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On the Dual Topology of a Class of Cartan Motion Groups

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Abstract. Let (G, K) be a compact Riemannian symmetric pair, and let G_0 be the associated Cartan motion group. Under some assumptions on the pair (G, K), we give a precise description of the set $(\widehat{G_0})_{\rm gen}$ of all equivalence classes of generic irreducible unitary representations of G_0. We also determine the topology of the space $(\mathfrak{g}_0^\ddagger/G_0)_{gen}$ of generic admissible coadjoint orbits of G_0 and we show that the bijection between $(\widehat{G_0})_{gen}$ and $(\mathfrak{g}_0^\ddagger/G_0)_{gen}$ is a homeomorphism. Furthermore, in the case where the pair (G, K) has rank one, we prove that the unitary dual $\widehat{G_0}$ is homeomorphic to the space $\mathfrak{g}_0^\ddagger/G_0$ of all admissible coadjoint orbits of G_0.
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1. Introduction

Let G be a locally compact group. By the unitary dual $\widehat{G}$ of G, we mean the set of all equivalence classes of irreducible unitary representations of G equipped with the Fell topology (see [5]). The first representation-theoretic question concerning the group G is the description of the set $\widehat{G}$. Apart this question, a significant importance is attached to the determination of the topology of $\widehat{G}$. If G is a Lie group with Lie algebra $\mathfrak{g}$, then the investigation of the relationship between $\widehat{G}$ and the space $\mathfrak{g}^*/G$ of G-coadjoint orbits turns out to be a deep mathematical problem. In this direction, it is well-known that for a simply connected nilpotent Lie group or, more generally, for an exponential solvable Lie group G, the unitary dual $\widehat{G}$ is homeomorphic to the orbit space $\mathfrak{g}^*/G$ (see [12]).

Let now (G, K) be a compact Riemannian symmetric pair, and let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the corresponding Cartan decomposition of the Lie algebra $\mathfrak{g}$ of G with $\mathfrak{k} = Lie(K)$. Since the subspace $\mathfrak{p}$ is $Ad(K)$-invariant, one can form the semidirect product $G_0 = K \ltimes \mathfrak{p}$ with respect to the adjoint action of K on $\mathfrak{p}$. The group G_0 is called the Cartan motion group associated to the pair (G, K). As an example of this group, we mention the Euclidean motion group $M_n = SO(n) \ltimes \mathbb{R}^n$ where $SO(n)$ acts on $\mathbb{R}^n$ by rotations. In this paper, we shall restrict ourselves to the

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case where G is semisimple and K is connected. Furthermore, if $\mathfrak{a}$ is a fixed maximal abelian subspace of $\mathfrak{p}$, then we shall assume that the centralizer M of $\mathfrak{a}$ in K is connected. Let us fix a positive Weyl chamber $C^+(\mathfrak{a})$ in $\mathfrak{a}$.! Applying Mackey's little group theory (see [14,15]), we obtain that every infinite dimensional irreducible unitary representation of G_0 is determined by a pair (μ, H), where μ is the highest weight of an irreducible representation of M and H is a non-zero vector in the closure $\overline{C^+(\mathfrak{a})}$. We denote such a representation by $\pi_{(\mu,H)}$. Apart from these infinite dimensional representations $\pi_{(\mu,H)}$, the finite dimensional unitary representations of K also yield finite dimensional unitary representations of G_0. If the vector H is contained in $C^+(\mathfrak{a})$, then the representation $\pi_{(\mu,H)}$ is said to be generic. We denote by $(\widehat{G_0})_{gen}$ the set of all equivalence classes of generic irreducible unitary representations of G_0.

Let $G_0(\psi)$ be the stabilizer in G_0 of a linear form $\psi \in \mathfrak{g}_0^*$. Then ψ is called admissible if there exists a unitary character χ of the identity component of $G_0(\psi)$ such that $d\chi = i\psi_{|\mathfrak{g}_0(\psi)}$. We denote by $\mathfrak{g}_0^\ddagger$ the set of all admissible linear forms on $\mathfrak{g}_0$. For $\psi \in \mathfrak{g}_0^\ddagger$, one can construct an irreducible unitary repesentation π_ψ by holomorphic induction. According to Lipsman (see [13]), every irreducible unitary representation of G_0 arises in this manner. Thus we obtain a map from the set $\mathfrak{g}_0^\ddagger$ onto the unitary dual $\widehat{G_0}$. By observing that π_ψ is equivalent to $\pi_{\psi'}$ if and only if ψ and ψ' lie in the same G_0-orbit, we get finally a bijection between the space $\mathfrak{g}_0^\ddagger/G_0$ of admissible coadjoint orbits and the unitary d! ual $\widehat{G_0}$. The natural question arises of whether this bijection is a homeomorphism. In the present work, we give an affirmative answer to this question in the case where the compact Riemannian symmetric pair (G, K) has rank one. This result is a generalization of analogous result in the case of the Euclidean motion group M_n (see [4]). We denote by $(\mathfrak{g}_0^\ddagger/G_0)_{gen}$ the set of generic admissible coadjoint orbits of G_0 corresponding to the set $(\widehat{G_0})_{gen}$. When the rank of the pair (G, K) is arbitrary, we prove that the correspondence between the topological spaces $(\mathfrak{g}_0^\ddagger/G_0)_{gen}$ and $(\widehat{G_0})_{gen}$ is a homeomorphism.

This paper is organized as follows. Section 2 reviews some facts about compact Riemannian symmetric pairs, mostly in order to fix our notations and terminology. Section 3 introduces the coadjoint orbits of a Cartan motion group G_0 associated to a compact Riemannian symmetric pair (G, K) with G semisimple and K connected. Section 4 deals with the description via Mackey's little group theory of the unitary dual $\widehat{G_0}$ of G_0. In the remaining sections of the paper, the subgroup $M \subset K$ defined above is assumed to be connected. Section 5 contains some results on the topology of $\widehat{G_0}$. Section 6 is devoted to the description of the space $\mathfrak{g}_0^\ddagger/G_0$ of admissible coadjoint orbits of G_0. In the last section, the convergence in the quotient space $\mathfrak{g}_0^\ddagger/G_0$ is studied and the main results of this work are derived.

2. Preliminaries

This section serves to fix notations and summarizes some facts about compact Riemannian symmetric pairs. We refer to the standard reference [9] for more details.

Let (G, K) be a compact Riemannian symmetric pair where G is semisimple and K is connected. This means that G is a compact connected semisimple Lie group and there exists an involutive analytic automorphism Θ of G such that K coincides with the identity component of the fixed point group of Θ. Let us denote by θ the differential of Θ. Then θ is an involution on the Lie algebra $\mathfrak{g}$ of G. Considering the eigenspaces of θ with respect to the eigenvalues 1 and -1, we obtain the direct sum decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ coincides with the Lie algebra of the subgroup K. It is easy to see that the vector space $\mathfrak{p}$ is $Ad(K)$-invariant. Furthermore, the following relations obviously hold:

$$[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \text{ and } [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

Let now $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. The dimension of the real vector space $\mathfrak{a}$ is called the rank of the Riemannian symmetric pair (G, K). An important fact worth mentioning here is that every adjoint orbit of K in $\mathfrak{p}$ intersects $\mathfrak{a}$ (see [9, p. 247]). Let $N_K(\mathfrak{a})$ and $Z_K(\mathfrak{a})$ denote respectively the normalizer and centralizer of $\mathfrak{a}$ in K, i.e.,

$$\begin{aligned} N_K(\mathfrak{a}) &= \{k \in K;\ Ad(k)\mathfrak{a} = \mathfrak{a}\}, \\ Z_K(\mathfrak{a}) &= \{k \in K;\ Ad(k)H = H,\ \forall H \in \mathfrak{a}\}. \end{aligned}$$

The quotient group $W(G, K) := N_K(\mathfrak{a})/Z_K(\mathfrak{a})$ is called the Weyl group of the pair (G, K). We shall denote the action of $W(G, K)$ on $\mathfrak{a}$ by $H \longmapsto s.H$ for $H \in \mathfrak{a}$ and $s \in W(G, K)$.

Let us take the subspaces $\tilde{\mathfrak{a}} := i\mathfrak{a}$, $\tilde{\mathfrak{p}} := i\mathfrak{p}$ and $\tilde{\mathfrak{g}} := \mathfrak{k} \oplus \tilde{\mathfrak{p}}$ of the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$. On the real semisimple lie algebra $\tilde{\mathfrak{g}}$, we fix the involution $\tilde{\theta}$ defined by

$$\tilde{\theta}(Y + iZ) = Y - iZ \quad \text{for } Y \in \mathfrak{k},\ Z \in \mathfrak{p}.$$

Then $(\tilde{\mathfrak{g}}, \tilde{\theta})$ is the orthogonal symmetric Lie algebra of the noncompact type which is dual to $(\mathfrak{g}, \theta)$. Given a linear form $\alpha \in \tilde{\mathfrak{a}}^*$, we set

$$\tilde{\mathfrak{g}}_\alpha = \{X \in \tilde{\mathfrak{g}};\ [\widetilde{H}, X] = \alpha(\widetilde{H})X,\ \forall \widetilde{H} \in \tilde{\mathfrak{a}}\}.$$

If $\alpha \neq 0$ and $\tilde{\mathfrak{g}}_\alpha \neq \{0\}$, the form α is said to be a restricted root of $\tilde{\mathfrak{g}}$. The set of all restricted roots is denoted by Σ. We obtain the restricted root space decomposition

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{g}}_0 \oplus \bigoplus_{\alpha \in \Sigma} \tilde{\mathfrak{g}}_\alpha$$

of the Lie algebra $\tilde{\mathfrak{g}}$. If $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$ denotes the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, then we also have the direct sum $\tilde{\mathfrak{g}}_0 = \mathfrak{m} \oplus \tilde{\mathfrak{a}}$. Let d_α be the dimension of the root space $\tilde{\mathfrak{g}}_\alpha$. Consider a basis $(X_{\alpha,1}, ..., X_{\alpha,d_\alpha})$ of $\tilde{\mathfrak{g}}_\alpha$ and set

$$Y_{\alpha,j} = X_{\alpha,j} + \tilde{\theta}(X_{\alpha,j}),\ \widetilde{Z}_{\alpha,j} = X_{\alpha,j} - \tilde{\theta}(X_{\alpha,j}) \text{ and } Z_{\alpha,j} = i\widetilde{Z}_{\alpha,j}.$$

We can define the subspaces

$$\mathfrak{k}_\alpha = \bigoplus_{j=1}^{d_\alpha} \mathbb{R}Y_{\alpha,j} \text{ and } \mathfrak{p}_\alpha = \bigoplus_{j=1}^{d_\alpha} \mathbb{R}Z_{\alpha,j}.$$

Endow the dual space $\tilde{\mathfrak{a}}^*$ with a lexicographic ordering and denote by Σ^+ the set of positive restricted roots. With respect to the Killing form B of $\mathfrak{g}$, we have the direct sum decompositions

$$\mathfrak{k} = \mathfrak{m} \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{k}_\alpha \quad \text{and} \quad \mathfrak{p} = \mathfrak{a} \oplus \bigoplus_{\alpha\in\Sigma^+} \mathfrak{p}_\alpha$$

(see [9, p. 335]). Setting

$$\mathfrak{l} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{k}_\alpha \quad \text{and} \quad \mathfrak{q} = \bigoplus_{\alpha\in\Sigma^+} \mathfrak{p}_\alpha,$$

we get the relations $[\mathfrak{a}, \mathfrak{l}] \subset \mathfrak{q}$ and $[\mathfrak{a}, \mathfrak{q}] \subset \mathfrak{l}$.

Next, we shall extend complex linearly the restricted roots to $\mathfrak{a}^{\mathbb{C}}$. An element $H \in \mathfrak{a}$ is said to be regular if $\alpha(H) \neq 0$ for all $\alpha \in \Sigma$. Fix a regular element $H_0 \in \mathfrak{a}$, and set $c_\alpha = \frac{1}{\alpha(H_0)}$ for $\alpha \in \Sigma^+$. We have the equalities

$$[H_0, c_\alpha Y_{\alpha,j}] = Z_{\alpha,j} \quad \text{and} \quad [H_0, c_\alpha Z_{\alpha,j}] = Y_{\alpha,j},$$

where $Y_{\alpha,j}$ and $Z_{\alpha,j}$ are as above. Consequently, we deduce that the linear maps $ad(H_0)|_{\mathfrak{l}} : \mathfrak{l} \longrightarrow \mathfrak{q}$ and $ad(H_0)|_{\mathfrak{q}} : \mathfrak{q} \longrightarrow \mathfrak{l}$ are surjective, where ad refers to the adjoint representation of $\mathfrak{g}$. Let S be the closure of the one-parameter subgroup $exp_G(\mathbb{R}H_0)$. Since S is a torus, its centralizer in G is connected [9, p. 287], and in fact has Lie algebra $\mathfrak{m} \oplus \mathfrak{a}$. On the other hand, the centralizer in G of the torus $A = exp_G(\mathfrak{a})$ is also connected and has Lie algebra $\mathfrak{m} \oplus \mathfrak{a}$ (see [9, p. 263]). This implies that $Z_G(H_0) = Z_G(\mathfrak{a})$, and hence $Z_K(H_0) = Z_K(\mathfrak{a})$.

A connected component of the set of regular elements in $\mathfrak{a}$ is called a Weyl chamber of the pair (G, K). As an example of Weyl chambers, let us set $C^+(\mathfrak{a}) := i\,\widetilde{C}^+(\tilde{\mathfrak{a}})$ with

$$\widetilde{C}^+(\tilde{\mathfrak{a}}) = \{\widetilde{H} \in \tilde{\mathfrak{a}};\ \alpha(\widetilde{H}) > 0,\ \forall \alpha \in \Sigma^+\}.$$

It is well-known that every $s \in W(G, K)$ permutes the Weyl chambers and that $W(G, K)$ acts simply transitively on the set of Weyl chambers (see [9, p. 288]). Furthermore, we have the following important result (see [9, p. 322]):

Theorem 2.1. *Let $C \subset \mathfrak{a}$ be a Weyl chamber. Each orbit of $W(G, K)$ in $\mathfrak{a}$ intersects the closure $\overline{C}$ in exactly one point.*

3. Cartan motion groups and their coadjoint orbits

Let (G, K) be a compact Riemannian symmetric pair with G semisimple and K connected. Then K is the fixed point group of an involutive analytic automorphism Θ of G. As before, the automorphism of the Lie algebra $\mathfrak{g}$ of G which is the differential of Θ is denoted by θ. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the decompositiopn of $\mathfrak{g}$ into ± 1 eigenspaces of θ, so that $\mathfrak{k} = Lie(K)$. The subgroup K acts on the vector space $\mathfrak{p}$ via the adjoint representation. The semidirect product $G_0 = K \ltimes \mathfrak{p}$

is called the Cartan motion group of the pair (G,K). The multiplication rule in this group is given by

$$(k_1,X_1)\cdot(k_2,X_2)=(k_1k_2,X_1+Ad(k_1)X_2).$$

As mentioned in the introduction, the group $M_n=SO(n)\ltimes\mathbb{R}^n$ is an example of Cartan motion groups. More precisely, M_n is the Cartan motion group associated to the compact Riemannian symmetric pair $(SO(n+1),SO(n))$.

Let Ad_0 and ad_0 denote respectively the adjoint representations of G_0 and its Lie algebra $\mathfrak{g}_0$. It follows easily from the group law in G_0 that

$$\begin{aligned} Ad_0((k,X))(U',X') &= (Ad(k)U',Ad(k)X'-[Ad(k)U',X]),\\ ad_0((U,X))(U',X') &= ([U,U'],[U,X']-[U',X]) \end{aligned}$$

for all $k\in K$, all $U,U'\in\mathfrak{k}$ and all $X,X'\in\mathfrak{p}$. Using the Killing form B of $\mathfrak{g}$, we define the following scalar product on $\mathfrak{g}_0$:

$$\langle (U,X),(U',X')\rangle=-B(U,U')-B(X,X'),$$

where $U,U'\in\mathfrak{k}$ and $X,X'\in\mathfrak{p}$. To an arbitrary element $\xi\in\mathfrak{g}_0$, we associate the natural linear form $F_\xi\in\mathfrak{g}_0^*$ given by $F_\xi(\eta)=\langle\xi,\eta\rangle$. In the sequel, we will use the map $\xi\longmapsto F_\xi$ to identify $\mathfrak{g}_0$ with its dual $\mathfrak{g}_0^*$. Let us now calculate the coadjoint representation Ad_0^* of G_0. For $(k_0,X_0)\in G_0$ and $(U,X),(U',X')\in\mathfrak{g}_0$, let us set

$$(\star)=\big[Ad_0^*((k_0,X_0))F_{(U,X)}\big](U',X').$$

So, we can write

$$\begin{aligned} (\star) &= F_{(U,X)}\big(Ad_0((k_0^{-1},-Ad(k_0^{-1})X_0))(U',X')\big)\\ &= F_{(U,X)}\big((Ad(k_0^{-1})U',Ad(k_0^{-1})X'-[Ad(k_0^{-1})U',-Ad(k_0^{-1})X_0])\big)\\ &= -B(U,Ad(k_0^{-1})U')-B(X,Ad(k_0^{-1})X'+Ad(k_0^{-1})[U',X_0])\\ &= -B(Ad(k_0)U+[X_0,Ad(k_0)X],U')-B(Ad(k_0)X,X')\\ &= F_{(Ad(k_0)U+[X_0,Ad(k_0)X],Ad(k_0)X)}(U',X'). \end{aligned}$$

Under the identification of $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$, we have

$$Ad_0^*((k_0,X_0))(U,X)=(Ad(k_0)U+[X_0,Ad(k_0)X],Ad(k_0)X).$$

Therefore, the coadjoint orbit of G_0 through (U,X) is given by

$$\begin{aligned} \mathcal{O}^{G_0}_{(U,X)} &= Ad_0^*(G_0)(U,X)\\ &= \{(Ad(k_0)U+[X_0,Ad(k_0)X],Ad(k_0)X);\ k_0\in K,\ X_0\in\mathfrak{p}\}. \end{aligned}$$

4. Dual spaces of Cartan motion groups

Let (G,K) be a compact Riemannian symmetric pair, and let $G_0=K\ltimes\mathfrak{p}$ be the associated Cartan motion group. We shall briefly review the description of the unitary dual of G_0 via Mackey's little group theory.

Let φ be a non-zero linear form on $\mathfrak{p}$. We denote by χ_φ the unitary character of the vector Lie group $\mathfrak{p}$ given by $\chi_\varphi = e^{i\varphi}$. Let K_φ be the stabilizer of φ under the coadjoint action of K on $\mathfrak{p}^*$, and let ρ be an irreducible unitary representation of K_φ on some Hilbert space $\mathcal{H}_\rho$. The map

$$\rho \otimes \chi_\varphi : (k, X) \longmapsto e^{i\varphi(X)}\rho(k)$$

is a representation of the semidirect product $K_\varphi \ltimes \mathfrak{p}$, which we may induce up so as to obtain a unitary representation of G_0. Let $L^2_\rho(K, \mathcal{H}_\rho)$ be the subspace of $L^2(K, \mathcal{H}_\rho)$ consisting of the maps f which satisfy the covariance condition

$$f(kk_0) = \rho(k_0^{-1})f(k)$$

for $k_0 \in K_\varphi$ and $k \in K$. The induced representation

$$\pi_{(\rho,\varphi)} := Ind_{K_\varphi \ltimes \mathfrak{p}}^{G_0}(\rho \otimes \chi_\varphi)$$

is realized on $L^2_\rho(K, \mathcal{H}_\rho)$ by

$$\pi_{(\rho,\varphi)}(k_0, X)f(k) = e^{i\varphi(Ad(k^{-1})X)}f(k_0^{-1}k),$$

where $(k_0, X) \in G_0$, $f \in L^2_\rho(K, \mathcal{H}_\rho)$ and $k \in K$. Mackey's theory tells us that the representation $\pi_{(\rho,\varphi)}$ is irreducible and that every infinite dimensional irreducible unitary representation of G_0 is equivalent to some $\pi_{(\rho,\varphi)}$. Furthermore, two representations $\pi_{(\rho,\varphi)}$ and $\pi_{(\rho',\varphi')}$ are equivalent if and only if φ and φ' lie in the same coadjoint orbit of K and the representations ρ and ρ' are equivalent under the identification of the conjugate subgroups K_φ and $K_{\varphi'}$. In this way, we obtain all irreducible representations of G_0 which are not trivial on the normal subgroup $\mathfrak{p}$. On the other hand, every irreducible unitary representation τ of K extends trivially to an irreducible representation, also denoted by τ, of G_0 by $\tau(k, X) := \tau(k)$ for $k \in K$ and $X \in \mathfrak{p}$.

Next, we shall provide a more precise description of the so-called "generic irreducible unitary representations" of G_0. Denote again by $\langle , \rangle$ the restriction to $\mathfrak{p} \times \mathfrak{p}$ of the $Ad(K)$-invariant scalar product $\langle , \rangle$ on $\mathfrak{g}_0$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and let M be the centralizer of $A = exp_G(\mathfrak{a})$ in K. In general, the compact Lie group M is not connected, and one can prove that $M = M_e \cdot (M \cap A)$ with M_e being the identity component of M. For the sake of convenience, we will give a short proof of the following well-known result.

Lemma 4.1. *Let $C \subset \mathfrak{a}$ be a Weyl chamber. Every adjoint orbit of K in $\mathfrak{p}$ intersects the closure $\overline{C}$ in exactly one point.*

Proof. Let X be a fixed element in $\mathfrak{p}$. Then X is $Ad(K)$-conjugate to some $H_0 \in \mathfrak{a}$. Let H be the unique point which belongs to the intersection of the orbit $W(G,K).H_0$ with the closure $\overline{C}$. Writing $H = Ad(k_0)H_0$ for some k_0 in the normalizer $N_K(\mathfrak{a})$, we see that X is $Ad(K)$-conjugate to H. If $H' \in \overline{C}$ is another element with the property that X is $Ad(K)$-conjugate to H', then there exists $k \in K$ such that $H' = Ad(k)H$. It follows that $H' = s.H$ for some $s \in W(G,K)$ (see [9, p. 285]). Using the result of Theorem 2.1, we deduce that $H = H'$ as desired. ∎

From the above lemma, we deduce that every infinite dimensional unitary representation of G_0 has the form $\pi_{(\rho,\varphi_H)}$, where H is a non-zero vector in $\overline{C^+(\mathfrak{a})}$ and φ_H is the linear form on $\mathfrak{p}$ given by $\varphi_H(X) = \langle H, X\rangle$. Observe that the isotropy group K_{φ_H} coincides with the centralizer $Z_K(H)$. Let us fix a regular element H in $\mathfrak{a}$. The subgroups K_{φ_H} and M of K are identical. If ρ is an irreducible representation of M, then the representation $\pi_{(\rho,\varphi_H)}$ corresponding to the pair (ρ, φ_H) is said to be generic. We denote by $(\widehat{G_0})_{gen}$ the set of all equivalence classes of generic irreducible unitary representations of G_0. Notice that $(\widehat{G_0})_{gen}$ has full Plancherel measure in the unitary dual $\widehat{G_0}$ (see [10]). Applying Mackey's ! analysis and the result of Lemma 1, we obtain the bijection

$$(\widehat{G_0})_{gen} \simeq \widehat{M} \times C^+(\mathfrak{a}).$$

In the particular case where the Riemannian symmetric pair (G, K) has rank one, we can find a vector $H_0 \in \mathfrak{a}$ such that $C^+(\mathfrak{a}) = \mathbb{R}_+^* H_0$. We derive in this case the bijections

$$(\widehat{G_0})_{gen} \simeq \widehat{M} \times \mathbb{R}_+^* \quad \text{and} \quad \widehat{G_0} \simeq \big(\widehat{M} \times \mathbb{R}_+^*\big) \cup \widehat{K}.$$

In the remainder of this paper, we shall assume that M is connected. Let ρ_μ be an irreducible representation of M with highest weight μ. For simplicity, we shall write $\pi_{(\mu,H)}$ instead of $\pi_{(\rho_\mu,\varphi_H)}$.

5. Convergence of irreducible representations of G_0

Let N be an abelian group, and assume that a compact Lie group K acts on the left on N by automorphisms. As sets, the semidirect product $K \ltimes N$ is the Cartesian product $K \times N$ and the group multiplication is given by

$$(k_1, x_1) \cdot (k_2, x_2) = (k_1k_2, x_1 + k_1x_2).$$

Let χ be a unitary character of N, and let K_χ be the stabilizer of χ under the action of K on $\widehat{N}$ defined by

$$(k \cdot \chi)(x) = \chi(k^{-1}x).$$

If ρ is an element of $\widehat{K_\chi}$, then the triple $(\chi, (K_\chi, \rho))$ is called a cataloguing triple. Following the notations of [4], we denote by $\pi(\chi, K_\chi, \rho)$ the induced representation $Ind_{K_\chi \ltimes N}^{K \ltimes N}(\rho \otimes \chi)$. Referring to a work of Baggett (see [2]), we have

Proposition 5.1. *The mapping* $(\chi, (K_\chi, \rho)) \longrightarrow \pi(\chi, K_\chi, \rho)$ *is onto* $\widehat{K \ltimes N}$.

Let $\mathcal{A}(K)$ be the set of all pairs (K', ρ'), where K' is a closed subgroup of K and ρ' is an irreducible representation of K'. We equip $\mathcal{A}(K)$ with the Fell topology (see [5]). Therefore, every element in $\widehat{K \ltimes N}$ can be catalogued by elements in the topological space $\widehat{N} \times \mathcal{A}(K)$. The following result of Baggett (see [2]) provides a precise and neat description of the topology of $\widehat{K \ltimes N}$.

Theorem 5.2. *Let Y be a subset of $\widehat{K \ltimes N}$ and π an element of $\widehat{K \ltimes N}$. Then π is weakly contained in Y if and only if there exist: a cataloguing triple $(\chi, (K_\chi, \rho))$ for π, an element $(K^{'}, \rho^{'})$ of $\mathcal{A}(K)$, and a net $\{(\chi_n, (K_{\chi_n}, \rho_n))\}$ of cataloguing triples such that:*

(i) for each n, the irreducible unitary representation $\pi(\chi_n, K_{\chi_n}, \rho_n)$ of $K \ltimes N$ is an element of Y;

(ii) the net $\{(\chi_n, (K_{\chi_n}, \rho_n))\}$ converges to $(\chi, (K^{'}, \rho^{'}))$;

(iii) K_χ contains $K^{'}$, and the induced representation $Ind_{K^{'}}^{K_\chi}(\rho^{'})$ contains ρ.

Let us now return to the context and notations of Section 4. To an irreducible representation ρ_μ of M with highest weight μ and a vector $H \in C^+(\mathfrak{a})$, we associate the generic representation $\pi_{(\mu,H)}$ of G_0 and its corresponding cataloguing triple $(\chi_{\varphi_H}, (M, \rho_\mu))$. Consider an irreducible representation τ_λ of K with highest weight λ. By $(0, (K, \tau_\lambda))$, we mean the cataloguing triple of the trivial extension of τ_λ to G_0. A direct application of Theorem 5.2 gives us the following results.

Proposition 5.3. *Let $(\pi_{(\mu^n, H_n)})_n$ be a sequence of generic irreducible representations of G_0. Then $(\pi_{(\mu^n, H_n)})_n$ converges to $\pi_{(\mu,H)}$ in $(\widehat{G_0})_{gen}$ if and only if $(H_n)_n$ tends to H as $n \longrightarrow +\infty$ and $\mu^n = \mu$ for n large enough.*

Proposition 5.4. *Let $(\pi_{(\mu^n, H_n)})_n$ be a sequence of generic irreducible representations of G_0. Then $(\pi_{(\mu^n, H_n)})_n$ converges to τ_λ in $\widehat{G_0}$ if and only if $(H_n)_n$ tends to 0 as $n \longrightarrow +\infty$ and ρ_{μ^n} occurs in the restriction $Res_M^K(\tau_\lambda)$ for n large enough.*

Remark 5.5. By Proposition 5.3, we immediately see that $(\widehat{G_0})_{gen}$ has a Hausdorff topology. Proposition 5.4 implies that sequences in $(\widehat{G_0})_{gen}$ which converge in $\widehat{K}$ have infinitely many different limit points.

6. Admissible coadjoint orbits of G_0

We shall freely use the notations of the previous sections. Let $\mathfrak{h}_\mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$, and let $\mathfrak{h}_\mathfrak{m} \subset \mathfrak{h}_\mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{m}$. Consider an irreducible representation ρ_μ of M with highest weight μ. We denote by U_μ the unique element of $\mathfrak{h}_\mathfrak{m}$ such that $B(U_\mu, U) = -i\mu(U)$ for all $U \in \mathfrak{h}_\mathfrak{m}$. Fix a vector $H \in C^+(\mathfrak{a})$. Under the identification of $\mathfrak{g}_0$ and $\mathfrak{g}_0^*$, we can define a linear form $\psi \in \mathfrak{g}_0^*$ by $\psi = (U_\mu, H)$. To simplify notation, we shall write $\mathcal{O}_{(\mu,H)}^{G_0}$ instead of $\mathcal{O}_{(U_\mu,H)}^{G_0}$. Such an orbit is called a generic coadjoint orbit of G_0.

Let $\mathfrak{l}$ be the orthogonal complement of $\mathfrak{m}$ in $\mathfrak{k}$ with respect to the Killing form of $\mathfrak{g}$. The stabilizer $G_0(\psi)$ of ψ in G_0 is given by

$$\begin{aligned} G_0(\psi) &= \big\{(k, X) \in G_0;\ (Ad(k)U_\mu + [X, Ad(k)H], Ad(k)H) = (U_\mu, H)\big\} \\ &= \big\{(k, X) \in G_0;\ k \in M,\ Ad(k)U_\mu + [X, H] = U_\mu\big\} \\ &= \big\{(k, X) \in G_0; X \in Z_\mathfrak{p}(H),\ k \in M,\ Ad(k)U_\mu = U_\mu\big\}, \end{aligned}$$

since $Ad(k)U_\mu \in \mathfrak{m}$ and $[X, H] \in \mathfrak{l}$ for all $k \in K$ and all $X \in \mathfrak{p}$. Thus, we have $G_0(\psi) = K(\psi) \ltimes \mathfrak{p}(\psi)$, and hence ψ is aligned (see [13]). A linear form $\psi \in \mathfrak{g}_0^*$ is called admissible if there exists a unitary character χ of the identity component of $G_0(\psi)$ such that $d\chi = i\psi_{|\mathfrak{g}_0(\psi)}$. Observe that the linear forms (U_μ, H) are all admissible. Then, according to Lipsman (see [13]), the representation of G_0 obtained by holomorphic induction from (U_μ, H) is equivalent to the generic representation $\pi_{(\mu,H)}$. Let $\mathfrak{g}_0^\ddagger \subset \mathfrak{g}_0^*$ be the set of all admissible linear forms on $\mathfrak{g}_0$. The orbit space $\mathfrak{g}_0^\ddagger/G_0$ is called the space of admissible coadjoint orbits of G_0. We denote by $\left(\mathfrak{g}_0^\ddagger/G_0\right)_{gen}$ the subspace of gene! ric admissible coadjoint orbits of G_0, that is the subspace formed by all the coadjoint orbits $\mathcal{O}^{G_0}_{(\mu,H)}$.

Let τ_λ be an irreducible representation of K with highest weight λ. We attach to τ_λ the linear form $(U_\lambda, 0)$ of $\mathfrak{g}_0^*$, where U_λ is the unique element of $\mathfrak{h}_\mathfrak{k}$ such that $B(U_\lambda, U) = -i\lambda(U)$ for all $U \in \mathfrak{h}_\mathfrak{k}$. The representation of G_0 obtained by holomorphic induction from $(U_\lambda, 0)$ is equivalent to τ_λ. We denote by $\mathcal{O}^{G_0}_\lambda$ the coadjoint orbit of $(U_\lambda, 0)$. It is clear that $\mathcal{O}^{G_0}_\lambda$ is an admissible coadjoint orbit of G_0. Furthermore, if the Riemannian symmetric pair (G, K) has rank one, then one can check that $\mathfrak{g}_0^\ddagger/G_0$ is the union of $\left(\mathfrak{g}_0^\ddagger/G_0\right)_{gen}$ and the set of all the coadjoint orbits $\mathcal{O}^{G_0}_\lambda$.

7. Convergence in the quotient space $\mathfrak{g}_0^\ddagger/G_0$

We continue to use the notations of the previous sections. Let T_K and T_M be maximal tori respectively in K and M such that $T_M \subset T_K$. The corresponding Lie algebras are denoted by $\mathfrak{h}_\mathfrak{k}$ and $\mathfrak{h}_\mathfrak{m}$. The Weyl groups of K and M associated respectively to the tori T_K and T_M are denoted by W_K and W_M. Let P_K be the integral weight lattice of T_K. Notice that every element λ in P_K takes pure imaginary values on $\mathfrak{h}_\mathfrak{k}$, hence can be considered as an element of $(i\mathfrak{h}_\mathfrak{k})^*$. Fix a positive Weyl chamber C_K^+ in $(i\mathfrak{h}_\mathfrak{k})^*$, and write $P_K^+ = P_K \cap C_K^+$ for the set of dominant integral weights of T_K. We recall that every W_K-orbit in $\mathfrak{k}^*$ intersects the closure $\overline{i!C_K^+} \subset \mathfrak{h}_\mathfrak{k}^*$ in exactly one point (see [3, p. 203]). For $\lambda \in P_K^+$, denote by $\mathcal{O}^K_\lambda$ the K-coadjoint orbit passing through the point $-i\lambda$. As proved by Kostant in [11], the projection of $\mathcal{O}^K_\lambda$ on $\mathfrak{h}_\mathfrak{k}^*$ is a convex polytope with vertices $-i(w.\lambda)$ for $w \in W_K$, that is the convex hull of $-i(W_K.\lambda)$. In a similar way, we fix a positive Weyl chamber C_M^+ in $\mathfrak{h}_\mathfrak{m}^*$ and we introduce the set P_M^+ of dominant integral weights of T_M.

Denote by q the $\mathbb{C}$-linear extension of both the natural projection of $\mathfrak{k}^*$ onto $\mathfrak{m}^*$ and the natural projection of $\mathfrak{h}_\mathfrak{k}^*$ onto $\mathfrak{h}_\mathfrak{m}^*$. Consider two irreducible representations $\tau_\lambda \in \widehat{K}$ and $\rho_\mu \in \widehat{M}$ with respective highest weights $\lambda \in P_K^+$ and $\mu \in P_M^+$. We have

Lemma 7.1. *If $\mu = q(w.\lambda)$ with $w \in W_K$, then ρ_μ occurs in the restriction $Res^K_M(\tau_\lambda)$.*

A proof of this lemma can be found in [1]. Let us take the coadjoint orbits $\mathcal{O}^K_\lambda$ and $\mathcal{O}^M_\mu$ of K and M passing through $-i\lambda$ and $-i\mu$, respectively. According to Guillemin and Sternberg [6,7] (compare [8, Theorem 7.5]), we have the following result.

Lemma 7.2. *If the restriction $Res_M^K(\tau_\lambda)$ contains ρ_μ, then the orbit $\mathcal{O}_\mu^M$ occurs in $q(\mathcal{O}_\lambda^K)$.*

It is well-known that $\widehat{K}$ (resp. $\widehat{M}$) is in bijective correspondence with P_K^+ (resp. P_M^+), and hence

$$\left(\mathfrak{g}_0^\ddagger/G_0\right)_{gen} \simeq P_M^+ \times C^+(\mathfrak{a}).$$

When the Riemannian symmetric pair (G,K) has rank one, we have

$$\left(\mathfrak{g}_0^\ddagger/G_0\right)_{gen} \simeq P_M^+ \times \mathbb{R}_+^* \text{ and } \mathfrak{g}_0^\ddagger/G_0 \simeq (P_M^+ \times \mathbb{R}_+^*) \cup P_K^+.$$

To study the convergence in the quotient space $\mathfrak{g}_0^\ddagger/G_0$, we need the following lemma (see [12]).

Lemma 7.3. *Let G be a unimodular Lie group with Lie algebra $\mathfrak{g}$ and let $\mathfrak{g}^*$ be the vector dual space of $\mathfrak{g}$. We denote $\mathfrak{g}^*/G$ the space of coadjoint orbits and by $p_G : \mathfrak{g}^* \longrightarrow \mathfrak{g}^*/G$ the canonical projection. We equip this space with the quotient topology, i.e., a subset V in $\mathfrak{g}^*/G$ is open if and only if $p_G^{-1}(V)$ is open in $\mathfrak{g}^*$. Therefore, a sequence $(\mathcal{O}_k^G)_k$ of elements in $\mathfrak{g}^*/G$ converges to the orbit $\mathcal{O}^G$ in $\mathfrak{g}^*/G$ if and only if for any $l \in \mathcal{O}^G$, there exist $l_k \in \mathcal{O}_k^G$, $k \in \mathbb{N}$, such that $l = \lim_{k\longrightarrow+\infty} l_k$.*

Now, we are in position to prove

Proposition 7.4. *Let $\left(\mathcal{O}^{G_0}_{(\mu^n,H_n)}\right)_n$ be a sequence of generic admissible coadjoint orbits of G_0. Then $\left(\mathcal{O}^{G_0}_{(\mu^n,H_n)}\right)_n$ converges to $\mathcal{O}^{G_0}_{(\mu,H)}$ in $(\mathfrak{g}_0^\ddagger/G_0)_{gen}$ if and only if $(H_n)_n$ tends to H as $n \longrightarrow +\infty$ and $\mu^n = \mu$ for n large enough.*

Proof. If $(H_n)_n$ tends to H as $n \longrightarrow +\infty$ and $\mu^n = \mu$ for n large enough, then we have $\lim_{n\longrightarrow+\infty}(U_{\mu^n}, H_n) = (U_\mu, H)$, and thus $\lim_{n\longrightarrow+\infty}\mathcal{O}^{G_0}_{(\mu^n,H_n)} = \mathcal{O}^{G_0}_{(\mu,H)}$.

Conversely, let us assume that $\left(\mathcal{O}^{G_0}_{(\mu^n,H_n)}\right)_n$ converges to $\mathcal{O}^{G_0}_{(\mu,H)}$. Then there exist two sequences $(k_n)_n \subset K$ and $(X_n)_n \subset \mathfrak{p}$ such that:

$$\begin{aligned}\lim_{n\longrightarrow+\infty}(Ad(k_n)U_{\mu^n} + [X_n, Ad(k_n)H_n]) &= U_\mu,\\ \lim_{n\longrightarrow+\infty} Ad(k_n)H_n &= H.\end{aligned}$$

Passing to a subsequence if necessary, we may assume that $\lim_{n\longrightarrow+\infty} k_n = k_0$. Therefore, we have $\lim_{n\longrightarrow+\infty} H_n = Ad(k_0^{-1})H$. Furthermore, we know that there exists $s \in W(G,K)$ such that $Ad(k_0^{-1})H = s.H$ (see [9, p. 285]). Since the element $s.H$ belongs to the intersection of the closure $\overline{C^+(\mathfrak{a})}$ with the orbit $W(G,K).H$, we obtain the equality $s.H = H$. We conclude that $\lim_{n\longrightarrow+\infty} H_n = H$ and $k_0 \in M$.

Setting $Y_n = [Ad(k_n^{-1})X_n, H_n]$, we can write

$$\lim_{n\longrightarrow+\infty} (U_{\mu^n} + Y_n) = Ad(k_0^{-1})U_\mu.$$

Consider the direct sum decomposition $\mathfrak{k} = \mathfrak{m}\oplus\mathfrak{l}$ with respect to the Killing form of $\mathfrak{g}$. For all n, it is clear that $Y_n \in \mathfrak{l}$. Thus, we deduce that $\lim_{n\longrightarrow+\infty} U_{\mu^n} = Ad(k_0^{-1})U_\mu$. On the other hand, we have $Ad(k_0^{-1})U_\mu = w.U_\mu$ for some w in the Weyl group W_M. By observing that $w.U_\mu = U_{w.\mu}$, we get $\lim_{n\longrightarrow+\infty} U_{\mu^n} = U_{w.\mu}$ and hence $\mu^n = w.\mu$ for n large enough. Since the weights μ^n and μ are contained in the set iC_M^+ and since every W_M-orbit in $\mathfrak{m}^*$ intersects the closure $\overline{iC_M^+}$ in exactly one point, it follows that $\mu^n = \mu$ for n large enough. ∎

Combining the results of Proposition 5.3 and Proposition 7.4, we obtain

Theorem 7.5. *The topological spaces $(\widehat{G_0})_{gen}$ and $(\mathfrak{g}_0^\ddagger/G_0)_{gen}$ are homeomorphic.*

Proposition 7.6. *Let $\left(\mathcal{O}^{G_0}_{(\mu^n,H_n)}\right)_n$ be a sequence of generic admissible coadjoint orbits of G_0. Then $\left(\mathcal{O}^{G_0}_{(\mu^n,H_n)}\right)_n$ converges to $\mathcal{O}^{G_0}_\lambda$ in $\mathfrak{g}_0^\ddagger/G_0$ if and only if $(H_n)_n$ tends to 0 as $n \longrightarrow +\infty$ and ρ_{μ^n} occurs in the restriction $Res^K_M(\tau_\lambda)$ for n large enough.*

Proof. Assume that $\left(\mathcal{O}^{G_0}_{(\mu^n,H_n)}\right)_n$ converges to $\mathcal{O}^{G_0}_\lambda$. There exist two sequences $(k_n)_n \subset K$ and $(X_n)_n \subset \mathfrak{p}$ such that:

$$\begin{aligned}\lim_{n\longrightarrow+\infty} (Ad(k_n)U_{\mu^n} + [X_n, Ad(k_n)H_n]) &= U_\lambda,\\ \lim_{n\longrightarrow+\infty} Ad(k_n)H_n &= 0.\end{aligned}$$

By compactness of K, we may assume that $\lim_{n\longrightarrow+\infty} k_n = k_0$. Then we easily see that $\lim_{n\longrightarrow+\infty} H_n = 0$. From the equality

$$\lim_{n\longrightarrow+\infty} (U_{\mu^n} + [Ad(k_n^{-1})X_n, H_n]) = Ad(k_0^{-1})U_\lambda,$$

we deduce that $\lim_{n\longrightarrow+\infty} U_{\mu^n}$ belongs to the projection of $Ad(K)U_\lambda \cap \mathfrak{h}_\mathfrak{k}$ onto $\mathfrak{h}_\mathfrak{m}$. Equivalently, this implies that $-i\mu^n$ belongs to the set $q(\mathcal{O}^K_\lambda \cap \mathfrak{h}^*_\mathfrak{k})$ for n large enough, i.e., $\mu^n \in q(W_K.\lambda)$ for n large enough. By Lemma 7.1, it follows that ρ_{μ^n} occurs in $Res^K_M(\tau_\lambda)$ for n large enough.

Conversely, assume that $\lim_{n\longrightarrow+\infty} H_n = 0$ and that ρ_{μ^n} occurs in $Res^K_M(\tau_\lambda)$ for n large enough. By Lemma 7.2, we know that the orbit $\mathcal{O}^M_{\mu^n}$ occurs in $q(\mathcal{O}^K_\lambda)$ for n large enough. Then for such n, there exist h_n in K and Y_n in the subspace $\mathfrak{l}$ of $\mathfrak{k}$ with $U_{\mu^n} + Y_n = Ad(h_n)U_\lambda$. Fix an element $(Ad(k)U_\lambda, 0)$ in $\mathcal{O}^{G_0}_\lambda$ and set $k_n = kh_n^{-1}$. Of course, we have $\lim_{n\longrightarrow+\infty} Ad(k_n)H_n = 0$. Since for every regular element H in $\mathfrak{a}$, the linear map $ad(H)|_\mathfrak{q} : \mathfrak{q} \longrightarrow \mathfrak{l}$ is surjective, we deduce for all

n there exists $Z_n \in \mathfrak{q}$ such that $[Z_n, H_n] = Y_n$. For $X_n = Ad(k_n)Z_n$ with n large enough, we have

$$Ad(k_n)U_{\mu^n} + [X_n, Ad(k_n)H_n] = Ad(k)U_\lambda.$$

We conclude that the sequence $\left(\mathcal{O}^{G_0}_{(\mu^n, H_n)}\right)_n$ converges to $\mathcal{O}^{G_0}_\lambda$ in $\mathfrak{g}_0^\ddagger/G_0$. This completes the proof of the proposition. ■

The above analysis allows us to derive the following theorem.

Theorem 7.7. *In the setting as above, assume that the compact Riemannian symmetric pair (G, K) has rank one. Then the unitary dual $\widehat{G_0}$ is homeomorphic to the space of admissible coadjoint orbits $\mathfrak{g}_0^\ddagger/G_0$.*

The special case of Theorem 7.7 where $(G, K) = (SO(n+1), SO(n))$ has been proved in [4]. The authors method of proof makes essential use of the classical branching rule from $SO(n)$ to $SO(n-1)$ for $n \geq 2$.

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