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Left-Invariant Lorentzian Flat Metrics on Lie Groups

Malika Ait Ben Haddou, Mohamed Boucetta and Hicham Lebzioui

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Abstract. We call the Lie algebra of a Lie group with a left invariant pseudo-Riemannian flat metric pseudo-Riemannian flat Lie algebra. We give a new proof of a classical result of Milnor on Riemannian flat Lie algebras. We reduce the study of Lorentzian flat Lie algebras to those with trivial center or those with degenerate center. We show that the double extension process can be used to construct all Lorentzian flat Lie algebras with degenerate center generalizing a result of Aubert-Medina on Lorentzian flat nilpotent Lie algebras. Finally, we give the list of Lorentzian flat Lie algebras with degenerate center up to dimension 6.

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1. Introduction

A *pseudo-Riemannian flat Lie group* is a Lie group with a left invariant pseudo-Riemannian flat metric. The Lie algebra of such a Lie group is called *pseudo-Riemannian flat Lie algebra*. If the metric on the Lie group is complete the Lie algebra is called complete. It is a well-known result that a pseudo-Riemannian flat Lie algebra is complete if and only if it is unimodular. A Riemannian (resp. Lorentzian) flat Lie group is a pseudo-Riemannian flat Lie group for which the metric is definite positive (respectively, of signature $(-,+\ldots+)$). In [6], Milnor showed that a Lie group is a Riemannian flat Lie group if and only if its Lie algebra is a semi-direct product of an abelian algebra $\mathfrak{b}$ with an abelian ideal $\mathfrak{u}$ and, for any $u \in \mathfrak{b}$, ad_u is skew-symmetric. The characterization of Lorentzian flat Lie algebras (eventually complete) is an open problem. It is a well-known result that a Lorentzian flat Lie algebra must be solvable (see [4]). On the other hand, in [1], Aubert and Medina showed that nilpotent Lorentzian flat Lie algebras are obtained by a double extension process from Riemannian abelian Lie algebras. In [3], a general method to build examples of Lorentzian flat Lie algebras is given. In this paper, we reduce the problem of finding Lorentzian flat Lie algebras to

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the determination of Lorentzian flat Lie algebras with degenerate center and those with trivial center. We show that the double extension process can be used to construct all Lorentzian flat Lie algebras with degenerate center from Riemannian flat Lie algebras and the Lie algebras obtained are unimodular and hence complete (see Theorem 2.1). This result generalizes Aubert-Medina's result. We give the list of Lorentzian flat Lie algebras with degenerate center up to dimension 6. The paper is organized as follows. In Section 2, we recall the double extension process and state our main result (Theorem 2.1). In Section 3, we revisit Milnor's theorem and give a new formulation and a new proof of this theorem using the Lie algebra of left invariant Killing vector fields on a pseudo-Riemannian flat Lie group (See Theorem 3.1). This Lie algebra will play a crucial role in the proof of our main result. Indeed, we will establish a key Lemma (See Lemma 3.1) involving this Lie algebra and, as a consequence, we reduce the problem of finding Lorentzian flat Lie algebras to the determination of Lorentzian flat Lie algebras with degenerate center and those with trivial center, we recover the keystone in the proof of Aubert-Medina's result (see [1] Lemma 1.1) and prove Theorem 2.1. In Section 4, we give some indications on how one can construct the tools used in the double extension process and we give the list of Lorentzian flat Lie algebras with degenerate center up to dimension 6.

2. Statement of the main result

A Lie group G together with a left-invariant pseudo-Riemannian metric is called *pseudo-Riemannian Lie group*. The left-invariant pseudo-Riemannian metric defines an inner product $\langle\ ,\ \rangle$ on the Lie algebra $\mathfrak{g}$ of G, and conversely, any inner product on $\mathfrak{g}$ gives rise to an unique left-invariant metric on G. The couple $(\mathfrak{g}, \langle\ ,\ \rangle)$ is called *pseudo-Riemannian Lie algebra*. We use the adjective Riemannian (resp. Lorentzian) instead of pseudo-Riemannian when the metric is definite positive (resp. of signature $(-,+\ldots+)$). For any endomorphism $D:\mathfrak{g}\longrightarrow\mathfrak{g}$ we denote by $D^*:\mathfrak{g}\longrightarrow\mathfrak{g}$ its adjoint with respect to $\langle\ ,\ \rangle$.
Let $(\mathfrak{g}, \langle\ ,\ \rangle)$ be a pseudo-Riemannian Lie algebra of dimension n. The Levi-Civita connection defines a product $(u,v)\mapsto uv$ on $\mathfrak{g}$ called *Levi-Civita product* given by Koszul's formula

$$2\langle uv, w\rangle = \langle [u,v], w\rangle + \langle [w,u], v\rangle + \langle [w,v], u\rangle. \tag{1}$$

For any $u\in\mathfrak{g}$, we denote by $\mathrm{L}_u:\mathfrak{g}\longrightarrow\mathfrak{g}$ and $\mathrm{R}_u:\mathfrak{g}\longrightarrow\mathfrak{g}$, respectively, the left multiplication and the right multiplication by u given by

$$\mathrm{L}_u v = uv \quad \text{and} \quad \mathrm{R}_u v = vu.$$

For any $u\in\mathfrak{g}$, L_u is skew-symmetric with respect to $\langle\ ,\ \rangle$ and

$$\mathrm{ad}_u = \mathrm{L}_u - \mathrm{R}_u, \tag{2}$$

where $\mathrm{ad}_u:\mathfrak{g}\longrightarrow\mathfrak{g}$ is given by $\mathrm{ad}_u v = [u,v]$. The *mean curvature vector* on $\mathfrak{g}$ is the vector given by

$$\langle H, u\rangle = \mathrm{tr}(\mathrm{ad_u}),\ \forall u\in\mathfrak{g}. \tag{3}$$

The Lie algebra $\mathfrak{g}$ is unimodular if and only if $H = 0$. The curvature of $\langle\ ,\ \rangle$ is given by

$$\mathrm{K}(u,v) = \mathrm{L}_{[u,v]} - [\mathrm{L}_u, \mathrm{L}_v].$$

The $(\mathfrak{g}, \langle\ ,\ \rangle)$ is called *pseudo-Riemannian flat Lie algebra* if K vanishes identically. This is equivalent to the fact that $\mathfrak{g}$ endowed with the Levi-Civita product is a left symmetric algebra, i.e., for any $u, v, w \in \mathfrak{g}$,

$$\mathrm{ass}(u,v,w) = \mathrm{ass}(v,u,w),$$

where $\mathrm{ass}(u,v,w) = (uv)w - u(vw)$. This relation is equivalent to

$$\mathrm{R}_{uv} - \mathrm{R}_v \circ \mathrm{R}_u = [\mathrm{L}_u, \mathrm{R}_v], \tag{4}$$

for any $u, v \in \mathfrak{g}$.

Let us recall now the double extension process and some related results as elaborated in [1]. In particular, Propositions 3.1-3.2 of the paper [1] are essential in the double extension process.
Let $(B, [\ ,\ ]_0, \langle\ ,\ \rangle_0)$ be a pseudo-Riemannian flat Lie algebra, $\xi, D : B \longrightarrow B$ two endomorphisms of B, $b_0 \in B$ and $\mu \in \mathbb{R}$ such that:

1. ξ is a 1-cocycle of $(B, [\ ,\ ]_0)$ with respect to the representation $\mathrm{L} : B \longrightarrow \mathrm{End}(B)$ defined by the left multiplication associated to the Levi-Civita product, i.e., for any $a, b \in B$,

$$\xi([a,b]) = \mathrm{L}_a \xi(b) - \mathrm{L}_b \xi(a), \tag{5}$$

2. D is a derivation of $(B, [\ ,\ ]_0)$,

3. $D - \xi$ is skew-symmetric with respect to $\langle\ ,\ \rangle_0$,

$$[D, \xi] = \xi^2 - \mu\xi - \mathrm{R}_{b_0}, \tag{6}$$

and for any $a, b \in B$

$$a\xi(b) - \xi(ab) = D(a)b + aD(b) - D(ab). \tag{7}$$

We call (ξ, D, μ, b_0) satisfying the three conditions above *admissible*.
Given (ξ, D, μ, b_0) admissible, we endow the vector space $\mathfrak{g} = \mathbb{R}z \oplus B \oplus \mathbb{R}\bar{z}$ with the inner product $\langle\ ,\ \rangle$ which extends $\langle\ ,\ \rangle_0$, for which $\mathrm{span}\{z, \bar{z}\}$ and B are orthogonal, $\langle z, z\rangle = \langle \bar{z}, \bar{z}\rangle = 0$ and $\langle z, \bar{z}\rangle = 1$. We define also on $\mathfrak{g}$ the bracket

$$[\bar{z}, z] = \mu z,\ [\bar{z}, a] = D(a) - \langle b_0, a\rangle_0 z \quad \text{and} \quad [a,b] = [a,b]_0 + \langle(\xi - \xi^*)(a), b\rangle_0 z, \tag{8}$$

where $a, b \in B$ and ξ^* is the adjoint of ξ with respect to $\langle\ ,\ \rangle_0$. Then $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is a pseudo-Riemannian flat Lie algebra called *double extension* of $(B, [\ ,\ ]_0, \langle\ ,\ \rangle_0)$ according to (ξ, D, μ, b_0). Moreover, any nilpotent Lorentzian flat Lie algebra is a double extension of a Riemannian abelian Lie algebra according to (ξ, D, μ, b_0) with $\xi = D$, $D^2 = 0$ and $\mu = 0$.
We can now state our main result.

Theorem 2.1. *A Lorentzian Lie algebra with degenerate center is flat if and only if it is a double extension of a Riemannian flat Lie algebra $(B, [\ ,\ ]_0, \langle\ ,\ \rangle_0)$ according to $(\xi, D, 0, b_0)$ with $(D, b_0) \neq (0,0)$. Moreover, a Lorentzian flat Lie algebra with degenerate center is unimodular and hence complete.*

3. The Lie algebra of left invariant Killing vector fields on a pseudo-Riemannian flat Lie group

The Riemannian case: Milnor's theorem revisited In this paragraph, we give a new formulation and a new proof of Milnor's theorem using the Lie algebra of left invariant Killing vector fields of a Riemannian flat Lie group. This Lie algebra will play a crucial role in the proof of our main result.

Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ be a pseudo-Riemannian Lie algebra. The Lie subalgebra

$$\mathfrak{L}(\mathfrak{g}) = \{u \in \mathfrak{g}, \mathrm{ad}_u + \mathrm{ad}_u^* = 0\} = \{u \in \mathfrak{g}, \mathrm{R}_u + \mathrm{R}_u^* = 0\} \tag{9}$$

is called *Killing* subalgebra of $\mathfrak{g}$. Indeed, if $\mathfrak{g}$ is the Lie algebra of left invariant vector fields of a pseudo-Riemannian Lie group then $\mathfrak{L}(\mathfrak{g})$ is the Lie algebra of left invariant Killing vector fields. On the other hand, one can see easily that the orthogonal of the derived ideal of $\mathfrak{g}$ is given by

$$\mathfrak{D}(\mathfrak{g})^{\perp} = \{u \in \mathfrak{g}, \mathrm{R}_u = \mathrm{R}_u^*\}. \tag{10}$$

Finally, we put

$$N_\ell(\mathfrak{g}) = \{u \in \mathfrak{g}, \mathrm{L}_u = 0\} \quad \text{and} \quad N_r(\mathfrak{g}) = \{u \in \mathfrak{g}, \mathrm{R}_u = 0\}.$$

We have obviously

$$N_r(\mathfrak{g}) = (\mathfrak{g}\mathfrak{g})^{\perp}. \tag{11}$$

Proposition 3.1. *Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ be a pseudo-Riemannian flat Lie algebra. Then:*

1. *For any $u \in \mathfrak{L}(\mathfrak{g})$, $\mathrm{R}_u^2 = 0$ and $[\mathrm{R}_u, \mathrm{L}_u] = 0$.*
2. *For any $u \in \mathfrak{D}(\mathfrak{g})^{\perp}$, R_u is nilpotent and $[\mathrm{R}_u, \mathrm{L}_u] = \mathrm{R}_u^2$.*
3. *The mean curvature vector satisfies $H \in \mathfrak{D}(\mathfrak{g}) \cap \mathfrak{D}(\mathfrak{g})^{\perp}$. In particular, if $\mathfrak{g}$ is non unimodular then $\mathfrak{D}(\mathfrak{g})$ is degenerate.*

Proof. By using (1) one can see easily that, for any $u \in \mathfrak{L}(\mathfrak{g}) \cup \mathfrak{D}(\mathfrak{g})^{\perp}$, $u.u = 0$ and deduce from (4) that

$$[\mathrm{R}_u, \mathrm{L}_u] = \mathrm{R}_u^2.$$

If $u \in \mathfrak{L}(\mathfrak{g})$ then R_u is skew-symmetric and, since L_u is always skew-symmetric, $[\mathrm{R}_u, \mathrm{L}_u]$ is skew-symmetric. But R_u^2 is symmetric which implies 1.
On the other hand, one can deduce by induction that for any $k \in \mathbb{N}^*$

$$[\mathrm{R}_u^k, \mathrm{L}_u] = k\mathrm{R}_u^{k+1}.$$

and hence $\mathrm{tr}(\mathrm{R}_u^k) = 0$ for any $k \geq 2$ which implies that R_u is nilpotent.
Since, for any $u, v \in \mathfrak{g}$, $\mathrm{tr}(\mathrm{ad}_{[u,v]}) = 0$, we deduce that $H \in \mathfrak{D}(\mathfrak{g})^\perp$. Now, for any $u \in \mathfrak{D}(\mathfrak{g})^\perp$, R_u is nilpotent and hence

$$\mathrm{tr}(\mathrm{ad}_u) = \mathrm{tr}(\mathrm{R}_u) = \langle H, u \rangle = 0,$$

which implies $H \in \mathfrak{D}(\mathfrak{g})$. ∎

Remark 1. *If $\mathfrak{g}$ is a Lorentzian flat Lie algebra, one can deduce from Proposition 3.1 that for any $u \in \mathfrak{D}(\mathfrak{g})^\perp$, $\mathrm{R}_u^3 = 0$. Moreover, if $\mathfrak{g}$ is non unimodular then*

$$\mathfrak{D}(\mathfrak{g}) \cap \mathfrak{D}(\mathfrak{g})^\perp = \mathbb{R}H.$$

We can give now a new formulation and a new proof of Milnor's theorem (see [6]). This new formulation appeared first in [2]. Recall that Milnor showed that a Lie group is a Riemannian flat Lie group if and only if its Lie algebra is a semi-direct product of an abelian algebra $\mathfrak{b}$ with an abelian ideal $\mathfrak{u}$ and, for any $u \in \mathfrak{b}$, ad_u is skew-symmetric.

Theorem 3.1. *Let G be a Riemannian Lie group. Then the curvature of G vanishes if and only if $\mathfrak{L}(\mathfrak{g})$ is abelian, $\mathfrak{D}(\mathfrak{g})$ is abelian and $\mathfrak{L}(\mathfrak{g})^\perp = \mathfrak{D}(\mathfrak{g})$. Moreover, in this case the dimension of $\mathfrak{D}(\mathfrak{g})$ is even and the Levi-Civita product is given by*

$$\mathrm{L}_a = \begin{cases} \mathrm{ad}_a & \text{if} \quad a \in \mathfrak{L}(\mathfrak{g}), \\ 0 & \text{if} \quad a \in \mathfrak{D}(\mathfrak{g}). \end{cases} \tag{12}$$

Proof. Suppose that G is a Riemannian flat Lie group and $\mathfrak{g}$ its Lie algebra. Let $(\mathfrak{D}(\mathfrak{g})^k)_{k \in \mathbb{N}}$ denote the commutator series of $\mathfrak{g}$ defined recursively by

$$\mathfrak{D}(\mathfrak{g})^0 = \mathfrak{g}, \quad \mathfrak{D}(\mathfrak{g})^1 = [\mathfrak{g}, \mathfrak{g}] \quad \text{and} \quad \mathfrak{D}(\mathfrak{g})^{k+1} = [\mathfrak{D}(\mathfrak{g})^k, \mathfrak{D}(\mathfrak{g})^k].$$

Since in positive definite context skew-symmetric or symmetric nilpotent endomorphism must vanish, we deduce from Proposition 3.1 that

$$\mathfrak{L}(\mathfrak{g}) = \mathfrak{D}(\mathfrak{g})^\perp = N_r(\mathfrak{g}) = (\mathfrak{g}\mathfrak{g})^\perp. \tag{13}$$

These relations imply that $\mathfrak{L}(\mathfrak{g})$ is abelian, $\mathfrak{D}(\mathfrak{g}) = \mathfrak{g}\mathfrak{g}$ and hence $\mathfrak{D}(\mathfrak{g})$ is a two-sided ideal of the Levi-Civita product so $\mathfrak{D}(\mathfrak{g})$ endowed with the restricted metric is a Riemannian flat Lie algebra and, by induction, for any $k \in \mathbb{N}$, $\mathfrak{D}(\mathfrak{g})^k$ is a Riemannian flat Lie algebra. On the other hand, it is known that a non null left symmetric algebra cannot be equal to its derived ideal (see [5] pp.31). So $\mathfrak{g}$ must be solvable and hence $\mathfrak{D}(\mathfrak{g})$ is nilpotent. If $\mathfrak{D}(\mathfrak{g})$ is non abelian then the splitting

$$\mathfrak{D}(\mathfrak{g}) = \mathfrak{L}(\mathfrak{D}(\mathfrak{g})) \oplus \mathfrak{D}^2(\mathfrak{g})$$

is non trivial. But the center of $\mathfrak{D}(\mathfrak{g})$ is contained in $\mathfrak{L}(\mathfrak{D}(\mathfrak{g}))$ and it intersects non trivially $\mathfrak{D}^2(\mathfrak{g})$ ($\mathfrak{D}(\mathfrak{g})$ is nilpotent) so $\mathfrak{D}(\mathfrak{g})$ must be abelian. This achieves the direct part of the theorem. The equation (12) is easy to establish and the

converse follows immediately from this equation.
Suppose that G is flat. Hence $\mathfrak{L}(\mathfrak{g})$ is abelian, $\mathfrak{D}(\mathfrak{g})$ is abelian and

$$\mathfrak{g} = \mathfrak{L}(\mathfrak{g}) \oplus \mathfrak{D}(\mathfrak{g}).$$

If $\mathfrak{L}(\mathfrak{g}) = \{0\}$ then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] = \{0\}$ and the result follows trivially. Suppose now that $\mathfrak{L}(\mathfrak{g}) \neq \{0\}$. Let $(s_1, ..., s_p)$ be a basis of $\mathfrak{L}(\mathfrak{g})$. The restriction of ad_{s_1} to $\mathfrak{D}(\mathfrak{g})$ is a skew-symmetric endomorphism, thus its kernel K_1 is of even codimension in $\mathfrak{D}(\mathfrak{g})$. Now, ad_{s_2} commutes with ad_{s_1} and K_1 is invariant by ad_{s_2}. By using the same argument as above, we deduce that $K_2 = K_1 \cap \ker \mathrm{ad}_{s_2}$ is of even codimension in K_1. Finally K_2 is of even codimension in $\mathfrak{D}(\mathfrak{g})$. Thus, by induction, we show that

$$K_p = \mathfrak{D}(\mathfrak{g}) \cap (\cap_{i=1}^{p} \ker \mathrm{ad}_{s_i})$$

is an even codimensional subspace of $\mathfrak{D}(\mathfrak{g})$. Now from its definition K_p is contained in the center of $\mathfrak{g}$ which is contained in $\mathfrak{L}(\mathfrak{g})$ and then $K_p = \{0\}$ and the second part of the theorem follows. ∎

The Lorentzian case It is known that a left invariant affine structure on a Lie group G is complete if and only if for any $u \in \mathfrak{g}$, R_u is nilpotent (see [8] for instance). If G is a Riemannian flat Lie group then the underlying left invariant affine structure is complete and one can deduce (13) immediately. We have avoided to use this argument in the proof of Theorem 3.1 and we have used arguments which are not specific to the Riemannian case. Unfortunately, in the Lorentzian case the argument used to prove (13) cannot be used since in the Lorentzian context there are non trivial skew-symmetric or symmetric nilpotent endomorphisms. However, in the following lemma, we show that a part of(13) is still valid in the Lorentzian case.

Lemma 3.1. *Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ be a Lorentzian flat Lie algebra. Then*

$$\mathfrak{L}(\mathfrak{g}) = N_r(\mathfrak{g}) = (\mathfrak{g}\mathfrak{g})^{\perp}.$$

Proof. Note first that we have always $N_r(\mathfrak{g}) \subset \mathfrak{L}(\mathfrak{g})$. Let $u \in \mathfrak{L}(\mathfrak{g})$. According to Proposition 3.1 1, $\mathrm{R}_u^2 = 0$ and since R_u is skew-symmetric we get that $\mathrm{Im}\mathrm{R}_u$ is a totally isotropic subspace and hence there exists an isotropic vector $e \in \mathfrak{g}$ and a covector $\alpha \in \mathfrak{g}^*$ such that $\mathrm{R}_u(v) = \alpha(v)e$ for any $v \in \mathfrak{g}$. Choose a basis $\{e, \bar{e}, f_1, \dots, f_{n-2}\}$ of $\mathfrak{g}$ such that $\mathrm{span}\{e, \bar{e}\}$ and $\mathrm{span}\{f_1, \dots, f_{n-2}\}$ are orthogonal, $\{f_1, \dots, f_{n-2}\}$ is orthonormal, $\bar{e}$ is isotropic and $\langle e, \bar{e}\rangle = 1$. We have, for any $i = 1, \dots, n-2$,

$$\begin{aligned}
\langle \mathrm{R}_u(e), \bar{e}\rangle &= \alpha(e) = -\langle e, \mathrm{R}_u(\bar{e})\rangle = 0,\\
\langle \mathrm{R}_u(\bar{e}), \bar{e}\rangle &= 0 = \alpha(\bar{e}),\\
\langle \mathrm{R}_u(f_i), \bar{e}\rangle &= \alpha(f_i) = -\langle f_i, \mathrm{R}_u(\bar{e})\rangle = 0,
\end{aligned}$$

hence $\alpha = 0$ and then $u \in N_r(\mathfrak{g})$ which achieves the proof of the lemma. ∎

Let $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ be a Lorentzian flat Lie algebra. There are some interesting consequences of Lemma 3.1:

1. We have
$$Z(\mathfrak{g}) = N_\ell(\mathfrak{g}) \cap N_r(\mathfrak{g}) \subset \mathfrak{L}(\mathfrak{g}) = (\mathfrak{g}\mathfrak{g})^\perp \subset \mathfrak{D}(\mathfrak{g})^\perp. \tag{14}$$
Thus $Z(\mathfrak{g})$ and $Z(\mathfrak{g})^\perp$ are ideals of $\mathfrak{g}$ and are two-sided ideals of $\mathfrak{g}$ endowed with the Levi-Civita product, so if $Z(\mathfrak{g})$ is non degenerate we have
$$\mathfrak{g} = Z(\mathfrak{g}) \oplus Z(\mathfrak{g})^\perp$$
and both $Z(\mathfrak{g})$ and $Z(\mathfrak{g})^\perp$ are flat when endowed with the restricted metric.
2. If $Z(\mathfrak{g})$ is degenerate, we have
$$Z(\mathfrak{g}) \cap \mathfrak{D}(\mathfrak{g}) \subset Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp \subset N_\ell(\mathfrak{g}) \cap N_r(\mathfrak{g}), \tag{15}$$
and hence $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$ is a one dimensional two-sided ideal and its orthogonal is also a two-sided ideal, so we can use the double extension process.
3. If $\mathfrak{g}$ is nilpotent then $Z(\mathfrak{g}) \cap \mathfrak{D}(\mathfrak{g}) \neq \{0\}$ and hence $Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp$ is one dimensional ideal contained in $N_\ell(\mathfrak{g})$ and Lemma 1.1 of [1] follows. Note that this lemma played a crucial role in the proof of Aubert-Medina's result and we give here a generalization and a new proof of this lemma.

In conclusion, we have shown that the problem of finding Lorentzian flat Lie algebras reduces to the determination of solvable Lorentzian flat Lie algebras with degenerate center and those with trivial center.

Proof of Theorem 2.1

Proof. Suppose that $(\mathfrak{g}, [\ ,\ ], \langle\ ,\ \rangle)$ is a Lorentzian flat Lie algebra and $Z(\mathfrak{g})$ is degenerate. Then $\mathfrak{I} = Z(\mathfrak{g}) \cap Z(\mathfrak{g})^\perp = \mathbb{R}z$ where z is an isotropic vector. Now from (14) we deduce that $\mathrm{L}_z = \mathrm{R}_z = 0$ and hence $\mathfrak{I}$ is a two-sided ideal for the Levi-Civita product. Moreover, the orthogonal $\mathfrak{I}^\perp$ is also a two-sided ideal. So, according to Proposition 3.1 of [1], $\mathfrak{g} = \mathbb{R}z \oplus B \oplus \mathbb{R}\bar{z}$ and it is a double extension of B according to (ξ, D, μ, b_0). The Lie bracket is given by
$$[\bar{z}, z] = \mu z,\ [\bar{z}, a] = D(a) - \langle b_0, a\rangle_0 z \quad \text{and} \quad [a, b] = [a, b]_0 + \langle(\xi - \xi^*)(a), b\rangle_0 z.$$
Since $z \in Z(\mathfrak{g})$ then $\mu = 0$. The converse is obviously true.
On the other hand, according to the brackets above and the fact that B is unimodular, $\mathfrak{g}$ is unimodular if and only if $\mathrm{tr}(D) = 0$. In Section 4, we will show that if $(\xi, D, 0, b_0)$ is admissible and B abelian then $D - \xi$ is skew-symmetric and ξ is nilpotent so $\mathrm{tr}(D) = 0$. When B is non abelian the relation $\mathrm{tr}(D) = 0$ follows from Proposition 4.2. This achieves the proof. ∎

Remark 2. *Let $\mathfrak{g}$ be a Lorentzian flat non unimodular Lie algebra. According to Theorem 2.1, $Z(\mathfrak{g})$ is non degenerate and hence $\mathfrak{g} = Z(\mathfrak{g}) \oplus Z(\mathfrak{g})^\perp$. Moreover, we have seen that $Z(\mathfrak{g})^\perp$ is a two-sided ideal with respect to Levi-Civita product so the restriction of the metric to $Z(\mathfrak{g})^\perp$ is Lorentzian and flat. Thus we reduce the study of Lorentzian flat non unimodular Lie algebras to those with trivial center. Moreover, if $\mathfrak{g}$ is such a Lie algebra then $\mathfrak{D}(\mathfrak{g}) \cap \mathfrak{D}(\mathfrak{g})^\perp = \mathbb{R}H$.*

4. Lorentzian flat Lie algebras with degenerate center up to dimension six

According to Theorem 2.1, one can determine entirely all Lorentzian flat Lie algebras with degenerate center if one can find all admissible $(\xi, D, 0, b_0)$ on Riemannian flat Lie algebras. In this section, we will give a general method to solve the equations satisfied by admissible $(\xi, D, 0, b_0)$ and we will use this method to give explicitly the solutions on Riemannian flat Lie algebras of dimension 2, 3 or 4. This will permit us to establish the list of all Lorentzian flat Lie algebras with degenerate center up to dimension six.

The abelian case Let B be a Riemannian flat abelian Lie algebra of dimension n. One can see easily that $(\xi, D, 0, b_0)$ is admissible if and only if $A = D - \xi$ is skew-symmetric and

$$[A, \xi] = \xi^2. \tag{16}$$

Let (A, ξ) be a solution of (16) with A is skew-symmetric. One can deduce by induction that, for any $k \in \mathbb{N}^*$,

$$[A, \xi^k] = k\xi^{k+1}, \tag{17}$$

and hence, for any $k \geq 2$,

$$\mathrm{tr}(\xi^k) = 0. \tag{18}$$

This implies that ξ is nilpotent. Thus there exists $q \leq \dim B$ such that

$$\{0\} \neq \ker \xi \subsetneqq \ker \xi^2 \subsetneqq \ldots \subsetneqq \ker \xi^q = B.$$

Then we have the orthogonal splitting of B

$$B = \bigoplus_{k=0}^{q-1} F_k, \tag{19}$$

where $F_0 = \ker \xi$ and, for any $k = 1, \ldots, q-1$, $F_k = \ker \xi^{k+1} \cap \left(\ker \xi^k\right)^\perp$. The key point is that (17) implies that $A(\ker \xi^k) \subset \ker \xi^k$ for any $k \in \mathbb{N}$ and since A is skew-symmetric, $A(F_k) \subset F_k$. By using an orthonormal basis which respect to the splitting (19), the matrix of A and ξ are simple and one can solve (16) easily. The following remarks can be used to simplify the computations when solving (16).

Remark 3. *Let (A, ξ) be a solution of (16) with A is skew-symmetric.*

1. *If $q = \dim B$ then for any $k = 0, \ldots, n-1$, $\dim F_k = 1$ and hence $A = 0$ so $\xi^2 = 0$ and then $\dim B = 2$. So if $\dim B \geq 3$ then $q < \dim B$.*
2. *One can deduce easily from (17) that $\ker A \subset \ker \xi^2$. Indeed, if $x \in \ker A$ then by (17), for any $k \in \mathbb{N}$, $A\xi^k(x) = k\xi^{k+1}(x)$ and since ξ is nilpotent there exists $p > 0$ such that $A^p\xi(x) = 0$. If $p = 1$ we get $A\xi(x) = \xi^2(x) = 0$ and hence $x \in \ker \xi^2$. If $p \geq 2$, then $A^{2p-2}\xi(x) = 0$ and, since A is skew-symmetric, $\langle A^{p-1}\xi(x), A^{p-1}\xi(x)\rangle = 0$. Thus $A^{p-1}\xi(x) = 0$. By repeating the argument above we get finally that $A\xi(x) = \xi^2(x) = 0$ and hence $x \in \ker \xi^2$.*

3. *For any* $1 \leq k \leq q-1$, *we have*

$$\dim F_k \leq \dim \ker \xi^j, \ j = 1, \ldots, k+1.$$

We can now use what we observed above to find all admissible $(\xi, D, 0, b_0)$ when $\dim B \leq 4$.

Proposition 4.1. *Let* B *be a Riemannian flat abelian Lie algebra. Then:*

1. *If* $\dim B = 2$ *then* $(\xi, D, 0, b_0)$ *is admissible if and only if* $\xi = D = 0$ *or there exists an orthonormal basis* $\{e_1, e_2\}$ *of* B *such that the matrices of* ξ *and* D *in this basis are*

$$\left(M(\xi) = M(D) = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\right) \quad or \quad \left(\xi = 0, \ M(D) = \begin{pmatrix} 0 & \lambda \\ -\lambda & 0 \end{pmatrix}\right),$$

with $a \neq 0$ *and* $\lambda > 0$.

2. *If* $\dim B = 3$ *then* $(\xi, D, 0, b_0)$ *is admissible if and only if* $\xi = D = 0$ *or there exists an orthonormal basis* $\{e_1, e_2, e_3\}$ *of* B *such that the matrices of* ξ *and* D *in this basis are*

$$\left(M(\xi) = M(D) = \begin{pmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right) \ or \ \left(\xi = 0, \ M(D) = \begin{pmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\right),$$

with $a \neq 0$ *and* $\lambda > 0$.

3. *If* $\dim B = 4$ *then* $(\xi, D, 0, b_0)$ *is admissible if and only if* $\xi = D = 0$ *or there exists an orthonormal basis* $\{e_1, e_2, e_3, e_4\}$ *of* B *such that the matrices of* ξ *and* D *in this basis have one of the following forms:*

$$(f_1) \quad M(\xi) = M(D) = \begin{pmatrix} 0 & 0 & a & b \\ 0 & 0 & 0 & c \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad ac \neq 0,$$

$$(f_2) \quad M(\xi) = M(D) = \begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad a \neq 0,$$

$$(f_3) \quad M(D) = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix}, \ \xi = 0, \ (a, b) \neq (0, 0),$$

$$(f_4) \quad M(D) = \begin{pmatrix} 0 & a & b & 0 \\ -a & 0 & 0 & b \\ 0 & 0 & 0 & a \\ 0 & 0 & -a & 0 \end{pmatrix}, \ M(\xi) = \begin{pmatrix} 0 & 0 & b & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ ab \neq 0,$$

$$(f_5) \quad M(D) = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ M(\xi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & 0 & 0 \end{pmatrix} \ a \neq 0, \ b \neq 0.$$

Proof. Note first that $(\xi, D, 0, b_0)$ is admissible if and only if $A = D - \xi$ is skew-symmetric and (A, ξ) is a solution of (16). If it is the case, we will use repeatedly the fact that A leaves invariant the splitting (19) and Remark 3. Let $(\xi, D, 0, b_0)$ be admissible such that $(\xi, D) \neq (0, 0)$. We consider the integer q defined in (19).

1. When $\dim B = 2$, there are two possibilities of q. If $q = 2$ then, according to (19), $B = \ker\xi \oplus F_1$ and hence $A = 0$. Thus $D = \xi$ and $\xi^2 = 0$. If $q = 1$ then $\xi = 0$ and D is skew-symmetric.

2. When $\dim B = 3$ then, according to Remark 3, there are also two possibilities of q. If $q = 1$ then $\xi = 0$ and D is skew-symmetric.
   If $q = 2$ then, according to (19) and Remark 3, $B = \ker\xi \oplus F_1$ with $\dim\ker\xi \geq \dim F_1$. Thus $\dim\ker\xi = 2$ and $\dim F_1 = 1$. So there exists an orthonormal basis (e_1, e_2, e_3) of B such that
   $$M(A) = \begin{pmatrix} 0 & a & 0 \\ -a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad M(\xi) = \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix},$$
   where $a \in \mathbb{R}$ and $(b, c) \neq (0, 0)$. A direct computation shows that (16) is equivalent to $A = 0$ and hence $D = \xi$. To get the desired form, one need to consider the orthonormal basis $(\frac{be_1+ce_2}{\sqrt{b^2+c^2}}, \frac{be_2-ce_1}{\sqrt{b^2+c^2}}, e_3)$.

3. When $\dim B = 4$ then, according to Remark 3, $q \leq 3$.

   - If $q = 1$ then $\xi = 0$ and D is skew-symmetric, this gives (f_3).
   - If $q = 2$ then $\xi \neq 0$, $B = \ker\xi \oplus F_1$ and $\dim\ker\xi \geq \dim F_1$. We distinguish two cases.
     First case: $\dim\ker\xi = 2$ and $\dim F_1 = 2$. Then there exists an orthonormal basis (e_1, e_2, e_3, e_4) such that
     $$M(A) = \begin{pmatrix} 0 & a & 0 & 0 \\ -a & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & -b & 0 \end{pmatrix} \quad \text{and} \quad M(\xi) = \begin{pmatrix} 0 & 0 & c & d \\ 0 & 0 & e & f \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$
     where $a, b \in \mathbb{R}$ and $cf - ed \neq 0$. A direct computation shows that (16) is equivalent to
     $$\begin{cases} ae + bd &= 0, \\ be + ad &= 0, \\ af - bc &= 0, \\ bf - ac &= 0. \end{cases}$$
     If $(a, b) = (0, 0)$, we recover (f_1) after the change of orthonormal basis
     $$(\frac{ce_1 + ee_2}{\sqrt{c^2 + e^2}}, \frac{ee_1 - ce_2}{\sqrt{c^2 + e^2}}, e_3, e_4).$$

If $(a,b)\neq(0,0)$ then, since $\xi\neq 0$, we get $a=b$ or $a=-b$. If $a=b$, we recover (f_4) after the change of orthonormal basis

$$(\frac{ce_1+ee_2}{\sqrt{c^2+e^2}},\frac{-ee_1+ce_2}{\sqrt{c^2+e^2}},e_3,e_4).$$

If $a=-b$ we recover also (f_4) after the change of orthonormal basis

$$(\frac{ee_1-ce_2}{\sqrt{c^2+e^2}},\frac{ce_1+ee_2}{\sqrt{c^2+e^2}},e_4,e_3).$$

Second case: $\dim\ker\xi=3$ and $\dim F_1=1$. Then there exists an orthonormal basis (e_1,e_2,e_3,e_4) such that

$$M(A)=\begin{pmatrix}0&a&0&0\\-a&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\quad\text{and}\quad M(\xi)=\begin{pmatrix}0&0&0&b\\0&0&0&c\\0&0&0&d\\0&0&0&0\end{pmatrix},$$

where $a\in\mathbb{R}$ and $(b,c,d)\neq(0,0,0)$. A direct computation shows that (16) is equivalent to $ab=0$ and $ac=0$. If $a=0$ we recover (f_2) after the change of basis $(\frac{be_1+ce_2+de_3}{\sqrt{b^2+c^2+d^2}},f_1,f_2,e_4)$ where $(\frac{be_1+ce_2+de_3}{\sqrt{b^2+c^2+d^2}},f_1,f_2)$ is any orthonormal basis of $\mathrm{span}\{e_1,e_2,e_3\}$. If $b=c=0$ we recover (f_5).

- If $q=3$ then $\xi^2\neq 0$, $B=\ker\xi\oplus F_1\oplus F_2$ with

$$\dim F_2\leq\dim(\ker\xi\oplus F_1)\quad\text{and}\quad\dim F_2\leq\dim\ker\xi.$$

Hence $\dim\ker\xi=2$ and $\dim\ker F_1=\dim\ker F_2=1$ then there exists an orthonormal basis such that

$$M(A)=\begin{pmatrix}0&a&0&0\\-a&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},\quad M(\xi)=\begin{pmatrix}0&0&b&c\\0&0&d&e\\0&0&0&f\\0&0&0&0\end{pmatrix}\quad\text{and}$$

$$M(\xi^2)=\begin{pmatrix}0&0&0&fb\\0&0&0&df\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

where $f\neq 0$. This case is impossible since $e_4\in\ker A$ and $e_4\notin\ker\xi^2$ which is in contradiction with Remark 3. ■

The non abelian case Let B be a Riemannian flat non abelian Lie algebra of dimension n. According to Theorem 3.1, $\mathfrak{L}(B)$ and $\mathfrak{D}(B)$ are abelian and

$$B=\mathfrak{L}(B)\oplus\mathfrak{D}(B).$$

Moreover,

$$\mathrm{L}_a=\begin{cases}\mathrm{ad}_a & \text{if}\quad a\in\mathfrak{L}(B),\\ 0 & \text{if}\quad a\in\mathfrak{D}(B).\end{cases}$$

Since $\mathfrak{L}(B)$ is abelian and acts on $\mathfrak{D}(B)$ by skew-symmetric endomorphisms, there exists a family of non vanishing vectors $u_1, \ldots, u_r \in \mathfrak{L}(B)$ and an orthonormal basis $(f_1, \ldots, f_{2r})$ of $\mathfrak{D}(B)$ such that, for any $j = 1, \ldots, r$ and any $s \in \mathfrak{L}(B)$,

$$[s, f_{2j-1}] = \langle s, u_j \rangle f_{2j} \quad \text{and} \quad [s, f_{2j}] = -\langle s, u_j \rangle f_{2j-1}. \tag{20}$$

If F is an endomorphism of B, we put, for any $u \in B$, $F(u) = F_1(u) + F_2(u)$ where $F_1(u) \in \mathfrak{L}(B)$ and $F_2(u) \in \mathfrak{D}(B)$, and we denote by $\overline{F}_1 \in \mathrm{End}(\mathfrak{L}(B))$ and $\overline{F}_2 \in \mathrm{End}(\mathfrak{D}(B))$, respectively, the restriction of F_1 to $\mathfrak{L}(B)$ and the restriction of F_2 to $\mathfrak{D}(B)$.

Proposition 4.2. *With the notations and hypothesis above, $(\xi, D, 0, b_0)$ is admissible if and only if $\overline{D}_1 - \overline{\xi}_1$ and $\overline{D}_2 - \overline{\xi}_2$ are skew-symmetric and, for any $a, b \in \mathfrak{L}(B)$ and any $c \in \mathfrak{D}(B)$,*

$$\begin{aligned}
D_{1|\mathfrak{D}(B)} &= \xi_{1|\mathfrak{D}(B)} = 0, \quad (\xi_2 - D_2)_{|\mathfrak{L}(B)} = 0, && (21)\\
0 &= [D_2(a), b] + [a, D_2(b)], && (22)\\
D_2([a, c]) &= [D_1(a), c] + [a, D_2(c)], && (23)\\
\xi_2([a, c]) &= [a, \xi_2(c)], && (24)\\
[\overline{D}_1, \overline{\xi}_1] &= \overline{\xi}_1^{\,2}, && (25)\\
[\overline{D}_2, \overline{\xi}_2] &= \overline{\xi}_2^{\,2}, && (26)\\
[D_2, \xi_2](a) &= \xi_2^2(a) + \xi_2 \circ D_1(a) + [b_0, a]. && (27)
\end{aligned}$$

Moreover, if $(\xi, D, 0, b_0)$ is admissible then $\mathrm{tr}(D) = 0$.

Proof. Recall that $(\xi, D, 0, b_0)$ is admissible if and only if D is a derivation of B, $D - \xi$ is skew-symmetric and (ξ, D, b_0) satisfy (5)-(7).
Now D is a derivation and (ξ, D) satisfy (5) and (7) if and only if, for any $a, b \in \mathfrak{L}(B)$ and any $c, d \in \mathfrak{D}(B)$,

$$\begin{aligned}
0 &= [D_2(a), b] + [a, D_2(b)],\\
0 &= [D_1(c), d] + [c, D_1(d)],\\
D_2([a, c]) &= [D_1(a), c] + [a, D_2(c)], \quad D_1([a, c]) = 0,\\
[a, \xi_2(b)] &= [b, \xi_2(a)],\\
\xi_2([a, c]) &= [a, \xi_2(c)],\ \xi_1([a, c]) = 0,\\
[a, \xi_2(b)] &= [a, D_2(b)], \qquad (*)\\
[a, \xi(c)] - \xi([a, c]) &= [D(a), c] + [a, D(c)] - D([a, c]),\\
[D_1(c), d] &= 0.
\end{aligned}$$

We get obviously that $D_{1|\mathfrak{D}(B)} = \xi_{1|\mathfrak{D}(B)} = 0$. Moreover, from $(*)$ we deduce that, for any $b \in \mathfrak{L}(B)$, $\xi_2(b) - D_2(b)$ is a central element and since the center of B is contained in $\mathfrak{L}(B)$, we deduce that $(\xi_2 - D_2)_{|\mathfrak{L}(B)} = 0$. On the other hand, one can see easily that if $D_{1|\mathfrak{D}(B)} = \xi_{1|\mathfrak{D}(B)} = 0$ and $(\xi_2 - D_2)_{|\mathfrak{L}(B)} = 0$ then $(\xi, D, 0, b_0)$ satisfies (6) if and only if

$$[D_1, \xi_1] = \xi_1^2 \quad \text{and} \quad [D_2, \xi_2] = \xi_2^2 + \xi_2 D_1 - \mathrm{R}_{b_0}.$$

By evaluating the second equation respectively on $\mathfrak{L}(B)$ and $\mathfrak{D}(B)$ one can conclude. Moreover, if $(\xi, D, 0, b_0)$ is admissible then one can deduce easily from what above that $\mathrm{tr}(D) = \mathrm{tr}(\overline{\xi}_1) + \mathrm{tr}(\overline{\xi}_2) = 0$ ($\overline{\xi}_1$ and $\overline{\xi}_2$ are nilpotent by virtue of (25) and (26)). ■

Let us use this proposition to find $(D, \xi, 0, b_0)$ admissible when $\dim B = 3$ or 4.

Proposition 4.3. *Let B be a Riemannian flat non abelian Lie algebra of dimension 3. Then there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of B such that $\mathfrak{L}(B) = \mathrm{span}\{e_1\}$, $\mathfrak{D}(B) = \mathrm{span}\{e_2, e_3\}$ and*

$$[e_1, e_2] = \lambda e_3 \quad \textit{and} \quad [e_1, e_3] = -\lambda e_2,$$

where $\lambda > 0$. Moreover, $(D, \xi, 0, b_0)$ is admissible if and only if

$$M(\xi) = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & 0 \\ b & 0 & 0 \end{pmatrix}, \ M(D) = \begin{pmatrix} 0 & 0 & 0 \\ a & 0 & c \\ b & -c & 0 \end{pmatrix} \quad \textit{and} \quad b_0 = b_1 e_1 + \frac{ca}{\lambda} e_2 + \frac{cb}{\lambda} e_3.$$

Proof. The existence of the orthonormal basis $\mathbb{B} = \{e_1, e_2, e_3\}$ in which the Lie bracket is given by the relations above is a consequence of Theorem 3.1.
Suppose now that $(D, \xi, 0, b_0)$ is admissible. Since $\mathfrak{L}(B)$ is a line, the equation (22) holds and we deduce from (25) and the fact $\overline{\xi}_1 - \overline{D}_1$ is skew-symmetric that $\overline{\xi}_1 = \overline{D}_1 = 0$ and hence $\xi_1 = D_1 = 0$. So D_2 and ξ_2 satisfy the same equation, namely (24). This equation is equivalent to

$$\begin{aligned} \xi_2([e_1, e_2]) &= [e_1, \xi_2(e_2)], \\ \lambda \xi_2(e_3) &= \lambda \langle \xi_2(e_2), e_2 \rangle e_3 - \lambda \langle \xi_2(e_2), e_3 \rangle e_2, \\ \xi_2([e_1, e_3]) &= [e_1, \xi_2(e_3)], \\ -\lambda \xi_2(e_2) &= \lambda \langle \xi_2(e_3), e_2 \rangle e_3 - \lambda \langle \xi_2(e_3), e_3 \rangle e_2. \end{aligned}$$

These equations are equivalent to

$$\langle \xi_2(e_3), e_2 \rangle = -\langle \xi_2(e_2), e_3 \rangle \quad \text{and} \quad \langle \xi_2(e_3), e_3 \rangle = \langle \xi_2(e_2), e_2 \rangle.$$

Thus, since $D - \xi$ is skew-symmetric, the matrices of D and ξ in the basis $\mathbb{B}$ are of the following form

$$M(\xi) = \begin{pmatrix} 0 & 0 & 0 \\ a & e & d \\ b & -d & e \end{pmatrix} \quad \text{and} \quad M(D) = \begin{pmatrix} 0 & 0 & 0 \\ a & e & c \\ b & -c & e \end{pmatrix}.$$

On the other hand, according to Proposition 4.1 1, the equation (26) and the fact that $\overline{\xi}_2 - \overline{D}_2$ is skew-symmetric is equivalent to

$$\big(M(\overline{\xi}_2){=}0 \text{ and } M(\overline{D}_2)^*{=} - M(\overline{D}_2)\big) \text{ or } \big(M(\overline{\xi}_2){=}M(\overline{D}_2) \text{ and } M(\overline{\xi}_2)^2 = 0\big). \quad (*)$$

Since $M(\overline{\xi}_2) = \begin{pmatrix} e & d \\ -d & e \end{pmatrix}$ and $M(\overline{D}_2) = \begin{pmatrix} e & c \\ -c & e \end{pmatrix}$. We deduce that $(*)$ is equivalent to $e = d = 0$. Now, the equation (27) is equivalent to

$$b_0 = b_1 e_1 + \frac{ca}{\lambda} e_2 + \frac{cb}{\lambda} e_3.$$

This achieves the proof. ■

Proposition 4.4. *Let B be a Riemannian flat non abelian Lie algebra of dimension 4. Then there exists an orthonormal basis $\{e_1, e_2, f_1, f_2\}$ of B such that $\mathfrak{L}(B) = \mathrm{span}\{e_1, e_2\}$, $\mathfrak{D}(B) = \mathrm{span}\{f_1, f_2\}$ and*

$$[e_1, f_1] = \lambda f_2,\ [e_1, f_2] = -\lambda f_1 \quad \text{and} \quad [e_2, f_1] = [e_2, f_2] = 0,$$

where $\lambda > 0$. Moreover, $(\xi, D, 0, b_0)$ is admissible if and only if

$$M(D) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & 0 & 0 & d \\ c & 0 & -d & 0 \end{pmatrix},\ M(\xi) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ b & 0 & 0 & 0 \\ c & 0 & 0 & 0 \end{pmatrix},$$

$$b_0 = b_1 e_1 + b_2 e_2 + \frac{bd}{\lambda} f_1 + \frac{cd}{\lambda} f_2.$$

Proof. According to Theorem 3.1 there exists a basis $\{e'_1, e'_2, f_1, f_2\}$ and $(\lambda_1, \lambda_2) \neq (0, 0)$ such that

$$[e'_i, f_1] = \lambda_i f_2 \quad \text{and} \quad [e'_i, f_2] = -\lambda_i f_1.$$

Put $\lambda = \sqrt{\lambda_1^2 + \lambda_2^2}$, $e_1 = \lambda^{-1}(\lambda_1 e'_1 + \lambda_2 e'_2)$ and $e_2 = \lambda^{-1}(\lambda_2 e'_1 - \lambda_1 e'_2)$. The Lie brackets of the elements of the orthonormal basis $\mathbb{B} = \{e_1, e_2, f_1, f_2\}$ satisfy the relations above.

Suppose now that $(D, \xi, 0, b_0)$ is admissible. The equation (22) is equivalent to

$$0 = [D_2(e_1), e_2] + [e_1, D_2(e_2)] = \lambda\langle D_2(e_2), f_1\rangle f_2 - \lambda\langle D_2(e_2), f_2\rangle f_1.$$

This is equivalent to existence of $(c, d) \in \mathbb{R}^2$ such that

$$D_2(e_1) = cf_1 + df_2 \quad \text{and} \quad D_2(e_2) = 0.$$

The equation (23) is equivalent to

$$\begin{aligned} \lambda D_2(f_2) &= [D_1(e_1), f_1] + [e_1, D_2(f_1)] \\ &= \lambda\langle D_1(e_1), e_1\rangle f_2 + \lambda\langle D_2(f_1), f_1\rangle f_2 - \lambda\langle D_2(f_1), f_2\rangle f_1, \\ 0 &= [D_1(e_2), f_1] + [e_2, D_2(f_1)] \\ &= \lambda\langle D_1(e_2), e_1\rangle f_2, \\ -\lambda D_2(f_1) &= [D_1(e_1), f_2] + [e_1, D_2(f_2)] \\ &= -\lambda\langle D_1(e_1), e_1\rangle f_1 + \lambda\langle D_2(f_2), f_1\rangle f_2 - \lambda\langle D_2(f_2), f_2\rangle f_1, \\ 0 &= [D_1(e_2), f_2] + [e_2, D_2(f_2)] \\ &= -\lambda\langle D_1(e_2), e_1\rangle f_1. \end{aligned}$$

This is equivalent to

$$\begin{aligned}\langle D_2(f_1), f_2\rangle &= -\langle D_2(f_2), f_1\rangle,\\ \langle D_2(f_2), f_2\rangle &= \langle D_1(e_1), e_1\rangle + \langle D_2(f_1), f_1\rangle,\\ 0 &= \langle D_1(e_2), e_1\rangle,\\ \langle D_2(f_1), f_1\rangle &= \langle D_1(e_1), e_1\rangle + \langle D_2(f_2), f_2\rangle.\end{aligned}$$

Which is equivalent to the existence of $(a, b) \in \mathbb{R}^2$ such that

$$\begin{aligned}\langle D_2(f_1), f_2\rangle &= -\langle D_2(f_2), f_1\rangle,\ \langle D_2(f_2), f_2\rangle = \langle D_2(f_1), f_1\rangle,\\ D_1(e_1) &= ae_2 \quad \text{and} \quad D_1(e_2) = be_2.\end{aligned}$$

The equation (24) is equivalent to

$$\begin{aligned}\lambda\xi_2(f_2) &= [e_1, \xi_2(f_1)] = \lambda\langle\xi_2(f_1), f_1\rangle f_2 - \lambda\langle\xi_2(f_1), f_2\rangle f_1,\\ -\lambda\xi_2(f_1) &= [e_1, \xi_2(f_2)] = \lambda\langle\xi_2(f_2), f_1\rangle f_2 - \lambda\langle\xi_2(f_2), f_2\rangle f_1.\end{aligned}$$

This is equivalent to

$$\langle\xi_2(f_2), f_1\rangle = -\langle\xi_2(f_1), f_2\rangle \quad \text{and} \quad \langle\xi_2(f_2), f_2\rangle = \langle\xi_2(f_1), f_1\rangle.$$

In conclusion, and since $D - \xi$ is skew-symmetric, we get that the matrices of D and ξ in the basis $\mathbb{B}$ have the following form

$$M(D) = \begin{pmatrix} 0 & 0 & 0 & 0\\ a & b & 0 & 0\\ c & 0 & \alpha & -\beta\\ d & 0 & \beta & \alpha\end{pmatrix} \quad \text{and} \quad M(\xi) = \begin{pmatrix} 0 & e & 0 & 0\\ a-e & b & 0 & 0\\ c & 0 & \alpha & -f-\beta\\ d & 0 & \beta+f & \alpha\end{pmatrix}.$$

On the other hand, according to Proposition 4.1 1, the equation (26) and the fact that $\overline{\xi}_2 - \overline{D}_2$ is skew-symmetric is equivalent to

$$\big(M(\overline{\xi}_2){=}0 \text{ and } M(\overline{D}_2)^*{=}-M(\overline{D}_2)\big) \text{ or } \big(M(\overline{\xi}_2){=}M(\overline{D}_2) \text{ and } M(\overline{\xi}_2)^2 = 0\big).(*)$$

Or $M(\overline{\xi}_2) = \begin{pmatrix} \alpha & -f-\beta\\ f+\beta & \alpha\end{pmatrix}$ and $M(\overline{D}_2) = \begin{pmatrix} \alpha & -\beta\\ \beta & \alpha\end{pmatrix}$. We deduce that $(*)$ is equivalent to $\alpha = f + \beta = 0$. In a similar way, the equation (25) and the fact that $\overline{\xi}_1 - \overline{D}_1$ is skew-symmetric is equivalent to

$$\big(M(\overline{\xi}_1) = 0 \quad \text{and} \quad M(\overline{D}_1)^* = -M(\overline{D}_1)\big) \text{ or } \big(M(\overline{\xi}_1) = M(\overline{D}_1) \quad \text{and} \quad M(\overline{\xi}_1)^2 = 0\big). \qquad (**)$$

Or $M(\overline{\xi}_1) = \begin{pmatrix} 0 & e\\ a-e & b\end{pmatrix}$ and $M(\overline{D}_1) = \begin{pmatrix} 0 & 0\\ a & b\end{pmatrix}$. A careful checking shows that $(**)$ is equivalent to $e = b = 0$. In conclusion

$$M(D) = \begin{pmatrix} 0 & 0 & 0 & 0\\ a & 0 & 0 & 0\\ c & 0 & 0 & f\\ d & 0 & -f & 0\end{pmatrix} \quad \text{and} \quad M(\xi) = \begin{pmatrix} 0 & 0 & 0 & 0\\ a & 0 & 0 & 0\\ c & 0 & 0 & 0\\ d & 0 & 0 & 0\end{pmatrix}.$$

Finally, the equation (27) is equivalent to $b_0 = b_1e_1 + b_2e_2 + \frac{fc}{\lambda}f_1 + \frac{fd}{\lambda}f_2$. ∎

By using Theorem 2.1 and Propositions 4.1, 4.3 and 4.4, let us give the list of non abelian Lorentzian flat Lie algebras with degenerate center up to dimension six. We proceed as follows:

1. We pick an admissible solution $(\xi, D, 0, b_0)$ found in Propositions 4.1, 4.3 and 4.4,
2. by using (8), we compute the Lie brackets and we make an appropriate change of basis to get a simple form of these brackets,
3. finally, we give the metric in the coordinates $\{z, \bar{z}, x_1, \ldots, x_n\}$ associated to the new basis which we continue to denote by $\{z, \bar{z}, e_1, \ldots, e_n\}$ $(n = 1, \ldots, 4)$. We change the parameters if it is necessary to get a simple form of the brackets and the metric.

1. **Dimension 3:**

   (a) • **Admissible solution:** $\xi = D = 0,\ b_0 \in \mathbb{R} \setminus \{0\}$.
   • **New basis:** $(|b_0|z, \bar{z}, -\epsilon e_1)$ where ϵ is the sign of b_0.
   • **Final non vanishing brackets and metric:**

   $$[\bar{z}, e_1] = z, \quad \langle\ ,\ \rangle = 2\alpha dz d\bar{z} + (dx_1)^2, \alpha > 0.$$

   The Lie algebra obtained is the 3-dimensional Heisenberg Lie algebra.

2. **Dimension 4:**

   (a) • **Admissible solution:** $\xi = D = 0,\ b_0 = (b_1, b_2)$ and $b_1 \neq 0$.
   • **New basis:** $(|b_1|z, \bar{z}, -\epsilon e_1, e_2 - b_2 b_1^{-1} e_1)$ where ϵ is the sign of b_1.
   • **Final non vanishing brackets and metric:**

   $$[\bar{z}, e_1] = z, \quad \langle\ ,\ \rangle = 2\alpha dz d\bar{z} + 2a dx_1 dx_2 + (dx_1)^2 + (1 + a^2)(dx_2)^2,$$

   $\alpha > 0, a \in \mathbb{R}$.
   The Lie algebra obtained is a trivial extension of the 3-dimensional Heisenberg Lie algebra.

   (b) • **Admissible solution:** Proposition 4.1 1, first case with $b_0 = (b_1, b_2)$.
   • **New basis:** $(a^2 z, \bar{z}, a e_1 - b_2 z, e_2)$.
   • **Final non vanishing brackets and metric:**

   $$[\bar{z}, e_1] = az,\ [\bar{z}, e_2] = e_1,\ [e_1, e_2] = -z,$$
   $$\langle\ ,\ \rangle = 2\alpha dz d\bar{z} + 2\beta d\bar{z} dx_1 + \alpha (dx_1)^2 + (dx_2)^2,$$

   $a, \beta \in \mathbb{R},\ \alpha > 0$.
   The Lie algebra obtained is 3-nilpotent.

   (c) • **Admissible solution:** Proposition 4.1 1, second case with $b_0 = (b_1, b_2)$.

- **New basis:** $(z, \lambda^{-1}\bar{z}, \lambda e_1 - b_2 z, -\lambda e_2 - b_1 z)$.
- **Final non vanishing brackets and metric:**

$$[\bar{z}, e_1] = e_2,\ [\bar{z}, e_2] = -e_1,$$
$$\langle\ ,\ \rangle = 2\alpha^{-1} dz d\bar{z} + 2d\bar{z}(\beta dx_1 + \gamma dx_2) + \alpha^2 \left((dx_1)^2 + (dx_2)^2\right),$$

$\beta, \gamma \in \mathbb{R},\ \alpha \neq 0$.
The Lie algebra obtained is 2-solvable and unimodular.

3. **Dimension 5:**

(a)
- **Admissible solution:** $\xi = D = 0,\ b_0 = (b_1, b_2, b_3)$ with $b_1 \neq 0$.
- **New basis:** $(|b_1|z, \bar{z}, -\epsilon e_1, e_2 - b_2 b_1^{-1} e_1, e_3 - b_3 b_1^{-1} e_1)$ where ϵ is the sign of b_1.
- **Final non vanishing brackets and metric:**

$$\begin{aligned} [\bar{z}, e_1] &= z, \\ \langle\ ,\ \rangle &= 2\alpha dz d\bar{z} + 2a dx_1 dx_2 + 2b dx_1 dx_3 + 2ab dx_2 dx_3 \\ &\quad + (dx_1)^2 + (1 + a^2)(dx_2)^2 + (1 + b^2)(dx_3)^2, \end{aligned}$$

$a, b \in \mathbb{R},\ \alpha > 0$.
The Lie algebra obtained is a trivial extension of the 3-dimensional Heisenberg Lie algebra.

(b)
- **Admissible solution:** Proposition 4.1 2, first case with $b_0 = (b_1, b_2, b_3)$.
- **New basis:** $(a^2 z, \bar{z}, a e_1 - b_3 z, e_2, e_3)$.
- **Final non vanishing brackets and metric:**

$$[\bar{z}, e_1] = az,\ [\bar{z}, e_2] = bz,\ [\bar{z}, e_3] = e_1,\ [e_1, e_3] = -z,$$
$$\langle\ ,\ \rangle = 2\alpha dz d\bar{z} + 2\beta d\bar{z} dx_1 + \alpha (dx_1)^2 + (dx_2)^2 + (dx_3)^2,$$

$\alpha > 0,\ a, b, \beta, \in \mathbb{R}$.
The Lie algebra obtained is 3-nilpotent.

(c)
- **Admissible solution:** Proposition 4.1 2, second case with $b_0 = (b_1, b_2, b_3)$.
- **New basis:** $(z, \lambda^{-1}\bar{z}, \lambda e_1 - b_2 z, -\lambda e_2 - b_1 z, e_3)$.
- **Final non vanishing brackets and metric:**

$$[\bar{z}, e_1] = e_2,\ [\bar{z}, e_2] = -e_1,\ [\bar{z}, e_3] = az,$$
$$\langle\ ,\ \rangle = 2\alpha^{-1} dz d\bar{z} + 2d\bar{z}(\beta dx_1 + \gamma dx_2) + \alpha^2 \left((dx_1)^2 + (dx_2)^2\right) + (dx_3)^2,$$

$\alpha > 0,\ a, \beta, \gamma \in \mathbb{R}$.
The Lie algebra obtained is 2-solvable and unimodular.

(d)
- **Admissible solution:** Proposition 4.3.
- **New basis:** $(z, \bar{z}, \lambda^{-1} e_1, \lambda e_2 - bz, \lambda e_3 + az)$.

- **Non vanishing brackets and metric:**

$$\begin{aligned}
[\bar{z},e_1] &= ae_2+be_3+cz,\ [\bar{z},e_2]=de_3,\ [\bar{z},e_3]=-de_2\\
[e_1,e_2] &= e_3,\ [e_1,e_3]=-e_2,\\
\langle\ ,\ \rangle &= 2dzd\bar{z}+2\alpha d\bar{z}(-bdx_2+adx_3)+\alpha^{-1}(dx_1)^2\\
&\quad +\alpha\left((dx_2)^2+(dx_3)^2\right),
\end{aligned}$$

$(a,b,c,d)\neq 0,\ \alpha>0$.
The Lie algebra obtained is 2-solvable and unimodular.

4. **Dimension 6:**

(a)
- **Admissible solution:** $\xi=D=0,\ b_0=(b_1,b_2,b_3,b_4)$ with $b_1\neq 0$.
- **New basis:** $(|b_1|z,\bar{z},-\epsilon e_1,e_2-b_2b_1^{-1}e_1,e_3-b_3b_1^{-1}e_1,e_4-b_4b_1^{-1}e_1)$ where ϵ is the sign of b_1.
- **Final non vanishing brackets and metric:**

$$\begin{aligned}
[\bar{z},e_1] &= z,\\
\langle\ ,\ \rangle &= 2\alpha dzd\bar{z}+2adx_1dx_2+2bdx_1dx_3+2cdx_1dx_4\\
&\quad +2abdx_2dx_3+2acdx_2dx_4+2bcdx_3dx_4+(dx_1)^2\\
&\quad +(1+a^2)(dx_2)^2+(1+b^2)(dx_3)^2+(1+c^2)(dx_4)^2,
\end{aligned}$$

$a,b,c\in\mathbb{R},\ \alpha>0$.
The Lie algebra obtained is a trivial extension of the 3-dimensional Heisenberg Lie algebra.

(b)
- **Admissible solution:** Proposition 4.1 3, (f_1) with $b_0=(b_1,b_2,b_3,b_4)$.
- **New basis:** $(z,\bar{z},ae_1-b_3z,be_1+ce_2-b_4z,e_3,e_4)$. Put $(b,c)=\rho(\cos\omega,\sin\omega)$. The condition $ac\neq 0$ is equivalent to $\omega\neq k\pi, k\in\mathbb{Z}$ and $a\neq 0$.
- **Non vanishing brackets and metric:**

$$\begin{aligned}
&[\bar{z},e_1]=az,\ ;[\bar{z},e_2]=bz,\ [\bar{z},e_3]=e_1,\\
&[\bar{z},e_4]=e_2,\ [e_1,e_3]=-c^2z,\ [e_2,e_4]=-\rho^2z,\\
&[e_1,e_4]=-c\rho\cos(\omega)z,\ [e_2,e_3]=-c\rho\cos(\omega)z,\\
&\langle\ ,\ \rangle=2dzd\bar{z}+2d\bar{z}\left(\beta dx_1+\gamma dx_2\right)+2c\rho\cos(\omega)dx_1dx_2\\
&+c^2(dx_1)^2+\rho^2(dx_2)^2+(dx_3)^2+(dx_4)^2,
\end{aligned}$$

$a,b,c,\beta,\gamma\in\mathbb{R},\ c\neq 0,\ \rho>0,\ \omega\neq k\pi,k\in\mathbb{Z}$.
The Lie algebra obtained is 3-nilpotent.

(c)
- **Admissible solution:** Proposition 4.1 3, (f_2) with $b_0=(b_1,b_2,b_3,b_4)$ and $a\neq 0$.
- **New basis:** $(a^2z,\bar{z},ae_1-b_4z,e_2,e_3,e_4)$.

- **Final non vanishing brackets and metric:**

$$\begin{aligned}&[\bar z, e_1] = az,\ ;[\bar z, e_2] = bz,\ [\bar z, e_3] = cz,\\ &[\bar z, e_4] = e_1,\ [e_1, e_4] = -z, ,\\ &\langle\ ,\ \rangle = 2\alpha dz d\bar z + 2\beta d\bar z dx_1\\ &+\alpha(dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2,\end{aligned}$$

$\alpha > 0,\ a, b, c, \beta \in \mathbb{R}$.
The Lie algebra obtained is 3-nilpotent.

(d)
- **Admissible solution:** Proposition 4.1 3, (f_3) $b_0 = (b_1, b_2, b_3, b_4)$, $a \neq 0$ and $b \neq 0$.
- **New basis:** $(z, a^{-1}\bar z, ae_1 - b_2 z, -ae_1 - b_1 z, be_3 - b_4 z, -be_4 - b_3 z)$.
- **Final non vanishing brackets and metric:**

$$\begin{aligned}&[\bar z, e_1] = e_2,\ [\bar z, e_2] = -e_1,\ [\bar z, e_3] = ae_4,\ [\bar z, e_4] = -ae_3,\\ &\langle\ ,\ \rangle = 2\alpha^{-1} dz d\bar z + 2d\bar z\left(\beta dx_1 + \gamma dx_2 + \mu dx_3 + \nu dx_4\right)\\ &+\alpha^2\left((dx_1)^2 + (dx_2)^2 + a^2(dx_3)^2 + a^2(dx_4)^2\right),\end{aligned}$$

$\alpha \neq 0,\ a \neq 0,\ \beta, \gamma, \mu, \nu \in \mathbb{R}$.
The Lie algebra obtained is 2-solvable and unimodular.

(e)
- **Admissible solution:** Proposition 4.1 3, (f_3) $b_0 = (b_1, b_2, b_3, b_4)$, $a \neq 0$ and $b = 0$.
- **New basis:** $(z, a^{-1}\bar z, ae_1 - b_2 z, -ae_2 - b_1 z, e_3, e_4)$.
- **Final non vanishing brackets and metric:**

$$\begin{aligned}&[\bar z, e_1] = e_2,\ [\bar z, e_2] = -e_1,\ [\bar z, e_3] = cz,\ [\bar z, e_4] = dz,\\ &\langle\ ,\ \rangle = 2\alpha^{-1} dz d\bar z + 2d\bar z\left(\beta dx_1 + \gamma dx_2\right)\\ &+\alpha^2\left((dx_1)^2 + (dx_2)^2\right) + (dx_3)^2 + (dx_4)^2,\end{aligned}$$

$\alpha \neq 0,\ c, d, \beta, \gamma \in \mathbb{R}$.
The Lie algebra obtained is 2-solvable and unimodular.

(f)
- **Admissible solution:** Proposition 4.1 3, (f_4) with $b_0 = (b_1, b_2, b_3, b_4)$.
- **New basis:** $(a^3 z, a^{-1}\bar z, ae_1 - b_2 z, -ae_2 - b_1 z, be_2 + ae_3 - b_4 z, be_1 - ae_4 - b_3 z)$.
- **Final non vanishing brackets and metric:**

$$\begin{aligned}&[\bar z, e_1] = e_2,\ \ [\bar z, e_2] = -e_1,\ [\bar z, e_3] = ae_1 + e_4,\\ &[\bar z, e_4] = ae_2 - e_3,\ [e_1, e_3] = -az,\ [e_2, e_4] = -az,\ [e_3, e_4] = 2a^2 z,\\ &\langle\ ,\ \rangle = 2dz d\bar z + 2d\bar z\left(\alpha dx_1 + \beta dx_2 + \gamma dx_3 + \mu dx_4\right)\\ &-2a dx_2 dx_3 + 2a dx_1 dx_4 + (dx_1)^2 + (dx_2)^2\\ &+(1 + a^2)\left((dx_3)^2 + (dx_4)^2\right),\end{aligned}$$

$a \neq 0,\ \alpha, \beta, \gamma, \mu \in \mathbb{R}$.
The Lie algebra obtained is 3-solvable and unimodular.

(g)
- **Admissible solution:** Proposition 4.1 3, (f_5) with $b_0 = (b_1, b_2, b_3, b_4)$.

- **New basis:**

$$(ab^2z, a^{-1}\bar{z}, ae_1 - b_2z, -ae_2 - b_1z, be_3 - b_4z, ae_4).$$

- **Final non vanishing brackets and metric:**

$$\begin{array}{l}[\bar{z}, e_1] = e_2,\ ; [\bar{z}, e_2] = -e_1,\ [\bar{z}, e_3] = az,\\ [\bar{z}, e_4] = e_3,\ [e_3, e_4] = -z,\\ \langle\ ,\ \rangle = 2\alpha dzd\bar{z} + 2d\bar{z}\,(\gamma dx_1 + \mu dx_2 + \nu dx_3)\\ +\beta(dx_1)^2 + \beta(dx_2)^2 + \alpha(dx_3)^2 + \beta(dx_4)^2,\end{array}$$

$\alpha, \beta > 0,\ a, \gamma, \mu, \nu \in \mathbb{R}$.
The Lie algebra obtained is 2-solvable and unimodular.

(h)
- **Admissible solution:** Proposition 4.4.
- **New basis:** $(z, \bar{z}, \lambda^{-1}e_1, e_2, f_1 - \frac{c}{\lambda}z, f_2 + \frac{b}{\lambda}z)$.
- **Final non vanishing brackets and metric:**

$$\begin{array}{l}[\bar{z}, e_1] = ae_2 + bf_1 + cf_2 + dz,\ [\bar{z}, e_2] = ez,\ [\bar{z}, f_1] = gf_2,\\ [\bar{z}, f_2] = -gf_1,\ [e_1, e_2] = az,\ [e_1, f_1] = f_2,\ [e_1, f_2] = -f_1,\\ \langle\ ,\ \rangle = 2dzd\bar{z} + 2d\bar{z}\,(-cdx_3 + bdx_4)\\ +\alpha^2(dx_1)^2 + (dx_2)^2 + (dx_3)^2 + (dx_4)^2,\end{array}$$

$\alpha > 0,\ a, b, c, d, e, g \in \mathbb{R}$.
The Lie algebra obtained is 2-solvable and unimodular.

Remark 4. *Two flat Lorentzian flat Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are called isomorphic if there exists an isomorphism of Lie algebras between $\mathfrak{g}_1$ and $\mathfrak{g}_2$ which is also an isometry. To complete the study of Lorentzian flat Lie algebras with degenerate center up to dimension 6, one needs to study the isomorphism classes of the Lorentzian flat Lie algebras listed above. This can be done in two steps:*

1. *in a given dimension, identify in the list the Lie algebras which are isomorphic as Lie algebras,*

2. *from the first step, one will get a list of non isomorphic Lie algebras and each one is endowed with a family of Lorentzian flat metrics and one must distinguish in this family which metrics are isomorphic.*

The first step is easy and one can deduce easily from the list above all the models of the Lie algebras up to dimension 6 which carry Lorentzian flat metrics. However, the second step seems difficult and needs material which is beyond the purpose of this paper.

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Malika Ait Ben Haddou
Faculté des Sciences Meknes
Meknes Morocco
maitbenha@gmail.com

Mohamed Boucetta
Université Cadi-Ayyad
Faculté des Sciences Gueliz
Marrakech (Morocco)
mboucetta2@yahoo.fr

Hicham Lebzioui
Faculté des Sciences Meknes
Meknes Morocco
hlebzioui@gmail.com

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