The Image of the Lepowsky Homomorphism for $SO(n, 1)$ and $SU(n, 1)$

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Communicated by E. Zelmanov

Abstract. Let $G_o$ be a classical rank one semisimple Lie group and let $K_o$ denote a maximal compact subgroup of $G_o$. Let $U(g)$ be the complex universal enveloping algebra of $G_o$ and let $U(g)^K$ denote the centralizer of $K_o$ in $U(g)$. Also let $P : U(g) \rightarrow U(t) \otimes U(a)$ be the projection map corresponding to the direct sum $U(g) = (U(t) \otimes U(a)) \oplus U(g)n$ associated to an Iwasawa decomposition of $G_o$ adapted to $K_o$. In this paper we give a characterization of the image of $U(g)^K$ under the injective antihomorphism $P : U(g)^K \rightarrow U(t)^M \otimes U(a)$ when $G_o$ is locally isomorphic to $SO(n, 1)$ and $SU(n, 1)$.

Mathematics Subject Classification 2000: 22E46, 16S30, 16U70.

Key Words and Phrases: Semisimple Lie groups; Universal enveloping algebra; Representation theory; Group invariants; Restriction theorem; Kostant degree.

1. Introduction

Let $G_o$ be a connected, noncompact, real semisimple Lie group with finite center, and let $K_o$ denote a maximal compact subgroup of $G_o$. We denote with $g_o$ and $k_o$ the Lie algebras of $G_o$ and $K_o$, and $k_o \subset g_o$ will denote the respective complexified Lie algebras. Let $U(g)$ be the universal enveloping algebra of $g$ and let $U(g)^K$ denote the centralizer of $K_o$ in $U(g)$.

Let $P : U(g) \rightarrow U(t) \otimes U(a)$ be the projection map corresponding to the direct sum $U(g) = (U(t) \otimes U(a)) \oplus U(g)n$, associated to an Iwasawa decomposition $g = k \oplus a \oplus n$ adapted to $k$. Let $G_o = K_oA_oN_o$ be the corresponding Iwasawa decomposition for $G_o$.

If $U(t)^M$ denotes the centralizer of $M_o$ in $U(k)$, $M_o$ being the centralizer of $A_o$ in $K_o$, then it is known (see [11]) that one has the exact sequence

$$0 \rightarrow U(g)^K \xrightarrow{P} U(t)^M \otimes U(a),$$

and that $P$ becomes an antihomomorphism of algebras if $U(t)^M \otimes U(a)$ is given the tensor product algebra structure. However, the image of $P$ is not yet well understood, we refer the reader to [10], [12] and [4] for further information.

* This research is partially supported by CONICET grant PIP 112-200801-01533.
In order to determine the actual image $P(U(g)^K)$, Tirao introduced in [12] a subalgebra $B$ of $U(t)^M \otimes U(a)$ defined by a set of linear equations derived from certain embeddings between Verma modules, and proved that $P(U(g)^K) = B^{W_\rho}$ when $G_0$ is locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$. Here, $W$ is the Weyl group of the pair $(g, a)$, $\rho$ is half the sum of the positive roots of $g$, and $B^{W_\rho}$ is the subalgebra of all elements in $B$ that are invariant under the tensor product action of $W$ on $U(t)^M$ and the translated action of $W$ on $U(a)$. Recently, in [4], we extended this result to the symplectic group $Sp(n, 1)$. In fact we obtained the following stronger result.

Theorem 1.1. If $G_o$ is locally isomorphic to $Sp(n, 1)$ then $P(U(g)^K) = B$.

We announced in [4] that the above result also holds for $SO(2n, 1)$, $SU(n, 1)$ and $F_4$. We did not know at that time whether $B = B^{W_\rho}$ for $SO(2n+1, 1)$ and we had not yet completed the proof for $F_4$. Now we know that the following theorem holds.

Theorem 1.2. If $G_o$ be locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$ it follows that

$$P(U(g)^K) = B,$$

and moreover $B = B^{W_\rho}$.

This paper is devoted to proving this theorem. As we mentioned above, it was proved in [12] that $P(U(g)^K) = B^{W_\rho}$ when $G_0$ is locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$. Thus the main contribution of this paper is that $B = B^{W_\rho}$ for $SO(n, 1)$ and $SU(n, 1)$. Additionally, we give a new and simpler proof of the fact that $P(U(g)^K) = B$. We are still working to complete the details for $F_4$.

The projection $P$ was originally introduced by Kostant long time ago in order to contribute to the understanding of the structure and representation theory of $U(g)^K$. The need for the study of the algebra $U(g)^K$ arises from the fundamental work of Harish-Chandra relating the infinite-dimensional representation theory of $G_o$ to the finite-dimensional representation theory of $U(g)^K$. Since then, there were a number of results on the structure of $U(g)^K$, see notably [7]. However, the study of $U(g)^K$ is acknowledged to be very difficult and the infinite-dimensional representation theory of $G_o$ has been approached by different means.

On the other hand, the algebra $B$ turns out to be an isomorphic copy of $U(g)^K$ strictly1 contained in $U(t)^M \otimes U(a)$ that is defined by a set of linear equations. The fact that we were able to prove that $B = B^{W_\rho}$ keeps alive the hope that it could help to understand the structure of $U(g)^K$.

2. The algebra $B$ and the image of $U(g)^K$

Assume that $G_o$ is a connected, noncompact real semisimple Lie group, with finite center and split rank one. Let $G_o = K_oA_oN_o$ be the an Iwasawa decomposition of $G_o$, let $\mathfrak{k}_o$, $\mathfrak{a}_o$ and $\mathfrak{n}_o$ be the corresponding Lie algebras and let $\mathfrak{k}$, $\mathfrak{a}$ and $\mathfrak{n}$ be their complexifications.

Let $\mathfrak{t}_o$ be a Cartan subalgebra of the Lie algebra $\mathfrak{m}_o$ of $M_o$. Set $\mathfrak{h}_o = \mathfrak{t}_o \oplus \mathfrak{a}_o$ and let $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{a}$ be the corresponding complexification. Then $\mathfrak{h}_o$ and $\mathfrak{h}$ are Cartan subalgebras of $\mathfrak{g}_o$ and $\mathfrak{g}$, respectively. Choose a Borel subalgebra $\mathfrak{t} \oplus \mathfrak{m}^+$ of the complexification $\mathfrak{m}$ of $\mathfrak{m}_o$ and take $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{m}^+ \oplus \mathfrak{n}$ as a Borel subalgebra of $\mathfrak{g}$.

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1See for example Theorem 2.2
Let $\Delta$ and $\Delta^+$ be, respectively, the corresponding sets of roots and positive roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. As usual $\rho$ is half the sum of the positive roots, $\theta$ the Cartan involution and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ the Cartan decomposition of $\mathfrak{g}$ corresponding to $(G_\alpha, K_\alpha)$. Also, if $\alpha \in \Delta$, $X_\alpha$ will be a nonzero root vector associated to $\alpha$ and $E_\alpha = X_\alpha - \theta X_\alpha$.

If $\langle , \rangle$ denotes the Killing form of $\mathfrak{g}$, for each $\alpha \in \Delta$ let $H_\alpha \in \mathfrak{h}$ be the unique element such that $\phi(H_\alpha) = 2\langle \phi, \alpha \rangle / \langle \alpha, \alpha \rangle$ for all $\phi \in \mathfrak{h}^*$, and let $\mathfrak{h}_\alpha$ be the real span of $\{H_\alpha : \alpha \in \Delta\}$. Also set $H_\alpha = Y_\alpha + Z_\alpha$ where $Y_\alpha \in \mathfrak{t}$ and $Z_\alpha \in \mathfrak{a}$, and let $P_+ = \{\alpha \in \Delta^+ : Z_\alpha \neq 0\}$. For each $\alpha \in P_+$ we can consider the elements in $U(\mathfrak{t}) \otimes U(\mathfrak{a})$ as polynomials in $Z_\alpha$ with coefficients in $U(\mathfrak{t})$. Then, let $B$ be the algebra of all $b \in U(\mathfrak{t})^m \otimes U(\mathfrak{a})$ that satisfy

$$E_\alpha^n b(n - Y_\alpha - 1) \equiv b(-n - Y_\alpha - 1)E_\alpha^n \mod (U(\mathfrak{t})m^+) \quad (1)$$

for all simple roots $\alpha \in P_+$ and all $n \in \mathbb{N}$. We know, from Theorem 5 and Corollary 6 of [12], that $P(U(\mathfrak{g})^K) \subset B$ for all rank one groups, and moreover, that $P(U(\mathfrak{g})^K) = B^{W_\rho}$ for $\text{SO} (n, 1)$ and $\text{SU} (n, 1)$.

Since in this paper we shall be concerned with $G_\alpha$ locally isomorphic to $\text{SO}(n, 1)$ or $\text{SU}(n, 1)$, we recall that in this case there is only one simple root in $P_+$ if $G_\alpha$ is locally isomorphic to $\text{SO}(n, 1)$ for $n > 3$, and there are two simple roots in $P_+$ if $G_\alpha$ is locally isomorphic to $\text{SO}(3, 1)$ or $\text{SU}(n, 1)$ for $n \geq 2$.

Let $G$ be the adjoint group of $\mathfrak{g}$ and let $K$ be the connected Lie subgroup of $G$ with Lie algebra $ad_\mathfrak{g}(\mathfrak{t})$. Also let $M = \text{Centr}_K(\mathfrak{a})$, $M' = \text{Norm}_K(\mathfrak{a})$ and $W = M'/M$. Let $\Gamma$ denote the set of all equivalence classes of irreducible holomorphic finite dimensional $K$-modules, such that $V^M_\gamma \neq 0$. Any $\gamma \in \Gamma$ can be realized as a submodule of all harmonic polynomial functions on $\mathfrak{p}$, homogeneous of degree $d$, for a uniquely determined $d = d(\gamma)$ (see [9]). If $V$ is any $K$-module and $\gamma \in \hat{K}$ then $V_\gamma$ will denote the isotypic component of $V$ corresponding to $\gamma$. Let $U(\mathfrak{t})_d = \bigoplus U(\mathfrak{t})^\gamma$, where the sum extends over all $\gamma \in \Gamma$ such that $d(\gamma) \leq d$. Then $U(\mathfrak{t})^M = \bigcup_{d \geq 0} U(\mathfrak{t})^M_d$ is an ascending filtration of $U(\mathfrak{t})^M$. If $b \in U(\mathfrak{t})^M$ define $d(b) = \min\{d \in \mathbb{N} : b \in U(\mathfrak{t})^M_d\}$ and call it the Kostant degree of $b$. Since we shall be mainly concerned with representations $\gamma \in \Gamma$ that occur as subrepresentations of $U(\mathfrak{t})$ we set,

$$\Gamma_1 = \{\gamma \in \Gamma : \gamma \text{ is a subrepresentation of } U(\mathfrak{t})\}. \quad (2)$$

If $0 \neq b \in U(\mathfrak{t}) \otimes U(\mathfrak{a})$ we write $b = b_m \otimes Z^m_\alpha + \cdots + b_0$ in a unique way with $b_j \in U(\mathfrak{t})$ for $0 \leq j \leq m$, and $b_m \neq 0$, for any simple root $\alpha \in P_+$. We shall refer to $m$ as the degree of $b$ and to $\tilde{b} = b_m \otimes Z^m_\alpha$ as the leading term of $b$. Let $(U(\mathfrak{t})^M \otimes U(\mathfrak{a}))^W$ denote the ring of $W$-invariants in $U(\mathfrak{t})^M \otimes U(\mathfrak{a})$ under the tensor product of the action of $W$ on $U(\mathfrak{t})^M$ and the action of $W$ on $U(\mathfrak{a})$. The following result was proved in Proposition 2.6 of [4] for any connected, noncompact, real semisimple Lie group $G_\alpha$, with finite center and split rank one.

**Proposition 2.1.** If $\tilde{b} = b_m \otimes Z^m_\alpha \in (U(\mathfrak{t})^M \otimes U(\mathfrak{a}))^W$ and $d(b_m) \leq m$, then there exits $u \in U(\mathfrak{g})^K$ such that $\tilde{b}$ is the leading term of $b = P(u)$.

From this result it follows that Theorem 1.2 is a consequence of the following theorem. We shall prove this statment in Proposition 2.3 below.
Theorem 2.2. If $b = b_m \otimes Z^n + \cdots + b_0 \in B$ and $b_m \neq 0$, then $d(b_m) \leq m$ and its leading term $\tilde{b} = b_m \otimes Z^n \in \big(U(\mathfrak{t})^M \otimes U(\mathfrak{a})\big)^W$.

Proposition 2.3. Theorem 2.2 implies Theorem 1.2.

Proof. We mentioned above that $P(U(\mathfrak{g})^K) = B^{W^r} \subset B$ for $SO(n, 1)$ and $SU(n, 1)$. Let us prove by induction on the degree $m$ of $b \in B$, that $B \subset P(U(\mathfrak{g})^K)$. If $m = 0$ we have $b = b_0 \in U(\mathfrak{t})^M$ and Theorem 2.2 implies that $d(b_0) = 0$. If $\gamma \in \Gamma_1$ and $d(\gamma) = 0$ then $\gamma$ can be realized by constant polynomial functions on $\mathfrak{p}$ and these functions are $K$-invariant. Thus $b_0 \in U(\mathfrak{t})^K$ and therefore $b = b_0 = P(b_0) \in P(U(\mathfrak{g})^K)$.

If $b \in B$ and $m > 0$, from Theorem 2.2 and Proposition 2.1 we know that there exists $v \in U(\mathfrak{g})^K$ such that $P(v) = \tilde{b}$. Then $b - P(v)$ lies in $B$ and the degree of $b - P(v)$ is strictly less than $m$. Hence, by the induction hypothesis, there exists $u \in U(\mathfrak{g})^K$ such that $P(u) = b - P(v)$ and $b = P(u + v) \in P(U(\mathfrak{g})^K)$. This completes the induction argument. Therefore we obtain that $B \subset P(U(\mathfrak{g})^K) = B^{W^r} \subset B$.

The rest of the paper will be devoted to proving Theorem 2.2 when $G_0$ is locally isomorphic to $SO(n, 1)$ or $SU(n, 1)$.

3. The equations defining $B$

To simplify the notation, for a given simple root $\alpha \in P_+$ set $E = E_\alpha$, $Y = Y_\alpha$ and $Z = Z_\alpha$. It follows from Lemma 29 of [12] that $[E, Y] = cE$, where $c = 1$ if $G_0$ is locally isomorphic to $SO(n, 1)$, and $c = \frac{3}{2}$ if $G_0$ is locally isomorphic to $SU(n, 1)$.

We identify $U(\mathfrak{t}) \otimes U(\mathfrak{a})$ with the polynomial ring in one variable $U(\mathfrak{t})[x]$, replacing $Z$ by the indeterminate $x$. To study the equation (1) we shall change the unknown $b(x) \in U(\mathfrak{t})[x]$ by $c(x) \in U(\mathfrak{t})[x]$ defined by

$$c(x) = b(x + H - 1),$$

(3)

where $H = 0$ if $c = 1$, and when $c = \frac{3}{2}$, $H$ is an appropriate vector in $\mathfrak{t}$ to be chosen later, depending on the simple root $\alpha \in P_+$ and such that $[H, E] = \frac{1}{2}E$ (see (10)). If $\tilde{Y} = Y + H$, we have $[E, \tilde{Y}] = E$. This is the main reason for introducing $H$, because it allow us to treat (1) in a unified way in both cases, $c = 1$, $\frac{3}{2}$.

Then $b(x) \in U(\mathfrak{t})[x]$ satisfies (1) if and only if $c(x) \in U(\mathfrak{t})[x]$ satisfies

$$E^n c(n - \tilde{Y}) \equiv c(-n - \tilde{Y}) E^n$$

(4)

for all $n \in \mathbb{N}$. Observe that (4) is an equation in the noncommutative ring $U(\mathfrak{t})$.

Now, if $p$ is a polynomial in one indeterminate $x$ with coefficients in a ring let $p^{(n)}$ denote the $n$-th discrete derivative of $p$. That is, $p^{(1)}(x) = p(x + \frac{1}{2}) - p(x - \frac{1}{2})$ and in general $p^{(n)}(x) = \sum_{j=0}^{n}(-1)^j \binom{n}{j} p(x + \frac{n}{2} - j)$. If $p = p_m x^m + \cdots + p_0$, then

$$p^{(n)}(x) = \begin{cases} 0, & \text{if } n > m \\ m!p_m, & \text{if } n = m. \end{cases}$$

(5)
Also, if \( X \in \mathfrak{k} \) we shall denote with \( \hat{X} \) the derivation of \( U(\mathfrak{k}) \) induced by \( \text{ad}(X) \). Moreover if \( D \) is a derivation of \( U(\mathfrak{k}) \) we shall denote with the same symbol the unique derivation of \( U(\mathfrak{k})[x] \) which extends \( D \) and such that \( Dx = 0 \). Thus for \( b \in U(\mathfrak{k})[x] \) and \( b = b_m x^m + \cdots + b_0 \), we have \( Db = (Db_m)x^m + \cdots + Db_0 \).

Observe that these derivations commute with the operation of taking the discrete derivative in \( U(\mathfrak{k})[x] \).

Next theorem gives a triangularized version of the system (1) that defines the algebra \( B \). The meaning of this will be clarified after the statement of the theorem. Its proof is contained in [2] where the system (4) is studied in a more abstract setting, in particular, an LU-decomposition of its coefficient matrix is obtained.

**Theorem 3.1.** Let \( c \in U(\mathfrak{k})[x] \). Then the following systems of equations are equivalent:

(i) \( E^n c(n - \tilde{Y}) \equiv c(-n - \tilde{Y}) E^n, \ (n \in \mathbb{N}_0) \);

(ii) \( E^{n+1}(c^{(n)})\left(\frac{n}{2} + 1 - \tilde{Y}\right) + \dot{E}^n(c^{(n+1)})(\frac{n}{2} - \tilde{Y})E \equiv 0, \ (n \in \mathbb{N}_0) \).

Moreover, if \( c \in U(\mathfrak{k})[x] \) is a solution of one of the above systems, then for all \( \ell, n \in \mathbb{N}_0 \) we have

(iii) \( (1)\dot{E}^\ell(c^{(n)})\left(-\frac{n}{2} + \ell - \tilde{Y}\right)E^n - (1)\dot{E}^n(c^{(\ell)})(-\frac{\ell}{2} + n - \tilde{Y})E^\ell \equiv 0 \).

Observe that if \( c \in U(\mathfrak{k})[x] \) is of degree \( m \) and \( c = c_m x^m + \cdots + c_0 \), then all the equations of the system (ii) corresponding to \( n > m \) are trivial because \( c^{(n)} = 0 \). Moreover, the equation corresponding to \( n = m \) reduces to \( \dot{E}^{m+1}(c_m) \equiv 0 \) and the equation associated to \( n = j \), for \( j < m \), only involves the coefficients \( c_m, \ldots, c_j \). In other words the system (ii) is a triangular system of \( m + 1 \) linear equations in the \( m + 1 \) unknowns \( c_m, \ldots, c_0 \).

Since we are going to use equations (iii) of Theorem 3.1, it is convenient to consider a basis of \( \mathbb{C}[x] \) that behaves well under the discrete derivative. Then let \( \{\varphi_n\}_{n \geq 0} \) be the basis of \( \mathbb{C}[x] \) defined by,

\[
\varphi_0 = 1, \quad \varphi_n(1) = \varphi_{n-1} \quad \text{if } n \geq 1, \quad \varphi_n(0) = 0 \quad \text{if } n \geq 1.
\]

The existence and uniqueness of the family \( \{\varphi_n\}_{n \geq 0} \) follows inductively from conditions (i), (ii) and (iii) above. Moreover it is easy to see that,

\[
\varphi_n(x) = \frac{1}{n!} x(x + \frac{n}{2} - 1)(x + \frac{n}{2} - 2)\cdots(x - \frac{n}{2} + 1), \quad n \geq 1.
\]

To simplify the notation from now on we shall write \( u \equiv v \mod (U(\mathfrak{k})\mathfrak{m}^+) \) for any \( u, v \in U(\mathfrak{k}) \).

**Lemma 3.2.** Let \( u \in U(\mathfrak{k}) \) and \( X \in \mathfrak{k} - \mathfrak{m}^+ \) be such that \( \hat{X}(\mathfrak{m}^+) \subset \mathfrak{m}^+ \). Then, if \( n \in \mathbb{N} \) and \( uX^n \equiv 0 \) we have \( u \equiv 0 \).

**Proof.** Choose a basis \( \{Z_1, \ldots, Z_q\} \) of \( \mathfrak{m}^+ \) and complete it to a basis of \( \mathfrak{k} \) by adding vectors \( X_1, \ldots, X_p \) with \( X_p = X \). Then by Poincaré-Birkhoff-Witt theorem the ordered monomials \( X^I = X_1^{i_1} \cdots X_p^{i_p}, \ I = \{i_1, \ldots, i_p\} \), and \( Z^J = Z_1^{j_1} \cdots Z_q^{j_q}, \ J = \{j_1, \ldots, j_q\} \), form a basis \( \{X^I Z^J\} \) of \( U(\mathfrak{k}) \).
It is enough to prove the lemma for \( n = 1 \). If \( u = \sum a_{I,J}X^I Z^J \) we have
\[
u X = \sum a_{I,J}X^I X Z^J - \sum a_{I,J}X^J X^I Z^J.
\]
Then, since \( \dot{X}(Z^J) \equiv 0 \) it follows that \( uX \equiv \sum a_{I,J}X^I X Z^J \). Therefore \( uX \equiv 0 \)
implies that \( a_{I,J} = 0 \) if \( J = 0 \). Hence the lemma follows.

The following result was proved in Theorem 3.11 of [3].

**Theorem 3.3.** Let \( G_o \) be locally isomorphic to \( SO(n,1) \) for \( n \geq 3 \), or to \( SU(n,1) \),
\( n \geq 2 \). Then, \( \sum_{j \geq 0} \dot{E}^j \left( U(\mathfrak{t})^M \right) \) is a direct sum and we have
\[
\left( \sum_{j \geq 0} \dot{E}^j \left( U(\mathfrak{t})^M \right) \right) \cap U(\mathfrak{t})m^+ = 0.
\]

**4. Representations in \( \Gamma \)**

It is well known that \((K,M)\) is a Gelfand pair when \( G_o \) is locally isomorphic to \( SO(n,1) \) or \( SU(n,1) \). In particular \( \dim(V_\gamma^M) = 1 \) for all \( \gamma \in \Gamma \). In these cases we have an alternative and convenient description of the Kostant degree of \( \gamma \in \Gamma \). In fact, given a simple root \( \alpha \in P_+ \) set \( E = X_\alpha + \theta X_\alpha \) for any \( X_\alpha \neq 0 \). Then if \( \gamma \in \Gamma \) define
\[
q(\gamma) = \max \{ q \in \mathbb{N} : E^q (V_\gamma^M) \neq 0 \}.
\]

The following propositions establish the relation between \( q(\gamma) \) and \( d(\gamma) \) for any \( \gamma \in \Gamma \) as well as other facts about the representations in \( \Gamma \). Some of these results where first established in [6], others were proved in [3] for \( G_o \) locally isomorphic to \( SO(n,1) \) or \( SU(n,1) \), and in [5] they were generalized to any real rank one semisimple Lie group.

**Proposition 4.1.** Let \( G_o \) be locally isomorphic to \( SO(n,1) \) for \( n \geq 3 \). Then there exists a Borel subalgebra \( \mathfrak{b}_t = \mathfrak{h}_t \oplus \mathfrak{t}^+ \) of \( \mathfrak{t} \) such that \( m^+ \subset \mathfrak{t}^+ \) and \( E \in \mathfrak{t}^+ \). For any such a Borel subalgebra there exists a fundamental weight \( \xi_o \) with the following properties:

(i) If \( \gamma \in \tilde{K} \) and \( \xi_o \) denotes its highest weight then \( \gamma \in \Gamma \) if and only if \( \xi_o = k\xi_o \)
when \( n \geq 4 \) and \( \xi_o = 2k\xi_o \) if \( n = 3 \), for some \( k \in \mathbb{N}_o \).

(ii) If \( \text{rank}(G_o) = \text{rank}(K_o) \) (that is, \( n \) is even) we have, \( \gamma \in \Gamma_1 \) if and only if \( \xi_o = k\xi_o \) with \( k \) even.

(iii) If \( \gamma \in \Gamma \) we have \( E^{q(\gamma)} (V_\gamma^M) = V_\gamma^M \), \( \xi_o = q(\gamma)\xi_o \) if \( n \geq 4 \), and \( \xi_o = 2q(\gamma)\xi_o \) if \( n = 3 \). Moreover \( d(\gamma) = q(\gamma) \).

As we indicated before if \( G_o \) is locally isomorphic to \( SU(n,1) \) there are two simple roots \( \alpha = \alpha_1, \alpha_n \) in \( P_+ \) (see Section 6 for more details). Hence, in this case we set \( E_1 = X_{-\alpha_1} + \theta X_{-\alpha_1} \) and \( E_2 = X_{-\alpha_n} + \theta X_{-\alpha_n} \). The following proposition summarizes some results about the representations \( \gamma \in \Gamma \) for the group \( SU(n,1) \).

**Proposition 4.2.** Let \( G_o \) be locally isomorphic to \( SU(n,1) \) for \( n \geq 2 \). Then for \( E = E_1 \) (respectively \( E = E_2 \)) there exists a Borel subalgebra \( \mathfrak{b}_t = \mathfrak{h}_t \oplus \mathfrak{t}^+ \) of \( \mathfrak{t} \) such that \( m^+ \subset \mathfrak{t}^+ \) and \( E_1 \in \mathfrak{t}^+ \) (respectively \( E_2 \in \mathfrak{t}^+ \)). Moreover:
(i) The Cartan complement $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, where $\mathfrak{p}_1$ and $\mathfrak{p}_2$ are irreducible $\mathfrak{k}$-modules and $\mathfrak{p}_1 = \mathfrak{p}_2^*$. 

(ii) If $\xi_1$ and $\xi_2$ are the highest weights of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ respectively, and $\xi_\gamma$ denotes the highest weight of any $\gamma \in \hat{K}$, then $\gamma \in \Gamma$ if and only if $\xi_\gamma = k_1 \xi_1 + k_2 \xi_2$ with $k_1, k_2 \in \mathbb{N}_o$, and $d(\gamma) = k_1 + k_2$. 

(iii) We have $\gamma \in \Gamma_1$ if and only if $\xi_\gamma = k(\xi_1 + \xi_2)$ for $k \in \mathbb{N}_o$. 

(iv) Let $\gamma \in \Gamma_1$, $E = E_1$ (respectively $E = E_2$) and let $q(\gamma)$ be as in (6). Then $E^{q(\gamma)}(\gamma) = V_\gamma^{\mathfrak{r}}$, $\xi_\gamma = q(\gamma)(\xi_1 + \xi_2)$ and $d(\gamma) = 2q(\gamma)$. 

(v) If we set $X = [E_1, E_2]$ then $X \neq 0$, $X \in \mathfrak{m}^+$ if $n \geq 3$ and $X \in \mathfrak{t} + 3(\mathfrak{t})$ if $n = 2$. Moreover $[X, E_1] = [X, E_2] = 0$ if $n \geq 3$. For $\gamma \in \Gamma_1$ let $0 \neq b \in V_\gamma^M$, then $E_2^k E_1^l(b) = E_1^l E_2^k(b)$ for all $k, l \geq 0$ and $E_2^{q(\gamma)} E_1^{q(\gamma)}(b) \neq 0$. 

For the construction of the Borel subalgebra $\mathfrak{b}$ of Propositions 4.1 and 4.2 we refer the reader to Section 3 of [5] and for the other statements of the above propositions we refer the reader to Proposition 4.4, Theorem 4.5 and Theorem 5.3 of [5].

The Weyl group $W = M'/M$ preserves the one dimensional space $V_\gamma^M$ for any $\gamma \in \Gamma$ and since $W = \{1, w_0\}$, it follows that $w_0$ is either the identity or minus the identity on $V_\gamma^M$. It is well known that if rank $(G_o) = \text{rank} (K_o)$ (that is, $G_o$ is locally isomorphic to $\text{SO}(2p, 1)$ or $\text{SU}(n, 1)$) the element $w_0$ acts as the identity on $\mathfrak{t}$ and therefore it acts as the identity on $V_\gamma^M$ for all $\gamma \in \Gamma_1$. On the other hand, if rank $(G_o) = \text{rank} (K_o) + 1$ we have $\Gamma = \Gamma_1$ and the following proposition describes the action of $w_0$ on $V_\gamma^M$.

**Proposition 4.3.** Let $G_o$ be locally isomorphic to $\text{SO}(2p + 1, 1)$ with $p \geq 1$ and let $\gamma \in \Gamma$ with $\xi_\gamma = k \xi_o$. Then $w_0$ is the identity on $V_\gamma^M$ if and only if $k$ even.

**Proof.** Since $\Gamma = \Gamma_1$ we may assume that $V_\gamma^M \subset U(\mathfrak{t})^M$ and let $v_0 \in V_\gamma^M$ be a non zero element. Since $U(\mathfrak{t})^M \simeq U(\mathfrak{t})_K \otimes U(\mathfrak{m})^M$ (see [7] and [13]) there exist unique $x_i \in U(\mathfrak{t})_K$ and $y_i \in U(\mathfrak{m})^M$ for $i = 1, \ldots, r$, such that

$$v_0 = \sum_{i=1}^r x_i y_i,$$

where $\{x_i\}$ is a linearly independent set in $U(\mathfrak{t})_K$. Then, $w_0 v_0 = \pm v_0$ if and only if $w_0 y_i = \pm y_i$ for all $i = 1, \ldots, r$. On the other hand,

$$y_i \equiv t_i \mod (U(\mathfrak{m}) \mathfrak{m}^+)$$

where $t_i \in U(\mathfrak{t})$ is the image of $y_i$ by the Harish-Chandra isomorphism $U(\mathfrak{m})^M \to U(\mathfrak{t})^W(\mathfrak{m}, \mathfrak{t}, \rho_m)$. Here $W(\mathfrak{m}, \mathfrak{t}, \rho_m)$ denotes the action of the Weyl group $W(\mathfrak{m}, \mathfrak{t})$ on $\mathfrak{t}$ translated by $\rho_m$.

If $\{T_1, T_2, \ldots, T_p\}$ is an orthonormal basis of $\mathfrak{t}$ with respect to the Killing form, the elements $q_k = \sum_{i=1}^p T_i^{2k}$ for $k = 1, \ldots, p - 1$, and $q_p = T_1 T_2 \ldots T_p$ are the generators of $S(\mathfrak{t})^W(\mathfrak{m}, \mathfrak{t}, \rho_m)$ (see [1]). Note that $q_k$ has even degree in $T_i$ for all $i = 1, \ldots, p$ and all $k = 1, \ldots, p - 1$, but $q_p$ has degree one in $T_i$ for all $i = 1, \ldots, p$. Let $\tilde{q}_k \in U(\mathfrak{t})^W(\mathfrak{m}, \mathfrak{t}, \rho_m)$ be the translated element by $\rho_m$ corresponding to $q_k$, for example $\tilde{q}_p = (T_1 + \rho_m(T_1)) \ldots (T_p + \rho_m(T_p))$. We know
that \( t_i = Q_i(\tilde{q}_1, \ldots, \tilde{q}_p) = Q'_i(T_1, \ldots, T_p) \), where \( Q_i \) and \( Q'_i \) are polynomials in \( \mathbb{C}[x_1, \ldots, x_p] \) for all \( i = 1, \ldots, p \).

It is not difficult to see that the basis \( \{T_1, T_2, \ldots, T_p\} \) and a representative of \( w_0 \) can be chosen so that,

(i) \( w_0 T_i = T_i \) for all \( i = 1, \ldots, p - 1 \) and \( w_0 T_p = -T_p \);

(ii) \( w_0 m^+ = m^+ \);

(iii) \( \dot{E}(T_1) = -E \) and \( \dot{E}(T_i) = 0 \) for \( i = 2, \ldots, p \).

Property (ii) implies that \( w_0 y_i = \pm y_i \) if and only if \( w_0 t_i = \pm t_i \), and property (i) implies that \( w_0 t_i = t_i \) (respectively \( w_0 t_i = -t_i \)) if and only if \( Q_i \) is an even (respectively odd) polynomial in \( \tilde{q}_p \).

Now assume that \( w_0 t_i = t_i \) for all \( i = 1, \ldots, r \). Then \( Q'_i \) has even degree in all the variables \( T_1, \ldots, T_p \). On the other hand, property (iii) implies that

\[
\dot{E}^s(T_j) = \left( \sum_{\ell=1}^s (-1)\ell (s\ell) (T_1 + \ell - s) \ell \right) E^s = \begin{cases} 0, & \text{if } s \neq j, \\ j! E^s, & \text{if } s = j, \end{cases}
\]

hence for \( s \in \mathbb{N}_0 \) and \( 1 \leq i \leq r \), there exists a polynomial \( \tilde{Q}'_i \in \mathbb{C}[x_1, \ldots, x_p] \) (that depends on \( s \)) such that \( \dot{E}^s(\tilde{Q}'_i(T_1, \ldots, T_p)) = \tilde{Q}'_i(T_1, \ldots, T_p) E^s \). Now, since \( v_0 \in V^M_\gamma \) and \( \xi_\gamma = k\xi_o \), from Proposition 4.1 we know that \( E^k(v_0) \neq 0 \) and \( \dot{E}^{k+1}(v_0) = 0 \). Then,

\[
0 = \dot{E}^{k+1}(v_0) = \sum_{i=1}^r x_i \dot{E}^{k+1}(y_i) = \sum_{i=1}^r x_i \dot{E}^{k+1}(Q'_i(T_1, \ldots, T_p)) = \sum_{i=1}^r x_i \tilde{Q}'_i(T_1, \ldots, T_p) E^{k+1}.
\]

Hence, in view of Lemma 3.2 this implies that

\[
\sum_{i=1}^r x_i \tilde{Q}'_i(T_1, \ldots, T_p) E^{k+1} = 0.
\]

Now, since \( \{x_i\} \) is a linearly independent set in \( U(t)K \) and \( \tilde{Q}'_i(T_1, \ldots, T_p) \in U(t) \), we obtain that \( \tilde{Q}'_i(T_1, \ldots, T_p) = 0 \) for \( i = 1, \ldots, r \) (see Proposition 13 of [13]). This implies that \( \dot{E}^{k+1}(Q'_i(T_1, \ldots, T_p)) = 0 \) for \( i = 1, \ldots, r \). On the other hand, since \( E^k(v_0) \neq 0 \) there exists some \( 1 \leq j \leq r \) such that \( E^k(Q'_j(T_1, \ldots, T_p)) \neq 0 \). These two results about \( Q'_j(T_1, \ldots, T_p) \), together with (7), imply that \( k \) is equal to the degree of \( Q'_i \) in the variable \( T_1 \) which we know is even.

Finally, if we assume that \( w_0 t_i = -t_i \) for all \( i = 1, \ldots, r \), we obtain that \( Q'_i \) has odd degree in all the variables \( T_1, \ldots, T_p \). Then the same argument as above shows that \( k \) is odd. This completes the proof of the proposition. •
5. The case SO(n,1)

In this section we shall prove Theorem 1.2 when $G_o$ is locally isomorphic to \( SO(n,1) \) with \( n \geq 3 \).

5.1. Preliminary results. As we pointed out before, there is only one simple root \( \alpha_1 \in P_+ \) if \( n \geq 4 \) and there are two \( \alpha_1, \alpha_2 \) if \( n = 3 \). In all cases we set \( \alpha = \alpha_1, \ E = E_o, \ Y = Y_o \) and \( Z = Z_o \). Also as in (3), to any \( b(x) \in U(\mathfrak{t})[x] \) we associate \( c(x) \in U(\mathfrak{t})[x] \) defined by \( c(x) = b(x-1) \). If \( b(x) \in U(\mathfrak{t})[x], \ b(x) \neq 0 \), we shall find it convenient to write, in a unique way, \( b = \sum_{j=0}^{m} b_j x^j \), \( b_j \in U(\mathfrak{t}) \), \( b_m \neq 0 \), and the corresponding \( c = \sum_{j=0}^{m} c_j \varphi_j \) with \( c_j \in U(\mathfrak{t}) \). Then the following result establishes the relation between the coefficients \( b_j \) and \( c_j \). Since its proof is straightforward we omit it.

**Lemma 5.1.** Let \( b = \sum_{j=0}^{m} b_j x^j \in U(\mathfrak{t})[x] \) and set \( c(x) = b(x-1) \). Then, if \( c = \sum_{j=0}^{m} c_j \varphi_j \) with \( c_j \in U(\mathfrak{t}) \) we have

\[
\begin{aligned}
    c_i &= \sum_{j=i}^{m} t_{ij} b_j & 0 \leq i \leq m,
\end{aligned}
\]

where \( t_{ij} \) are rational numbers and \( t_{ii} = i! \). In other words, the vectors \( (b_0, \ldots, b_m)^t \) and \( (c_0, \ldots, c_m)^t \) are related by a rational nonsingular upper triangular matrix.

**Lemma 5.2.** If \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) then \( \hat{E}^{m+1}(b_j) = 0 \) for all \( 0 \leq j \leq m \), and thus \( \hat{E}^{m+1}(b_j) = 0 \) for all \( 0 \leq j \leq m \).

**Proof.** We regard \( b \) as a polynomial \( b = \sum_{j=0}^{m} b_j x^j \) with \( b_j \in U(\mathfrak{t})^M \) and let \( c(x) = b(x-1) = \sum_{j=0}^{m} c_j \varphi_j(x) \) with \( c_j \in U(\mathfrak{t})^M \). Then, since \( b \in B \), \( c \) satisfies the system of equations (i) of Theorem 3.1 with \( \tilde{Y} = Y \). Therefore \( c \) satisfies equations (iii) of Theorem 3.1 for all \( \ell, n \in \mathbb{N}_o \).

Hence, since \( c^{(m+1)} = 0 \), if we consider \( \ell = m + 1 \) in equation (iii) of Theorem 3.1 and we use Lemma 3.2 with \( X = E \) we obtain

\[
\sum_{j=n}^{m} \hat{E}^{m+1}(c_j) \varphi_{j-n}(\frac{2m+2-n}{2} - Y) \equiv 0,
\]

for \( 0 \leq n \leq m \). Now, taking into account that right multiplication by \( Y \) leaves invariant the left ideal \( U(\mathfrak{t})^m \) because \( Y \in \mathfrak{t} \), (9) together with decreasing induction on \( n \) starting from \( n = m \) implies that \( \hat{E}^{m+1}(c_j) \equiv 0 \) for all \( 0 \leq j \leq m \).

From this, applying \( \hat{E}^{m+1} \) to (8) and making use of Theorem 3.3, the theorem follows because the matrix \( (t_{ij}) \) is a nonsingular scalar matrix.

5.2. Bound for the Kostant degree. We are now ready to prove the boundedness condition on the Kostant degree required in Theorem 2.2 for \( G_o \) locally isomorphic to \( SO(n,1) \).

**Theorem 5.3.** Assume that \( G_o \) is locally isomorphic to \( SO(n,1) \) for \( n \geq 3 \) and let \( b = b_m \otimes Z^m + \cdots + b_0 \in B \), then \( d(b_j) \leq m \) for all \( 0 \leq j \leq m \).
Proof. Let \( b = b_m \otimes Z^m + \cdots + b_0 \in B \), then it follows from Lemma 5.2 that \( \bar{E}^{m+1}(b_j) = 0 \) for all \( 0 \leq j \leq m \). In view of (6) and (iii) of Proposition 4.1 this implies that \( b_j \in \bigoplus U(\mathfrak{t})^W \), where the sum extends over all \( \gamma \in \Gamma \) such that \( d(\gamma) \leq m \). Therefore \( d(b_j) \leq m \) for all \( 0 \leq j \leq m \), as we wanted to prove. \( \blacksquare \)

5.3. Weyl group invariance of the leading term. Our next goal is to prove the \( W \)-invariance condition of Theorem 2.2. That is, if \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) and \( b_m \neq 0 \), then its leading term \( b_m \otimes Z^m \in (U(\mathfrak{t})^M \otimes U(\mathfrak{a}))^W \). Recall that \( (U(\mathfrak{t})^M \otimes U(\mathfrak{a}))^W \) denotes the ring of \( W \)-invariants in \( U(\mathfrak{t})^M \otimes U(\mathfrak{a}) \) under the tensor product of the action of \( W \) on \( U(\mathfrak{t})^M \) and the action of \( W \) on \( U(\mathfrak{a}) \).

If \( G_o \) is locally isomorphic to \( SO(2p,1) \) the Weyl group \( W \) acts trivially on \( \mathfrak{t} \). On the other hand, if \( G_o \) is locally isomorphic to \( SO(2p+1,1) \) with \( p \geq 1 \), recall that we can choose an orthonormal basis \( \{ T_1, T_2, \ldots, T_p \} \) of \( \mathfrak{t} \) and a representative of \( w_0 \) such that,

\[
\begin{align*}
(1) & \quad w_0 T_i = T_i \text{ for all } i = 1, \ldots, p - 1 \text{ and } w_0 T_p = -T_p; \\
(2) & \quad w_0 m^+ = m^+; \\
(3) & \quad \bar{E}(T_1) = -E \text{ and } \bar{E}(T_i) = 0 \text{ for } i = 2, \ldots, p.
\end{align*}
\]

Moreover, this choice can be made in such a way that \( w_0 E = -E \) and \( Y = -T_1 \). Hence, if we extend the action of \( W \) in \( U(\mathfrak{t})^M \) to \( U(\mathfrak{t})^M \otimes U(\mathfrak{a}) \) by letting it act trivially on \( U(\mathfrak{a}) \), it is clear that \( W \) preserves the algebra \( B \) and thus \( B = B_1 \oplus B_{-1} \), where \( B_{\pm 1} = \{ b \in B : w_0 b = \pm b \} \).

Lemma 5.4. If \( u \in U(\mathfrak{t})^M \) the following statements hold,

\[
\begin{align*}
(1) & \quad \text{If } w_0 u = u \text{ and } \bar{E}^{2t}(u) = 0 \text{ for } t \in \mathbb{N}, \text{ then } \bar{E}^{2t-1}(u) = 0. \\
(2) & \quad \text{If } w_0 u = -u \text{ and } \bar{E}^{2t+1}(u) = 0 \text{ for } t \in \mathbb{N}, \text{ then } \bar{E}^{2t}(u) = 0.
\end{align*}
\]

Proof. We may assume that \( u \in V_\gamma^M \subset U(\mathfrak{t})^M \) for \( \gamma \in \Gamma_1 \). We begin by proving (1). If \( \bar{E}^{2t-1}(u) \neq 0 \) then \( \bar{E}^{2t-1}(u) \) would be a highest weight vector of weight \( \xi = (2t - 1)\xi_0 \). This contradicts (ii) of Proposition 4.1 if \( G_o \) is locally isomorphic to \( SO(2p,1) \), or contradicts Proposition 4.3 if \( G_o \) is locally isomorphic to \( SO(2p+1,1) \), because we are assuming that \( w_0 \) acts as the identity on \( V_\gamma^M \).

The proof of (2) is similar: if \( \bar{E}^{2t}(u) \neq 0 \) then \( \bar{E}^{2t}(u) \) would be a highest weight vector of weight \( \xi = 2t\xi_0 \) but this contradicts Proposition 4.3 as in the previous case. \( \blacksquare \)

Theorem 5.5. If \( G_o \) is locally isomorphic to \( SO(n,1) \) with \( n \geq 3 \) and \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) with \( b_m \neq 0 \), then its leading term \( \bar{b} = b_m \otimes Z^m \in (U(\mathfrak{t})^M \otimes U(\mathfrak{a}))^W \).

Proof. We shall prove first that if \( b = b_m \otimes Z^m + \cdots + b_0 \in B_1 \) (respectively \( b \in B_{-1} \)) then \( m \) is even (respectively odd). Let \( b = b_m \otimes Z^m + \cdots + b_0 \in B_1 \) with \( b_m \neq 0 \), and assume that \( m \) is odd. From Lemma 5.2 it follows that \( \bar{E}^{m+1}(b_j) = 0 \) for all \( 0 \leq j \leq m \). Then, since \( m + 1 \) is even and \( w_0 b_j = b_j \) for all \( 0 \leq j \leq m \),
from (1) of Lemma 5.4 it follows that $\dot{E}^m(b_j) = 0$ for all $0 \leq j \leq m$. Hence, from (8) we get $\dot{E}^m(c_j) = 0$ for all $0 \leq j \leq m$. Now, if we consider $t = m$ and $n = 0$ in equation (iii) of Theorem 3.1 we get

$$
\sum_{j=0}^{m} \dot{E}^m(c_j) \varphi_j(m - Y) - ml b_mE^m \equiv 0,
$$

which implies that $b_m \equiv 0$, and therefore $b_m = 0$ (Theorem 3.3). This is a contradiction therefore $m$ is even, as we wanted to prove. A similar argument proves that if $b = b_m \otimes Z^m + \cdots + b_0 \in B_{-1}$ and $b_m \neq 0$, then $m$ is odd. Observe that in both cases (ie. $b \in B_1$ or $b \in B_{-1}$) we have $\tilde{b} = b_m \otimes Z^m \in (U(t)^M \otimes U(a))^W$.

Now consider $b = b_m \otimes Z^m + \cdots + b_0 \in B$ with $b_m \neq 0$. Since $B = B_1 \oplus B_{-1}$ we can write $\tilde{b} = \tilde{b}^{(1)} + \tilde{b}^{(-1)}$ with $\tilde{b}^{(1)} \in B_1$ and $\tilde{b}^{(-1)} \in B_{-1}$. Then the leading term of $\tilde{b}$ is either $\tilde{b}^{(1)}$ or $\tilde{b}^{(-1)}$, the leading terms of $b^{(1)}$ and $b^{(-1)}$ respectively. Hence, by above the observation, in either case we conclude that $\tilde{b} = b_m \otimes Z^m \in (U(t)^M \otimes U(a))^W$, as we wanted to prove.

**Remark 5.1.** When $G_o$ is locally isomorphic to SO(3,1) we have used only one of the equations that define the algebra $B$. In other words, if for each simple root $\alpha \in P_+$ we define $B_\alpha$ as the subalgebra of all elements $b \in U(t) \otimes U(a)$ that satisfy (1) for all $n \in \mathbb{N}$, then we have proved that $P(U(g)^K) = B_{W^\alpha} = B_{aW^\alpha}$. Moreover, taking advantage that in this case the elements of the algebra $B$ satisfy two different equations, it is not difficult to see that $B_{W^\alpha} = B$.

This completes the proof of theorem 2.2 when $G_0$ is locally isomorphic to SO($n,1$).

6. The case SU($n,1$)

In this section we prove Theorem 1.2 when $G_o$ is locally isomorphic to SU($n,1$) for $n \geq 2$. Although some results of this section are contained in [12], we include them here for completeness and to prove that $B_{W^\alpha} = B$ which is a new result.

**6.1. Preliminary results.** We can choose an orthonormal basis $\{e_i\}_{i=1}^{n+1}$ of $(\mathfrak{h}_\mathbb{R} \oplus \mathbb{R})^*$ in such a way that $\mathfrak{h}_\mathbb{R} = \{H \in \mathfrak{h}_\mathbb{R} \oplus \mathbb{R} : (e_1 + \cdots + e_{n+1})(H) = 0\}$, $\alpha_i = e_i - e_{i+1}$ if $1 \leq i \leq n$, $e_i' = -e_i$ if $2 \leq i \leq n$ and $e'_1 = -e_{n+1}$. Then from the Dynkin-Satake diagram of $\mathfrak{g}$ we obtain that

$$
\Delta^+(\mathfrak{g}, \mathfrak{h}) = \{e_i - e_j : 1 \leq i < j \leq n + 1\},
$$

$$
P_+ = \{e_1 - e_j, e_j - e_{n+1} : 2 \leq j \leq n\} \cup \{e_1 - e_{n+1}\},
$$

$$
P_- = \{e_i - e_j : 2 \leq i < j \leq n\},
$$

where $P_-$ denotes the set of roots in $\Delta^+(\mathfrak{g}, \mathfrak{h})$ that vanish on $\mathfrak{a}$.

In this case there are two simple roots $\alpha = \alpha_1, \alpha_n$ in $P_+$; in both cases $\mathfrak{y}_\alpha \neq 0$. Set $E_1 = X_{-\alpha_1} + \theta X_{-\alpha_1}$, $E_2 = X_{-\alpha_n} + \theta X_{-\alpha_n}$, $Y_1 = Y_{\alpha_1}$, $Y_2 = Y_{\alpha_n}$ and $Z = Z_{\alpha_1} = Z_{\alpha_n}$. Let $T \in \mathfrak{t}_\mathbb{R}$ be defined by $\epsilon_2(T) = \cdots = \epsilon_n(T) = 2 \frac{2}{n+1}$. Then $T \in \mathfrak{z}(\mathfrak{m})$ and $\dim(\mathfrak{z}(\mathfrak{m})) = 1$. Since $\epsilon_1(T) = \epsilon_{n+1}(T)$ and $(\epsilon_1 + \cdots + \epsilon_{n+1})(T) = 0$
we get \( \epsilon_2(T) - \epsilon_1(T) = \epsilon_n(T) - \epsilon_{n+1}(T) = 1 \); thus \([T,E_1] = E_1\) and \([T,E_2] = -E_2\).

Now define the vector \( H \) considered in (3) as follows,

\[
H = \begin{cases} 
\frac{1}{2}T, & \text{if } \alpha = \alpha_1 \\
-\frac{1}{2}T, & \text{if } \alpha = \alpha_n,
\end{cases}
\]  

(10)

and we write generically \( E, Y \), and \( \tilde{Y} = Y + H \) for the corresponding vectors associated to a simple root \( \alpha \in P_+ \). Then \( \tilde{E}(H) = -\frac{1}{2}E \), and thus \( \tilde{E}(\tilde{Y}) = E \).

Also as in (3), to any \( b(x) \in U(\mathfrak{t})[x] \) associate \( c(x) \in U(\mathfrak{t})[x] \) defined by \( c(x) = b(x + H - 1) \). If \( b(x) \in U(\mathfrak{t})[x] \), \( b(x) \neq 0 \), we shall find it convenient to write, in a unique way, \( b = \sum_{j=0}^m b_j x^j \), \( b_j \in U(\mathfrak{t}) \), \( b_m \neq 0 \), and the corresponding \( c = \sum_{j=0}^m c_j \varphi_j \) with \( c_j \in U(\mathfrak{t}) \). Then the following lemma establishes the relation between the coefficients \( b_j \) and \( c_j \).

**Lemma 6.1.** Let \( b = \sum_{j=0}^m b_j x^j \in U(\mathfrak{t})[x] \) and set \( c(x) = b(x + H - 1) \). Then, if \( c = \sum_{j=0}^m c_j \varphi_j \) with \( c_j \in U(\mathfrak{t}) \) we have

\[
c_i = \sum_{j=0}^m b_j t_{ij} \quad 0 \leq i \leq m,
\]  

(11)

where \( t_{ij} = \sum_{k=0}^i (-1)^k \binom{i}{k} (H + \frac{i}{2} - 1 - k)^j \in \mathfrak{z}(U(\mathfrak{m})) \). Thus \( t_{ii} = i! \), \( t_{ij} \) is a polynomial in \( H \) of degree \( j - i \), and

\[
\tilde{E}^{j-i}(t_{ij}) = \left(-\frac{1}{2}\right)^{j-i} j! E^{j-i}.
\]

Moreover if \( b_j \in U(\mathfrak{t})^M \) for \( 0 \leq j \leq m \), then \( c_j \in U(\mathfrak{t})^M \) for \( 0 \leq j \leq m \).

**Proof.** Since almost all the results follow from straightforward computations, we only prove that \( \tilde{E}^{j-i}(t_{ij}) = \left(-\frac{1}{2}\right)^{j-i} j! E^{j-i} \).

It follows by induction that if \( \tilde{H}(E) = cE \) and \( a \in \mathbb{C} \), then

\[
\tilde{E}^m(H + a)^j = E^m \sum_{\ell=0}^m (-1)^{\ell} \binom{m}{\ell} (H + a + c\ell)^j.
\]  

(12)

This implies that

\[
\tilde{E}^{j-i}(H^{j-i}) = E^{j-i} \sum_{\ell=0}^{j-i} (-1)^{\ell} \binom{j-i}{\ell} \left( H + \frac{\ell}{2} \right)^{j-i} = \left(-\frac{1}{2}\right)^{j-i} (j-i)! E^{j-i}.
\]

Now, since

\[
t_{ij} = \sum_{k=0}^i (-1)^k \binom{i}{k} \sum_{\ell=0}^j \binom{j}{\ell} \left( \frac{i}{2} - 1 - k \right)^\ell H^{j-\ell}
\]

\[
= \sum_{\ell=0}^j \left( \sum_{k=0}^i (-1)^k \binom{i}{k} \left( \frac{i}{2} - 1 - k \right)^\ell \right) \binom{j}{\ell} H^{j-\ell}
\]

\[
= \frac{j!}{(j-i)!} H^{j-i} + \cdots,
\]
it follows that
\[
\hat{E}^{j-i}(t_{ij}) = \frac{j!}{(j-i)!} \hat{E}^{j-i}(H^{j-i}) = (-\frac{1}{2})^{j-i} j! E^{j-i}.
\]

\section*{Theorem 6.2.} If \( b = b_m \otimes Z^m + \cdots + b_0 \in B \), then \( \hat{E}^{m+1}(c_j) = 0 \) for all \( 0 \leq j \leq m \).

\textbf{Proof.} Since \( b \in B \), \( c \) satisfies the system of equations (i) of Theorem 3.1 with \( \tilde{Y} = Y + H \). Therefore \( c \) satisfies equations (iii) of Theorem 3.1 for all \( \ell, n \in \mathbb{N}_0 \). Hence, since \( c^{(m+1)} = 0 \), if we consider \( \ell = m + 1 \) in equation (iii) of Theorem 3.1 and we use Lemma 3.2 with \( X = E \) we obtain
\[
\sum_{j=n}^{m} \hat{E}^{m+1}(c_j) \varphi_{j-n} \left( \frac{2m+2-n}{2} - \tilde{Y} \right) = 0,
\]
for \( 0 \leq n \leq m \). Now, taking into account that right multiplication by \( \tilde{Y} \) leaves invariant the left ideal \( U(t) \mathfrak{m}^+ \) because \( \tilde{Y} \in t \), (13) together with decreasing induction on \( n \) starting from \( n = m \) implies that \( \hat{E}^{m+1}(c_j) = 0 \). Hence using Lemma 6.1 and Theorem 3.3 it follows that \( \hat{E}^{m+1}(c_j) = 0 \) for all \( 0 \leq j \leq m \).

\section*{Corollary 6.3.} If \( b = b_m \otimes Z^m + \cdots + b_0 \in B \), then \( \hat{E}^{2m+1-j}(b_j) = 0 \) for all \( 0 \leq j \leq m \).

\textbf{Proof.} For \( j = m \) the assertion follows directly from Theorem 6.2 since \( c_m = m! b_m \) (Lemma 6.1). Now we proceed by decreasing induction on \( j \). Thus let \( 0 \leq j < m \) and assume that \( \hat{E}^{2m+1-k}(b_k) = 0 \) for all \( j < k \leq m \). Then, since \( m + 1 < 2m + 1 - j \), using Leibniz rule, Lemma 6.1 and the inductive hypothesis we obtain
\[
\hat{E}^{2m+1-j}(c_j) = \hat{E}^{2m+1-j} \left( \sum_{k=j}^{m} b_k t_{jk} \right) = j! \hat{E}^{2m+1-j}(b_j).
\]
Since \( \hat{E}^{2m+1-j}(c_j) = 0 \) the proof of the corollary is completed.

The following result was proved in Theorem 30 of [12], but in a different way. Here we derive this theorem directly from Theorem 3.1.

\section*{Theorem 6.4.} Let \( m, w, \alpha \in \mathbb{Z} \), \( 0 \leq w, \alpha \leq m \), \( \alpha + w \geq m + 1 \). If \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) and \( \hat{E}^{m+\alpha+1-j}(b_j) = 0 \) for all \( 0 \leq j \leq m \), then
\[
\sum_{j=m-w}^{m} (-2)^{-j} j! \left( \frac{\alpha + w}{j + w - m} \right) \hat{E}^{m+\alpha-j}(b_j) E^j = 0.
\]

\textbf{Proof.} From the previous theorem we know that \( \hat{E}^{m+1}(c_j) = 0 \) for every \( 0 \leq j \leq m \). Since \( w \geq 1 \) we have \( \hat{E}^{\alpha+w}(c_{m-w}) = 0 \). Now using the Leibnitz rule
and Lemma 6.1 we compute
\[
\dot{E}^{\alpha+w}(c_{m-w}) = \dot{E}^{\alpha+w}\left(\sum_{j=m-w}^{m} b_j t_{m-w,j}\right)
\]
\[
= \sum_{j=m-w}^{m} \left(\alpha + w\right) \dot{E}^{m+\alpha-j}(b_j) \dot{E}^{j+w-m}(t_{m-w,j})
\]
\[
= \sum_{j=m-w}^{m} \left(\alpha + w\right) (-2)^{-(j+w-m)} j! \dot{E}^{m+\alpha-j}(b_j) E^{j+w-m}.
\]

Therefore
\[
\sum_{j=m-w}^{m} \left(\alpha + w\right) (-2)^{-(j+w-m)} j! \dot{E}^{m+\alpha-j}(b_j) E^{j+w-m} \equiv 0.
\]

If we multiply this last equation on the right by \((-2)^{w-m} E^{m-w}\) we obtain the stated result.

Lemma 6.5. Let \(k \in \mathbb{N}_0\) and \(u \in U(\mathfrak{t})^M\). Then, \(\dot{E}^{k}_i(u) \equiv 0\) for \(i = 1\) or \(i = 2\) if and only if \(\dot{E}^{k}_{i'}(u) = 0\) for every \(i' \in \{1, 2\}\).

Proof. Let us assume that \(\dot{E}^{k}_i(u) \equiv 0\) for \(k \geq 1\). Then Theorem 3.3 implies that \(\dot{E}^{k}_i(u) = 0\). Hence, in view of Proposition 4.2, it follows that \(u \in \bigoplus U(\mathfrak{t})_i^M\) where the sum extends over all \(\gamma \in \Gamma_1\) such that \(q(\gamma) \leq k - 1\). Then since \(q(\gamma)\) is independent of the choice of the simple root \(\alpha = \alpha_1\) or \(\alpha = \alpha_n\), we obtain \(\dot{E}^{k}_{\gamma}(u) = 0\) which completes the proof.

For further reference we now recall Lemma 1 of [13].

Lemma 6.6. Let \(G_o\) be locally isomorphic to \(SU(2,1)\) and set \(Y = Y_{a_1} = -Y_{a_2}\). Also let \(0 \neq D \in \mathfrak{z}(\mathfrak{t})\) and let \(\zeta\) denote the Casimir element of \([\mathfrak{t},\mathfrak{t}]\). Then \(\{\zeta D^j\}_{j \geq 0}\) is a basis of \(\mathfrak{z}(U(\mathfrak{t}))\) and \(\{\zeta^i D^j Y^k\}_{i,j,k \geq 0}\) is a basis of \(U(\mathfrak{t})^M\).

The following theorem plays a crucial role in the proof of Theorem 2.2 because it allows us to obtain from Theorem 6.4 two systems of linear equations and therefore doubling the number of equations.

Theorem 6.7. Let \(G_o\) be locally isomorphic to \(SU(n,1)\) for \(n \geq 2\). Also let \(m, k \in \mathbb{N}_0\), \(m \leq k\), and let \(b_j \in U(\mathfrak{t})^M\) be such that \(\dot{E}^{k+1-j}(b_j) \equiv 0\) for all \(0 \leq j \leq m\) and for \(E = E_1\) or \(E = E_2\). Then,

(i) If \(\sum_{j=0}^{m} \dot{E}^{k-j}(b_j) E^j \equiv 0\) for \(E = E_1\) and \(E = E_2\) we obtain
\[
\sum_{0 \leq j \leq m} \dot{E}^{k-j}(b_j) E^j = 0 = \sum_{0 \leq j \leq m} \dot{E}^{k-j}(b_j) E^j.
\]

(ii) If \(\sum_{j=0}^{m} \dot{E}^{k-j}(b_j) E^j \equiv 0\) for \(E = E_1\) or \(E = E_2\) we have
\[
\sum_{j=0}^{m} \dot{E}^{k-j}(b_j) E^j = 0 = \sum_{j=0}^{m} (-1)^j \dot{E}^{k-j}(b_j) E^j.
\]

for \(i' \neq i\) and \(i, i' \in \{1, 2\}\).
Proof. The statement in (i) is the same as that of Theorem 32 of [12], and its proof for \( n \geq 3 \) can be found there. Here we will prove (i) for \( n = 2 \) and will also prove (ii). To do this we recall the following equality obtained in Theorem 32 of [12] for \( n \geq 2 \),

\[
\sum_{j=0}^{m} \dot{E}^{k-j}(b_j)E^j = \dot{E}^k \left( \sum_{j=0}^{m} \binom{k}{j}^{-1} (-\epsilon)^j (j!)^{-1} b_j T^j \right),
\]

where \( \epsilon = 1 \) if \( E = E_1 \), \( \epsilon = -1 \) if \( E = E_2 \) and \( T \in \mathfrak{z}(m) \) was defined at the beginning of this section. Since \( \sum_{j=0}^{m} \binom{k}{j}^{-1} (-\epsilon)^j (j!)^{-1} b_j T^j \in U(\mathfrak{t})^M \), if we assume that the hypothesis in (ii) holds, applying Lemma 6.5 we obtain (ii) for every \( n \geq 2 \).

On the other hand, if we assume that the hypothesis in (i) holds, applying Theorem 3.3 (or Lemma 6.5) to (14) we obtain that,

\[
\sum_{j=0}^{m} \dot{E}^{k-j}(b_j)E^j = 0, \quad \text{for } E = E_1 \text{ and } E = E_2.
\]

Also, since \( \dot{E}^{k+1-j}(b_j) \equiv 0 \) for \( 0 \leq j \leq m \) and for \( E = E_1 \) or \( E = E_2 \), it follows from Lemma 6.5 that \( \dot{E}^{k+1-j}(b_j) = 0 \) for \( E = E_1 \) and \( E = E_2 \).

Assume now that \( n = 2 \). It follows from Lemma 6.6 that we can write, in a unique way, \( b_j = \sum_i a_{i,j} Y^i \) with \( a_{i,j} \in \mathfrak{z}(U(\mathfrak{t})) \) and \( 0 \leq j \leq m \). On the other hand, from the definition of \( Y \) in Lemma 6.6 and the comment at the beginning of Section 3, we have \( \dot{E}(Y) = \frac{3\epsilon}{2} E \). Hence,

\[
\dot{E}^t(Y^i) = \begin{cases} 0, & \text{if } t > i \\ t! \left( \frac{3\epsilon}{2} \right)^t E^i, & \text{if } t = i. \end{cases}
\]

Then, since \( \dot{E}^{k+1-j}(b_j) = 0 \) for \( E = E_1 \) and \( E = E_2 \), using (16) we obtain that \( b_j = \sum_{i=0}^{k-j} a_{i,j} Y^i \). Therefore

\[
\sum_{j=0}^{m} \dot{E}^{k-j}(b_j)E^j = E^k \sum_{j=0}^{m} \left( \frac{3\epsilon}{2} \right)^{k-j} a_{k-j,j}
\]

for both \( E = E_1 \) and \( E = E_2 \). Then using (15) and (17) for \( E = E_1 \) and \( E = E_2 \) we obtain (i) for \( n = 2 \).

Taking into account Theorems 6.4 and 6.7 we are led to consider, for each \( 1 \leq \alpha \leq m \), the following systems of linear equations

\[
\sum_{\substack{m-w \leq j \leq m \\text{ (even)} \rule{0pt}{1.2em} \text{ (odd)}}} (-2)^{-j} j! \left( \binom{\alpha + w}{j + w - m} \right) \dot{E}^{-m+\alpha-j}(b_j)E^j = 0,
\]

for \( m + 1 - \alpha \leq w \leq m \).
If we set $x_j = \frac{(-2)^{-j}j!}{(\alpha + m - j)!} E_m^{\alpha+j}(b_j) E^{j+w-m}$ and multiply (18) by $\frac{1}{(\alpha + w)!}$ we obtain
\begin{equation}
\sum_{\substack{m-w \leq j \leq m \\text{even (odd)}}} \frac{1}{(j+w-m)!} x_j = 0.
\end{equation}

Now if we make the change of indices $j = 2r - \delta$, $m - w + \delta = s$ and set $y_r = \frac{r + \frac{s}{2}}{2r}$ the systems (19) become
\begin{equation}
\sum_{\delta \leq r \leq \left[\frac{m-d}{2}\right]} \binom{2r}{s} y_r = 0,
\end{equation}
for $\delta \leq s \leq \alpha + \delta - 1$ and $\delta \in \{0,1\}$.

**Proposition 6.8.** For $\delta \in \{0,1\}$ let $M_\delta$ be the matrix with entries defined by $M_{rs} = \binom{2r}{s}$ for $\delta \leq r, s \leq k$. Then
$$\det(M_\delta) = 2^{k(k+1)/2}.$$  

**Proof.** For each $\delta \leq s \leq k$ we let $\binom{2r}{s}$ denote the $s$-column of $M_\delta$ and we consider the determinant of $M_\delta$ as a multilinear function of its columns. Thus
$$\det(M_\delta) = \det\left(\binom{2r}{\delta}, \binom{2r}{\delta+1}, \ldots, \binom{2r}{k}\right).$$
If we view the binomial coefficient $\binom{2r}{s}$ as a polynomial in the variable $r$ of degree $s$ we can write, in a unique way,
$$\binom{2r}{s} = 2^s \binom{r}{s} + a_{s-1} \binom{r}{s-1} + \cdots + a_0,$$ 
where $a_j = 0$ for $j < \frac{s}{2}$. Then
$$\det(M) = \det\left(2^\delta \binom{r}{\delta}, 2^{\delta+1} \binom{r}{\delta+1}, \ldots, 2^k \binom{r}{k}\right) = 2^{k(k+1)/2}.$$ 
This completes the proof of the proposition. \(\blacksquare\)

**6.2. Bound for the Kostant degree.** We are now almost ready to prove the first part of Theorem 2.2 when $G_o$ is locally isomorphic to SU$(n,1)$. We need the following proposition.

**Proposition 6.9.** Let $G_o$ be locally isomorphic to SU$(n,1)$ with $n \geq 2$. If $b = b_m \otimes Z^m + \cdots + b_0 \in B$, then $E^j [m]+m+1-j(b_j) = 0$ for all $0 \leq j \leq m$.

**Proof.** We will prove by decreasing induction on $\alpha$ in the interval $\left[\frac{m}{2}\right] \leq \alpha \leq m$ that $\dot{E}^{\alpha+m+1-j}(b_j) = 0$ for all $0 \leq j \leq m$. For $\alpha = m$ this result follows from Corollary 6.3 and Theorem 3.3. Thus assume that $\left[\frac{m}{2}\right] < \alpha \leq m$ and that $\dot{E}^{\alpha+m+1-j}(b_j) = 0$ for all $0 \leq j \leq m$. Then in view of Theorems 6.4 and 6.7 we know that the systems of linear equations (18) and their equivalent versions (19) and (20) hold.
Since \( \left[ \frac{m}{2} \right] + 1 \leq \alpha \) the number of unknowns in the system (20) is less or equal than the number of equations. Moreover, it follows from Proposition 6.8 that when \( \delta = 0 \) the rank of the coefficient matrix of the system (20) is \( \left[ \frac{m}{2} \right] + 1 \) which it is equal to the number of unknowns. Thus \( \hat{E}^{m+m-j}(b_j) = 0 \) for \( 0 \leq j \leq m \) and \( j \) even. Similarly, when \( \delta = 1 \) the rank of the coefficient matrix is \( \left[ \frac{m+1}{2} \right] \) which it is also equal to the number of unknowns. Therefore \( \hat{E}^{m+m-j}(b_j) = 0 \) for \( 0 \leq j \leq m \) and \( j \) odd. Then the inductive step is completed and the proposition is proved.

\[ \square \]

**Theorem 6.10.** Let \( G_o \) be locally isomorphic to \( SU(n, 1) \) with \( n \geq 2. \) If \( b = b_m \otimes Z^m + \cdots + b_0 \in B, \) then \( d(b_j) \leq 3m - 2j \) for all \( 0 \leq j \leq m. \) In particular \( d(b_m) \leq m. \)

**Proof.** Let \( b = b_m \otimes Z^m + \cdots + b_0 \in B, \) then it follows from Proposition 6.9 that \( \hat{E}^{[m/2]+1-j}(b_j) = 0 \) for all \( 0 \leq j \leq m. \) Hence in view of (6) and Proposition 4.2 it follows that \( b_j \in \bigoplus U(\mathfrak{g})^M, \) where the sum extends over all \( \gamma \in \Gamma_1 \) such that \( d(\gamma) \leq 3m - 2j. \) Therefore \( d(b_j) \leq 3m - 2j \) as we wanted to prove.

\[ \square \]

### 6.3. Weyl group invariance of the leading term.

We shall now prove the second condition required by Theorem 2.2. That is, we need to show that if \( b \in B \) then its leading term \( \delta = b_m \otimes Z^m \in (U(\mathfrak{t})^M \otimes U(\mathfrak{a}))^W. \) As in the case \( SO(2p, 1), \) since the non trivial element of \( W \) can be represented by an element in \( M'_o \) which acts on \( g \) as the Cartan involution, it is enough to prove that \( m \) is even.

As before, to any \( b(x) \in U(\mathfrak{t})[x] \) we associate \( c(x) \in U(\mathfrak{t})[x] \) defined by \( c(x) = b(x + H - 1) \) where \( H \) is defined in (10). Recall that if \( b(x) \in U(\mathfrak{t})^M[x] \) then \( c(x) \in U(\mathfrak{t})^M[x] \) (see Lemma 6.1). Whenever necessary we shall refer to \( c(x) \) as \( c_1(x) \) or \( c_2(x) \) according as \( \alpha = \alpha_1 \) or \( \alpha = \alpha_n. \) On the other hand, \( c(x) \) will generically stand for \( c_1(x) \) or \( c_2(x). \) Also, as before we shall find it convenient to write \( c_i(x) = \sum_{j=0}^{m-1} c_{i,j}\phi_j(x) \) with \( c_{i,j} \in U(\mathfrak{t}) \) for \( i = 1, 2. \)

**Proposition 6.11.** Let \( r \in \mathbb{N}_o, \) \( 0 \leq r \leq m. \) If \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) and \( \hat{E}^{m+r+1-j}(c_{1,j}) = \hat{E}^{m+r+1-j}(c_{2,j}) = 0 \) for \( r + 1 \leq j \leq m \) then

\[ \hat{E}^{m-j}(c_{1,r+j})E_1^j = (-1)^{m-r}\hat{E}^{r+j}(c_{1,m-j})E_1^{m-r-j} \]

and

\[ \hat{E}^{m-j}(c_{2,r+j})E_1^j = \hat{E}^{r+j}(c_{2,m-j})E_1^{m-r-j} \]

for \( j = 0, \ldots, \left[ \frac{m-r}{2} \right]. \)

**Proof.** If we set \( \ell = m - j \) and \( n = r + j \) in equation (iii) of Theorem 3.1 we get,

\[ \hat{E}^{m-j}(c_{1,\ell+j})(-\frac{\ell+j}{2} + m - j - \tilde{Y}_1)E_1^{r+j} \]

\[ - (-1)^{m-r}\hat{E}^{r+j}(c_{1,m-j})(-\frac{m-j}{2} + r + j - \tilde{Y}_1)E_1^{m-j} \equiv 0. \]

By hypothesis \( \hat{E}^{m-j}(c_{1,\ell+j}) = \sum_k \hat{E}^{m-j}(c_{1,k})\varphi_{k-r-j} = \hat{E}^{m-j}(c_{1,\ell+j}), \) and the first assertion follows from Theorem 6.7 (i).
In a similar way we obtain
\[ \hat{E}^{m-j}_2(c_{2,r+j})E^j_2 = (-1)^{m-r} \hat{E}^{r+j}_2(c_{2,m-j})E^{m-r-j}_2. \]

Then the second assertion is a direct consequence of Theorem 6.7 (ii). \[ \Box \]

In order to get a better insight of Proposition 6.11, for \( r = 0, \ldots, m+1 \) we introduce the column vectors \( \sigma_r = \sigma_r(b) \) and \( \tau_r = \tau_r(b) \) of \( m+r+1 \) entries defined by
\[
\sigma_r = (0, \ldots, 0, \hat{E}^r_1(c_{1,m})E^{m-r}_1, \ldots, \hat{E}^{m-1}_1(c_{1,r+1})E_1, \hat{E}^m_1(c_{1,r}), 0, \ldots, 0)^t, \\
\tau_r = (0, \ldots, 0, \hat{E}^r_1(c_{2,m})E^{m-r}_1, \ldots, \hat{E}^{m-1}_1(c_{2,r+1})E_1, \hat{E}^m_1(c_{2,r}), 0, \ldots, 0)^t.
\]

Let us observe that by definition \( \sigma_{m+1} = \tau_{m+1} = 0 \), and that the last \( m+1 \) entries of \( \sigma_r \) and \( \tau_r \) are respectively of the form \( \hat{E}^{r+j}_1(c_{1,m-j})E^{m-r-j}_1 \) and \( \hat{E}^{r+j}_1(c_{2,m-j})E^{m-r-j}_1 \) for \( 0 \leq j \leq m \), see Theorem 6.3 and Lemma 6.7.

Let \( J_s \) be the \( (s+1) \times (s+1) \) matrix with ones in the skew diagonal and zeros everywhere else, thus
\[
J_s = \begin{pmatrix} 0 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 0 \end{pmatrix}.
\]

(21)

In the following corollary we rephrase Proposition 6.11 in terms of the vectors \( \sigma_r \) and \( \tau_r \).

**Corollary 6.12.** Let \( r \in \mathbb{N}_o, \ 0 \leq r \leq m \). If \( b = b_m \otimes Z^m + \cdots + b_0 \in B \) and \( \sigma_{r+1} = \tau_{r+1} = 0 \) then
\[
J_{m+r} \sigma_r = (-1)^{m+r} \sigma_r \quad \text{and} \quad J_{m+r} \tau_r = \tau_r.
\]

The vectors \( \sigma_r \) and \( \tau_r \) are nicely related by a Pascal matrix. Let \( P_k \) denote the following \( (k+1) \times (k+1) \) lower triangular matrix
\[
P_k = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & k \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \binom{k}{s} \end{pmatrix}.
\]

(22)

**Proposition 6.13.** If \( r \in \mathbb{N}_o, \ 0 \leq r \leq m \) and \( \sigma_{r+1} = 0 \), then \( P_{m+r} \sigma_r = \tau_r \).

**Proof.** Since \( c_2(x) = c_1(x-T) \), for any \( 0 \leq j \leq m-r \) we have
\[
c_{2,r+j} = c_2^{(r+j)}(0) = c_1^{(r+j)}(-T) = \sum_{s=0}^{m-r-j} c_{1,r+j+s} \varphi_s(-T).
\]

On the other hand, since \( \hat{E}_1(T) = -E_1 \), it follows that \( \hat{E}_1^k((-T)^k) = k! E_1^k \) and \( \hat{E}_1^k((-T)^k) = 0 \) if \( t > k \). Therefore, since \( \varphi_k(-T) = \frac{1}{k!}(-T)^k + \cdots \), where
the dots stand for lower degree terms in $T$, we have $\hat{E}_1^k(\varphi_k(-T)) = E_1^k$ and $\hat{E}_1^l(\varphi_k(-T)) = 0$ if $t > k$. Now the hypothesis $\sigma_{r+1} = 0$ together with Theorem 6.2 imply that $\hat{E}_1^m+\tau+1−1(c_{1,i}) = 0$ for every $0 \leq i \leq m$. Hence, for any $−r \leq j \leq m−r$ using the Leibnitz rule we obtain

$$E_1^m−j(c_{2,r+j})E_1^j = \sum_{s=0}^{m−r−j} \hat{E}_1^{m−j}(c_{1,r+j+s}\varphi_s(-T))E_1^j$$

$$= \sum_{s=0}^{m−r−j} \sum_{\ell=0}^{m−j} \binom{m−j}{\ell} \hat{E}_1^{m−j−\ell}(c_{1,r+j+s})\hat{E}_1^\ell(\varphi_s(-T))E_1^j$$

$$= \sum_{s=0}^{m−r−j} \binom{m−j}{s} \hat{E}_1^{m−j−s}(c_{1,r+j+s})E_1^{s+j},$$

which implies that the last $m+1$ components of $P_{m+r}\sigma_r$ and $\tau_r$ are equal. Since by definition the first $r$ components of $P_{m+r}\sigma_r$ and $\tau_r$ are equal to 0 the proposition follows.

For $t \in \mathbb{N}_0$ we shall be interested in considering certain $(t+1) \times (t+1)$ submatrices of a Pascal matrix $P_n$ formed by any choice of $t+1$ consecutive rows and $t+1$ consecutive columns of $P_n$, with the only condition that the submatrix does not have zeros in its main diagonal. To be precise, for any $0 \leq a, b \leq n$, $a, b \in \mathbb{N}_0$ such that $b \leq a$ we shall be interested in submatrices $A$ of $P_n$ of the following form

$$A = \begin{pmatrix}
\begin{pmatrix} a \\ b \\ (a+1) \\ (b+1) \\ \vdots \\ (a+t) \\ b \\ (a+t+1) \\ b+1 \\ (a+t+2) \\ (b+1) \\
\end{pmatrix} & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\end{pmatrix}.$$ (23)

In the following proposition we collect some results that will be very important in the proof of our goal, that is, that the algebra $B$ does not contain elements of odd degree. The proof of this proposition will be given in an appendix at the end of this section.

**Proposition 6.14.** If $J_n$ and $P_n$ are as in (21) and (22) we have,

(i) If $v \in \mathbb{C}^{n+1}$ satisfies $J_nv = (-1)^nv$ and $J_nP_nv = P_nv$ then $v$ begins and ends with the same number of coordinates, say $k$, equal to zero. Moreover, $k$ is even or odd according as $n$ is even or odd, respectively.

(ii) If $A$ is a $(t+1) \times (t+1)$ submatrix of $P_n$ of the form (23) then $A$ is nonsingular.

**Lemma 6.15.** Let $n \in \mathbb{N}_0$ be an even number and let $v \in U(t)^M$ be such that $E_t^{t+1}(v) = 0$. If $n \geq 2t$ then there exists $b \in B$ of degree $n$ with $b_n = v$ and $\sigma_{t+1}(b) = 0$.

**Proof.** The proof will be by induction on $n$. If $n = 0$ the assertion follows from Proposition 4.2 and Proposition 2.1. Let us now take $n > 0$ even and
consider $S = \{ b \in B : \deg(b) = n \text{ and } b_n = v \}$. From Proposition 2.1 we know that $S$ is nonempty, because from Proposition 4.2 we obtain $d(v) \leq 2t \leq n$. For each $b \in S$ let $r(b) \in \mathbb{N}_0$ be such that $\sigma_{r(b)+1}(b) = 0$ and $\sigma_{r(b)}(b) \neq 0$, and let $r = \min\{ r(b) : b \in S \}$. We want to prove that $r \leq t$.

Let us assume that $r > t$ and let us take $b \in S$ such that $r(b) = r$. We have

$$
\sigma_r(b) = (0, \ldots, 0, \hat{E}_r^1(c_{1,n})E_1^{n-r}, \ldots, \hat{E}_r^{n-1}(c_{1,r+1})E_1, \hat{E}_r^n(c_{1,r}), 0, \ldots, 0)^t,
$$

$$
J_{n+r}\sigma_r(b) = (-1)^{n+r}\sigma_r(b) \quad \text{and} \quad J_{n+r}P_{n+r}\sigma_r(b) = P_{n+r}\sigma_r(b).
$$

Since $r > t$ the hypothesis $\hat{E}_{t+1}(v) = 0$ implies that the number of zeros with which $\sigma_r(b)$ starts is of the form $r + j_0$ with $j_0 \geq 1$. Thus we have

$$
\sigma_r(b) = (0, \ldots, 0, \hat{E}_{r+j_0}^1(c_{1,n-j_0})E_1^{n-j_0-r}, \ldots, \hat{E}_{r+j_0}^{n-j_0}(c_{1,r+j_0})E_1^{j_0}, 0, \ldots, 0)^t,
$$

with $j_0$ even. From $\sigma_r(b) \neq 0$ we get $n + 1 - r - 2j_0 > 0$ and from the definition of $j_0$ we obtain $\hat{E}_{r+j_0}^1(c_{1,n-j_0}) \neq 0$. Among all $b \in S$ with $\sigma_r(b) \neq 0$ we choose one with the largest $j_0$.

Let $\nu' = n - j_0$, $t' = r + j_0$, $v' = c_{1,n-j_0}$. Since $\sigma_{r+1}(b) = 0$ we have $\hat{E}_{t'+1}(v') = 0$. Now we shall consider the following two possibilities: $\nu' \geq 2t'$ and $\nu' < 2t'$, in both cases we will get a contradiction that will finish the proof of the lemma.

If $\nu' \geq 2t'$ then the inductive hypothesis implies that there exists $b' \in B$ of degree $\nu'$ such that $b_{\nu'}' = v'$ and $\sigma_{\nu'+1}(b') = 0$, thus

$$
(0, \ldots, 0, \hat{E}_{t'+1}^1(c_{1,n-j_0})E_1^{n-j_0-r-1}, \ldots, \hat{E}_{t'+1}^{n-j_0}(c_{1,r+j_0+1}), 0, \ldots, 0)^t = 0.
$$

Therefore $\sigma_{\nu'+1}(b - b') = 0$. This is a contradiction because either $\sigma_r(b - b')$ starts with more zeros than $\sigma_r(b)$ or $r(b - b') < r$.

On the other hand if $\nu' < 2t'$ then $n - r - 2j_0 < r + j_0$. Let $A$ be the submatrix of $P_{n+r}$ formed by the elements in the last $n + 1 - r - 2j_0$ rows and in the $n + 1 - r - 2j_0$ central columns of $P_{n+r}$. From Proposition 6.14 we know that $A$ is nonsingular.

Since $P_{n+r}\sigma_r(b) = \tau_r(b)$, $\tau_r(b)$ starts with $r + j_0$ zeros, and $J_{n+r}\tau_r(b) = \tau_r(b)$ implies that the last $r + j_0$ coordinates of $\tau_r(b)$ are also zeros. Therefore the equation $P_{n+r}\sigma_r(b) = \tau_r(b)$ implies that the vector $u$ formed by the $n + 1 - r - 2j_0$ central coordinates of $\sigma_r(b)$ satisfies $Au = 0$, since $n + 1 - r - 2j_0 \leq r + j_0$. This is a contradiction because $\sigma_r(b) \neq 0$. This completes the proof of the lemma.

We are now in a position to prove that the algebra $B$ does not have elements of odd degree, which will complete the proof of the Theorem 1.2 when $G_o$ is locally isomorphic to $SU(n,1)$, $n \geq 2$.

**Theorem 6.16.** If $G_o$ is locally isomorphic to $SU(n,1)$ with $n \geq 2$, and $b = b_m \otimes Z^m + \cdots + b_0 \in B$ with $m$ odd, then $b_m = 0$. That is, $B$ does not contain odd degree elements.
Proof. Let \( B_0 = \{ b \in B : \deg(b) \text{ is odd} \} \) and let us assume that \( B_0 \) is not empty. Now define \( r = \min \{ t \in \mathbb{N}_0 : \sigma_{t+1}(b) = 0 \text{ and } b \in B_0 \} \) and take \( b \in B_0 \) such that \( \sigma_{r+1}(b) = 0 \); clearly \( \sigma_r(b) \neq 0 \). Let \( m = m(b) \) denote the degree of \( b \). Then in view of Corollary 6.12 and Proposition 6.13 we have

\[
J_{m+r}\sigma_r(b) = (-1)^{m+r}\sigma_r(b) \quad \text{and} \quad J_{m+r}P_{m+r}\sigma_r(b) = P_{m+r}\sigma_r(b). \tag{24}
\]

Hence the vector \( \sigma_r(b) \) satisfies the conditions of part (i) of Proposition 6.14, therefore if \( r \) is even \( \sigma_r(b) \) begins (and ends) with an odd number of coordinates equal to zero and, on the other hand, if \( r \) is odd \( \sigma_r(b) \) begins (and ends) with an even number of coordinates equal to zero.

We recall that the first and the last \( r \) coordinates of \( \sigma_r(b) \) are zero and that the others are

\[
\dot{E}_t^{r+j}(c_{1,m-j})E_1^{m-r-j}, \quad j = 0, \ldots, m - r.
\]

Therefore \( \dot{E}_1^{r}(c_{1,m}) = 0 \). Let \( j_0(b) = \max\{ j \in \mathbb{N}_0 : \dot{E}_t^{r+j}(c_{1,m-l}) = 0 \text{ for all } 0 \leq t < j \leq m - r - 1 \} \). Then we know that \( j_0(b) \) is even and that \( m - r - 2j_0(b) - 1 > 0 \) because \( \sigma_r(b) \neq 0 \), \( \sigma_r(b) \) starts with \( r + j_0(b) + 1 \) zeros and \( J_{m+r}\sigma_r(b) = (-1)^{m+r}\sigma_r(b) \).

Among all \( b \in B_0 \) such that \( \sigma_{r+1}(b) = 0 \) we choose one such that \( j_0 = j_0(b) \) is the largest possible. We also have \( m - j_0 - 1 < 2(r + j_0 + 1) \), because from \( m - j_0 - 1 \geq 2(r + j_0 + 1) \) and \( \sigma_{r+1}(b) = 0 \) we would obtain \( d(c_{1,m-j_0-1}) = 2(r + j_0 + 1) \leq m - j_0 - 1 \). Hence from Lemma 6.15 we would know that there exist \( b' = c_{1,m-j_0-1} \otimes Z^{m-j_0-1} + \cdots \in B \) such that \( \sigma_{r+j_0+2}(b') = 0 \) and the element \( b - b' \in B_0 \) would contradict the maximality of \( j_0 \).

Let \( A \) be the submatrix of \( P_{m+r} \) formed by the elements in the last \( m - r - 2j_0 - 1 \) rows and in the \( m - r - 2j_0 - 1 \) central columns of \( P_{m+r} \). From Proposition 6.14 we know that \( A \) is nonsingular. Since \( P_{m+r}\sigma_r(b) = \tau_r(b) \), \( \tau_r(b) \) starts with \( r + j_0 + 1 \) zeros and since \( J_{m+r}\tau_r(b) = \tau_r(b) \) the last \( r + j_0 + 1 \) coordinates of \( \tau_r(b) \) are also zeros. Therefore the equation \( P_{m+r}\sigma_r(b) = \tau_r(b) \) implies that the vector \( u \) formed by the \( m - r - 2j_0 - 1 \) central coordinates of \( \sigma_r(b) \) satisfies \( Au = 0 \), since \( m - r - 2j_0 - 1 \leq r + j_0 + 1 \). This is a contradiction because \( \sigma_r(b) \neq 0 \). This completes the proof of the theorem.

\[7. \text{ Appendix}\]

Our goal in this appendix is to prove Proposition 6.14. For any \( n \in \mathbb{N}_0 \) let \( J_n \) and \( P_n \) be the \( (n + 1) \times (n + 1) \) matrices defined in (21) and (22), and let \( H_n \) be the following \( (n + 1) \times (n + 1) \) diagonal matrix

\[
H_n = \begin{pmatrix}
(-1)^n & (-1)^{n-1} & \cdots & 1 \\
(-1)^{n-1} & \cdots & 1 \\
\vdots & \ddots & \vdots \\
1 & \cdots & (-1) \\
\end{pmatrix}.
\]

Let \( V \) denote the vector space over \( \mathbb{C} \) of all polynomials in \( \mathbb{C}[X] \) of degree less or equal to \( n \). Then \( P_n, H_n \) and \( J_n \) are respectively the matrices of the linear
operators on $V$ given by

\[ f(X) \mapsto f(X+1), \quad f(X) \mapsto f(-X), \quad f(X) \mapsto X^n f(1/X), \quad (25) \]

with respect to the ordered basis $\{(\nchoose{n}{0}) X^n, (\nchoose{n}{1}) X^{n-1}, \ldots, (\nchoose{n}{n})\}$. In the next lemma we summarize some basic properties of the matrices $P_n$, $H_n$ and $J_n$. The proof of this lemma follows from simple calculations with the operators given in (25).

**Lemma 7.1.**

(i) $J_n^2 = H_n^2 = I$ and $J_n H_n = (-1)^n H_n J_n$.

(ii) $P_n^{-1} = H_n P_n H_n$.

(iii) $J_n$ and $P_n H_n$ are conjugate, in fact $J_n = (J_n P_n H_n)^{-1} P_n H_n (J_n P_n H_n)$. Hence the eigenvectors of $P_n H_n$ associated to the eigenvalue $\lambda = \pm 1$ are all of the form $J_n P_n H_n(v)$ where $v$ is an eigenvector of $J_n$ associated to the eigenvalue $\lambda$.

Now, let $k \in \mathbb{N}_o$ and let $v = (v_o, \ldots, v_n)$ be a vector in $\mathbb{C}^{n+1}$. We shall say that $v$ begins with $k$ coordinates equal to zero if $v_o = v_1 = \cdots = v_{k-1} = 0$ and $v_k \neq 0$. Similarly we shall say that $v$ ends with $k$ coordinates equal to zero if $v_{n-k+1} = v_{n-k+2} = \cdots = v_n = 0$ and $v_k \neq 0$. Also via the ordered basis $\{(\nchoose{n}{0}) X^n, (\nchoose{n}{1}) X^{n-1}, \ldots, (\nchoose{n}{n})\}$ we shall identify any vector $v = (v_o, \ldots, v_n) \in \mathbb{C}^{n+1}$ with the polynomial $f_v(X) = v_o (\nchoose{n}{0}) X^n + v_1 (\nchoose{n}{1}) X^{n-1} + \cdots + v_n$. In particular observe that $v$ begins with $k$ coordinates equal to zero if and only if the degree of $f_v$ is $n-k$. In the following lemma we prove part (i) of Proposition 6.14.

**Lemma 7.2.** If $v \in \mathbb{C}^{n+1}$ satisfies $J_n v = (-1)^n v$ and $J_n P_n v = P_n v$ then $v$ begins and ends with the same number of coordinates, say $k$, equal to zero. Moreover, $k$ is even or odd according as $n$ is even or odd, respectively.

**Proof.** Let $v \in \mathbb{C}^{n+1}$ be as in the statement of the lemma and assume that $v$ begins with $k$ coordinates equal to zero. If we identify $v$ with the polynomial $f_v$ defined above we claim that the degree of $f_v$ is even. In fact from Lemma 7.1 it follows that $H_n(v)$ is an eigenvector of $J_n$ associated to the eigenvalue 1, and that $J_n P_n H_n(v) = J_n P_n v = P_n v$ is an eigenvector of $P_n H_n$ associated to the eigenvalue 1. Then $P_n H_n(v) = P_n v$, which implies that $H_n P_n v = v$ or, equivalently, that $f_v(1-X) = f_v(X)$. Now if we define $g(X) = f_v(X + \frac{1}{2})$ we obtain $g(X) = g(-X)$, which in particular implies that the degree of $g$ is even. Hence the degree of $f_v$ is even. The other assertion is a direct consequence of $J_n v = (-1)^n v$. \hfill $\blacksquare$

We shall now prove part (ii) of Proposition 6.14. Let $t, a, b \in \mathbb{N}_o$ be such that $b \leq a \leq n$ and let $A$ be the $(t+1) \times (t+1)$ submatrix, of the Pascal matrix $P_t$, defined in (23). We want to prove that $A$ is nonsingular. Associated to the parameters $t, a, b$ we shall consider a $(t+1) \times (t+1)$ diagonal matrix $D_x$ defined for $x \in \mathbb{N}_o$, $x \geq b$, as follows

\[
D_x = \begin{pmatrix}
\binom{t}{b} & \binom{t+1}{b} & \cdots & \binom{t+1}{t} \\
\cdots & \cdots & \cdots & \cdots \\
\binom{t}{x} & \cdots & \binom{t+1}{x} & \cdots & \binom{t+1}{t}
\end{pmatrix},
\]
and a \((t + 1) \times (t + 1)\) matrix \(A_0\) of the following form

\[
A_0 = \begin{pmatrix}
\binom{a - b}{0} & \cdots & \binom{a - b}{t} \\
\vdots & \ddots & \vdots \\
\binom{a - b + t}{0} & \cdots & \binom{a - b + t}{t}
\end{pmatrix}.
\]  \quad (26)

The following lemma establishes the desired result about \(A\).

**Lemma 7.3.** Let \(t, a, b \in \mathbb{N}_0\) be such that \(b \leq a \leq n\) and let \(A, D_x\) and \(A_0\) be as above. Then

(i) \(A = D_A A_0 D_b^{-1}\),

(ii) \(\det A = \prod_{i=0}^t \binom{a+i}{b} \binom{b+i}{b}^{-1}\) and therefore \(A\) is nonsingular.

**Proof.**

(i) For \(0 \leq i, j \leq t\) let \(A_{i,j}\) denote the \((i, j)\) entry of the matrix \(A\), then we have

\[
A_{i,j} = \binom{a + i}{b + j} = \frac{(a + i)!}{(b + j)! (a - b + i - j)!} \frac{(a - b + i)!}{(a + i)!} \frac{b! j!}{(b - a + i)! j! (a - b + i - j)! (b + j)!} = \left( \frac{a + i}{b} \right) \left( \frac{a - b + i}{j} \right) \left( \frac{b + j}{b} \right)^{-1}.
\]

Since the right hand side of this equality is the \((i, j)\) entry of the product \(D_A A_0 D_b^{-1}\) (i) follows.

In order to prove (ii) it is enough to show that \(\det A_0 = 1\) for any matrix \(A_0\) as in (26). We proceed by induction on \(t\). It is clear that the result holds for \(t = 0\), so let us assume that it holds for any matrix as in (26) of size \(t \times t\) and let \(A_0\) be the \((t + 1) \times (t + 1)\) matrix defined in (26). Let \(C_0, C_1, \ldots, C_t\) denote the rows of \(A_0\). Since for any \(0 \leq j \leq t - 1\) we have

\[
\binom{a - b + j + 1}{i} - \binom{a - b + j}{i} = \begin{cases} 0, & \text{if } i = 0 \\ (-1)^{i-1} \binom{a - b + j}{i-1}, & \text{if } 1 \leq i \leq t,
\end{cases}
\]

we obtain for any \(0 \leq j \leq t - 1\) that

\[
C_{j+1} - C_j = \left( \begin{array}{c} a - b + j + 1 \\ 0 \\ \vdots \\ a - b + j \\ 0 \\ \vdots \\ 0 \end{array} \right) - \left( \begin{array}{c} a - b + j \\ 0 \\ \vdots \\ a - b + j \\ 0 \\ \vdots \\ 0 \end{array} \right) = \left( \begin{array}{c} 0 \\ \vdots \\ 0 \\ a - b + j \\ \vdots \\ \vdots \\ a - b + j \end{array} \right),
\]

Then if we regard \(\det A_0\) as a multilinear function of the rows of \(A_0\) we get,

\[
\det A_0 = \det \left( C_0, C_1 - C_0, \ldots, C_t - C_{t-1} \right) = \det \left( \begin{array}{ccc}
\binom{a-b}{0} & \cdots & \binom{a-b}{t} \\
0 & \cdots & \binom{a-b}{t-1} \\
\vdots & \ddots & \vdots \\
0 & \cdots & \binom{a-b}{t-1}
\end{array} \right) = \det \left( \begin{array}{ccc}
\binom{a-b}{0} & \cdots & \binom{a-b}{t-1} \\
0 & \cdots & \binom{a-b}{t-1}
\end{array} \right) = 1,
\]

by the inductive hypothesis. This completes the proof of the lemma. \(\blacksquare\)
References


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Received August 30, 2010
and in final form October 28, 2010