_0, \dots, c_m)^t$ are related by a rational nonsingular upper triangular matrix.

Lemma 5.2. *If $b = b_m \otimes Z^m + \cdots + b_0 \in B$ then $\dot{E}^{m+1}(b_j) \equiv 0$ for all $0 \leq j \leq m$, and thus $\dot{E}^{m+1}(b_j) = 0$ for all $0 \leq j \leq m$.*

Proof. We regard b as a polynomial $b = \sum_{j=0}^m b_j x^j$ with $b_j \in U(\mathfrak{k})^M$ and let $c(x) = b(x-1) = \sum_{j=0}^m c_j \varphi_j(x)$ with $c_j \in U(\mathfrak{k})^M$. Then, since $b \in B$, c satisfies the system of equations (i) of Theorem 3.1 with $\widetilde{Y} = Y$. Therefore c satisfies equations (iii) of Theorem 3.1 for all $\ell, n \in \mathbb{N}_o$.

Hence, since $c^{(m+1)} = 0$, if we consider $\ell = m+1$ in equation (iii) of Theorem 3.1 and we use Lemma 3.2 with $X = E$ we obtain

$$\sum_{j=n}^{m} \dot{E}^{m+1}(c_j)\varphi_{j-n}\left(\tfrac{2m+2-n}{2} - Y\right) \equiv 0, \tag{9}$$

for $0 \leq n \leq m$. Now, taking into account that right multiplication by Y leaves invariant the left ideal $U(\mathfrak{k})\mathfrak{m}^+$ because $Y \in \mathfrak{t}$, (9) together with decreasing induction on n starting from $n = m$ implies that $\dot{E}^{m+1}(c_j) \equiv 0$ for all $0 \leq j \leq m$. From this, applying $\dot{E}^{m+1}$ to (8) and making use of Theorem 3.3, the theorem follows because the matrix (t_{ij}) is a nonsingular scalar matrix. ■

5.2. Bound for the Kostant degree. We are now ready to prove the boundeness condition on the Kostant degree required in Theorem 2.2 for G_o locally isomorphic to $\mathrm{SO}(n,1)$.

Theorem 5.3. *Assume that G_o is locally isomorphic to $SO(n,1)$ for $n \geq 3$ and let $b = b_m \otimes Z^m + \cdots + b_0 \in B$, then $d(b_j) \leq m$ for all $0 \leq j \leq m$.*

Proof. Let $b = b_m \otimes Z^m + \cdots + b_0 \in B$, then it follows from Lemma 5.2 that $\dot{E}^{m+1}(b_j) = 0$ for all $0 \le j \le m$. In view of (6) and (iii) of Proposition 4.1 this implies that $b_j \in \bigoplus U(\mathfrak{k})^M_\gamma$, where the sum extends over all $\gamma \in \Gamma$ such that $d(\gamma) \le m$. Therefore $d(b_j) \le m$ for all $0 \le j \le m$, as we wanted to prove. ■

5.3. Weyl group invariance of the leading term. Our next goal is to prove the W-invariance condition of Theorem 2.2. That is, if $b = b_m \otimes Z^m + \cdots + b_0 \in B$ and $b_m \neq 0$, then its leading term $b_m \otimes Z^m \in \big(U(\mathfrak{k})^M \otimes U(\mathfrak{a})\big)^W$. Recall that $\big(U(\mathfrak{k})^M \otimes U(\mathfrak{a})\big)^W$ denotes the ring of W-invariants in $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ under the tensor product of the action of W on $U(\mathfrak{k})^M$ and the action of W on $U(\mathfrak{a})$

If G_o is locally isomorphic to $\mathrm{SO}(2p,1)$ the Weyl group W acts trivially on $\mathfrak{k}$. On the other hand, if G_o is locally isomorphic to $\mathrm{SO}(2p+1,1)$ with $p \ge 1$, recall that we can choose an orthonormal basis $\{T_1, T_2, \ldots, T_p\}$ of $\mathfrak{t}$ and a representative of w_0 such that,

(i) $w_0 T_i = T_i$ for all $i = 1, \ldots, p-1$ and $w_0 T_p = -T_p$;

(ii) $w_0 \mathfrak{m}^+ = \mathfrak{m}^+$;

(iii) $\dot{E}(T_1) = -E$ and $\dot{E}(T_i) = 0$ for $i = 2, \ldots, p$.

Moreover, this choice can be made in such a way that $w_0 E = -E$ and $Y = -T_1$. Hence, if we extend the action of W in $U(\mathfrak{k})^M$ to $U(\mathfrak{k})^M \otimes U(\mathfrak{a})$ by letting it act trivially on $U(\mathfrak{a})$, it is clear that W preserves the algebra B and thus $B = B_1 \oplus B_{-1}$, where $B_{\pm 1} = \{b \in B : w_0 b = \pm b\}$.

Lemma 5.4. *If $u \in U(\mathfrak{k})^M$ the following statements hold,*

(1) If $w_0 u = u$ and $\dot{E}^{2t}(u) = 0$ for $t \in \mathbb{N}$, then $\dot{E}^{2t-1}(u) = 0$.

(2) If $w_0 u = -u$ and $\dot{E}^{2t+1}(u) = 0$ for $t \in \mathbb{N}$, then $\dot{E}^{2t}(u) = 0$.

Proof. We may assume that $u \in V_\gamma^M \subset U(\mathfrak{k})^M$ for $\gamma \in \Gamma_1$. We begin by proving (1). If $\dot{E}^{2t-1}(u) \neq 0$ then $\dot{E}^{2t-1}(u)$ would be a highest weight vector of weight $\xi = (2t-1)\xi_o$. This contradicts (ii) of Proposition 4.1 if G_o is locally isomorphic to SO(2p,1), or contradicts Proposition 4.3 if G_o is locally isomorphic to SO(2p+1,1), because we are assuming that w_0 acts as the identity on V_γ^M. The proof of (2) is similar: if $\dot{E}^{2t}(u) \neq 0$ then $\dot{E}^{2t}(u)$ would be a highest weight vector of weight $\xi = 2t\xi_o$ but this contradicts Proposition 4.3 as in the previous case. ■

Theorem 5.5. *If G_o is locally isomorphic to $SO(n,1)$ with $n \ge 3$ and $b = b_m \otimes Z^m + \cdots + b_0 \in B$ with $b_m \neq 0$, then its leading term $\widetilde{b} = b_m \otimes Z^m \in \big(U(\mathfrak{k})^M \otimes U(\mathfrak{a})\big)^W$.*

Proof. We shall prove first that if $b = b_m \otimes Z^m + \cdots + b_0 \in B_1$ (respectively $b \in B_{-1}$) then m is even (respectively odd). Let $b = b_m \otimes Z^m + \cdots + b_0 \in B_1$ with $b_m \neq 0$, and assume that m is odd. From Lemma 5.2 it follows that $\dot{E}^{m+1}(b_j) = 0$ for all $0 \le j \le m$. Then, since $m+1$ is even and $w_0 b_j = b_j$ for all $0 \le j \le m$, from (1) of Lemma 5.4 it follows that $\dot{E}^m(b_j)=0$ for all $0\le j\le m$. Hence, from (8) we get $\dot{E}^m(c_j)=0$ for all $0\le j\le m$. Now, if we consider $\ell=m$ and $n=0$ in equation (iii) of Theorem 3.1 we get

$$\sum_{j=0}^{m}\dot{E}^m(c_j)\varphi_j(m-Y)-m!b_mE^m\equiv 0,$$

which implies that $b_m\equiv 0$, and therefore $b_m=0$ (Theorem 3.3). This is a contradiction therefore m is even, as we wanted to prove. A similar argument proves that if $b=b_m\otimes Z^m+\cdots+b_0\in B_{-1}$ and $b_m\neq 0$, then m is odd. Observe that in both cases (ie. $b\in B_1$ or $b\in B_{-1}$) we have $\widetilde{b}=b_m\otimes Z^m\in\left(U(\mathfrak{k})^M\otimes U(\mathfrak{a})\right)^W$.

Now consider $b=b_m\otimes Z^m+\cdots+b_0\in B$ with $b_m\neq 0$. Since $B=B_1\oplus B_{-1}$ we can write $b=b^{(1)}+b^{(-1)}$ with $b^{(1)}\in B_1$ and $b^{(-1)}\in B_{-1}$. Then the leading term of b is either $\widetilde{b^{(1)}}$ or $\widetilde{b^{(-1)}}$, the leading terms of $b^{(1)}$ and $b^{(-1)}$ respectively. Hence, by above the observation, in either case we conclude that $\widetilde{b}=b_m\otimes Z^m\in\left(U(\mathfrak{k})^M\otimes U(\mathfrak{a})\right)^W$, as we wanted to prove. ∎

Remark 5.1. When G_o is locally isomorphic to $\mathrm{SO}(3,1)$ we have used only one of the equations that define the algebra B. In other words, if for each simple root $\alpha\in P_+$ we define B_α as the subalgebra of all elements $b\in U(\mathfrak{k})\otimes U(\mathfrak{a})$ that satisfy (1) for all $n\in\mathbb{N}$, then we have proved that $P\left(U(\mathfrak{g})^K\right)=B^{W_\rho}=B_\alpha^{W_\rho}$. Moreover, taking advantage that in this case the elements of the algebra B satisfy two different equations, it is not difficult to see that $B^{W_\rho}=B$.

This completes the proof of theorem 2.2 when G_0 is locally isomorphic to $\mathrm{SO}(n,1)$.

6. The case SU(n,1)

In this section we prove Theorem 1.2 when G_o is locally isomorphic to $\mathrm{SU}(n,1)$ for $n\ge 2$. Although some results of this section are contained in [12], we include them here for completeness and to prove that $B^{W_\rho}=B$ which is a new result.

6.1. Preliminary results. We can choose an orthonormal basis $\{\epsilon_i\}_{i=1}^{n+1}$ of $(\mathfrak{h}_\mathbb{R}\oplus\mathbb{R})^*$ in such a way that $\mathfrak{h}_\mathbb{R}=\{H\in\mathfrak{h}_\mathbb{R}\oplus\mathbb{R}:(\epsilon_1+\cdots+\epsilon_{n+1})(H)=0\}$, $\alpha_i=\epsilon_i-\epsilon_{i+1}$ if $1\le i\le n$, $\epsilon_i^\sigma=-\epsilon_i$ if $2\le i\le n$ and $\epsilon_1^\sigma=-\epsilon_{n+1}$. Then from the Dynkin-Satake diagram of $\mathfrak{g}$ we obtain that

$$\begin{aligned}\Delta^+(\mathfrak{g},\mathfrak{h})&=\{\epsilon_i-\epsilon_j:1\le i<j\le n+1\},\\ P_+&=\{\epsilon_1-\epsilon_j,\epsilon_j-\epsilon_{n+1}:2\le j\le n\}\cup\{\epsilon_1-\epsilon_{n+1}\},\\ P_-&=\{\epsilon_i-\epsilon_j:2\le i<j\le n\},\end{aligned}$$

where P_- denotes the set of roots in $\Delta^+(\mathfrak{g},\mathfrak{h})$ that vanish on $\mathfrak{a}$.

In this case there are two simple roots $\alpha=\alpha_1,\alpha_n$ in P_+; in both cases $Y_\alpha\neq 0$. Set $E_1=X_{-\alpha_1}+\theta X_{-\alpha_1}$, $E_2=X_{-\alpha_n}+\theta X_{-\alpha_n}$, $Y_1=Y_{\alpha_1}$, $Y_2=Y_{\alpha_n}$ and $Z=Z_{\alpha_1}=Z_{\alpha_n}$. Let $T\in\mathfrak{t}_\mathbb{R}$ be defined by $\epsilon_2(T)=\cdots=\epsilon_n(T)=\frac{2}{n+1}$. Then $T\in\mathfrak{z}(\mathfrak{m})$ and $\dim(\mathfrak{z}(\mathfrak{m}))=1$. Since $\epsilon_1(T)=\epsilon_{n+1}(T)$ and $(\epsilon_1+\cdots+\epsilon_{n+1})(T)=0$

we get $\epsilon_2(T)-\epsilon_1(T)=\epsilon_n(T)-\epsilon_{n+1}(T)=1$; thus $[T,E_1]=E_1$ and $[T,E_2]=-E_2$. Now define the vector H considered in (3) as follows,

$$H=\begin{cases}\frac{1}{2}T, & \text{if } \alpha=\alpha_1\\ -\frac{1}{2}T, & \text{if } \alpha=\alpha_n,\end{cases} \tag{10}$$

and we write generically E, Y, and $\widetilde{Y}=Y+H$ for the corresponding vectors associated to a simple root $\alpha\in P_+$. Then $\dot{E}(H)=-\frac{1}{2}E$, and thus $\dot{E}(\widetilde{Y})=E$.

Also as in (3), to any $b(x)\in U(\mathfrak{k})[x]$ associate $c(x)\in U(\mathfrak{k})[x]$ defined by $c(x)=b(x+H-1)$. If $b(x)\in U(\mathfrak{k})[x]$, $b(x)\neq 0$, we shall find it convenient to write, in a unique way, $b=\sum_{j=0}^m b_jx^j$, $b_j\in U(\mathfrak{k})$, $b_m\neq 0$, and the corresponding $c=\sum_{j=0}^m c_j\varphi_j$ with $c_j\in U(\mathfrak{k})$. Then the following lemma establishes the relation between the coefficients b_j and c_j.

Lemma 6.1. *Let $b=\sum_{j=0}^m b_jx^j\in U(\mathfrak{k})[x]$ and set $c(x)=b(x+H-1)$. Then, if $c=\sum_{j=0}^m c_j\varphi_j$ with $c_j\in U(\mathfrak{k})$ we have*

$$c_i=\sum_{j=i}^m b_jt_{ij} \qquad 0\le i\le m, \tag{11}$$

where $t_{ij}=\sum_{k=0}^i(-1)^k\binom{i}{k}(H+\frac{i}{2}-1-k)^j\in\mathfrak{z}(U(\mathfrak{m}))$. Thus $t_{ii}=i!$, t_{ij} is a polynomial in H of degree $j-i$, and

$$\dot{E}^{j-i}(t_{ij})=\left(-\tfrac{1}{2}\right)^{j-i}j!E^{j-i}.$$

Moreover if $b_j\in U(\mathfrak{k})^M$ for $0\le j\le m$, then $c_j\in U(\mathfrak{k})^M$ for $0\le j\le m$.

Proof. Since almost all the results follow from straightforward computations, we only prove that $\dot{E}^{j-i}(t_{ij})=\left(-\frac{1}{2}\right)^{j-i}j!E^{j-i}$.

It follows by induction that if $\dot{H}(E)=cE$ and $a\in\mathbb{C}$, then

$$\dot{E}^m(H+a)^j=E^m\sum_{\ell=0}^m(-1)^\ell\binom{m}{\ell}(H+a+c\ell)^j. \tag{12}$$

This implies that

$$\dot{E}^{j-i}(H^{j-i})=E^{j-i}\sum_{\ell=0}^{j-i}(-1)^\ell\binom{j-i}{\ell}\left(H+\frac{\ell}{2}\right)^{j-i}=\left(-\tfrac{1}{2}\right)^{j-i}(j-i)!E^{j-i}.$$

Now, since

$$\begin{aligned}t_{ij}&=\sum_{k=0}^i(-1)^k\binom{i}{k}\sum_{\ell=0}^j\binom{j}{\ell}\left(\frac{i}{2}-1-k\right)^\ell H^{j-\ell}\\&=\sum_{\ell=i}^j\left(\sum_{k=0}^i(-1)^k\binom{i}{k}\left(\frac{i}{2}-1-k\right)^\ell\right)\binom{j}{\ell}H^{j-\ell}\\&=\frac{j!}{(j-i)!}H^{j-i}+\cdots,\end{aligned}$$

it follows that

$$\dot{E}^{j-i}(t_{ij}) = \frac{j!}{(j-i)!}\dot{E}^{j-i}(H^{j-i}) = \left(-\tfrac{1}{2}\right)^{j-i} j! E^{j-i}.$$

■

Theorem 6.2. *If $b = b_m \otimes Z^m + \cdots + b_0 \in B$, then $\dot{E}^{m+1}(c_j) = 0$ for all $0 \le j \le m$.*

Proof. Since $b \in B$, c satisfies the system of equations (i) of Theorem 3.1 with $\widetilde{Y} = Y + H$. Therefore c satisfies equations (iii) of Theorem 3.1 for all $\ell, n \in \mathbb{N}_o$. Hence, since $c^{(m+1)} = 0$, if we consider $\ell = m+1$ in equation (iii) of Theorem 3.1 and we use Lemma 3.2 with $X = E$ we obtain

$$\sum_{j=n}^{m} \dot{E}^{m+1}(c_j)\varphi_{j-n}\left(\frac{2m+2-n}{2} - \widetilde{Y}\right) \equiv 0, \tag{13}$$

for $0 \le n \le m$. Now, taking into account that right multiplication by $\widetilde{Y}$ leaves invariant the left ideal $U(\mathfrak{k})\mathfrak{m}^+$ because $\widetilde{Y} \in \mathfrak{t}$, (13) together with decreasing induction on n starting from $n = m$ implies that $\dot{E}^{m+1}(c_j) \equiv 0$. Hence using Lemma 6.1 and Theorem 3.3 it follows that $\dot{E}^{m+1}(c_j) = 0$ for all $0 \le j \le m$. ■

Corollary 6.3. *If $b = b_m \otimes Z^m + \cdots + b_0 \in B$, then $\dot{E}^{2m+1-j}(b_j) = 0$ for all $0 \le j \le m$.*

Proof. For $j = m$ the assertion follows directly from Theorem 6.2 since $c_m = m! b_m$ (Lemma 6.1). Now we proceed by decreasing induction on j. Thus let $0 \le j < m$ and assume that $\dot{E}^{2m+1-k}(b_k) = 0$ for all $j < k \le m$. Then, since $m+1 < 2m+1-j$, using Leibnitz rule, Lemma 6.1 and the inductive hypothesis we obtain

$$\dot{E}^{2m+1-j}(c_j) = \dot{E}^{2m+1-j}\Big(\sum_{k=j}^{m} b_k t_{jk}\Big) = j!\dot{E}^{2m+1-j}(b_j).$$

Since $\dot{E}^{2m+1-j}(c_j) = 0$ the proof of the corollary is completed. ■

The following result was proved in Theorem 30 of [12], but in a different way. Here we derive this theorem directly from Theorem 3.1.

Theorem 6.4. *Let $m, w, \alpha \in \mathbb{Z}$, $0 \le w, \alpha \le m$, $\alpha + w \ge m+1$. If $b = b_m \otimes Z^m + \cdots + b_0 \in B$ and $\dot{E}^{m+\alpha+1-j}(b_j) \equiv 0$ for all $0 \le j \le m$, then*

$$\sum_{j=m-w}^{m} (-2)^{-j} j! \binom{\alpha+w}{j+w-m} \dot{E}^{m+\alpha-j}(b_j)E^j \equiv 0.$$

Proof. From the previous theorem we know that $\dot{E}^{m+1}(c_j) \equiv 0$ for every $0 \le j \le m$. Since $w \ge 1$ we have $\dot{E}^{\alpha+w}(c_{m-w}) = 0$. Now using the Leibnitz rule and Lemma 6.1 we compute

$$\begin{aligned}\dot{E}^{\alpha+w}(c_{m-w}) &= \dot{E}^{\alpha+w}\Big(\sum_{j=m-w}^{m} b_j t_{m-w,j}\Big)\\ &\equiv \sum_{j=m-w}^{m}\binom{\alpha+w}{j+w-m}\dot{E}^{m+\alpha-j}(b_j)\dot{E}^{j+w-m}(t_{m-w,j})\\ &= \sum_{j=m-w}^{m}\binom{\alpha+w}{j+w-m}(-2)^{-(j+w-m)}j!\dot{E}^{m+\alpha-j}(b_j)E^{j+w-m}.\end{aligned}$$

Therefore

$$\sum_{j=m-w}^{m}\binom{\alpha+w}{j+w-m}(-2)^{-(j+w-m)}j!\dot{E}^{m+\alpha-j}(b_j)E^{j+w-m}\equiv 0.$$

If we multiply this last equation on the right by $(-2)^{w-m}E^{m-w}$ we obtain the stated result. ■

Lemma 6.5. *Let $k\in\mathbb{N}_o$ and $u\in U(\mathfrak{k})^M$. Then, $\dot{E}_i^k(u)\equiv 0$ for $i=1$ or $i=2$ if and only if $\dot{E}_i^k(u)=0$ for every $i\in\{1,2\}$.*

Proof. Let us assume that $\dot{E}_1^k(u)\equiv 0$ for $k\geq 1$. Then Theorem 3.3 implies that $\dot{E}_1^k(u)=0$. Hence, in view of Proposition 4.2, it follows that $u\in\bigoplus U(\mathfrak{k})_\gamma^M$ where the sum extends over all $\gamma\in\Gamma_1$ such that $q(\gamma)\leq k-1$. Then since $q(\gamma)$ is independent of the choice of the simple root $\alpha=\alpha_1$ or $\alpha=\alpha_n$, we obtain $\dot{E}_2^k(u)=0$ which completes the proof. ■

For further reference we now recall Lemma 1 of [13].

Lemma 6.6. *Let G_o be locally isomorphic to $SU(2,1)$ and set $Y=Y_{\alpha_1}=-Y_{\alpha_2}$. Also let $0\neq D\in\mathfrak{z}(\mathfrak{k})$ and let ζ denote the Casimir element of $[\mathfrak{k},\mathfrak{k}]$. Then $\{\zeta^iD^j\}_{i,j\geq 0}$ is a basis of $\mathfrak{z}(U(\mathfrak{k}))$ and $\{\zeta^iD^jY^k\}_{i,j,k\geq 0}$ is a basis of $U(\mathfrak{k})^M$.*

The following theorem plays a crucial role in the proof of Theorem 2.2 because it allows us to obtain from Theorem 6.4 two systems of linear equations and therefore doubling the number of equations.

Theorem 6.7. *Let G_o be locally isomorphic to $SU(n,1)$ for $n\geq 2$. Also let $m,k\in\mathbb{N}_o$, $m\leq k$, and let $b_j\in U(\mathfrak{k})^M$ be such that $\dot{E}^{k+1-j}(b_j)\equiv 0$ for all $0\leq j\leq m$ and for $E=E_1$ or $E=E_2$. Then,*

(i) If $\sum_{j=0}^{m}\dot{E}^{k-j}(b_j)E^j\equiv 0$ for $E=E_1$ and $E=E_2$ we obtain

$$\sum_{\substack{0\leq j\leq m\\ j\text{ even}}}\dot{E}^{k-j}(b_j)E^j=0=\sum_{\substack{0\leq j\leq m\\ j\text{ odd}}}\dot{E}^{k-j}(b_j)E^j.$$

(ii) If $\sum_{j=0}^{m}\dot{E}^{k-j}(b_j)E^j\equiv 0$ for $E=E_1$ or $E=E_2$ we have

$$\sum_{j=0}^{m}\dot{E}_i^{k-j}(b_j)E_i^j=0=\sum_{j=0}^{m}(-1)^j\dot{E}_{i'}^{k-j}(b_j)E_{i'}^j,$$

for $i'\neq i$ and $i,i'\in\{1,2\}$.

Proof. The statement in (i) is the same as that of Theorem 32 of [12], and its proof for $n \geq 3$ can be found there. Here we will prove (i) for $n = 2$ and will also prove (ii). To do this we recall the following equality obtained in Theorem 32 of [12] for $n \geq 2$,

$$\sum_{j=0}^{m} \dot{E}^{k-j}(b_j)E^j = \dot{E}^k\Big(\sum_{j=0}^{m}\binom{k}{j}^{-1}(-\epsilon)^j(j!)^{-1}b_jT^j\Big), \tag{14}$$

where $\epsilon = 1$ if $E = E_1$, $\epsilon = -1$ if $E = E_2$ and $T \in \mathfrak{z}(\mathfrak{m})$ was defined at the beginning of this section. Since $\sum_{j=0}^{m}\binom{k}{j}^{-1}(-\epsilon)^j(j!)^{-1}b_jT^j \in U(\mathfrak{k})^M$, if we assume that the hypothesis in (ii) holds, applying Lemma 6.5 we obtain (ii) for every $n \geq 2$.

On the other hand, if we assume that the hypothesis in (i) holds, applying Theorem 3.3 (or Lemma 6.5) to (14) we obtain that,

$$\sum_{j=0}^{m} \dot{E}^{k-j}(b_j)E^j = 0, \quad \text{for} \quad E = E_1 \quad \text{and} \quad E = E_2. \tag{15}$$

Also, since $\dot{E}^{k+1-j}(b_j) \equiv 0$ for $0 \leq j \leq m$ and for $E = E_1$ or $E = E_2$, it follows from Lemma 6.5 that $\dot{E}^{k+1-j}(b_j) = 0$ for $E = E_1$ and $E = E_2$.

Assume now that $n = 2$. It follows from Lemma 6.6 that we can write, in a unique way, $b_j = \sum_i a_{i,j}Y^i$ with $a_{i,j} \in \mathfrak{z}(U(\mathfrak{k}))$ and $0 \leq j \leq m$. On the other hand, from the definition of Y in Lemma 6.6 and the comment at the beginning of Section 3, we have $\dot{E}(Y) = \frac{3\epsilon}{2}E$. Hence,

$$\dot{E}^t(Y^i) = \begin{cases} 0, & \text{if } t > i \\ t!\left(\frac{3\epsilon}{2}\right)^t E^t, & \text{if } t = i. \end{cases} \tag{16}$$

Then, since $\dot{E}^{k+1-j}(b_j) = 0$ for $E = E_1$ and $E = E_2$, using (16) we obtain that $b_j = \sum_{i=0}^{k-j} a_{i,j}Y^i$. Therefore

$$\sum_{j=0}^{m} \dot{E}^{k-j}(b_j)E^j = E^k \sum_{j=0}^{m}\left(\tfrac{3\epsilon}{2}\right)^{k-j} a_{k-j,j} \tag{17}$$

for both $E = E_1$ and $E = E_2$. Then using (15) and (17) for $E = E_1$ and $E = E_2$ we obtain (i) for $n = 2$. ■

Taking into account Theorems 6.4 and 6.7 we are led to consider, for each $1 \leq \alpha \leq m$, the following systems of linear equations

$$\sum_{\substack{m-w\leq j\leq m \\ j \text{ even (odd)}}} (-2)^{-j}j!\binom{\alpha+w}{j+w-m}\dot{E}^{m+\alpha-j}(b_j)E^j = 0, \tag{18}$$

for $m + 1 - \alpha \leq w \leq m$.

If we set $x_j = \frac{(-2)^{-j} j!}{(\alpha+m-j)!} \dot{E}^{m+\alpha-j}(b_j) E^{j+w-m}$ and multiply (18) by $\frac{1}{(\alpha+w)!}$ we obtain

$$\sum_{\substack{m-w \le j \le m \\ j \text{ even (odd)}}} \frac{1}{(j+w-m)!} x_j = 0. \tag{19}$$

Now if we make the change of indices $j = 2r - \delta$, $m - w + \delta = s$ and set $y_r = \frac{x_{2r-\delta}}{(2r)!}$ the systems (19) become

$$\sum_{\delta \le r \le [\frac{m+\delta}{2}]} \binom{2r}{s} y_r = 0, \tag{20}$$

for $\delta \le s \le \alpha + \delta - 1$ and $\delta \in \{0, 1\}$.

Proposition 6.8. *For $\delta \in \{0, 1\}$ let M_δ be the matrix with entries defined by $M_{rs} = \binom{2r}{s}$ for $\delta \le r, s \le k$. Then*

$$\det(M_\delta) = 2^{k(k+1)/2}.$$

Proof. For each $\delta \le s \le k$ we let $\binom{2r}{s}$ denote the s-column of M_δ and we consider the determinant of M_δ as a multilinear function of its columns. Thus

$$\det(M_\delta) = \det\left(\binom{2r}{\delta}, \binom{2r}{\delta+1}, \dots, \binom{2r}{k}\right).$$

If we view the binomial coefficient $\binom{2r}{s}$ as a polynomial in the variable r of degree s we can write, in a unique way,

$$\binom{2r}{s} = 2^s \binom{r}{s} + a_{s-1} \binom{r}{s-1} + \dots + a_0,$$

where $a_j = 0$ for $j < \frac{s}{2}$. Then

$$\det(M) = \det\left(2^\delta \binom{r}{\delta}, 2^{\delta+1} \binom{r}{\delta+1}, \dots, 2^k \binom{r}{k}\right) = 2^{k(k+1)/2}.$$

This completes the proof of the proposition. ∎

6.2. Bound for the Kostant degree. We are now almost ready to prove the first part of Theorem 2.2 when G_o is locally isomorphic to $\mathrm{SU}(n, 1)$. We need the following proposition.

Proposition 6.9. *Let G_o be locally isomorphic to $SU(n, 1)$ with $n \ge 2$. If $b = b_m \otimes Z^m + \dots + b_0 \in B$, then $\dot{E}^{[\frac{m}{2}]+m+1-j}(b_j) = 0$ for all $0 \le j \le m$.*

Proof. We will prove by decreasing induction on α in the interval $[\frac{m}{2}] \le \alpha \le m$ that $\dot{E}^{\alpha+m+1-j}(b_j) = 0$ for all $0 \le j \le m$. For $\alpha = m$ this result follows from Corollary 6.3 and Theorem 3.3. Thus assume that $[\frac{m}{2}] < \alpha \le m$ and that $\dot{E}^{\alpha+m+1-j}(b_j) = 0$ for all $0 \le j \le m$. Then in view of Theorems 6.4 and 6.7 we know that the systems of linear equations (18) and their equivalent versions (19) and (20) hold.

Since $[\frac{m}{2}]+1\le\alpha$ the number of unknowns in the system (20) is less or equal than the number of equations. Moreover, it follows from Proposition 6.8 that when $\delta=0$ the rank of the coefficient matrix of the system (20) is $[\frac{m}{2}]+1$ which it is equal to the number of unknowns. Thus $\dot E^{\alpha+m-j}(b_j)=0$ for $0\le j\le m$ and j even. Similarly , when $\delta=1$ the rank of the coefficient matrix is $[\frac{m+1}{2}]$ which it is also equal to the number of unknowns. Therefore $\dot E^{\alpha+m-j}(b_j)=0$ for $0\le j\le m$ and j odd. Then the inductive step is completed and the proposition is proved. ∎

Theorem 6.10. *Let G_o be locally isomorphic to $SU(n,1)$ with $n\ge 2$.If $b=b_m\otimes Z^m+\cdots+b_0\in B$, then $d(b_j)\le 3m-2j$ for all $0\le j\le m$. In particular $d(b_m)\le m$.*

Proof. Let $b=b_m\otimes Z^m+\cdots+b_0\in B$, then it follows from Proposition 6.9 that $\dot E^{[m/2]+m+1-j}(b_j)=0$ for all $0\le j\le m$. Hence in view of (6) and Proposition 4.2 it follows that $b_j\in\bigoplus U(\mathfrak{k})^M_\gamma$, where the sum extends over all $\gamma\in\Gamma_1$ such that $d(\gamma)\le 3m-2j$. Therefore $d(b_j)\le 3m-2j$ as we wanted to prove. ∎

6.3. Weyl group invariance of the leading term. We shall now prove the second condition required by Theorem 2.2. That is, we need to show that if $b\in B$ then its leading term $\widetilde b=b_m\otimes Z^m\in\big(U(\mathfrak{k})^M\otimes U(\mathfrak{a})\big)^W$. As in the case $\mathrm{SO}(2p,1)$, since the non trivial element of W can be represented by an element in M_o' which acts on $\mathfrak{g}$ as the Cartan involution, it is enough to prove that m is even.

As before, to any $b(x)\in U(\mathfrak{k})[x]$ we associate $c(x)\in U(\mathfrak{k})[x]$ defined by $c(x)=b(x+H-1)$ where H is defined in (10). Recall that if $b(x)\in U(\mathfrak{k})^M[x]$ then $c(x)\in U(\mathfrak{k})^M[x]$ (see Lemma 6.1). Whenever necessary we shall refer to $c(x)$ as $c_1(x)$ or $c_2(x)$ according as $\alpha=\alpha_1$ or $\alpha=\alpha_n$. On the other hand, $c(x)$ will generically stand for $c_1(x)$ or $c_2(x)$. Also, as before we shall find it convenient to write $c_i(x)=\sum_{j=0}^m c_{i,j}\varphi_j(x)$ with $c_{i,j}\in U(\mathfrak{k})$ for $i=1,2$.

Proposition 6.11. *Let $r\in\mathbb{N}_o$, $0\le r\le m$. If $b=b_m\otimes Z^m+\cdots+b_0\in B$ and $\dot E_1^{m+r+1-j}(c_{1,j})=\dot E_1^{m+r+1-j}(c_{2,j})=0$ for $r+1\le j\le m$ then*

$$\dot E_1^{m-j}(c_{1,r+j})E_1^j=(-1)^{m-r}\dot E_1^{r+j}(c_{1,m-j})E_1^{m-r-j}$$

and

$$\dot E_1^{m-j}(c_{2,r+j})E_1^j=\dot E_1^{r+j}(c_{2,m-j})E_1^{m-r-j}$$

for $j=0,\dots,\left[\frac{m-r}{2}\right]$.

Proof. If we set $\ell=m-j$ and $n=r+j$ in equation (iii) of Theorem 3.1 we get,

$$\begin{aligned}&\dot E_1^{m-j}\big(c_1^{(r+j)}\big)(-\tfrac{r+j}{2}+m-j-\widetilde Y_1)E_1^{r+j}\\&\qquad-(-1)^{m-r}\dot E_1^{r+j}\big(c_1^{(m-j)}\big)(-\tfrac{m-j}{2}+r+j-\widetilde Y_1)E_1^{m-j}\equiv 0.\end{aligned}$$

By hypothesis $\dot E_1^{m-j}\big(c_1^{(r+j)}\big)=\sum_k\dot E_1^{m-j}(c_{1,k})\varphi_{k-r-j}=\dot E_1^{m-j}(c_{1,r+j})$, and the first assertion follows from Theorem 6.7 (i).

In a similar way we obtain

$$\dot{E}_2^{m-j}(c_{2,r+j})E_2^j = (-1)^{m-r}\dot{E}_2^{r+j}(c_{2,m-j})E_2^{m-r-j}.$$

Then the second assertion is a direct consequence of Theorem 6.7 (ii). ∎

In order to get a better insight of Proposition 6.11, for $r = 0, \dots, m+1$ we introduce the column vectors $\sigma_r = \sigma_r(b)$ and $\tau_r = \tau_r(b)$ of $m+r+1$ entries defined by

$$\sigma_r = (0, \dots, 0, \dot{E}_1^r(c_{1,m})E_1^{m-r}, \dots, \dot{E}_1^{m-1}(c_{1,r+1})E_1, \dot{E}_1^m(c_{1,r}), 0, \dots, 0)^t,$$

$$\tau_r = (\underbrace{0, \dots, 0}_{r}, \underbrace{\dot{E}_1^r(c_{2,m})E_1^{m-r}, \dots, \dot{E}_1^{m-1}(c_{2,r+1})E_1, \dot{E}_1^m(c_{2,r})}_{m+1-r}, \underbrace{0, \dots, 0}_{r})^t.$$

Let us observe that by definition $\sigma_{m+1} = \tau_{m+1} = 0$, and that the last $m+1$ entries of σ_r and τ_r are respectively of the form $\dot{E}_1^{r+j}(c_{1,m-j})E_1^{m-r-j}$ and $\dot{E}_1^{r+j}(c_{2,m-j})E_1^{m-r-j}$ for $0 \le j \le m$, see Theorem 6.3 and Lemma 6.7.

Let J_s be the $(s+1)\times(s+1)$ matrix with ones in the skew diagonal and zeros everywhere else, thus

$$J_s = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}. \tag{21}$$

In the following corollary we rephrase Proposition 6.11 in terms of the vectors σ_r and τ_r.

Corollary 6.12. *Let $r \in \mathbb{N}_o$, $0 \le r \le m$. If $b = b_m \otimes Z^m + \cdots + b_0 \in B$ and $\sigma_{r+1} = \tau_{r+1} = 0$ then*

$$J_{m+r}\sigma_r = (-1)^{m+r}\sigma_r \qquad \textit{and} \qquad J_{m+r}\tau_r = \tau_r.$$

The vectors σ_r and τ_r are nicely related by a Pascal matrix. Let P_k denote the following $(k+1)\times(k+1)$ lower triangular matrix

$$P_k = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 2 & 1 & & \\ \cdot & & & \cdot & \\ \cdot & & & & \cdot \\ \cdot & & & & & \cdot \\ \binom{k}{0} & \cdot & \cdot & \cdots & \binom{k}{k} \end{pmatrix}. \tag{22}$$

Proposition 6.13. *If $r \in \mathbb{N}_o$, $0 \le r \le m$ and $\sigma_{r+1} = 0$, then $P_{m+r}\sigma_r = \tau_r$.*

Proof. Since $c_2(x) = c_1(x-T)$, for any $0 \le j \le m-r$ we have

$$c_{2,r+j} = c_2^{(r+j)}(0) = c_1^{(r+j)}(-T) = \sum_{s=0}^{m-r-j} c_{1,r+j+s}\varphi_s(-T).$$

On the other hand, since $\dot{E}_1(T) = -E_1$, it follows that $\dot{E}_1^k((-T)^k) = k!E_1^k$ and $\dot{E}_1^t((-T)^k) = 0$ if $t > k$. Therefore, since $\varphi_k(-T) = \frac{1}{k!}(-T)^k + \cdots$, where the dots stand for lower degree terms in T, we have $\dot{E}_1^k(\varphi_k(-T)) = E_1^k$ and $\dot{E}_1^t(\varphi_k(-T)) = 0$ if $t > k$. Now the hypothesis $\sigma_{r+1} = 0$ together with Theorem 6.2 imply that $\dot{E}_1^{m+r+1-i}(c_{1,i}) = 0$ for every $0 \le i \le m$. Hence, for any $-r \le j \le m-r$ using the Leibnitz rule we obtain

$$\begin{aligned}
\dot{E}_1^{m-j}(c_{2,r+j})E_1^j &= \sum_{s=0}^{m-r-j} \dot{E}_1^{m-j}(c_{1,r+j+s}\varphi_s(-T))E_1^j \\
&= \sum_{s=0}^{m-r-j}\sum_{\ell=0}^{m-j} \binom{m-j}{\ell} \dot{E}_1^{m-j-\ell}(c_{1,r+j+s})\dot{E}_1^{\ell}(\varphi_s(-T))E_1^j \\
&= \sum_{s=0}^{m-r-j} \binom{m-j}{s} \dot{E}_1^{m-j-s}(c_{1,r+j+s})E_1^{s+j},
\end{aligned}$$

which implies that the last $m+1$ components of $P_{m+r}\sigma_r$ and τ_r are equal. Since by definition the first r components of $P_{m+r}\sigma_r$ and τ_r are equal to 0 the proposition follows. ∎

For $t \in \mathbb{N}_o$ we shall be interested in considering certain $(t+1) \times (t+1)$ submatrices of a Pascal matrix P_n formed by any choice of $t+1$ consecutive rows and $t+1$ consecutive columns of P_n, with the only condition that the submatrix does not have zeros in its main diagonal. To be precise, for any $0 \le a, b \le n$, $a, b \in \mathbb{N}_o$ such that $b \le a$ we shall be interested in submatrices A of P_n of the following form

$$A = \begin{pmatrix} \binom{a}{b} & \binom{a}{b+1} & \cdot\ \cdot\ \cdot & \binom{a}{b+t} \\ \binom{a+1}{b} & \binom{a+1}{b+1} & \cdot\ \cdot\ \cdot & \binom{a+1}{b+t} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \binom{a+t}{b} & \binom{a+t}{b+1} & \cdot\ \cdot\ \cdot & \binom{a+t}{b+t} \end{pmatrix}. \tag{23}$$

In the following proposition we collect some results that will be very important in the proof of our goal, that is, that the algebra B does not contain elements of odd degree. The proof of this proposition will be given in an appendix at the end of this section.

Proposition 6.14. *If J_n and P_n are as in* (21) *and* (22) *we have,*
(i) If $v \in \mathbb{C}^{n+1}$ satisfies $J_n v = (-1)^n v$ and $J_n P_n v = P_n v$ then v begins and ends with the same number of coordinates, say k, equal to zero. Moreover, k is even or odd according as n is even or odd, respectively.
(ii) If A is a $(t+1) \times (t+1)$ submatrix of P_n of the form (23) then A is non-singular.

Lemma 6.15. *Let $n \in \mathbb{N}_0$ be an even number and let $v \in U(\mathfrak{k})^M$ be such that $\dot{E}^{t+1}(v) = 0$. If $n \ge 2t$ then there exists $b \in B$ of degree n with $b_n = v$ and $\sigma_{t+1}(b) = 0$.*

Proof. The proof will be by induction on n. If $n = 0$ the assertion follows from Proposition 4.2 and Proposition 2.1. Let us now take $n > 0$ even and consider $S = \{b \in B : \deg(b) = n \text{ and } b_n = v\}$. From Proposition 2.1 we know that S is nonempty, because from Proposition 4.2 we obtain $d(v) \leq 2t \leq n$. For each $b \in S$ let $r(b) \in \mathbb{N}_o$ be such that $\sigma_{r(b)+1}(b) = 0$ and $\sigma_{r(b)}(b) \neq 0$, and let $r = \min\{r(b) : b \in S\}$. We want to prove that $r \leq t$.

Let us assume that $r > t$ and let us take $b \in S$ such that $r(b) = r$. We have

$$\sigma_r(b) = (\underbrace{0, \dots, 0}_{r}, \underbrace{\dot{E}_1^r(c_{1,n})E_1^{n-r}, \dots, \dot{E}_1^{n-1}(c_{1,r+1})E_1, \dot{E}_1^n(c_{1,r})}_{n+1-r}, \underbrace{0, \dots, 0}_{r})^t,$$

$$J_{n+r}\sigma_r(b) = (-1)^{n+r}\sigma_r(b) \qquad \text{and} \qquad J_{n+r}P_{n+r}\sigma_r(b) = P_{n+r}\sigma_r(b).$$

Since $r > t$ the hypothesis $\dot{E}^{t+1}(v) = 0$ implies that the number of zeros with which $\sigma_r(b)$ starts is of the form $r + j_0$ with $j_0 \geq 1$. Thus we have

$$\sigma_r(b) = (\underbrace{0, \dots, 0}_{r+j_0}, \underbrace{\dot{E}_1^{r+j_0}(c_{1,n-j_0})E_1^{n-j_0-r}, \dots, \dot{E}_1^{n-j_0}(c_{1,r+j_0})E_1^{j_0}}_{n+1-r-2j_0}, \underbrace{0, \dots, 0}_{r+j_0})^t,$$

with j_0 even. From $\sigma_r(b) \neq 0$ we get $n + 1 - r - 2j_0 > 0$ and from the definition of j_0 we obtain $\dot{E}_1^{r+j_0}(c_{1,n-j_0}) \neq 0$. Among all $b \in S$ with $\sigma_r(b) \neq 0$ we choose one with the largest j_0.

Let $n' = n - j_0$, $t' = r + j_0$, $v' = c_{1,n-j_0}$. Since $\sigma_{r+1}(b) = 0$ we have $\dot{E}_1^{t'+1}(v') = 0$. Now we shall consider the following two possibilities: $n' \geq 2t'$ and $n' < 2t'$, in both cases we will get a contradiction that will finish the prove of the lemma.

If $n' \geq 2t'$ then the inductive hypothesis implies that there exists $b' \in B$ of degree n' such that $b'_{n'} = v'$ and $\sigma_{t'+1}(b') = 0$, thus

$$(\underbrace{0, \dots, 0}_{r+j_0+1}, \underbrace{\dot{E}_1^{r+j_0+1}(c'_{1,n-j_0})E_1^{n-2j_0-r-1}, \dots, \dot{E}_1^{n-j_0}(c'_{1,r+j_0+1})}_{n-r-2j_0}, \underbrace{0, \dots, 0}_{r+j_0+1})^t = 0.$$

Therefore $\sigma_{r+1}(b - b') = 0$. This is a contradiction because either $\sigma_r(b - b')$ starts with more zeros than $\sigma_r(b)$ or $r(b - b') < r$.

On the other hand if $n' < 2t'$ then $n - r - 2j_0 < r + j_0$. Let A be the submatrix of P_{n+r} formed by the elements in the last $n + 1 - r - 2j_0$ rows and in the $n + 1 - r - 2j_0$ central columns of P_{n+r}. From Proposition 6.14 we know that A is nonsingular.

Since $P_{n+r}\sigma_r(b) = \tau_r(b)$, $\tau_r(b)$ starts with $r + j_0$ zeros, and $J_{n+r}\tau_r(b) = \tau_r(b)$ implies that the last $r + j_0$ coordinates of $\tau_r(b)$ are also zeros. Therefore the equation $P_{n+r}\sigma_r(b) = \tau_r(b)$ implies that the vector u formed by the $n + 1 - r - 2j_0$ central coordinates of $\sigma_r(b)$ satisfies $Au = 0$, since $n + 1 - r - 2j_0 \leq r + j_0$. This is a contradiction because $\sigma_r(b) \neq 0$. This completes the proof of the lemma. ■

We are now in a position to prove that the algebra B does not have elements of odd degree, which will complete the proof of the Theorem 1.2 when G_o is locally isomorphic to $\mathrm{SU}(n, 1)$, $n \geq 2$.

Theorem 6.16. *If G_o is locally isomorphic to $SU(n, 1)$ with $n \geq 2$, and $b = b_m \otimes Z^m + \cdots + b_0 \in B$ with m odd, then $b_m = 0$. That is, B does not contain odd degree elements.*

Proof. Let $B_o = \{b \in B : \deg(b) \text{ is odd}\}$ and let us assume that B_o is not empty. Now define $r = \min\{t \in \mathbb{N}_o : \sigma_{t+1}(b) = 0 \text{ and } b \in B_o\}$ and take $b \in B_o$ such that $\sigma_{r+1}(b) = 0$; clearly $\sigma_r(b) \neq 0$. Let $m = m(b)$ denote the degree of b. Then in view of Corollary 6.12 and Proposition 6.13 we have

$$J_{m+r}\sigma_r(b) = (-1)^{m+r}\sigma_r(b) \quad \text{and} \quad J_{m+r}P_{m+r}\sigma_r(b) = P_{m+r}\sigma_r(b). \tag{24}$$

Hence the vector $\sigma_r(b)$ satisfies the conditions of part (i) of Proposition 6.14, therefore if r is even $\sigma_r(b)$ begins (and ends) with an odd number of coordinates equal to zero and, on the other hand, if r is odd $\sigma_r(b)$ begins (and ends) with an even number of coordinates equal to zero.

We recall that the first and the last r coordinates of $\sigma_r(b)$ are zero and that the others are

$$\dot{E}_1^{r+j}(c_{1,m-j})E_1^{m-r-j}, \qquad j = 0, \dots, m-r.$$

Therefore $\dot{E}_1^r(c_{1,m}) = 0$. Let $j_0(b) = \max\{j \in \mathbb{N}_o : \dot{E}_1^{r+t}(c_{1,m-t}) = 0 \text{ for all } 0 \leq t \leq j \leq m-r-1\}$. Then we know that $j_0(b)$ is even and that $m - r - 2j_0(b) - 1 > 0$ because $\sigma_r(b) \neq 0$, $\sigma_r(b)$ starts with $r + j_0(b) + 1$ zeros and $J_{m+r}\sigma_r(b) = (-1)^{m+r}\sigma_r(b)$.

Among all $b \in B_o$ such that $\sigma_{r+1}(b) = 0$ we choose one such that $j_0 = j_0(b)$ is the largest possible. We also have $m - j_0 - 1 < 2(r + j_0 + 1)$, because from $m - j_0 - 1 \geq 2(r + j_0 + 1)$ and $\sigma_{r+1}(b) = 0$ we would obtain $d(c_{1,m-j_0-1}) = 2(r + j_0 + 1) \leq m - j_0 - 1$. Hence from Lemma 6.15 we would know that there exist $b' = c_{1,m-j_0-1} \otimes Z^{m-j_0-1} + \cdots \in B$ such that $\sigma_{r+j_0+2}(b') = 0$ and the element $b - b' \in B_o$ would contradict the maximality of j_0.

Let A be the submatrix of P_{m+r} formed by the elements in the last $m-r-2j_0-1$ rows and in the $m-r-2j_0-1$ central columns of P_{m+r}. From Proposition 6.14 we know that A is nonsingular. Since $P_{m+r}\sigma_r(b) = \tau_r(b)$, $\tau_r(b)$ starts with $r+j_0+1$ zeros and since $J_{m+r}\tau_r(b) = \tau_r(b)$ the last $r+j_0+1$ coordinates of $\tau_r(b)$ are also zeros. Therefore the equation $P_{m+r}\sigma_r(b) = \tau_r(b)$ implies that the vector u formed by the $m-r-2j_0-1$ central coordinates of $\sigma_r(b)$ satisfies $Au = 0$, since $m-r-2j_0-1 \leq r+j_0+1$. This is a contradiction because $\sigma_r(b) \neq 0$. This completes the proof of the theorem. ∎

7. Appendix

Our goal in this appendix is to prove Proposition 6.14. For any $n \in \mathbb{N}_o$ let J_n and P_n be the $(n+1) \times (n+1)$ matrices defined in (21) and (22), and let H_n be the following $(n+1) \times (n+1)$ diagonal matrix

$$H_n = \begin{pmatrix} (-1)^n & & & & \\ & (-1)^{n-1} & & & \\ & & \ddots & & \\ & & & -1 & \\ & & & & 1 \end{pmatrix}.$$

Let V denote the vector space over $\mathbb{C}$ of all polynomials in $\mathbb{C}[X]$ of degree less or equal to n. Then P_n, H_n and J_n are respectively the matrices of the linear operators on V given by

$$f(X) \mapsto f(X+1), \quad f(X) \mapsto f(-X), \quad f(X) \mapsto X^n f(1/X), \tag{25}$$

with respect to the ordered basis $\{\binom{n}{n}X^n, \binom{n}{n-1}X^{n-1}, \dots, \binom{n}{0}\}$. In the next lemma we summarize some basic properties of the matrices P_n, H_n and J_n. The proof of this lemma follows from simple calculations with the operators given in (25).

Lemma 7.1. *(i) $J_n^2 = H_n^2 = I$ and $J_nH_n = (-1)^nH_nJ_n$.*
(ii) $P_n^{-1} = H_nP_nH_n$.
(iii) J_n and P_nH_n are conjugate, in fact $J_n = (J_nP_nH_n)^{-1}P_nH_n(J_nP_nH_n)$. Hence the eigenvectors of P_nH_n associated to the eigenvalue $\lambda = \pm 1$ are all of the form $J_nP_nH_n(v)$ where v is an eigenvector of J_n associated to the eigenvalue λ.

Now, let $k \in \mathbb{N}_o$ and let $v = (v_o, \dots, v_n)$ be a vector in $\mathbb{C}^{n+1}$. We shall say that v begins with k coordinates equal to zero if $v_o = v_1 = \cdots = v_{k-1} = 0$ and $v_k \neq 0$. Similarly we shall say that v ends with k coordinates equal to zero if $v_{n-k+1} = v_{n-k+2} = \cdots = v_n = 0$ and $v_{n-k} \neq 0$. Also via the ordered basis $\{\binom{n}{n}X^n, \binom{n}{n-1}X^{n-1}, \dots, \binom{n}{0}\}$ we shall identify any vector $v = (v_o, \dots, v_n) \in \mathbb{C}^{n+1}$ with the polynomial $f_v(X) = v_0\binom{n}{n}X^n + v_1\binom{n}{n-1}X^{n-1} + \cdots + v_n$. In particular observe that v begins with k coordinates equal to zero if and only if the degree of f_v is $n-k$. In the following lemma we prove part (i) of Proposition 6.14.

Lemma 7.2. *If $v \in \mathbb{C}^{n+1}$ satisfies $J_nv = (-1)^nv$ and $J_nP_nv = P_nv$ then v begins and ends with the same number of coordinates, say k, equal to zero. Moreover, k is even or odd according as n is even or odd, respectively.*

Proof. Let $v \in \mathbb{C}^{n+1}$ be as in the statement of the lemma and assume that v begins with k coordinates equal to zero. If we identify v with the polynomial f_v defined above we claim that the degree of f_v is even. In fact from Lemma 7.1 it follows that $H_n(v)$ is an eigenvector of J_n associated to the eigenvalue 1, and that $J_nP_nH_n(H_nv) = J_nP_nv = P_nv$ is an eigenvector of P_nH_n associated to the eigenvalue 1. Then $P_nH_n(P_nv) = P_nv$, which implies that $H_nP_nv = v$ or, equivalently, that $f_v(1-X) = f_v(X)$. Now if we define $g(X) = f_v(X + \frac{1}{2})$ we obtain $g(X) = g(-X)$, which in particular implies that the degree of g is even. Hence the degree of f_v is even. The other assertion is a direct consequence of $J_nv = (-1)^nv$. ■

We shall now prove part (ii) of Proposition 6.14. Let $t, a, b \in \mathbb{N}_o$ be such that $b \leq a \leq n$ and let A be the $(t+1)\times(t+1)$ submatrix, of the Pascal matrix P_n, defined in (23). We want to prove that A is nonsingular. Associated to the parameters t, a, b we shall consider a $(t+1)\times(t+1)$ diagonal matrix D_x defined for $x \in \mathbb{N}_o$, $x \geq b$, as follows

$$D_x = \begin{pmatrix} \binom{x}{b} & & & \\ & \binom{x+1}{b} & & \\ & & \ddots & \\ & & & \binom{x+t}{b} \end{pmatrix},$$

and a $(t+1)\times(t+1)$ matrix A_0 of the following form

$$A_0 = \begin{pmatrix} \binom{a-b}{0} & \cdots & \binom{a-b}{t} \\ \vdots & & \vdots \\ \binom{a-b+t}{0} & \cdots & \binom{a-b+t}{t} \end{pmatrix}. \tag{26}$$

The following lemma establishes the desired result about A.

Lemma 7.3. *Let $t, a, b \in \mathbb{N}_o$ be such that $b \le a \le n$ and let A, D_x and A_0 be as above. Then*
(i) $A = D_a A_0 D_b^{-1}$,
(ii) $\det A = \prod_{i=0}^{t} \binom{a+i}{b}\binom{b+i}{b}^{-1}$ and therefore A is nonsingular.

Proof. (i) For $0 \le i, j \le t$ let $A_{i,j}$ denote the (i,j) entry of the matrix A, then we have

$$\begin{aligned} A_{i,j} = \binom{a+i}{b+j} &= \frac{(a+i)!}{(b+j)!(a-b+i-j)!} \\ &= \frac{(a+i)!}{b!(a-b+i)!}\frac{(a-b+i)!}{j!(a-b+i-j)!}\frac{b!j!}{(b+j)!} \\ &= \binom{a+i}{b}\binom{a-b+i}{j}\binom{b+j}{b}^{-1}. \end{aligned}$$

Since the right hand side of this equality is the (i,j) entry of the product $D_a A_0 D_b^{-1}$ (i) follows.

In order to prove (ii) it is enough to show that $\det A_0 = 1$ for any matrix A_0 as in (26). We proceed by induction on t. It is clear that the result holds for $t=0$, so let us assume that it holds for any matrix as in (26) of size $t\times t$ and let A_0 be the $(t+1)\times(t+1)$ matrix defined in (26). Let $C_0, C_1, \dots, C_t$ denote the rows of A_0. Since for any $0 \le j \le t-1$ we have

$$\binom{a-b+j+1}{i} - \binom{a-b+j}{i} = \begin{cases} 0, & \text{if } i = 0 \\ \binom{a-b+j}{i-1}, & \text{if } 1 \le i \le t, \end{cases}$$

we obtain for any $0 \le j \le t-1$ that

$$C_{j+1} - C_j = \left(0, \binom{a-b+j}{0}, \dots, \binom{a-b+j}{t-1}\right).$$

Then if we regard $\det A_0$ as a multilinear function of the rows of A_0 we get,

$$\begin{aligned} \det A_0 &= \det\left(C_0, C_1 - C_0, \dots, C_t - C_{t-1}\right) \\ &= \det \begin{pmatrix} \binom{a-b}{0} & \binom{a-b}{1} & \cdots & \binom{a-b}{t} \\ 0 & \binom{a-b}{0} & \cdots & \binom{a-b}{t-1} \\ \vdots & \vdots & & \vdots \\ 0 & \binom{a-b+t-1}{0} & \cdots & \binom{a-b+t-1}{t-1} \end{pmatrix} = \det \begin{pmatrix} \binom{a-b}{0} & \cdots & \binom{a-b}{t-1} \\ \vdots & & \vdots \\ \binom{a-b+t-1}{0} & \cdots & \binom{a-b+t-1}{t-1} \end{pmatrix} = 1, \end{aligned}$$

by the inductive hypothesis. This completes the proof of the lemma. ■

References

[1] Bourbaki, N., *Groupes et algèbre de Lie, Chap. 7 et 8,* Masson, Paris, 1990.

[2] Brega, A., and L. Cagliero, *LU-decomposition of a noncommutative linear system and Jacobi polynomials*, J. of Lie Theory **19** (2009), 463–481.

[3] Brega, A., and J. Tirao, *A transversality property of a derivation of the universal enveloping algebra U(k), for SO(n,1) and SU(n,1),* Manuscripta math. **74** (1992), 195–215.

[4] Brega, A., L. Cagliero, and J. Tirao, *The image of the Lepowsky homomorphism for the split rank one symplectic group*, J. of Algebra **320** (2008), 996–1050.

[5] Cagliero, L., and J. Tirao, *(K,M)-spherical modules of a rank one semisimple Lie group* G_o, Manuscripta math. **113** (2004), 107–124.

[6] Johnson, K., and N. Wallach, *Composition series and intertwining operators for the spherical principal series I*, Trans. Amer. Math. Soc. **229** (1977), 137–173.

[7] Knop, F., *A Harish-Chandra homomorphism for reductive group actions*, Ann. of Math. **140** (1994), 253–288.

[8] Kostant, B., *On the centralizer of* K *in* $U(\mathfrak{g})$, J. of Algebra **313** (2007), 252–267.

[9] Kostant, B., and S. Rallis, *Orbits and representations associated with symmetric spaces*, Amer. J. Math. **93** (1971), 753–809.

[10] Kostant, B., and J. Tirao, *On the structure of certain subalgebras of a universal enveloping algebra*, Trans. Amer. Math. Soc. **218** (1976), 133–154.

[11] Lepowsky, J., *Algebraic results on representations of semisimple Lie groups*, Trans. Amer.Math. Soc. **176** (1973), 1–44.

[12] Tirao, J., *On the centralizer of* K *in the universal enveloping algebra of* $\mathrm{SO}(n,1)$ *and* $\mathrm{SU}(n,1)$, Manuscripta math. **85** (1994), 119–139.

[13] —, *On the structure of the classifying ring of* $\mathrm{SO}(n,1)$ *and* $\mathrm{SU}(n,1)$, Rev. Unión Mat. Arg., **40** (1996), 15–31.

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