Quasitriangular Hom-Lie Bialgebras

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Abstract. Recently certain twisted Lie algebras, so-called Hom-Lie algebras, and their duals have been considered in the literature. In this paper we investigate boundary and quasi-triangular Hom-Lie bialgebras further. In particular, we characterize the quasi-triangularity of boundary Hom-Lie bialgebras in terms of both a certain Hom-Lie algebra morphism and a certain Hom-Lie coalgebra morphism. We also give a necessary and sufficient condition for a given Hom-Lie algebra and a given 2-tensor to admit a coboundary Hom-Lie bialgebra structure. Finally, we generalize the Drinfeld double of a Lie bialgebra to Hom-Lie bialgebras and discuss the dual codouble.

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Introduction

As generalizations of Lie algebras, Hom-Lie algebras were motivated by applications to physics and to deformations of Lie algebras, especially Lie algebras of vector fields. The notion of Hom-Lie algebras was firstly introduced by Hartwig, Larsson and Silvestrov in [10] to describe the structure of certain $q$-deformations of the Witt and the Virasoro algebras. Indeed, Hom-Lie algebras are different from Lie algebras as the Jacobi identity is replaced by a twisted form using a morphism. This twisted Jacobi identity is called Hom-Jacobi identity given by

$$\left[\alpha(x), [y, z]\right] + \left[\alpha(y), [z, x]\right] + \left[\alpha(z), [x, y]\right] = 0.$$ 

Recently, Hom-Lie structures have been studied extensively in a series of papers [1, 2, 3, 11, 12, 13, 17, 21, 23, 24, 25] by many scholars, including Hom-Lie bialgebras, quasi-Hom-Lie algebras, Hom-Lie superalgebras, Hom-Lie color algebras, Hom-Lie admissible Hom-algebras, Hom-Nambu-Lie algebras and so on.

The twisting of parts of the defining identities was transferred to other algebraic structures. In this way many Hom-structures were introduced, such
as Hom-associative algebras, Hom-Hopf algebras, Hom-alternative algebras, Hom-Jordan algebras, Hom-Poisson algebras, Hom-Leibniz algebras, infinitesimal Hom-bialgebras, Hom-power associative algebras, quasi-triangular Hom-bialgebras in [6, 7, 8, 9, 14, 15, 16, 22, 23].

In [23] Yau generalized the Yang-Baxter equation (YBE) to a Hom-type identity, the so-called Hom-Yang-Baxter equation (HYBE). The HYBE states

\[(\alpha \otimes B) \circ (B \otimes \alpha) \circ (\alpha \otimes B) = (B \otimes \alpha) \circ (\alpha \otimes B) \circ (B \otimes \alpha),\]

where \(\alpha\) is an endomorphism of the vector space \(V\), and \(B : V^\otimes 2 \rightarrow V^\otimes 2\) is a bilinear map that commutes with \(\alpha^{\otimes 2}\). Meanwhile, Yau defined the classical Hom-Yang-Baxter equation (abbreviated to CHYBE) in the same manner and studied Hom-Lie bialgebras in [25]. In fact, the quasi-element of quasi-triangular Hom-Lie bialgebras is a solution of CHYBE.

In [4], Drinfel’d showed that a Lie algebra \(L\) with a comultiplication is a Lie bialgebra if and only if the double space \(D(L) = L^* \oplus L\) is a Lie algebra. Majid introduced the classical double Lie bialgebra and proved that it is a quasi-triangular Lie bialgebra in [18].

Motivated by these results, we prove related results for Hom-Lie bialgebras. This paper is organized as follows. In Section 1, we recall some basic definitions for Hom-Lie (co)algebras. In Section 2, we recall some concepts and results about Hom-Lie bialgebras and show that Hom-Lie bialgebras are self-dual. Meanwhile, we investigate boundary and quasi-triangular Hom-Lie bialgebras further. We also give a necessary and sufficient condition for a given Hom-Lie algebra and a given 2-tensor to admit a coboundary Hom-Lie bialgebra structure. In Section 3 we introduce the concept of a double Hom-Lie bialgebra, which generalizes double Lie bialgebras in [18], and prove that the double is indeed a quasi-triangular Hom-Lie bialgebra. As an immediate application, by example, we investigate the quasi-triangular structure on the double Hom-Lie bialgebra \(D(sl(2)_\alpha)\). Finally, we discuss the co-quasi-triangular structure on the codouble Hom-Lie bialgebra \(D(L)^*\).

Throughout this paper, let \(k\) be a field of characteristic zero. Unless otherwise specified, vector spaces, algebras, linearity, modules and \(\otimes\) are all meant over \(k\). Sum symbols are always omitted and we write \(\Delta(x) = x_1 \otimes x_2\) in which \(\Delta\) is a comultiplication. Let \(\xi\) be the cyclic permutation \((1\ 2\ 3)\). Then we denote the sum over id, \(\xi\) and \(\xi^2\) applied to a 3-tensor by the symbol \(\odot\). Namely, we denote the Hom-Jacobi identity by \(\odot [\alpha(x), [y, z]] = 0\) in place of

\[[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0.\]

1. Preliminaries

In this section we recall some concepts and notations that will be useful in the rest of the paper.

**Definition 1.1.** A multiplicative Hom-Lie algebra is a triple \((L, [\cdot, \cdot, \cdot], \alpha)\) consisting of a vector space \(L\), a linear map \([\cdot, \cdot, \cdot] : L^\otimes 3 \rightarrow L\), and a linear endomorphism \(\alpha : L \rightarrow L\) satisfying the following conditions:
(1.1.1) \([x, y] + [y, x] = 0\) (anti-symmetry),
(1.1.2) \(\circ [\alpha(x), [y, z]] = 0\) (Hom-Jacobi identity),
(1.1.3) \(\alpha[x, y] = [\alpha(x), \alpha(y)]\) (multiplicativity),

for all \(x, y, z \in L\).

For convenience we will use in this paper the term Hom-Lie algebra instead of multiplicative Hom-Lie algebra. This should not lead to any confusion as we only consider the latter. A Hom-Lie algebra \(L\) with twist \(\alpha\) is called involutive if \(\alpha^2 = \text{id}_L\).

A subspace \(M\) is a sub-Hom-Lie algebra of \(L\) if \(M\) is also a Hom-Lie algebra with the restriction maps \([-,-]|_M : M \otimes M \to M, \alpha|_M : M \to M\). A morphism of Hom-Lie algebras \(f : (L, [-,-], \alpha) \to (L', [-,-]', \alpha')\) is a linear map such that \(\alpha' \circ f = f \circ \alpha\) and \(f([-,-]) = [-,-]' \circ f^\otimes 2\).

For every Lie algebra \((L, [-,-])\), we can construct a Hom-Lie algebra \(L_\alpha := (L, [-,-], \alpha := \alpha \circ [-,-], \alpha)\) via twisting with any Lie algebra endomorphism \(\alpha : L \to L\). In fact, this result can be found in [22, Corollary 2.6]. Then, some well-known examples of Hom-Lie algebras can be obtained in this way.

**Example 1.2.** Consider the one-sided Witt Lie algebra \(W_1\) (see for example [19] or [20]) on the vector space with basis \(\{x_i\}_{i=-1}^{\infty}\), whose Lie bracket is defined by

\([x_i, x_j] = (j - i)x_{i+j}\),

for all integers \(i, j \geq -1\). \(W_1\) may be identified with Der\((k[x])\), the Lie algebra of \(k\)-derivations of the polynomial algebra \(k[x]\) in the indeterminate \(x\) with coefficients in \(k\), where \(x_i\) can be identified with the differential operator \(x^{i+1}(d/dx)\).

Define a linear map

\(\alpha : \{x_i\}_{i=-1}^{\infty} \to \{x_i\}_{i=-2}^{\infty}, \quad \alpha(x_i) \mapsto \frac{1}{2}x_{2i}\),

where \(x_{-2} := 0\).

In fact, \(\alpha\) is a Lie algebra homomorphism. Then we obtain a Hom-Lie algebra \((W_1, [-,-], \alpha)\) called one-sided Witt Hom-Lie algebra.

In the following, let \(\tau\) denote the twist isomorphism given by \(\tau(x \otimes y) = y \otimes x\). The next Definition is due to Yau [25, Definition 3.2].

**Definition 1.3.** A Hom-Lie coalgebra is a triple \((\Gamma, \Delta, \alpha)\) consisting of a vector space \(\Gamma\), a linear map \(\Delta : \Gamma \to \Gamma^\otimes 2\) and a linear endomorphism \(\alpha : \Gamma \to \Gamma\) satisfying the following conditions:

(1.3.1) \(\Delta + \tau \circ \Delta = 0\) (anti-symmetry),
(1.3.2) \(\circ (\alpha \otimes \Delta) \circ \Delta = 0\) (Hom-coJacobi identity),
(1.3.3) \(\Delta \circ \alpha = \alpha^\otimes 2 \circ \Delta\) (co-multiplicativity).
The definition of \emph{sub-Hom-Lie coalgebras} is analogous to sub-Hom-Lie algebras. A \emph{morphism of Hom-Lie coalgebras} $f : (\Gamma, \Delta, \alpha) \rightarrow (\Gamma', \Delta', \alpha')$ is a linear map such that $\alpha' \circ f = f \circ \alpha$ and $\Delta' \circ f = f^{\otimes 2} \circ \Delta$.

Let $(L, [-, -], \alpha)$ be a Hom-Lie algebra. For any $x \in L$ and any integer $n \geq 2$, we define the \emph{adjoint diagonal action} $\text{ad}_x : L^\otimes n \rightarrow L^\otimes n$ by

$$\text{ad}_x(y_1 \otimes \cdots \otimes y_n) = \sum_{i=1}^{n} \alpha(y_1) \otimes \cdots \otimes \alpha(y_{i-1}) \otimes [x, y_i] \otimes \alpha(y_{i+1}) \cdots \otimes \alpha(y_n).$$

In particular, for $n = 2$, we have

$$\text{ad}_x(y_1 \otimes y_2) = [x, y_1] \otimes \alpha(y_2) + \alpha(y_1) \otimes [x, y_2].$$

\section{Hom-Lie bialgebras}

In this section, we investigate boundary and quasi-triangular Hom-Lie bialgebras further. We also give a necessary and sufficient condition for a given Hom-Lie algebra and a given 2-tensor to admit a coboundary Hom-Lie bialgebra structure.

We begin this section by recalling the definition of a Hom-Lie bialgebra as introduced by Yau in [25, Definition 3.3]:

**Definition 2.1.** A \emph{Hom-Lie bialgebra} is a quadruple $(L, [-, -], \Delta, \alpha)$ in which $(L, [-, -], \alpha)$ is a Hom-Lie algebra and $(L, \Delta, \alpha)$ is a Hom-Lie coalgebra such that the following compatibility condition holds for all $x, y \in L$:

$$\Delta([x, y]) = \text{ad}_x(\Delta(y)) - \text{ad}_y(\Delta(x)).$$

(2.1)

Explicitly, the compatibility condition can be restated as

$$\Delta([x, y]) = [\alpha(x), y_1] \otimes \alpha(y_2) + \alpha(y_1) \otimes [\alpha(x), y_2] - [\alpha(y), x_1] \otimes \alpha(x_2) - \alpha(x_1) \otimes [\alpha(y), x_2].$$

A Lie bialgebra is a Hom-Lie bialgebra with the trivial twist $\alpha = \text{id}$. Similarly to Lie bialgebras, the compatibility condition for Hom-Lie bialgebras states exactly that $\Delta \in C^1(L, L \otimes L)$ is a 1-cocycle in Hom-Lie algebra cohomology (see [25, Remark 3.4]).

Let $(\Gamma, \Delta, \alpha)$ be a Hom-Lie coalgebra. Then, by a straightforward computation, it can be seen that the dual space $\Gamma^* := \text{Hom}(\Gamma, k)$ of $\Gamma$ is a Hom-Lie algebra via the bracket $[\cdot, \cdot]^\circ$ and twist $\alpha^*$ defined by

$$[\phi, \varphi]^\circ := (\phi \otimes \varphi) \circ \Delta, \quad \alpha^*(\phi) := \phi \circ \alpha,$$

for all $\phi, \varphi \in \Gamma^*$.

Conversely, we consider the restricted or continuous dual of a Hom-Lie algebra. Let $(L, [-, -], \alpha)$ be a Hom-Lie algebra. Then consider the linear maps $[-, -]^\ast : L^* \rightarrow (L \otimes L)^*$ defined by $[-, -]^\ast(\phi) := \phi \circ [-, -]$ and $\alpha^* : L^* \rightarrow L^*$ defined by $\alpha^*(\phi) := \phi \circ \alpha$ for every $\phi \in L^*$. A subspace $M$ of $L^*$ is called \emph{good} if $[-, -]^\ast(M) \subseteq M \otimes M$ and $\alpha^*(M) \subseteq M$, where $M \otimes M \subseteq L^* \otimes L^* \subseteq (L \otimes L)^*$. Let
Let \( L^o \) denote the sum of all good subspaces of \( L^* \). Then \([-,-]^o([L^o]) \subseteq L^o \otimes L^o \) and \( \alpha^*(L^o) \subseteq L^o \) and the triple \((L^o, \Delta^o, \alpha^o)\) is a Hom-Lie coalgebra, where \( \Delta^o \) is the restriction map of \([-,-]^o \) to \( L^o \) and \( \alpha^o \) is the restriction map of \( \alpha^* \) to \( L^o \). We obtain the following generalization of [25, Theorem 3.9] from finite dimensional Hom-Lie bialgebras to arbitrary dimensions:

**Theorem 2.2.** If \((L, [-,-], \Delta, \alpha)\) is a Hom-Lie bialgebra, then the quadruple \((L^o, [-,-]^o, \Delta^o, \alpha^o)\) defined as above is again a Hom-Lie bialgebra.

**Proof.** Since \( L^o \) is a good subspace of \( L^* \), \( L^o \) is both a Hom-Lie algebra and a Hom-Lie coalgebra. And the compatibility condition (2.1) for \( L^o \) is exactly the same as the one for \( L^* \) in the proof of Theorem 3.9 in [25].

Note that Theorem 2.2 shows that the concept of a Hom-Lie bialgebra is self-dual generalizing the self-duality of Lie bialgebras (see [18, Proposition 8.1.2]). If the underlying vector space is finite dimensional, the concept of a Hom-Lie bialgebra can be dualized in the usual way without using the concept of good subspaces.

Now we recall the definition of the classical Hom-Yang-Baxter equation (CHYBE) for a Hom-Lie algebra \((L, [-,-], \alpha)\) introduced by Yau [25, (1.0.3)]. For any 2-tensor \( r = r_1 \otimes r_2 \) in \( L \otimes L \) we set

\[
[r_{12}, r_{13}] := [r_1, r'_1] \otimes \alpha(r_2) \otimes \alpha(r'_2),
\]

\[
[r_{12}, r_{23}] := \alpha(r_1) \otimes [r_2, r'_1] \otimes \alpha(r'_2),
\]

\[
[r_{13}, r_{23}] := \alpha(r_1) \otimes \alpha(r'_1) \otimes [r_2, r'_2],
\]

where \( r' = r'_1 \otimes r'_2 \) is a copy of \( r \). Then

\[
\text{CHYB}(r) := [r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0
\]

is called the classical Hom-Yang-Baxter equation. Now we are ready to introduce coboundary Hom-Lie bialgebras and quasi-triangular Hom-Lie bialgebras as defined by Yau in [25, Definition 4.1].

**Definition 2.3.** A Hom-Lie bialgebra \((L, [-,-], \Delta, \alpha)\) is a **coboundary Hom-Lie bialgebra** if there exists an element \( r \in L \otimes L \) such that \( \alpha^ {(\otimes 2)}(r) = r \) and \( \Delta(x) = \text{ad}_x(r) \) for every \( x \in L \). A **quasi-triangular Hom-Lie bialgebra** is a coboundary Hom-Lie bialgebra such that \( \text{CHYB}(r) = 0 \).

Note that for a coboundary Hom-Lie bialgebra \((L, [-,-], \Delta, \alpha, r)\), the symmetric part \( r + \tau(r) \) of \( r \) is adjoint invariant, that is, \( \text{ad}_x(r + \tau(r)) = 0 \) for every \( x \in L \). This is equivalent to \( \Delta \) being anti-symmetric.

In the following result we characterize the quasi-triangularity of boundary Hom-Lie bialgebras in terms of both a certain Hom-Lie algebra morphism and a certain Hom-Lie coalgebra morphism. The dual pairing of \( L^* \) and \( L \) will be denoted by \( \langle -,- \rangle \).
Proposition 2.4. Let \((L, [-, -], \Delta, \alpha, r)\) be an involutive coboundary Hom-Lie bialgebra with \(r = r_1 \otimes r_2\). Then \(L\) is a quasi-triangular Hom-Lie bialgebra if and only if \(s_1 : L^* \to L\) defined by \(s_1(\phi) = \langle \phi, \alpha(r_1) \rangle r_2\) is a Hom-Lie algebra morphism. Likewise, if and only if \(s_2 : L^* \to L\) defined by \(s_2(\phi) = r_1 \langle \phi, \alpha(r_2) \rangle\) is a Hom-Lie coalgebra morphism.

Proof. Since we are given an involutive coboundary Hom-Lie bialgebra, we know that
\[
\alpha \circ s_1(\phi) = \langle \phi, \alpha(r_1) \rangle \alpha(r_2) = \langle \phi, r_1 \rangle r_2
\]
for all \(\phi \in L^*\).

From the fact \(\alpha(r_1) \otimes \alpha(r_2) = r_1 \otimes r_2\) and \(L\) is involutive, we have
\[
\alpha(r_1) \otimes r_2 = r_1 \otimes \alpha(r_2), \tag{2.2}
\]
which is used in the following proof.

To show that \(L\) is quasi-triangular if and only if \(s_1\) is a Hom-Lie algebra morphism, we are equivalent to show that \(\text{CHYB}(r) = 0\) if and only if \(s_1([\phi, \varphi]) = [s_1(\phi), s_1(\varphi)]\), for all \(\phi, \varphi \in L^*\). Indeed,
\[
s_1([\phi, \varphi]) - [s_1(\phi), s_1(\varphi)] = \langle \phi, \varphi \rangle, \alpha(r_1)) r_2 - \langle \phi, \alpha(r_1) \rangle \alpha(r_2) [r_2, r_2']
\]
\[
= \langle \phi \otimes \varphi \otimes \alpha, \Delta(\alpha(r_1)) \otimes r_2 - \alpha(r_1) \otimes \alpha(r_2) \otimes [r_2, r_2']
\]
\[
= \langle \phi \otimes \varphi \otimes \alpha, [\alpha(r_1), r_1'] \otimes \alpha(r_2) \otimes r_2 + \alpha(r_1) \otimes [\alpha(r_1), r_2] \otimes r_2
\]
\[
- \alpha(r_1) \otimes \alpha(r_2) \otimes [r_2, r_2']
\]
\[
= \langle \phi \otimes \varphi \otimes \alpha, [r_1, r_1'] \otimes \alpha(r_2) \otimes r_2 + \alpha(r_1) \otimes [r_2, r_2'] \otimes r_2
\]
\[
- \alpha(r_1) \otimes \alpha(r_2) \otimes [r_2, r_2']
\]
\[
= \langle \phi \otimes \varphi \otimes \alpha, -\text{CHYB}(r)\rangle,
\]
where \(r'\) is another copy of \(r\).

The proof for \(s_2\) is strictly analogous. \(\blacksquare\)

Proposition 2.5. Let \((L, [-, -], \alpha)\) be an involutive Hom-Lie algebra and \(r = r_1 \otimes r_2 \in L \otimes L\) such that \(\alpha^{\otimes 2}(r) = r, r = -\tau(r)\). Set
\[
\Delta(x) = \text{ad}_x(r) = [x, r_1] \otimes \alpha(r_2) + \alpha(r_1) \otimes [x, r_2].
\]

Then, for all \(x \in L\),
\[
\Diamond (\alpha \otimes \Delta) \circ \Delta(x) = \text{ad}_{\alpha(x)}(\text{CHYB}(r)).
\]

Proof. According to (2.2) and \(\alpha^{\otimes 2}(r) = r\), for any \(x \in L\), we have
\[
\text{ad}_{\alpha(x)}(\text{CHYB}(r)) = [\alpha(x), [r_1, r_1'] \otimes r_2 \otimes r_2' + \alpha([r_1, r_1']) \otimes [\alpha(x), \alpha(r_2)] \otimes r_2'
\]
\[
+ \alpha([r_1, r_1']) \otimes r_2 \otimes [\alpha(x), \alpha(r_2)] + [\alpha(x), \alpha(r_1)] \otimes [r_2, r_2'] \otimes r_2'
\]
\[
+ r_1 \otimes [\alpha(x), [r_2, r_2']] \otimes r_2' + r_1 \otimes \alpha([r_2, r_2']) \otimes [\alpha(x), \alpha(r_2')]
\]
\[
+ [\alpha(x), \alpha(r_1) \otimes r_2 \otimes [\alpha(x), \alpha(r_2) + r_1 \otimes [\alpha(x), \alpha(r_1) \otimes [r_2, r_2']
\]
\[
+ r_1 \otimes r_2 \otimes [\alpha(x), [r_2, r_2']].
\]
\[ = [\alpha(x), [r_1, r'_1]] \otimes r_2 \otimes r'_2 + [r_1, \alpha(r'_1)] \otimes [\alpha(x), r_2] \otimes r'_2 + [\alpha(r_1), r'_1] \otimes [\alpha(x), r_2] \]
\[ + [\alpha(x), r_1] \otimes [r_2, \alpha(r'_1)] \otimes r'_2 + r_1 \otimes [\alpha(x), [r_2, r'_1]] \otimes r'_2 + r_1 \otimes [\alpha(r_1), r'_1] \otimes [\alpha(x), r_2] \]
\[ + [\alpha(x), r_1] \otimes r'_1 \otimes [r_2, \alpha(r'_2)] + r_1 \otimes [\alpha(x), r'_1] \otimes [\alpha(r_2), r'_2] + r_1 \otimes r'_1 \otimes [\alpha(x), [r_2, r'_2]]. \]

Meanwhile,

\[
\circledast (\alpha \otimes \Delta) \circ \Delta(x) = \circledast (\alpha \otimes \Delta)([x, r_1] \otimes \alpha(r_2) + \alpha(r_1) \otimes [x, r_2])
\[ = \circledast (\alpha([x, r_1]) \otimes [\alpha(r_2), r'_1] \otimes \alpha(r'_2) + \alpha([x, r_1]) \otimes \alpha(r'_1) \otimes [\alpha(r_2), r'_2]
\]
\[ + r_1 \otimes [[x, r_2], r'_1] \otimes \alpha(r'_2) + r_1 \otimes \alpha(r'_1) \otimes [[x, r_2], r'_2])
\[ = \circledast ([\alpha(x), r_1] \otimes [r_2, r'_1] \otimes \alpha(r'_2) + [\alpha(x), r_1] \otimes \alpha(r'_1) \otimes [r_2, r'_2]
\]
\[ + r_1 \otimes [[x, r_2], r'_1] \otimes \alpha(r'_2) + r_1 \otimes \alpha(r'_1) \otimes [[x, r_2], r'_2])
\]

\[
= [\alpha(x), r_1] \otimes [r_2, r'_1] \otimes \alpha(r'_2) + [\alpha(x), r_1] \otimes \alpha(r'_1) \otimes [r_2, r'_2]
\]
\[ + r_1 \otimes [[x, r_2], r'_1] \otimes \alpha(r'_2) + r_1 \otimes \alpha(r'_1) \otimes [[x, r_2], r'_2] \]
\[ + [r_2, r'_1] \otimes \alpha(r'_2) \otimes [\alpha(x), r_1] + [\alpha(r'_1) \otimes [r_2, r'_2] \otimes [\alpha(x), r_1]
\]
\[ + [x, r_2], r'_1] \otimes \alpha(r'_2) \otimes r_1 + [\alpha(r'_1) \otimes [x, r_2], r'_2] \otimes r_1
\]
\[ + \alpha(r'_2) \otimes [\alpha(x), r_1] \otimes [r_2, r'_1] + [r_2, r'_2] \otimes [\alpha(x), r_1] \otimes \alpha(r'_1)
\]
\[ + \alpha(r'_2) \otimes r_1 \otimes [[x, r_2], r'_1] + [[x, r_2], r'_2] \otimes r_1 \otimes \alpha(r'_1)
\]

\[ = [\alpha(x), r_1] \otimes [r_2, r'_1] \otimes \alpha(r'_2) + [\alpha(x), r_1] \otimes \alpha(r'_1) \otimes [r_2, r'_2] + r_1 \otimes \alpha([x, r_2], r'_1) \otimes r'_2
\]
\[ + r_1 \otimes r'_1 \otimes [\alpha(x), [r_2, r'_2]] + [r_2, r'_1] \otimes \alpha(r'_2) \otimes [\alpha(x), r_1] + [\alpha(r'_1) \otimes [r_2, r'_2] \otimes [\alpha(x), r_1]
\]
\[ + [\alpha(x), [r_1, r'_1]] \otimes r_2 \otimes r'_2 + \alpha(r'_2) \otimes [\alpha(x), r_1] \otimes [r_2, r'_1] + [r_2, r'_2] \otimes [\alpha(x), r_1] \otimes \alpha(r'_1). \]

We break these twelve terms into nine groups, which is equal to the nine terms of \( \text{ad}_{\alpha(x)}(\text{CHYB}(r)) \) respectively.

From Proposition 2.2 of [3] and Proposition 2.5, we have the main result of this section, which generalizes the result in [4]. It gives a necessary and sufficient condition under which a Hom-Lie algebra becomes a coboundary Hom-Lie algebra. Indeed, it’s also a direct consequence of [25, Theorem 4.5], which gives a sufficient condition only but the necessity follows from the equalities in the proof.
Theorem 2.6. Let \((L, [-,-], \alpha)\) be an involutive multiplicative Hom-Lie algebra over \(k\) and \(r \in L \otimes L\) such that \(\alpha^2 (r) = r\). Then the map \(\Delta_r(x) := \text{ad}_x (r)\) for any \(x \in L\) yields a coboundary Hom-Lie bialgebra on \(L\) if and only if the following conditions are satisfied:

(i) \(\text{ad}_x (r + \tau(r)) = 0\) for every \(x \in L\),
(ii) \(\text{ad}_x (\text{CHYB}(r)) = 0\) for every \(x \in L\).

Remarks (1) It is enough to assume that the characteristic of \(k\) is not 2.
(2) According to [25, Theorem 4.5], it is also sufficient to assume that \(\alpha\) is injective (or even that only \(\alpha^3\) is injective) instead of \(\alpha^2 = \text{id}\).
(3) Theorem 4.5 of [25] is a version of the above Theorem for triangular Hom-Lie bialgebra structures on \(L\) (for the definition see the bottom of p. 20 in [25]).

3. The double Hom-Lie bialgebra

In this section, we generalize the Drinfel’d double of a Lie bialgebra to Hom-Lie bialgebras and show that the Drinfel’d double of a Hom-Lie bialgebra is indeed a quasi-triangular Hom-Lie bialgebra.

Theorem 3.1. Let \((L, [-,-], \Delta, \alpha)\) be a finite dimensional involutive Hom-Lie bialgebra with the dual \(L^*\) given by the “*” dual. Then, there is a quasi-triangular Hom-Lie bialgebra \((D(L) = L^* \oplus L, [-,-], \Delta_D, \alpha_D, r)\) called a Drinfel’d double of Hom-Lie bialgebra, built on \(L^{\text{op}} \oplus L\) as a vector space, with the following structures,

\[
\begin{align*}
\{ \phi \otimes x, \varphi \otimes y \} & = \{ \varphi, \phi \} + \varphi_1 \langle \varphi_2, x \rangle - \phi_1 \langle \phi_2, y \rangle \\
\oplus & \{ x, y \} + x_1 \langle \varphi, x_2 \rangle - y_1 \langle \phi, y_2 \rangle,
\end{align*}
\]

\[
\begin{align*}
\Delta_D(\phi \otimes x) & = \phi_1 \otimes \phi_2 + x_1 \otimes x_2,
\alpha_D(\phi \otimes x) & = \alpha^*(\phi) + \alpha(x),
\end{align*}
\]

\[
r = \frac{1}{2} \sum_a (f^a \otimes \alpha(e_a) + \alpha^*(f^a) \otimes e_a),
\]

for all \(\phi, \varphi \in L^*\) and \(x, y \in L\). Here \(L^{\text{op}}, L\) are sub-Hom-Lie bialgebras of the Drinfel’d double Hom-Lie bialgebra, where \((-)^{\text{op}}\) denotes the opposite Lie bracket. The set \(\{e_a\}\) is a basis of \(L\) and \(\{f^a\}\) is the dual basis.

Proof. Noting that every element of direct sum has a unique decomposition into a vector in \(L^*\) and a vector in \(L\), and from the definition of \(D(L)\) we know

\[
\begin{align*}
\{ \phi, \varphi \} & = -[\phi, \varphi], [x, y]_D = [x, y], \\
\{ x, \phi \} & = \phi_1 \langle \phi_2, x \rangle + x_1 \langle \phi, x_2 \rangle,
\end{align*}
\]

\[
\begin{align*}
\Delta_D(\phi) & = \Delta(\phi), \Delta_D(x) = \Delta(x),
\end{align*}
\]

\[
\begin{align*}
\alpha_D(\phi) & = \alpha^*(\phi), \alpha_D(x) = \alpha(x),
\end{align*}
\]
for all $\phi, \varphi \in L^*$ and $x, y \in L$, where the right hands of the above equalities are in terms of the structures of $L^*$ and $L$.

By the definition, it is clear that $[-,-]_D$ is anti-symmetric and the Hom-Jacobi identity holds when we restrict all the elements to $L^*$ or to $L$. So we need to check the cross brackets. For all $\phi, \varphi \in L^*$ and $x \in L$,

$$
\begin{align*}
\langle [\alpha(x), [\varphi, \phi]]_D \rangle & = -\langle \phi, \varphi, x \rangle_D - \alpha(x) \lbrack \phi, \varphi, x \rbrack_D - \alpha(x) \lbrack \phi, \varphi, x \rbrack_D \\
& = \langle [\alpha^*(\phi), [\varphi, \phi]]_D \rangle - \langle \phi, \varphi, x \rangle_D - \alpha(x) \lbrack \phi, \varphi, x \rbrack_D - \alpha(x) \lbrack \phi, \varphi, x \rbrack_D \\
& = -\langle \phi, \varphi, x \rangle_D - \alpha(x) \lbrack \phi, \varphi, x \rbrack_D - \alpha(x) \lbrack \phi, \varphi, x \rbrack_D \\
& = -\alpha^*(\phi) \lbrack \phi, \varphi, x \rbrack_D - \alpha(x) \lbrack \phi, \varphi, x \rbrack_D - \alpha(x) \lbrack \phi, \varphi, x \rbrack_D \\
& = -\alpha^*(\phi) \lbrack \phi, \varphi, x \rbrack_D - \alpha(x) \lbrack \phi, \varphi, x \rbrack_D - \alpha(x) \lbrack \phi, \varphi, x \rbrack_D .
\end{align*}
$$

where $\phi \leftrightarrow \varphi$ means swapping $\phi$ for $\varphi$ in the forward expression. On the other hand,

$$
\begin{align*}
\lbrack \alpha^*(\phi), [\varphi, x]_D \rbrack_D & = \langle \alpha^*(\phi), \varphi, x \rangle_D - \langle \varphi, \phi, x \rangle_D \\
& = \langle \alpha^*(\phi), \varphi, x \rangle_D - \langle \varphi, \phi, x \rangle_D + \langle \phi, \varphi, x \rangle_D - \langle \phi, \varphi, x \rangle_D \\
& = \langle \alpha^*(\phi), \varphi, x \rangle_D - \langle \varphi, \phi, x \rangle_D + \langle \phi, \varphi, x \rangle_D - \langle \phi, \varphi, x \rangle_D \\
& = \langle \alpha^*(\phi), \varphi, x \rangle_D - \langle \varphi, \phi, x \rangle_D + \langle \phi, \varphi, x \rangle_D - \langle \phi, \varphi, x \rangle_D .
\end{align*}
$$

Then, from the above two equalities we have

$$
\begin{align*}
\langle [\alpha^*(\phi), [\varphi, x]_D]_D \rangle & = \langle [\alpha^*(\phi), [\varphi, x]_D]_D \rangle - \langle \phi, \varphi, x \rangle_D \\
& = \langle [\alpha^*(\phi), [\varphi, x]_D]_D \rangle - \langle \phi, \varphi, x \rangle_D + \langle \phi, \varphi, x \rangle_D - \langle \phi, \varphi, x \rangle_D \\
& = \langle [\alpha^*(\phi), [\varphi, x]_D]_D \rangle - \langle \phi, \varphi, x \rangle_D + \langle \phi, \varphi, x \rangle_D - \langle \phi, \varphi, x \rangle_D \\
& = \langle [\alpha^*(\phi), [\varphi, x]_D]_D \rangle - \langle \phi, \varphi, x \rangle_D + \langle \phi, \varphi, x \rangle_D - \langle \phi, \varphi, x \rangle_D .
\end{align*}
$$

So $\circ [\alpha_D(x), [\phi, \varphi]_D]_D = 0$ from the Hom-Jacobi identity for $L$. Similarly, $\circ [\alpha_D(x), [y, \phi]_D]_D = 0$ from the Hom-coJacobi identity for $L^*$. Thus, $(D(L), [-,-]_D, \alpha_D)$ is a Hom-Lie algebra.

In addition, from the definition of $\Delta_D$, we know that it satisfies the anti-symmetry and the Hom-coJacobi identity, so $(D(L), \Delta_D, \alpha_D)$ is a Hom-Lie coalgebra.

In the following proof, we need two very useful identities:

$$
\begin{align*}
\alpha^*(f^a) \otimes \alpha(e_{a_1}) \langle \phi, \alpha(e_{a_2}) \rangle & = [f^a, \phi] \otimes e_a, \quad (3.1) \\
f^a \langle f^a_2, x \rangle \otimes e_a & = \alpha^*(f^a) \otimes [\alpha(e_a), x], \quad (3.2)
\end{align*}
$$

for all $\phi \in L^*, x \in L$. These are true by using the duality pairing $f^a \langle \phi, e_a \rangle = \phi$ and $\langle f^a, x \rangle e_a = x$. In fact, for any $\varphi \in L^*$,

$$
\begin{align*}
[f^a, \phi] \langle \varphi, e_a \rangle & = \varphi \phi = f^a \langle [\varphi, \phi], e_a \rangle \\
& = \alpha^*(f^a) \langle \alpha^* \varphi, \phi, e_a \rangle \\
& = \alpha^*(f^a) \langle \varphi, \alpha(e_{a_1}) \rangle \langle \phi, \alpha(e_{a_2}) \rangle .
\end{align*}
$$
So 
\[ \alpha^*(f^a) \otimes \alpha(e_{a_1}) \langle \phi, \alpha(e_{a_2}) \rangle = [f^a, \phi] \otimes e_a. \]

In the same way, (3.2) holds too.

By identity (3.1), we have \( \text{ad}_\phi(r) = \)
\[
\begin{align*}
&= \frac{1}{2} \text{ad}_\phi \left( f^a \otimes \alpha(e_a) + \alpha^*(f^a) \otimes e_a \right) \\
&= \frac{1}{2} \left( [\phi, f^a] \right) \otimes e_a + \alpha^*(f^a) \otimes \alpha(e_a) + f^a \otimes [\phi, e_a] \\
&= \frac{1}{2} [f^a, \phi] \otimes e_a + \alpha^*(f^a) \otimes \alpha(e_a) + f^a \otimes \alpha(e_a)
\end{align*}
\]
for any \( \phi \in L^*. \) Meanwhile, by identity (3.2), we get \( \text{ad}_x(r) = \)
\[
\begin{align*}
&= \frac{1}{2} \text{ad}_x \left( f^a \otimes \alpha(e_a) + \alpha^*(f^a) \otimes e_a \right) \\
&= \frac{1}{2} \left( [x, f^a] \right) \otimes e_a + \alpha^*(f^a) \otimes [x, \alpha(e_a)] + f^a \otimes [x, e_a] \\
&= \frac{1}{2} [f^a, x] \otimes e_a + \alpha^*(f^a) \otimes [x, \alpha(e_a)] + f^a \otimes \alpha(e_a) + f^a \otimes [x, e_a]
\end{align*}
\]
for any \( x \in L. \) So \( \Delta_D(d) = \text{ad}_d(r) \), for any \( d \in D(L). \)

In addition,
\[
\alpha_D^\otimes(r) = \alpha_D^\otimes(f^a \otimes \alpha(e_a) + \alpha^*(f^a) \otimes e_a) = r,
\]
so the compatibility of the Hom-Lie bialgebra holds from [3, Proposition 2.2]. Hence, \( (D(L), [-, -], \Delta_D, \alpha_D, r) \) is a coboundary Hom-Lie bialgebra.

Finally, \( r \) obeys the CHYBE. Since \( r = \frac{1}{2} (f^a \otimes \alpha(e_a) + \alpha^*(f^a) \otimes e_a) \), we have
\[ \text{CHYB}(r) = \]
\[
\begin{align*}
&= \frac{1}{4} \left( [f^a, f^b] \otimes e_a \otimes e_b + [f^a, \alpha^*(f^b)] \otimes e_a \otimes \alpha(e_b) \right) \\
&\quad + \alpha^*(f^a) \otimes [\alpha(e_a), \alpha(f^b)] \otimes \alpha(e_b) + [f^a, \alpha^*(f^b)] \otimes \alpha(e_a) \otimes \alpha(e_b) \\
&\quad + \alpha^*(f^a) \otimes \alpha(e_a) \otimes \alpha(f^b) \otimes e_b + \alpha^*(f^a) \otimes \alpha(e_a) \otimes \alpha(f^b) \otimes e_b \\
&\quad + f^a \otimes [e_a, f^b] \otimes e_b + f^a \otimes [e_a, \alpha^*(f^b)] \otimes \alpha(e_b) \\
&\quad + \alpha^*(f^a) \otimes f^b \otimes [\alpha(e_a), \alpha(e_b)] + \alpha^*(f^a) \otimes f^b \otimes [\alpha(e_a), e_b] \\
&\quad + f^a \otimes \alpha^*(f^b) \otimes [\alpha(e_a), \alpha(e_b)] + f^a \otimes f^b \otimes [e_a, e_b] \\
\end{align*}
\]
which can be divided into four groups as above. In the group (1), from (3.1) and (3.2), we get
\[
[f^a, f^b]_D \otimes e_a \otimes e_b + \alpha^*(f^a) \otimes [\alpha(e_a), f^b]_D \otimes e_b + \alpha^*(f^a) \otimes \alpha^*(f^b) \otimes [\alpha(e_a), \alpha(e_b)]_D \\
= -[f^a, f^b]_D \otimes e_a \otimes e_b + \alpha^*(f^a) \otimes [\alpha(e_a), f^b]_D \otimes e_b \\
+ \alpha^*(f^a) \otimes f^b_1(f^b_2, \alpha(e_a)) \otimes e_b + \alpha^*(f^a) \otimes \alpha^*(f^b) \otimes [\alpha(e_a), \alpha(e_b)] \\
= 0.
\]

In the same way, the other three groups are all zero too. So CHYB\((r) = 0 \) and \( D(L) \) is a quasi-triangular Hom-Lie bialgebra.

The double of Hom-Lie bialgebras is different from Drinfel’d’s original construction (see [18, Proposition 8.2.1]) in the quasi-triangular structure.

**Example 3.2.** Let sl(2)\(\alpha\) be the Hom-Lie bialgebra introduced in [25, Proposition 3.10] and sl(2)\(\alpha^*\) the dual Hom-Lie bialgebra, where \(\alpha(H) = H\), \(\alpha(X_\pm) = -X_\pm\). In this situation, the structure maps of sl(2)\(\alpha\) are given by
\[
[H, X_\pm]_\alpha = \mp 2X_\pm, \ [X_+, X_-]_\alpha = H;
\]
\[
\Delta_\alpha(H) = 0, \ \Delta_\alpha(X_\pm) = -\frac{1}{2}(X_\pm \otimes H - H \otimes X_\pm).
\]
And respectively, the structures of sl(2)\(\alpha^*\) are as follows
\[
\alpha^*(H^*) = H^*, \ \alpha^*(X^*_\pm) = -X^*_\pm;
\]
\[
[X^*_\pm, H^*]_\alpha = -\frac{1}{2}X^*_\pm, \ [X^*_+, X^*_-]_\alpha = 0;
\]
\[
\Delta_\alpha(X^*_\pm) = \mp 2(H^* \otimes X^*_\pm - X^*_\pm \otimes H^*), \Delta_\alpha(H^*) = X^*_+ \otimes X_- - X^*_- \otimes X^*_+.
\]
From direct computation, we obtain the double Hom-Lie bialgebra \(D(sl(2)\alpha)\) built on the vector space sl(2)\(\alpha^*\) \(\oplus sl(2)\alpha\) with the structures \([-\cdot, -\cdot]_D, \Delta_D, \alpha_D\) defined by
\[
[X^*_\pm, H^*]_D = \frac{1}{2}X^*_\pm, \ [X^*_+, X^*_-]_D = 0, \ [H, X^*_\pm]_D = \mp 2X^*_\pm, \ [X^*_+, X^*_-]_D = H, \ [H, H^*]_D = 0, \ [X^*_+, X^*_-]_D = -2H^* + \frac{1}{2}H, \ [X^*_+, X^*_-]_D = 2H^* + \frac{1}{2}H, \ [X^*_-, X^*_+]_D = [X^*_-, X^*_+]_D = 0, \ [H, X^*_+]_D = \pm 2X^*_+;
\]
\[
[X^*_\pm, H^*]_D = -\frac{1}{2}X^*_\pm \mp X^*_\pm;
\]
\[
\Delta_D(X^*_\pm) = \mp 2(H^* \otimes X^*_\pm - X^*_\pm \otimes H^*), \ \Delta_D(H^*) = X^*_+ \otimes X_- - X^*_- \otimes X^*_+;
\]
\[
\Delta_D(H) = 0, \ \Delta_D(X^*_\pm) = -\frac{1}{2}(X^*_\pm \otimes H - H \otimes X^*_\pm);
\]
\[
\alpha_D(H^*) = H^*, \ \alpha_D(X^*_\pm) = -X^*_\pm, \ \alpha_D(H) = H, \ \alpha_D(X^*_\pm) = -X^*_\pm.
\]
In addition,
\[
r = \frac{1}{2}(H^* \otimes \alpha(H) + \alpha^*(H^*) \otimes H + X^*_+ \otimes \alpha(X^*_+) \\
+ \alpha^*(X^*_+) \otimes X^*_+ + X^*_- \otimes \alpha(X^*_-) + \alpha^*(X^*_-) \otimes X^-) \\
= H^* \otimes H - X^*_+ \otimes X^*_+ - X^*_- \otimes X^-.
\]
Then, we have the double Hom-Lie bialgebra \((D(sl(2)_\alpha), [-,-]_D, \Delta_D, \alpha_D, r)\) which is quasi-triangular.

Furthermore, working over the complex number field, we note that \(sl(2)_\alpha\) and \(sl(2)_\alpha^*\) have another pair of dual bases

\[
e_1 = -\frac{i}{2}(X_+ + X_-), e_2 = -\frac{1}{2}(X_+ - X_-), e_3 = -\frac{i}{2}H,
\]

\[
f^1 = i(X_+^* + X_-^*), f^2 = -(X_+^* - X_-^*), f^3 = 2iH^*.
\]

We can check easily that \(\langle f^a, e_b \rangle = \delta^a_b\) given the duality pairing relation. Then we construct another quasi-triangular Hom-Lie bialgebra on \(D(sl(2)_\alpha)\) with \([-,-]_D, \Delta_D, \alpha_D\) defined as above and \(r'\) given by \(r' = \)

\[
= \frac{1}{2} \sum_a (f^a \otimes \alpha(e_a) + \alpha^*(f^a) \otimes e_a) \\
= \frac{1}{2} (f^1 \otimes \alpha(e_1) + \alpha^*(f^1) \otimes e_1 + f^2 \otimes \alpha(e_2) + \alpha^*(f^2) \otimes e_2 + f^3 \otimes \alpha(e_3) + \alpha^*(f^3) \otimes e_3) \\
= -\frac{1}{2} ((X_+^* + X_-^*) \otimes (X_+ + X_-) + (X_+^* - X_-^*) \otimes (X_+ - X_-) - 2H^* \otimes H) \\
= H^* \otimes H - X_+^* \otimes X_+ - X_-^* \otimes X_-.
\]

Obviously, \(r = r'\). So we find that though \(sl(2)_\alpha\) and \(sl(2)_\alpha^*\) have different dual bases, there is the same quasi-triangular structure on \(D(sl(2)_\alpha)\).

**Definition 3.3.** A Hom-Lie bialgebra \((L, [-,-], \Delta, \alpha)\) is a co-quasi-triangular Hom-Lie bialgebra if there exists a linear map \(\sigma : L \otimes L \to k\) such that the Lie bracket has a special form

\[
[x, y] = x_1 \sigma(x_2, \alpha(y)) + y_1 \sigma(\alpha(x), y_2),
\]

and obeys the CHYBE in the dual form

\[
\sigma(x_1, \alpha(y))\sigma(x_2, \alpha(z)) + \sigma(\alpha(x), y_1)\sigma(y_2, \alpha(z)) + \sigma(\alpha(x), z_1)\sigma(\alpha(y), z_2) = 0,
\]

for all \(x, y, z \in L\).

Next we discuss the dual codouble Hom-Lie bialgebra \(D(L)^*\) built on the vector space \(L^{\text{op}} \oplus L^*\). \(D(L)^*\) has the direct sum Hom-Lie algebra structure and a complicated Lie cobracket, which is analogous to the codouble Lie bialgebra in [18, Section 8, p. 370]. In addition, the twist of the dual codouble Hom-Lie bialgebra \(D(L)^*\) is \(\alpha + \alpha^*\). Then we have the following result.

**Proposition 3.4.** Let \((L, [-,-], \Delta, \alpha)\) be a finite dimensional involutive Hom-Lie bialgebra. From the Lie cobracket of direct sum Hom-Lie bialgebra \(L^{\text{op}} \oplus L^*\), we define a perturbed Lie cobracket \(\Delta_{L^{\text{op}} \oplus L^*} + \text{ad}(t)\) which is exactly the Lie cobracket \(\Delta_{D(L)^*}\) of codouble Hom-Lie bialgebra \(D(L)^*\), where

\[
t = \frac{1}{2} \sum_a (\alpha^*(f^a) \otimes e_a - e_a \otimes \alpha^*(f^a) + f^a \otimes \alpha(e_a) - \alpha(e_a) \otimes f^a).
\]

Here \(\{e_a\}\) is a basis of \(L\) and \(\{f^a\}\) is the dual basis, and \(L^{\text{op}}\) denotes the opposite cobracket.

In particular, the codouble Hom-Lie bialgebra \(D(L)^*\) is a co-quasi-triangular Hom-Lie bialgebra.
Proof. With the twist \( \alpha_{L^{\text{cop}} \oplus L^*}(x \oplus \phi) = \alpha(x) \oplus \alpha^*(\phi) \), the direct sum Hom-Lie algebra structure on \( L^{\text{cop}} \oplus L^* \) means that
\[
[x \oplus \phi, y \oplus \varphi] = [x, y] \oplus [\phi, \varphi],
\]
for all \( x, y \in L \) and \( \phi, \varphi \in L^* \), or equivalently that \( L, L^* \) are sub-Hom-Lie algebras with \( [x, \phi] = 0 \) for the Lie bracket between them. Dually, the direct sum Hom-Lie coalgebra structure on \( L^{\text{cop}} \oplus L^* \) means that \( \alpha_{L^{\text{cop}} \oplus L^*}(x \oplus \phi) = \alpha(x) \oplus \alpha^*(\phi) \), \( \Delta_{L^{\text{cop}} \oplus L^*}(x) = -\Delta(x) \), and \( \Delta_{L^{\text{cop}} \oplus L^*}(\phi) = \Delta(\phi) \).

The duality pairing between \( D(L)^* \) and \( D(L) \) is given by
\[
\langle x \oplus \phi, \psi \oplus y \rangle = \langle x, \psi \rangle + \langle \phi, y \rangle.
\]
And using this, we can obtain the Lie cobracket of the codouble as follows
\[
\langle \Delta_{L^{\text{cop}} \oplus L^*}(x \oplus \phi) + \text{ad}_{x \oplus \varphi}(t), (\varphi \oplus y) \otimes (\psi \oplus z) \rangle
\]
\[
= \langle \Delta_{L^{\text{cop}} \oplus L^*}(x \oplus \phi), (\varphi \oplus y) \otimes (\psi \oplus z) \rangle + \langle \text{ad}_{x \oplus \varphi}(t), (\varphi \oplus y) \otimes (\psi \oplus z) \rangle
\]
\[
= \langle -\Delta(x) + \Delta(\phi), (\varphi \oplus y) \otimes (\psi \oplus z) \rangle
\]
\[
+ \left( \frac{1}{2} (\langle \varphi, \alpha^*(f^a) \rangle \otimes \alpha(e_a) + f^a \otimes [x, e_a] - [x, e_a] \otimes f^a - \alpha(e_a) \otimes [\phi, \alpha^*(f^a)] + \langle \phi, f^a \rangle \otimes \alpha^*(f^a) \otimes [x, \alpha(e_a)] - [x, \alpha(e_a)] \otimes \alpha^*(f^a)) - e_a \otimes [\phi, \alpha^*(f^a)] \right, (\varphi \oplus y) \otimes (\psi \oplus z) \rangle
\]
\[
= \langle \Delta(x), \varphi \otimes \psi \rangle + \langle \Delta(\phi), y \otimes z \rangle
\]
\[
+ \frac{1}{2} \left( \langle [\phi, \alpha^*(f^a)] \otimes \alpha(e_a) + [\phi, f^a] \otimes e_a, y \otimes \psi \rangle + \langle f^a \otimes [x, e_a] + \alpha^*(f^a) \otimes [x, \alpha(e_a)], y \otimes \psi \rangle - \langle [x, e_a] \otimes f^a + [x, \alpha(e_a)] \otimes \alpha^*(f^a), \varphi \otimes z \rangle - \langle \alpha(e_a) \otimes [\phi, \alpha^*(f^a)] + e_a \otimes [\phi, f^a], \varphi \otimes z \rangle \right) \]
\[
= \langle \Delta(x), \varphi \otimes \psi \rangle + \langle \Delta(\phi), y \otimes z \rangle
\]
\[
+ \langle \phi, \psi \rangle, y \rangle + \langle [x, y], \psi \rangle - \langle [x, z], \varphi \rangle - \langle [\phi, \varphi], z \rangle,
\]
for all \( x, y, z \in L \) and \( \phi, \varphi, \psi \in L^* \). So \( \Delta_{D(L)^*} = \Delta_{L^{\text{cop}} \oplus L^*} + \text{ad}(t) \).

From direct computation,
\[
\text{CHYB}(t) + \circ (\alpha_{L^{\text{cop}} \oplus L^*} \otimes \Delta_{L^{\text{cop}} \oplus L^*})(t) = 0,
\]
then the codouble Hom-Lie bialgebra \( D(L)^* \) is a Hom-Lie bialgebra from Theorem 5.1 in [25].

Since the linear dual of any finite dimensional quasi-triangular Hom-Lie bialgebra is a co-quasi-triangular Hom-Lie bialgebra, \( D(L)^* \) is a co-quasi-triangular Hom-Lie bialgebra by Theorem 3.1, which completes the proof.

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