A Cubic $E_6$-Generalization of the Classical Theorem on Harmonic Polynomials

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Abstract. Classical harmonic analysis says that the spaces of homogeneous harmonic polynomials (solutions of Laplace equation) are irreducible modules of the corresponding orthogonal Lie group (algebra) and the whole polynomial algebra is a free module over the invariant polynomials generated by harmonic polynomials. Dickson invariant trilinear form is the unique fundamental invariant in the polynomial algebra over the basic irreducible module of $E_6$. In this paper, we prove that the space of homogeneous polynomial solutions with degree $m$ for the dual cubic Dickson invariant differential operator is exactly a direct sum of $[m/2] + 1$ explicitly determined irreducible $E_6$-submodules and the whole polynomial algebra is a free module over the polynomial algebra in the Dickson invariant generated by these solutions. Thus we obtain a cubic $E_6$-generalization of the above classical theorem on harmonic polynomials.

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1. Introduction

The $E_6$ Lie algebra and group are popular mathematical objects with broad applications. Dickson [D] (1901) first realized that there exists an $E_6$-invariant trilinear form on its 27-dimensional basic irreducible module, whose corresponding cubic polynomial invariant and constant-coefficient differential operator will also be the main objects in this paper. The 78-dimensional simple Lie algebra of type $E_6$ can be realized by all the derivations and multiplication operators with trace zero on the 27-dimensional exceptional simple Jordan algebra (e.g., cf. [T], [Ad]). Aschbacher [As] used the Dickson form to study the subgroup structure of the group $E_6$. Bion-Nadal [B-N] proved that the $E_6$ Coxeter graph can be realized as a principal graph of subfactor of the hyperfinite $\Pi_1$ factor. Brylinski and Kostant [BK] obtained a generalized Capelli identity on the minimal representation of $E_6$. Binegar and Zierau [BZ] found a singular representation of $E_6$. Ginzburg [G] proved that the twisted partial $L$-function on the 27-dimensional representation...
of $GE_6(\mathbb{C})$ is entire except the points 0 and 1. Ilyakov [I] showed that the field of invariant rational functions of $E_6$ on the direct sum of finite copies of the basic module and its dual is purely transcendental. Suzuki and Wakui [SW] studied the Turaev-Viro-Ocneanu invariant of 3-manifolds derived from the $E_6$-subfactor. Moreover, Cerchiai and Scotti [CS] investigated the mapping geometry of the $E_6$ group. Furthermore, the $(A_2,G_2)$ duality in $E_6$ was obtained by Rubenthaler [R].

Okamoto and Marshak [OM] constructed a grand unification preon model with $E_6$ metacolor. The $E_6$ Lie algebra was used in [HH] to explain the degeneracies encountered in the genetic code as the result of a sequence of symmetry breakings that have occurred during its evolution. Wang [W] identified Geoner’s model with twisted LG model and $E_6$ singlets. Morrison, Pieruschka and Wybourne [MPW] constructed the $E_6$ interacting boson model. Berglund, Candelas et al. [BCDH] studied instanton contributions to the masses and couplings of $E_6$ singles. Haba and Matsuoka [HM] found large lepton flavor mixing in the $E_6$-type unification models. Ghezelbash, Shafiekhani and Abolbasani [GSA] derived explicitly a set of Picard-Fuchs equations of $N = 2$ supersymmetric $E_6$ Yang-Mills theory. Anderson and Blažek [AB1-AB3] found certain Clebsch-Gordan coefficients in connection with $E_6$ unification model building. Fernández-Núñez, Garcia-Fuertes and Perelomov [FGP] used the quantum Calogero-Sutherland model corresponding to the root system of $E_6$ to calculate Clebach-Gordan series for this algebra. Howl and King [HK] proposed a minimal $E_6$ supersymmetric standard model which allows Planck scale unification, provides a solution to the $\mu$ problem and predicts a new $Z'$. Das and Laparashvili [DL] studied Preon model related to family replicated $E_6$ unification.

Classical harmonic analysis says that the spaces of homogeneous harmonic polynomials (solutions of Laplace equation) are irreducible modules of the corresponding orthogonal Lie group (algebra) and the whole polynomial algebra is a free module over the invariant polynomials generated by harmonic polynomials. Cao [C] proved that the subspaces of homogeneous polynomial vector solutions of the $n$-dimensional Navier equations in elasticity are exactly direct sums of three explicitly given irreducible submodules when $n \neq 4$ and direct sums of four explicitly given irreducible submodules if $n = 4$ of the corresponding orthogonal Lie group (algebra), and the whole polynomial vector space is also a free module over the invariant polynomials generated these solutions. Moreover, he solved the initial value problem for the Navier equations. In particular, Cao’s work can be viewed as a supplement to Olver’s well known work [O] on algebraic study of linear elasticity. It is a quadratic vector generalization of the classical theorem on harmonic polynomials.

The purpose of this paper is to prove a cubic $E_6$-generalization of the classical theorem on harmonic polynomials. It is well known that Dickson invariant trilinear form is the unique fundamental invariant in the polynomial algebra over the basic irreducible module of $E_6$. We prove that the space of homogeneous polynomial solutions with degree $m$ for the dual cubic Dickson invariant differential operator is exactly a direct sum of $\lceil m/2 \rceil + 1$ explicitly determined irreducible $E_6$-submodules and the whole polynomial algebra is a free module over the polynomial algebra in the Dickson invariant generated by these solutions. Below we give a more
detailed introduction to our results.

Denote by \( E_{r,s} \) the square matrix with 1 as its \((r,s)\)-entry and 0 as the others. The orthogonal Lie algebra

\[
o(n, \mathbb{R}) = \sum_{1 \leq r < s \leq n} \mathbb{R}(E_{r,s} - E_{s,r}).
\]

(1)

It acts on the polynomial algebra \( \mathcal{A} = \mathbb{R}[x_1, ..., x_n] \) by

\[
(E_{r,s} - E_{s,r})|_{\mathcal{A}} = x_r \partial_{x_s} - x_s \partial_{x_r}.
\]

(2)

Denote by \( \mathcal{A}_k \) the subspace of homogeneous polynomials in \( \mathcal{A} \) with degree \( k \). When \( n \geq 3 \), it is well known that the subspace of harmonic polynomials

\[
\mathcal{H}_k = \{ f \in \mathcal{A}_k \mid (\partial^2_{x_1} + \cdots + \partial^2_{x_n})(f) = 0 \}
\]

(3)

forms an irreducible \( o(n, \mathbb{R}) \)-module and

\[
\mathcal{A}_k = \mathcal{H}_k \oplus (x_1^2 + x_2^2 + \cdots + x_n^2)\mathcal{A}_{k-2}.
\]

(4)

Navier equations

\[
\iota_1 \Delta(\vec{u}) + (\iota_1 + \iota_2)(\nabla \cdot \nabla)(\vec{u}) = 0
\]

(5)

are used to describe the deformation of a homogeneous, isotropic and linear elastic medium in the absence of body forces, where \( \vec{u} \) is an \( n \)-dimensional vector-valued function, \( \Delta = \partial^2_{x_1} + \partial^2_{x_2} + \cdots + \partial^2_{x_n} \) is the Laplace operator, \( \nabla = (\partial_{x_1}, \partial_{x_2}, ..., \partial_{x_n}) \) is the gradient operator, \( \iota_1 \) and \( \iota_2 \) are Lamé constants with \( \iota_1 > 0 \), \( 2\iota_1 + \iota_2 > 0 \) and \( \iota_1 + \iota_2 \neq 0 \). In fact, \( \nabla \cdot \nabla \) is the well-known Hessian operator. Mathematically, the above system is a natural vector \( O(n, \mathbb{R}) \)-invariant generalization of the Laplace equation in (1.3).

Denote

\[
\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{f}(\vec{x}) = \begin{pmatrix} f_1(\vec{x}) \\ \vdots \\ f_n(\vec{x}) \end{pmatrix},
\]

(6)

\[
\mathcal{A} = \bigoplus_{k=0}^{\infty} \mathcal{A}_k \quad \text{with} \quad \mathcal{A}_k = \{ \vec{f} \mid f_j \in \mathcal{A}_k \}.
\]

(7)

Moreover, we define

\[
\mathcal{H}_k = \{ \vec{f} \in \mathcal{A}_k \mid \iota_1 \Delta(\vec{f}) + (\iota_1 + \iota_2)(\nabla^T \cdot \nabla)(\vec{f}) = 0 \}.
\]

(8)

Cao [C] proved that the subspace \( \mathcal{H}_k \) is a direct sum of three explicitly given irreducible \( o(n, \mathbb{R}) \)-submodules when \( n \neq 4 \) and a direct sum of four explicitly given irreducible \( o(4, \mathbb{R}) \)-submodules if \( n = 4 \). Moreover,

\[
\mathcal{A}_k = \mathcal{H}_k \oplus (x_1^2 + \cdots + x_n^2)\mathcal{A}_{k-2}.
\]

(9)

The Dynkin diagram of \( E_6 \) is as follows:
Denote by $\lambda_i$ the $i$th fundamental weight of $E_6$ with respect to the above labeling. Let $V$ be the 27-dimensional irreducible $E_6$-module of highest weight $\lambda_1$. Denote by $\mathcal{A}$ the polynomial algebra (equivalently, symmetric tensor) over $V$ and by $\mathcal{A}_m$ the subspace of homogeneous polynomial with degree $m$. A singular vector in $\mathcal{A}$ is a weight vector annihilated by positive root vectors. We explicitly construct a linear singular vector $x_1$ of weight $\lambda_1$ in (2.25), a quadratic singular vector $\zeta_1$ of weight $\lambda_6$ in (3.6) and a cubic singular vector $\eta$ of weight 0 in (3.47), where $\eta$ is the unique fundamental invariant corresponding to the Dickson trilinear form. The following is the main theorem of this paper:

**Main Theorem.** Denote by $L(m_1, m_2)$ the irreducible $E_6$-submodule generated by $x_1^{m_1} \zeta_1^{m_2}$ with highest weight $m_1 \lambda_1 + m_2 \lambda_6$. Let $D$ be the unique constant-coefficient fundamental invariant differential operator dual to $\eta$. Then

$$\Phi_m = \{ f \in \mathcal{A}_m \mid D(f) = 0 \} = \bigoplus_{i=0}^{[m/2]} L(m - 2i, i)$$

(10)

and

$$\mathcal{A}_m = \Phi_m \oplus \eta \mathcal{A}_{m-3}.$$  

(11)

Note that (1.11) is exactly a cubic generalization of the quadratic one in (1.4) and (1.9). The fundamental difference is that our subspace $\Phi_m$ of homogeneous polynomial solutions is a sum of $[m/2] + 1$ irreducible submodules.

In Section 2, we explicitly construct the 27-dimensional basic representation of $E_6$ in terms of differential operators via the root lattice construction of the $E_7$ simple Lie algebra. The proof of the main theorem is given in Section 3.

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### 2. Basic Representation of $E_6$

In this section, we will explicitly construct the 27-dimensional basic irreducible representation of $E_6$.

For convenience, we will use the notion

$$\overline{i, i+j} = \{i, i+1, i+2, ..., i+j\}$$

(12)

for integer $i$ and positive integer $j$ throughout this paper. We start with the root lattice construction of the simple Lie algebra of type $E_7$. As we all known, the Dynkin diagram of $E_7$ is as follows:
Let \( \{ \alpha_i \mid i \in \{1, 7\} \} \) be the simple positive roots corresponding to the vertices in the diagram, and let \( \Phi_{E_7} \) be the root system of \( E_7 \). Set

\[
Q_{E_7} = \sum_{i=1}^{7} \mathbb{Z} \alpha_i,
\]

the root lattice of type \( E_7 \). Denote by \((\cdot, \cdot)\) the symmetric \( \mathbb{Z} \)-bilinear form on \( Q_{E_7} \) such that

\[
\Phi_{E_7} = \{ \alpha \in Q_{E_7} \mid (\alpha, \alpha) = 2 \}.
\]

Define \( F(\cdot, \cdot) : Q_{E_7} \times Q_{E_7} \to \{ \pm 1 \} \) by

\[
F\left( \sum_{i=1}^{7} k_i \alpha_i, \sum_{j=1}^{7} l_j \alpha_j \right) = (-1)^{P_{i>j} k_i l_i + P_{i \geq j} k_i l_j (\alpha_i, \alpha_j)}, \quad k_i, l_j \in \mathbb{Z}.
\]

Then for \( \alpha, \beta, \gamma \in Q_{E_7} \),

\[
F(\alpha + \beta, \gamma) = F(\alpha, \gamma) F(\beta, \gamma), \quad F(\alpha, \beta + \gamma) = F(\alpha, \beta) F(\alpha, \gamma),
\]

\[
F(\alpha, \beta) F(\beta, \alpha)^{-1} = (-1)^{(\alpha, \beta)}, \quad F(\alpha, \alpha) = (-1)^{(\alpha, \alpha)/2}.
\]

In particular,

\[
F(\alpha, \beta) = -F(\beta, \alpha) \text{ if } \alpha, \beta, \alpha + \beta \in \Phi_{E_7}.
\]

Denote

\[
H_{E_7} = \sum_{i=1}^{7} \mathbb{R} \alpha_i.
\]

The simple Lie algebra of type \( E_7 \) is

\[
G^{E_7} = H_{E_7} \oplus \bigoplus_{\alpha \in \Phi_{E_7}} \mathbb{R} E_\alpha
\]

with the Lie bracket \([\cdot, \cdot]\) determined by:

\[
[H_{E_7}, H_{E_7}] = 0, \quad [h, E_\alpha] = (h, \alpha) E_\alpha, \quad [E_\alpha, E_-\alpha] = -\alpha,
\]

\[
[E_\alpha, E_\beta] = \begin{cases} 
0 & \text{if } \alpha + \beta \notin \Phi_{E_7}, \\
F(\alpha, \beta) E_{\alpha + \beta} & \text{if } \alpha + \beta \in \Phi_{E_7}
\end{cases}
\]

for \( \alpha, \beta \in \Phi_{E_7} \) and \( h \in H_{E_7} \).

Note that

\[
Q_{E_6} = \sum_{i=1}^{6} \mathbb{Z} \alpha_i \subset Q_{E_7}
\]
is the root lattice of $E_6$ and
\[ \Phi_{E_6} = Q_{E_6} \cap \Phi_{E_7} \] (24)
is the root system of $E_6$. Set
\[ H_{E_6} = \sum_{i=1}^{6} \mathbb{R} \alpha_i. \] (25)
Then the subalgebra
\[ G_{E_6} = H_{E_6} \oplus \bigoplus_{\alpha \in \Phi_{E_6}} \mathbb{R} E_{\alpha} \] (26)
of $G_{E_7}$ is exactly the simple Lie algebra of type $E_6$. Denote by $\Phi_{E_6}^+$ the set of positive roots of $E_6$ and by $\Phi_{E_7}^+$ the set of positive roots of $E_7$. The elements of $\Phi_{E_6}^+$ are:
\[ \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6, \] (27)
\[ \{ \alpha_1 + \sum_{r=3}^{j} \alpha_r \mid j \in \{2,6\} \} \bigcup \{ \sum_{r=1}^{j} \alpha_r \mid 2 \leq i < j \leq 6 \}, \] (28)
\[ \{ \sum_{s=2}^{j} \alpha_s + \sum_{t=4}^{k} \alpha_t \mid 2 \leq j < k \leq 6 \} \] (29)
and
\[ \{ \sum_{i=1}^{j} \alpha_i + \sum_{s=3}^{j} \alpha_s + \sum_{t=4}^{k} \alpha_t \mid 2 \leq i < j < k \leq 6 \}. \] (30)
Denote by $\Phi_{E_7}^+$ the set of the following positive roots:
\[ \alpha_1 + \sum_{r=3}^{7} \alpha_r, \quad \alpha_3 + 2\alpha_4 + \alpha_5 + \sum_{i=1}^{6} \alpha_i + \sum_{r=1}^{7} \alpha_r, \] (31)
\[ \{ 2 \sum_{s=1}^{6} \alpha_s - \alpha_1 + \alpha_4 - \alpha_6 + \sum_{r=i+1}^{7} \alpha_r \mid i \in \{1,6\} \}, \quad \{ \sum_{r=i+1}^{7} \alpha_r \mid i \in \{2,6\} \}, \] (32)
\[ \{ \sum_{s=2}^{j} \alpha_s + \sum_{t=4}^{7} \alpha_t \mid j \in \{2,6\} \}, \quad \{ \sum_{i=1}^{l} \alpha_i + \sum_{s=3}^{i} \alpha_s + \sum_{t=4}^{7} \alpha_t \mid 2 \leq i < j \leq 6 \}. \] (33)
Then
\[ \Phi_{E_7}^+ = \Phi_{E_6}^+ \bigcup \Phi_{E_7}^+. \] (34)
In particular,
\[ V = \sum_{\beta \in \Phi_{E_7}^+} \mathbb{R} E_{\beta} \] (35)
forms the 27-dimensional basic $G_{E_6}$-module of highest weight $\lambda_1$ with the representation $\text{ad}_{G_{E_7}}$. Denote
\[ x_1 = E_{\alpha_3 + 2\alpha_4 + \alpha_5 + \sum_{i=1}^{6} \alpha_i + \sum_{r=3}^{7} \alpha_r}, \quad x_2 = E_{2\sum_{s=1}^{6} \alpha_s - \alpha_1 + \alpha_4 - \alpha_6 + \sum_{r=3}^{7} \alpha_r}, \] (36)
Under the above basis

\begin{align*}
x_3 &= E_2 \sum_{r=4}^{6} \alpha_r - \alpha_1 + \alpha_4 - \alpha_6 + \sum_{r=4}^{7} \alpha_r,
x_4 &= E_2 \sum_{r=3}^{6} \alpha_r - \alpha_1 - \alpha_6 + \sum_{r=4}^{7} \alpha_r, \\
x_5 &= E_2 \sum_{r=1}^{6} \alpha_r - \alpha_1 + \alpha_6 - \alpha_4 + \alpha_7, \\
x_6 &= E_2 \sum_{r=1}^{6} \alpha_r + \sum_{r=4}^{7} \alpha_r, \\
x_7 &= E_4 \sum_{r=3}^{6} \alpha_r + \sum_{i=1}^{7} \alpha_i, \\
x_8 &= E_2 \sum_{r=1}^{6} \alpha_r - \alpha_1 + \alpha_6 + \alpha_4 + \alpha_7, \\
x_9 &= E_4 \sum_{r=2}^{6} \alpha_r + \sum_{i=1}^{7} \alpha_i, \\
x_{10} &= E_4 \sum_{r=3}^{6} \alpha_r + \sum_{i=1}^{7} \alpha_i, \\
x_{11} &= E_6 \sum_{r=4}^{6} \alpha_r + \sum_{i=1}^{7} \alpha_i, \\
x_{12} &= E_5 \sum_{r=3}^{6} \alpha_r + \sum_{i=1}^{7} \alpha_i, \\
x_{13} &= E_{3+4} \sum_{r=3}^{6} \alpha_r + \sum_{i=1}^{7} \alpha_i, \\
x_{14} &= E_4 \sum_{r=2}^{6} \alpha_r + \sum_{i=1}^{7} \alpha_i, \\
x_{15} &= E_{4+5} \sum_{r=1}^{7} \alpha_r, \\
x_{16} &= E_{4+3} \sum_{i=1}^{7} \alpha_i, \\
x_{17} &= E_{4+5} \sum_{i=2}^{7} \alpha_i, \\
x_{18} &= E_4 \sum_{i=1}^{7} \alpha_i, \\
x_{19} &= E_{4+3} \sum_{i=2}^{7} \alpha_i, \\
x_{20} &= E_3 \sum_{r=3}^{4} \alpha_r, \\
x_{21} &= E_7 \sum_{i=2}^{7} \alpha_i, \\
x_{22} &= E_2 \sum_{r=4}^{5} \alpha_r, \\
x_{23} &= E_{4+7} \sum_{i=3}^{7} \alpha_i, \\
x_{24} &= E_7 \sum_{r=4}^{7} \alpha_r, \\
x_{25} &= E_8 \sum_{i=5}^{7} \alpha_i, \\
x_{26} &= E_{6+7} \sum_{i=5}^{7} \alpha_i, \\
x_{27} &= E_{6+7} \sum_{i=5}^{7} \alpha_i. \\
\end{align*}

Then \( \{ x_i \mid i \in [1, 27] \} \) forms a basis of \( V \).

Under the above basis

\begin{align*}
E_{\alpha_1} V &= -x_1 \partial_{x_2} + x_{11} \partial_{x_{14}} + x_{15} \partial_{x_{17}} + x_{16} \partial_{x_{19}} + x_{18} \partial_{x_{21}} + x_{20} \partial_{x_{23}}, \\
E_{\alpha_2} V &= -x_2 \partial_{x_6} - x_5 \partial_{x_7} - x_9 \partial_{x_{10}} - x_{18} \partial_{x_{20}} + x_{21} \partial_{x_{23}} + x_{22} \partial_{x_{24}}, \\
E_{\alpha_3} V &= -x_2 \partial_{x_3} + x_9 \partial_{x_{11}} + x_{12} \partial_{x_{15}} + x_{13} \partial_{x_{16}} + x_{21} \partial_{x_{22}} + x_{23} \partial_{x_{24}}, \\
E_{\alpha_4} V &= -x_3 \partial_{x_4} - x_7 \partial_{x_{9}} - x_{10} \partial_{x_{12}} - x_{16} \partial_{x_{18}} - x_{19} \partial_{x_{21}} + x_{23} \partial_{x_{25}}, \\
E_{\alpha_5} V &= -x_4 \partial_{x_5} - x_6 \partial_{x_{7}} - x_{12} \partial_{x_{13}} - x_{15} \partial_{x_{16}} - x_{17} \partial_{x_{19}} + x_{25} \partial_{x_{26}}, \\
E_{\alpha_6} V &= -x_5 \partial_{x_8} - x_7 \partial_{x_{10}} - x_{11} \partial_{x_{15}} - x_{14} \partial_{x_{17}} + x_{26} \partial_{x_{27}}, \\
E_{\alpha_{1+3}} V &= -x_1 \partial_{x_3} - x_9 \partial_{x_{14}} - x_{12} \partial_{x_{17}} - x_{13} \partial_{x_{19}} + x_{18} \partial_{x_{22}} + x_{20} \partial_{x_{24}}, \\
E_{\alpha_{2+4}} V &= -x_3 \partial_{x_6} + x_5 \partial_{x_9} + x_8 \partial_{x_{12}} + x_{16} \partial_{x_{20}} + x_{19} \partial_{x_{23}} + x_{22} \partial_{x_{25}}, \\
E_{\alpha_{3+4}} V &= x_2 \partial_{x_4} + x_7 \partial_{x_{11}} + x_{10} \partial_{x_{15}} - x_{13} \partial_{x_{18}} + x_{19} \partial_{x_{22}} + x_{23} \partial_{x_{25}}, \\
E_{\alpha_{4+5}} V &= x_3 \partial_{x_5} - x_6 \partial_{x_{7}} + x_{10} \partial_{x_{13}} - x_{15} \partial_{x_{18}} - x_{17} \partial_{x_{21}} + x_{24} \partial_{x_{26}}, \\
E_{\alpha_{5+6}} V &= x_4 \partial_{x_8} + x_6 \partial_{x_{10}} - x_9 \partial_{x_{13}} - x_{15} \partial_{x_{16}} - x_{14} \partial_{x_{19}} + x_{25} \partial_{x_{27}}, \\
E_{\alpha_{1+3+4}} V &= -x_1 \partial_{x_4} - x_7 \partial_{x_{14}} - x_{10} \partial_{x_{17}} + x_{13} \partial_{x_{21}} + x_{16} \partial_{x_{22}} + x_{20} \partial_{x_{25}}, \\
E_{\alpha_{2+3+4}} V &= x_2 \partial_{x_5} - x_5 \partial_{x_{11}} - x_8 \partial_{x_{15}} + x_{13} \partial_{x_{20}} - x_{19} \partial_{x_{24}} + x_{21} \partial_{x_{25}}, \\
E_{\alpha_{2+4+5}} V &= x_3 \partial_{x_7} + x_4 \partial_{x_{9}} - x_8 \partial_{x_{12}} + x_{15} \partial_{x_{20}} + x_{17} \partial_{x_{23}} + x_{22} \partial_{x_{26}}, \\
E_{\alpha_{3+4+5}} V &= -x_2 \partial_{x_5} + x_6 \partial_{x_{11}} - x_{10} \partial_{x_{16}} - x_{12} \partial_{x_{19}} + x_{17} \partial_{x_{22}} + x_{23} \partial_{x_{26}}, \\
E_{\alpha_{4+5+6}} V &= -x_3 \partial_{x_8} + x_6 \partial_{x_{12}} + x_7 \partial_{x_{13}} - x_{11} \partial_{x_{18}} - x_{14} \partial_{x_{21}} + x_{24} \partial_{x_{27}}, \\
E_{\alpha_{1+5+6}} V &= -x_1 \partial_{x_6} + x_5 \partial_{x_{14}} + x_8 \partial_{x_{17}} - x_{13} \partial_{x_{23}} - x_{16} \partial_{x_{24}} + x_{18} \partial_{x_{25}}, \\
E_{\alpha_{1+\sum^5_{i=3} \alpha_i}} V &= x_1 \partial_{x_5} - x_5 \partial_{x_{14}} + x_{10} \partial_{x_{19}} + x_{12} \partial_{x_{21}} + x_{15} \partial_{x_{22}} + x_{20} \partial_{x_{26}}, \\
E_{\sum^5_{i=2} \alpha_i} V &= -x_2 \partial_{x_7} - x_4 \partial_{x_{11}} + x_8 \partial_{x_{16}} + x_{12} \partial_{x_{20}} - x_{17} \partial_{x_{24}} + x_{21} \partial_{x_{26}}, \\
E_{\alpha_{3+\sum^4_{i=3} \alpha_i}} V &= -x_3 \partial_{x_{10}} - x_4 \partial_{x_{12}} - x_5 \partial_{x_{13}} + x_{11} \partial_{x_{20}} + x_{14} \partial_{x_{22}} + x_{22} \partial_{x_{27}}, \\
E_{\sum^6_{i=3} \alpha_i} V &= x_2 \partial_{x_8} - x_6 \partial_{x_{15}} - x_7 \partial_{x_{16}} - x_9 \partial_{x_{18}} + x_{14} \partial_{x_{22}} + x_{23} \partial_{x_{27}}, \\
\end{align*}
\[ E_{\sum_{i=1}^{\infty} a_i} |V = x_1 \partial_{x_7} + x_4 \partial_{x_{14}} - x_8 \partial_{x_{19}} - x_{12} \partial_{x_{23}} - x_{15} \partial_{x_{24}} + x_{18} \partial_{x_{26}}, \]  
(67)

\[ E_{\sum_{i=1}^{\infty} a_i} |V = -x_1 \partial_{x_8} + x_6 \partial_{x_{17}} + x_7 \partial_{x_{19}} + x_9 \partial_{x_{21}} + x_{11} \partial_{x_{22}} + x_{20} \partial_{x_{27}}, \]  
(68)

\[ E_{\sum_{i=2}^{\infty} a_i} |V = x_2 \partial_{x_9} - x_3 \partial_{x_{11}} - x_8 \partial_{x_{18}} + x_{10} \partial_{x_{20}} - x_{17} \partial_{x_{25}} + x_{19} \partial_{x_{26}}, \]  
(69)

\[ E_{\sum_{i=2}^{\infty} a_i} |V = x_2 \partial_{x_{10}} + x_4 \partial_{x_{15}} + x_5 \partial_{x_{16}} + x_9 \partial_{x_{20}} - x_{14} \partial_{x_{24}} + x_{21} \partial_{x_{27}}, \]  
(70)

\[ E_{\sum_{i=1}^{\infty} a_i} |V = -x_1 \partial_{x_9} + x_3 \partial_{x_{14}} + x_8 \partial_{x_{21}} - x_{10} \partial_{x_{23}} - x_{15} \partial_{x_{25}} + x_{16} \partial_{x_{26}}, \]  
(71)

\[ E_{\sum_{i=1}^{\infty} a_i} |V = -x_1 \partial_{x_{10}} - x_4 \partial_{x_{17}} - x_5 \partial_{x_{19}} - x_9 \partial_{x_{23}} - x_{11} \partial_{x_{24}} + x_{18} \partial_{x_{27}}, \]  
(72)

\[ E_{\sum_{i=2}^{\infty} a_i} |V = -x_2 \partial_{x_{12}} + x_3 \partial_{x_{15}} - x_5 \partial_{x_{16}} + x_7 \partial_{x_{20}} - x_{14} \partial_{x_{25}} + x_{19} \partial_{x_{27}}, \]  
(73)

\[ E_{\sum_{i=1}^{\infty} a_i} |V = x_1 \partial_{x_{11}} - x_2 \partial_{x_{14}} - x_8 \partial_{x_{22}} + x_{10} \partial_{x_{24}} - x_{12} \partial_{x_{25}} + x_{13} \partial_{x_{26}}, \]  
(74)

\[ E_{\sum_{i=1}^{\infty} a_i} |V = x_1 \partial_{x_{12}} - x_3 \partial_{x_{17}} + x_5 \partial_{x_{21}} - x_7 \partial_{x_{23}} - x_{11} \partial_{x_{25}} + x_{16} \partial_{x_{27}}, \]  
(75)

\[ E_{\sum_{i=2}^{\infty} a_i} |V = x_2 \partial_{x_{13}} - x_3 \partial_{x_{16}} - x_4 \partial_{x_{18}} + x_6 \partial_{x_{20}} - x_{14} \partial_{x_{26}} + x_{17} \partial_{x_{27}}, \]  
(76)

\[ E_{\sum_{i=1}^{\infty} a_i} |V = -x_1 \partial_{x_{15}} + x_2 \partial_{x_{17}} - x_3 \partial_{x_{22}} + x_7 \partial_{x_{24}} - x_{16} \partial_{x_{25}} + x_{13} \partial_{x_{27}}, \]  
(77)

\[ E_{\sum_{i=1}^{\infty} a_i} |V = -x_1 \partial_{x_{13}} + x_3 \partial_{x_{19}} + x_4 \partial_{x_{21}} - x_6 \partial_{x_{23}} - x_{11} \partial_{x_{26}} + x_{15} \partial_{x_{27}}, \]  
(78)

\[ E_{\sum_{i=1}^{\infty} a_i} |V = x_1 \partial_{x_{16}} - x_2 \partial_{x_{19}} - x_3 \partial_{x_{22}} + x_6 \partial_{x_{24}} - x_{10} \partial_{x_{26}} + x_{12} \partial_{x_{27}}, \]  
(79)

\[ E_{\sum_{i=1}^{\infty} a_i} |V = x_1 \partial_{x_{18}} - x_2 \partial_{x_{21}} + x_3 \partial_{x_{22}} - x_6 \partial_{x_{25}} + x_7 \partial_{x_{26}} - x_{10} \partial_{x_{27}}, \]  
(80)

\[ E_{\sum_{i=1}^{\infty} a_i} |V = x_1 \partial_{x_{20}} - x_2 \partial_{x_{23}} + x_3 \partial_{x_{24}} - x_4 \partial_{x_{25}} - x_5 \partial_{x_{26}} - x_8 \partial_{x_{27}}. \]  
(81)

Recall that we also view \(a_i\) as the elements of \(G^{E_7}\) (cf. (2.8) and (2.9)).

We write

\[ [a_j, x_i] = a_{i,j} x_i \text{ for } i \in \Gamma, j \in \Gamma, 6. \]  
(82)

Then the weight of \(x_i\) is \(\sum_{j=1}^{6} a_{i,j} \lambda_j\), where \(\lambda_j\) is the \(j\)th fundamental weight of \(G^{E_6}\). We calculate the following table:

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<th>(a_{1,5})</th>
<th>(a_{1,6})</th>
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<th>(a_{1,3})</th>
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</tbody>
</table>
The last equation in (3.3) implies $E_c$ naturally holds by (3.1) and (2.40). Taking (2.35)-(2.74)

$A = \text{Now}$

By (2.35),

Furthermore, (2.37) gives

In particular,

In addition, (2.38) yields

$E_{-\alpha}|_V = -\psi(E_{\alpha}|_V)$ for $\alpha \in \Phi^+_E$ (85)

by the second equations in (2.5) and (2.6). In particular,

$E_{-\alpha_1}|_V = x_2\partial_{x_1} - x_{14}\partial_{x_{11}} - x_{17}\partial_{x_{15}} - x_{19}\partial_{x_{16}} - x_{21}\partial_{x_{18}} - x_{23}\partial_{x_{20}},$ (86)

$E_{-\alpha_2}|_V = x_5\partial_{x_1} + x_7\partial_{x_2} - x_{10}\partial_{x_3} - x_{20}\partial_{x_{16}} - x_{23}\partial_{x_{21}} - x_{24}\partial_{x_{22}},$ (87)

$E_{-\alpha_3}|_V = x_3\partial_{x_2} - x_{11}\partial_{x_9} - x_{15}\partial_{x_12} - x_{16}\partial_{x_{13}} - x_{22}\partial_{x_{21}} - x_{24}\partial_{x_{23}},$ (88)

$E_{-\alpha_4}|_V = x_4\partial_{x_8} + x_9\partial_{x_7} + x_{12}\partial_{x_{10}} + x_{18}\partial_{x_{16}} + x_{21}\partial_{x_{19}} - x_{25}\partial_{x_{24}},$ (89)

$E_{-\alpha_5}|_V = x_5\partial_{x_8} + x_7\partial_{x_6} + x_{13}\partial_{x_{12}} + x_{16}\partial_{x_{15}} + x_{19}\partial_{x_{17}} - x_{26}\partial_{x_{25}},$ (90)

$E_{-\alpha_6}|_V = x_8\partial_{x_5} + x_{10}\partial_{x_7} + x_{12}\partial_{x_9} + x_{15}\partial_{x_{11}} + x_{17}\partial_{x_{14}} - x_{27}\partial_{x_{26}}.$ (91)

3. Proof of the Main Theorem

Now $\mathcal{A} = \mathbb{R}[x_1, ..., x_{27}]$ becomes a $G^E$-module via the differential operators in (2.35)-(2.74)

According to Table 1, we look for a singular vector of the form:

$$\zeta_1 = c_1x_1x_{14} + c_2x_2x_{11} + c_3x_3x_9 + c_4x_4x_7 + c_5x_5x_6.$$ (92)

By (2.35),

$$0 = E_{\alpha_1}(\zeta_1) = (c_1 - c_2)x_1x_{11} \implies c_1 = c_2.$$ (93)

Moreover, (2.36) implies

$$0 = E_{\alpha_2}(\zeta_1) = -(c_4 + c_5)x_4x_5 \implies c_5 = -c_4.$$ (94)

Furthermore, (2.37) gives

$$0 = E_{\alpha_3}(\zeta_1) = (c_2 - c_3)x_2x_9 \implies c_2 = c_3.$$ (95)

In addition, (2.38) yields

$$0 = E_{\alpha_4}(\zeta_1) = -(c_3 + c_4)x_3x_7 \implies c_4 = -c_3.$$ (96)

The last equation in (3.3) implies $E_{\alpha_5}(\zeta_1) = 0$ by (2.39). Besides, $E_{\alpha_6}(\zeta_1) = 0$ naturally holds by (3.1) and (2.40). Taking $c_1 = 1$, we have the singular vector

$$\zeta_1 = x_1x_{14} + x_2x_{11} + x_3x_9 - x_4x_7 + x_5x_6.$$ (97)
of weight $\lambda_6$.

According to (2.75)-(2.80), we set

\[
\zeta_2 = E_{-\alpha_6}(\zeta_1) = x_1x_{17} + x_2x_{15} + x_3x_{12} - x_4x_{10} + x_6x_8,
\]

\[
\zeta_3 = E_{-\alpha_5}(\zeta_2) = x_1x_{19} + x_2x_{16} + x_3x_{13} - x_5x_{10} + x_7x_8,
\]

\[
\zeta_4 = E_{-\alpha_4}(\zeta_3) = x_1x_{21} + x_2x_{18} + x_4x_{13} - x_5x_{12} + x_8x_9,
\]

\[
\zeta_5 = E_{-\alpha_3}(\zeta_4) = -x_1x_{22} + x_3x_{18} - x_4x_{16} + x_5x_{15} - x_8x_{11},
\]

\[
\zeta_6 = E_{-\alpha_2}(\zeta_5) = -x_1x_{23} - x_2x_{20} + x_6x_{13} - x_7x_{12} + x_9x_{10},
\]

\[
\zeta_7 = E_{-\alpha_1}(\zeta_6) = x_1x_{24} - x_3x_{20} - x_6x_{16} + x_7x_{15} - x_9x_{11},
\]

\[
\zeta_8 = E_{-\alpha_1}(\zeta_5) = -x_2x_{22} - x_3x_{21} + x_4x_{19} - x_5x_{17} + x_8x_{14},
\]

\[
\zeta_9 = E_{-\alpha_4}(\zeta_7) = -x_1x_{25} - x_4x_{20} - x_6x_{18} + x_9x_{15} - x_{11}x_{12},
\]

\[
\zeta_{10} = E_{-\alpha_1}(\zeta_7) = x_2x_{24} + x_3x_{23} + x_6x_{19} - x_7x_{17} + x_{10}x_{14},
\]

\[
\zeta_{11} = -E_{-\alpha_5}(\zeta_9) = -x_1x_{26} + x_5x_{20} + x_7x_{18} - x_9x_{16} + x_{11}x_{13},
\]

\[
\zeta_{12} = E_{-\alpha_4}(\zeta_{10}) = -x_2x_{25} + x_4x_{23} + x_6x_{21} - x_9x_{17} + x_{12}x_{14},
\]

\[
\zeta_{13} = E_{-\alpha_3}(\zeta_{12}) = -x_3x_{25} - x_4x_{23} + x_6x_{21} + x_9x_{17} - x_{12}x_{14},
\]

\[
\zeta_{14} = -E_{-\alpha_6}(\zeta_{11}) = -x_1x_{27} - x_3x_{20} - x_{10}x_{18} + x_{12}x_{16} - x_{13}x_{15},
\]

\[
\zeta_{15} = -E_{-\alpha_5}(\zeta_{12}) = -x_2x_{26} - x_5x_{23} - x_7x_{21} + x_9x_{19} - x_{13}x_{14}.
\]

Define a map $\iota : \mathbb{1}, 27 \rightarrow \mathbb{1}, 27$ by

\[
\iota(13) = 13, \quad \iota(14) = 14, \quad \iota(15) = 15,
\]

\[
\iota(i) = 28 - i \text{ for } i \in \mathbb{1}, 27 \setminus \{13, 14, 15\}.
\]

Let $\tau$ be an algebraic automorphism of $\mathcal{A}$ determined by

\[
\tau(x_i) = x_{\iota(i)} \text{ for } i \in \mathbb{1}, 27.
\]

Now we set

\[
\zeta_i = \tau(\zeta_{28-i}) \text{ for } i \in \mathbb{16, 27}.
\]

It can be verified that

\[
\bar{V} = \sum_{r=1}^{27} \mathbb{R}\zeta_r
\]

an irreducible $G_{E_6}$-submodule and $\{\zeta_r \mid r \in \mathbb{1}, 27\}$ forms a basis of $\bar{V}$.

From the Dynkin diagram of $E_6$, we have the following automorphism of $Q_{E_6}$:

\[
\sigma(\sum_{i=1}^{6} k_i\alpha_i) = k_6\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4 + k_5\alpha_5 + k_1\alpha_6
\]

for $\sum_{i=1}^{6} k_i\alpha_i \in Q_{E_6}$. Let $\nu$ be an associative algebra homomorphism of the associative algebra

\[
\mathbb{A} = \sum_{i_1, \ldots, i_{27}=0}^{\infty} \mathcal{A}\partial_{x_1}^{i_1} \cdots \partial_{x_{27}}^{i_{27}}
\]
of differential operators to itself determined by
\[ \nu(x_i) = \zeta_i, \quad \nu(\partial_{x_i}) = \partial_{\zeta_i} \text{ for } i \in \Gamma, 27. \]  

(119)

It can be proved that
\[ E_{\alpha|\tilde{\nu}} = \nu(E_{\sigma(a)}|\tilde{\nu}) \text{ for } \alpha \in \Phi_{E_6}^+. \]  

(120)

Moreover,
\[ \alpha_j|\tilde{\nu} = \sum_{i=1}^{27} b_{i,j} \zeta_i \partial_{\zeta_i}, \]  

(121)

where
\[ b_{i,1} = a_{i,6}, \quad b_{i,3} = a_{i,5}, \quad b_{i,2} = a_{i,2}, \quad b_{i,4} = a_{i,4}. \]  

(122)

Thus we have the following table:

**Table 2**

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</table>

According to Table 1 and Table 2, we look for an invariant of the form
\[ \eta = \sum_{i=1}^{12} (d_i x_i \zeta_{28-i} + d_{28-i} x_{28-i} \zeta_i) + d_{13} x_{13} \zeta_{13} + d_{14} x_{14} \zeta_{14} + d_{15} x_{15} \zeta_{15}, \]  

(123)

where $d_i \in \mathbb{R}$. By (2.35), (2.40) and (3.29), we have

\begin{align*}
0 &= E_{a_1}(\eta) \\
&= -d_2 x_1 \zeta_{26} + d_{14} x_{11} \zeta_{14} + d_{17} x_{15} \zeta_{11} + d_{19} x_{16} \zeta_{9} + d_{21} x_{18} \zeta_{7} + d_{23} x_{20} \zeta_{5} \\
&\quad -d_{20} x_{20} \zeta_{5} - d_{18} x_{18} \zeta_{7} - d_{16} x_{16} \zeta_{9} - d_{15} x_{15} \zeta_{11} - d_{11} x_{11} \zeta_{14} + d_{1} x_{1} \zeta_{26}. \quad (124)
\end{align*}

\begin{align*}
0 &= E_{a_6}(\eta) \\
&= -d_{8} x_{5} \zeta_{20} - d_{10} x_{7} \zeta_{18} - d_{12} x_{9} \zeta_{16} - d_{15} x_{11} \zeta_{15} - d_{17} x_{14} \zeta_{11} + d_{27} x_{26} \zeta_{1} \\
&\quad -d_{26} x_{26} \zeta_{1} + d_{14} x_{14} \zeta_{11} + d_{11} x_{11} \zeta_{15} + d_{9} x_{9} \zeta_{16} + d_{7} x_{7} \zeta_{18} + d_{5} x_{5} \zeta_{20}. \quad (125)
\end{align*}
Therefore, we have the following invariant
\[0 = E_{\alpha_3}(\eta)\]
\[= -d_5x_2\zeta_{25} + d_{11}x_9\zeta_{17} + d_{15}x_{12}\zeta_{15} + d_{16}x_{13}\zeta_{12} + d_{22}x_{21}\zeta_6 + d_{24}x_{23}\zeta_4 - d_{23}x_{23}\zeta_4 - d_{21}x_{21}\zeta_6 - d_{13}x_{13}\zeta_{12} - d_{12}x_{12}\zeta_{15} - d_{19}x_9\zeta_{17} + d_{22}x_2\zeta_{25}, \]  
(128)
Moreover, (2.37), (2.39) and (3.29) imply
\[0 = E_{\alpha_5}(\eta)\]
\[= -d_5x_4\zeta_{23} - d_7x_6\zeta_{21} - d_{13}x_{12}\zeta_{13} - d_{16}x_{15}\zeta_{12} - d_{19}x_{17}\zeta_9 + d_{26}x_{25}\zeta_2 - d_{25}x_{25}\zeta_2 + d_{17}x_{17}\zeta_9 + d_{15}x_{15}\zeta_{12} + d_{12}x_{12}\zeta_{13} + d_6x_6\zeta_{21} + d_4x_4\zeta_{23}. \]  
(129)
Hence we get
\[d_3 = d_2, \quad d_{11} = d_9, \quad d_{15} = d_{12}, \quad d_{16} = d_{13}, \quad d_{22} = d_{21}, \quad d_{24} = d_{23}, \]
(130)
\[d_5 = d_4, \quad d_7 = d_6, \quad d_{13} = d_{12}, \quad d_{16} = d_{15}, \quad d_{19} = d_{17}, \quad d_{26} = d_{25}. \]  
(131)
Furthermore, (2.36), (2.38) and (3.29) yield
\[0 = E_{\alpha_2}(\eta)\]
\[= -d_6x_4\zeta_{22} - d_{22}x_{22}\zeta_4 - d_7x_5\zeta_{21} - d_{21}x_{21}\zeta_5 - d_{10}x_8\zeta_{18} - d_{18}x_{18}\zeta_8 + d_{20}x_{18}\zeta_8 + d_8x_8\zeta_{18} + d_{23}x_{21}\zeta_5 + d_5x_5\zeta_{21} + d_{24}x_{22}\zeta_4 + d_4x_4\zeta_{22}, \]  
(132)
Hence we get
\[d_6 = d_4, \quad d_{24} = d_{22}, \quad d_7 = d_5, \quad d_{23} = d_{21}, \quad d_{10} = d_8, \quad d_{20} = d_{18}, \]
(134)
\[d_4 = d_3, \quad d_{25} = d_{24}, \quad d_9 = -d_7, \quad d_{21} = -d_{19}, \quad d_{12} = -d_{10}, \quad d_{18} = -d_{16}. \]  
(135)
By (3.35), (3.36), (3.39), (3.40), (3.42) and (3.43), we have
\[d_1 = d_2 = d_3 = d_4 = d_5 = d_6 = d_7 = d_8 = -d_9 = d_10 = -d_{11} \]
(136)
\[= -d_{12} = -d_{13} = -d_{14} = -d_{15} = -d_{16} = -d_{17} = d_{18} \]
(137)
\[= -d_{19} = d_{20} = d_{21} = d_{22} = d_{23} = d_{24} = d_{25} = d_{26} = d_{27}. \]
Therefore, we have the following invariant
\[\eta = \sum_{i=1}^{8} (x_i\zeta_{28-i} + x_{28-i}\zeta_i) + x_{10}\zeta_{18} - \sum_{r=9,11,12} (x_r\zeta_{28-r} + x_{28-r}\zeta_r) - \sum_{s=13}^{15} x_s\zeta_s. \]  
(138)
According to (3.6)-(3.24), \( \eta = \)
\[
3[(x_1 x_{14} + x_2 x_{11} + x_3 x_9)x_{27} + (x_1 x_{17} + x_2 x_{15} + x_3 x_{12})x_{26} + (x_1 x_{19} + x_2 x_{16} + x_3 x_{13})x_{25} + (x_4 x_{15} - x_5 x_{12} + x_8 x_9)x_{24} - (x_4 x_{16} - x_5 x_{15} + x_8 x_{11})x_{23} + (x_6 x_{13} - x_7 x_{12} + x_9 x_{10})x_{22} + (x_7 x_{15} + x_8 x_{16} - x_{10} x_{11})x_{21} + (x_4 x_{19} - x_5 x_{17} + x_8 x_{14})x_{20} + (x_6 x_{18} - x_9 x_{15} + x_{11} x_{12})x_{19} + (x_{10} x_{14} - x_7 x_{17})x_{18} + (x_9 x_{16} - x_{11} x_{13})x_{17} - x_{12} x_{14} x_{16} + x_{14} x_{15} x_{13} + (x_4 x_{17} - x_5 x_6) x_{27} + (x_4 x_{10} - x_5 x_8) x_{26} + (x_5 x_{10} - x_7 x_8) x_{25} + (x_1 x_{21} + x_2 x_{18}) x_{24} + (x_3 x_{18} - x_1 x_{22}) x_{23} - (x_2 x_{22} + x_3 x_{21}) x_{20}. \]

(139)

**Lemma 3.1.**  Any homogeneous singular vector in \( \mathcal{A} \) is a monomial in \( x_1, \zeta_1 \) and \( \eta \).

**Proof.**  Note that
\[
x_1 x_{14} = \zeta_1 - x_2 x_{11} - x_3 x_9 + x_4 x_7 - x_5 x_6 \quad (140)
\]
\[
x_1 x_{17} = \zeta_2 - x_2 x_{15} - x_3 x_{12} + x_4 x_{10} - x_6 x_8, \quad (141)
\]
\[
x_1 x_{19} = \zeta_3 - x_2 x_{16} - x_3 x_{13} + x_5 x_{10} - x_7 x_8, \quad (142)
\]
\[
x_1 x_{21} = \zeta_4 - x_2 x_{18} - x_3 x_{15} + x_5 x_{12} - x_8 x_9, \quad (143)
\]
\[
x_1 x_{22} = -\zeta_5 - x_3 x_{18} - x_4 x_{16} + x_5 x_{15} - x_8 x_{11}, \quad (144)
\]
\[
x_1 x_{23} = -\zeta_6 - x_2 x_{20} + x_6 x_{13} - x_7 x_{12} + x_9 x_{10}, \quad (145)
\]
\[
x_1 x_{24} = \zeta_7 - x_3 x_{20} + x_6 x_{16} - x_7 x_{15} + x_{10} x_{11}, \quad (146)
\]
\[
x_1 x_{25} = -\zeta_9 - x_4 x_{20} - x_6 x_{18} + x_9 x_{15} - x_{11} x_{12}, \quad (147)
\]
\[
x_1 x_{26} = -\zeta_{11} + x_5 x_{20} + x_7 x_{18} - x_9 x_{16} + x_{11} x_{13}, \quad (148)
\]
by (3.6)-(3.12), (3.14) and (3.16). Moreover, (3.47) can be written as
\[
(3 x_1 x_{14} + 3 x_2 x_{11} + 3 x_3 x_9 + x_4 x_7 - x_5 x_6)x_{27} = \eta - 3[(x_1 x_{17} + x_2 x_{15} + x_3 x_{12})x_{26} + (x_1 x_{19} + x_2 x_{16} + x_3 x_{13})x_{25} + (x_4 x_{13} - x_5 x_{12} + x_8 x_9)x_{24} - (x_4 x_{16} - x_5 x_{15} + x_8 x_{11})x_{23} + (x_6 x_{13} - x_7 x_{12} + x_9 x_{10})x_{22} + (x_7 x_{15} + x_8 x_{16} - x_{10} x_{11})x_{21} + (x_4 x_{19} - x_5 x_{17} + x_8 x_{14})x_{20} + (x_6 x_{18} - x_9 x_{15} + x_{11} x_{12})x_{19} + (x_{10} x_{14} - x_7 x_{17})x_{18} + (x_9 x_{16} - x_{11} x_{13})x_{17} - x_{12} x_{14} x_{16} + x_{14} x_{15} x_{13} - (x_4 x_{10} - x_5 x_8) x_{26} - (x_5 x_{10} - x_7 x_8) x_{25} - (x_1 x_{21} + x_2 x_{18}) x_{24} - (x_3 x_{18} - x_1 x_{22}) x_{23} + (x_2 x_{22} + x_3 x_{21}) x_{20}. \]

(149)

Let \( f \) be any homogenous singular vector in \( \mathcal{A} \). According to the above equations, \( f \) can be written as a rational function \( f_1 \) in
\[
\{x_i, \zeta_r, \eta \mid i \in \{1, 13, 15, 16, 18, 20\}; r \in \{17, 9, 11\}\}. \quad (150)
\]
By (2.63)-(2.70), (3.28) and (3.29),
\[
0 = E_{a_1 + a_4 + \sum_{i=1}^s a_i}(f_1) = x_1 \partial_{x_{11}}(f_1), \quad (151)
\]
\[0 = E_{a_4 + \sum_{i=1}^{6} a_i} (f_1) = x_1 \partial_{x_12} (f_1), \quad (152)\]
\[0 = E_{a_4 + a_5 + \sum_{i=2}^{6} a_i} (f_1) = x_2 \partial_{x_13} (f_1) + \zeta_1 \partial_{\zeta_11} (f_1), \quad (153)\]
\[0 = E_{a_3 + a_4 + \sum_{i=1}^{6} a_i} (f_1) = -x_1 \partial_{x_13} (f_1), \quad (154)\]
\[0 = E_{a_4 + a_5 + \sum_{i=1}^{6} a_i} |V| = -x_1 \partial_{x_13} (f_1), \quad (155)\]
\[0 = E_{\sum_{r=3}^{5} a_r + \sum_{i=1}^{6} a_i} |V| = x_1 \partial_{x_16} (f_1), \quad (156)\]
\[0 = E_{a_4 + \sum_{r=3}^{5} a_r + \sum_{i=1}^{6} a_i} |V| = x_1 \partial_{x_18} (f_1), \quad (157)\]
\[0 = E_{a_2 + a_4 + \sum_{r=3}^{5} a_r + \sum_{i=1}^{6} a_i} (f_1) = x_1 \partial_{x_20} (f_1). \quad (158)\]

So \( f_1 \) is independent of \( x_{11}, x_{12}, x_{13}, x_{15}, x_{16}, x_{18}, x_{20} \) and \( \zeta_{11} \), that is, \( f_1 \) is a rational function in
\[\{x_i, \zeta_r, \eta \mid i \in \{1, 10\}; r \in \{1, 7, 9\}\}. \quad (159)\]

Next (2.56)-(2.62), (3.28) and (3.29) imply that
\[0 = E_{\sum_{i=1}^{6} a_i} (f_1) = x_1 \partial_{x_7} (f_1), \quad (160)\]
\[0 = E_{a_1 + \sum_{i=3}^{6} a_i} (f_1) = -x_1 \partial_{x_8} (f_1), \quad (161)\]
\[0 = E_{a_4 + \sum_{i=2}^{6} a_i} (f_1) = x_2 \partial_{x_9} (f_1) + \zeta_2 \partial_{\zeta_9} (f_1), \quad (162)\]
\[0 = E_{\sum_{i=2}^{6} a_i} (f_1) = x_2 \partial_{x_10} (f_1) + \zeta_1 \partial_{\zeta_10} (f_1), \quad (163)\]
\[0 = E_{a_4 + \sum_{r=1}^{5} a_r} (f_1) = -x_1 \partial_{x_9} (f_1), \quad (164)\]
\[0 = E_{\sum_{r=1}^{5} a_r} (f_1) = -x_1 \partial_{x_10} (f_1). \quad (165)\]

Hence \( f_1 \) is independent of \( x_7, x_8, x_9, x_{10}, \zeta_7 \) and \( \zeta_9 \), that is, \( f_1 \) is a rational function in
\[\{x_i, \zeta_r, \eta \mid i, r \in \{1, 6\}\}. \quad (166)\]

Now (2.41), (2.45)-(2.50), (2.52), (2.55), (3.28) and (3.29) give that
\[0 = E_{a_1 + a_3} (f_1) = x_1 \partial_{x_3} (f_1), \quad (167)\]
\[0 = E_{a_5 + a_6} (f_1) = \zeta_1 \partial_{\zeta_1} (f_1), \quad (168)\]
\[0 = E_{a_1 + a_3 + a_4} (f_1) = -x_1 \partial_{x_4} (f_1), \quad (169)\]
\[0 = E_{a_2 + a_3 + a_4} (f_1) = x_2 \partial_{x_4} (f_1), \quad (170)\]
\[0 = E_{a_2 + a_4 + a_5} |V| = \zeta_2 \partial_{\zeta_4} (f_1), \quad (171)\]
\[0 = E_{a_3 + a_4 + a_5} (f_1) = -x_2 \partial_{x_5} (f_1) - \zeta_2 \partial_{\zeta_5} (f_1), \quad (172)\]
\[0 = E_{a_4 + a_5 + a_6} (f_1) = -\zeta_1 \partial_{\zeta_4} (f_1), \quad (173)\]
\[0 = E_{a_1 + \sum_{i=3}^{6} a_i} (f_1) = x_1 \partial_{x_5} (f_1). \quad (174)\]

Thus \( f_1 \) is independent of \( \{x_i, \zeta_i \mid i \in \{3, 6\}\} \), that is, \( f_1 \) is a rational function in \( \{x_1, x_2, \zeta_1, \zeta_2, \eta\} \). Finally, (2.35), (2.40), (3.28) and (3.29) yield
\[0 = E_{a_1} (f_1) = -x_1 \partial_{x_12} (f_1), \quad 0 = E_{a_6} (f_1) = -\zeta_1 \partial_{\zeta_1} (f_1). \quad (175)\]

Therefore, \( f_1 \) is independent of \( x_2 \) and \( \zeta_2 \), that is, \( f = f_1 \) is a rational function in \( x_1, \zeta_1 \) and \( \eta \). By (3.48) and (3.57), it must be a polynomial in \( x_1, \zeta_1 \) and \( \eta \). Recall that the weights of \( x_1, \zeta_1 \) and \( \eta \) are \( \lambda_1, \lambda_6 \) and 0, respectively. The homogeneity of \( f \) implies that it must be a monomial in \( x_1, \zeta_1 \) and \( \eta \). 

\( \blacksquare \)
Let \( L(m_1, m_2, m_3) \) be the \( G_{E_6} \)-submodule generated by \( x_1^{m_1} \zeta_2^{m_2} \eta^{m_3} \). Note that (2.15) is a Cartan root space decomposition over \( \mathbb{R} \). Moreover, (2.71) implies that \( \mathcal{A} \) is a direct sum of weigh subspaces of \( G_{E_6} \) and subspaces of homogeneous polynomials are finite-dimensional \( G_{E_6} \)-submodules. Thus \( L(m_1, m_2, m_3) \) is a finite-dimensional irreducible submodule of highest weight \( m_1 \lambda_1 + m_2 \lambda_6 \). By the Weyl’s theorem of completely reducibility and the above lemma, we have

\[
\mathcal{A} = \sum_{m_1, m_2, m_3 = 0}^{\infty} L(m_1, m_2, m_3). \tag{176}
\]

Recall we denote by \( V(\lambda) \) the finite-dimensional irreducible module of highest weight \( \lambda \). The above equation implies

\[
\frac{1}{(1 - q)^{27}} = \frac{1}{1 - q^3} \sum_{m_1, m_2 = 0}^{\infty} (\dim V(m_1 \lambda_1 + m_2 \lambda_6)) q^{m_1 + 2m_2}. \tag{177}
\]

Equivalently, we have:

**Lemma 3.2.** The following dimensional property of irreducible \( G_{E_6} \)-modules holds:

\[
(1 - q)^{26} \sum_{m_1, m_2 = 0}^{\infty} (\dim V(m_1 \lambda_1 + m_2 \lambda_6)) q^{m_1 + 2m_2} = 1 + q + q^2. \tag{178}
\]

Set

\[
W = \sum_{i=1}^{27} \mathbb{R} \partial_{x_i}. \tag{179}
\]

Then \( W \) isomorphic to the module of linear functions on \( V \) via \( \partial_{x_i}(x_j) = \delta_{i,j} \).

Indeed, the linear map determined by \( \partial_{x_i} \mapsto \zeta_i \) (cf. (3.21), (3.22)) is a \( G_{E_6} \)-module isomorphism. We define a linear map \( \mathfrak{A} : \mathcal{A} \to \mathbb{R}[\partial_{x_1}, \ldots, \partial_{x_{27}}] \) by

\[
\mathfrak{A}(x_1^{\alpha_1}x_2^{\alpha_2} \cdots x_{27}^{\alpha_{27}}) = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_{27}}^{\alpha_{27}}. \tag{180}
\]

Set

\[
\mathcal{D} = \mathfrak{A}(\eta), \quad \mathcal{D}_1 = \sum_{i=1}^{27} x_i \partial_{x_i}, \quad \mathcal{D}_2 = \sum_{i=1}^{27} \zeta_i \mathfrak{A}(\zeta_i). \tag{181}
\]

Then \( \mathcal{D}, \mathcal{D}_1 \) and \( \mathcal{D}_2 \) are invariant differential operators, that is,

\[
(\mathcal{D}\xi)|_{\mathcal{A}} = (\xi \mathcal{D})|_{\mathcal{A}}, \quad (\mathcal{D}_r \xi)|_{\mathcal{A}} = (\xi \mathcal{D}_r)|_{\mathcal{A}} \text{ for } \xi \in G_{E_6}. \tag{182}
\]

Note that Lemma 3.1 implies

\[
V^2 = L(2, 0, 0) + L(0, 1, 0). \tag{183}
\]

Symmetrically,

\[
W^2 = L'(0, 2, 0) + L'(1, 0, 0), \tag{184}
\]

where \( L'(0, 2, 0) \) is a module generated by the highest weight vector \( \partial_{x_{27}}^2 \) with weight \( 2\lambda_6 \) and \( L'(1, 0, 0) \) is a module generated by the highest weight vector.
\[ \mathfrak{S}(\zeta_{27}) \] with weight \( \lambda_1 \). Thus the subspace of invariants (the trivial submodule) in \( V^2W^2 \) is two-dimensional. The trivial submodule of \( L(0, 1, 0)L'(1, 0, 0) \) is \( \mathbb{R}D_2 \). In \( L(2, 0, 0)L'(0, 2, 0) \), there exists an invariant \( D_3 \) with a term \( x_1^2 \partial_{x_1}^2 \). So any invariant in \( V^2W^2 \) must be in \( \mathbb{R}D_2 + \mathbb{R}D_3 \). In particular, the invariant differential operator

\[ [D, \eta] = D\eta - \eta D = b_0 + b_1D_1 + b_2D_2 + b_3D_3 \]  

(185)

for some \( b_s \in \mathbb{R} \). According to (3.47), \( \eta \) does not contain \( x_1^2 \). So \( b_3 = 0 \). Moreover, (3.47) also implies \( b_0 = 111 \).

According to (3.57), the coefficient of \( x_27\partial_{x_{27}} \) in \([D_0, \eta]\) must be 11, which implies \( b_1 = 1 \). Observe that there exists a unique monomial in \( \eta \) containing \( x_1x_{14} \), which is \( 3x_1x_{14}x_{27} \). Thus the coefficient of \( x_1x_{14}\partial_{x_1}\partial_{x_{14}} \) in \([D, \eta]\) must be 9, that is, \( b_2 = 9 \). So we have:

Lemma 3.3. As operators on \( A \),

\[ [D, \eta] = 111 + 11D_1 + 9D_2. \]  

(186)

Let \( m_1 \) and \( m_2 \) be nonnegative integers. If \( D(x_1^{m_1}\zeta_1^{m_2}) \neq 0 \), then it is also a singular of degree \( m_1 + 2m_2 - 3 \) with the same weight \( m_1\lambda_1 + m_2\lambda_6 \). But Lemma 3.1 implies that any singular vector with weight \( m_1\lambda_1 + m_2\lambda_6 \) must has degree \( \geq m_1 + 2m_2 \). This leads a contradiction. Thus

\[ D(x_1^{m_1}\zeta_1^{m_2}) = 0 \text{ for } m_1, m_2 \in \mathbb{N}. \]  

(187)

Moreover, (3.90) implies

\[ D(L(m_1, m_2, 0)) = \{0\} \text{ for } m_1, m_2 \in \mathbb{N}. \]  

(188)

Since \( D_2(x_1^{m_1}\zeta_1^{m_2}) \) is also a singular vector of degree \( m_1 + 2m_2 \) with the same weight \( m_1\lambda_1 + m_2\lambda_6 \), we have

\[ D_2(x_1^{m_1}\zeta_1^{m_2}) = cx_1^{m_1}\zeta_1^{m_2} \]  

(189)

for some \( c \in \mathbb{R} \). Let

\[ x_i = 0 \text{ for } 1, 14 \neq i \in \{1, 27\} \]  

(190)

in (3.97) and we get \( cx_1^{m_1+m_2}x_{14}^{m_2} = \)

\[ \lim_{x_i \to 0; 8, 10 \neq i \in \{1, 27\}} x_1x_{14}(\partial_{x_1}\partial_{x_{14}} + \partial_{x_2}\partial_{x_{11}} + \partial_{x_3}\partial_{x_9} - \partial_{x_4}\partial_{x_7} + \partial_{x_5}\partial_{x_6})[x_1^{m_1} \times (x_1x_{14} + x_2x_{11} + x_3x_9 - x_4x_7 + x_5x_6)^{m_2}] \]  

(191)

\[ = m_2(m_1 + m_2 + 4)x_1^{m_1+m_2}x_{14}^{m_2} \]  

(192)

by (3.6)-(3.24), that is, \( c = m_2(m_1 + m_2 + 4) \). We get:

Lemma 3.4. For \( m_1, m_2 \in \mathbb{N} \),

\[ D_2(x_1^{m_1}\zeta_1^{m_2}) = m_2(m_1 + m_2 + 4)x_1^{m_1}\zeta_1^{m_2}. \]  

(193)
According to Lemma 3.1,
\[ V^4 = L(4, 0, 0) + L(2, 1, 0) + L(1, 0, 1). \] (194)
Moreover, \( L(1, 0, 1) = \eta V \). Thus the invariants in \( V^4W \) are \( \mathbb{R} \eta D_1 \). Hence
\[ [D_2, \eta] = c_1 \eta + c_2 \eta D_1 \] for some \( c_1, c_2 \in \mathbb{R} \). (195)

Letting the above equation act on 1, we have
\[ D_2(\eta) = c_1 \eta. \] (196)

By (3.6)-(3.24) and (3.47), \( 3c_1 x_1 x_{14} x_{27} = \lim_{x_i \to 0; 14 \neq i \in \mathbb{Z}} \frac{\partial 
abla x_{14} x_{27}}{\partial x_{14} x_{27} + x_1 x_{27} \partial x_1 x_{27}}(x_1 x_{14} x_{27}) = 9 x_1 x_{14} x_{27} \). (197)
So \( c_1 = 3 \). Letting (3.102) act on \( x_1 \), we have:
\[ D_2(\eta x_1) = (3 + c_2) \eta x_1. \] (198)

As (3.104),
\[ 3(3 + c_2)x_1^2 x_{14} x_{27} = \lim_{x_i \to 0; 14 \neq i \in \mathbb{Z}} (3 + c_2) \eta x_1 = \lim_{x_i \to 0; 14 \neq i \in \mathbb{Z}} D_2(\eta x_1) \]
\[ = 3(x_1 x_{14} \partial_{x_1} x_{14} + x_{14} x_{27} \partial_{x_{14}} x_{27} + x_1 x_{27} \partial_{x_1} x_{27})(x_1^2 x_{14} x_{27}) = 15 x_1^2 x_{14} x_{27}. \] (199)
Hence \( c_2 = 2 \). We get:

**Lemma 3.5.** As operators on \( \mathcal{A} \),
\[ [D_2, \eta] = \eta(3 + 2D_1). \] (200)

For \( m, m_1, m_2 \in \mathbb{N} \) with \( m > 0 \), we have
\[ D(\eta^{m_1} x_1^{m_1} \zeta_1^{m_2}) = \sum_{s=1}^{m} s(33 + 9(3s + m_1 + 2m_2)) \eta^{m_1-1} x_1^{m_1} \zeta_1^{m_2} \neq 0 \] (201)
by Lemmas 3.3-3.5. According to (3.84) and (3.108), we have:

**Lemma 3.6.** For any \( 0 \neq f \in \mathcal{A} \),
\[ D(\eta f) \neq 0. \] (202)

The above lemma implies that
\[ \{ f \in \mathcal{A} \mid D(f) \} = \sum_{m_1, m_2=0}^{\infty} L(m_1, m_2, 0). \] (203)
Recall that \( \mathcal{A}_m \) be the subspace of homogeneous polynomials of degree \( m \) in \( \mathcal{A} \). Denote
\[ \Phi_m = \{ f \in \mathcal{A}_m \mid D(f) = 0 \}. \] (204)
In summary, we have the following version of the main theorem.
Theorem 3.7. The set \( \{ x_1^{m_1} \xi_1^{m_2} \eta^{m_3} \mid n_1, m_2, m_3 \in \mathbb{N} \} \) is the set of all singular vectors in \( A \) up to a scalar multiple. In particular, \( \eta \) is the unique fundamental invariant (up to a scalar multiple) and the identity

\[
(1 - q)^{26} \sum_{m_1, m_2 = 0}^{\infty} (\dim V(m_1 \lambda_1 + m_2 \lambda_6)) q^{m_1 + 2m_2} = 1 + q + q^2 
\]

holds. Furthermore,

\[
A_k = \Phi_k \oplus \eta A_{k-3} \quad \text{for} \quad k \in \mathbb{N} \tag{206}
\]

and

\[
\Phi_m = \sum_{i=0}^{\lfloor m/2 \rfloor} L(m - 2i, i, 0) \quad \text{for} \quad m \in \mathbb{N}, \tag{207}
\]

where we treat \( A_r = \{0\} \) if \( r < 0 \).

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