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Generalized Bessel Function Associated with Dihedral Groups

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Abstract. Motivated by Dunkl operators theory, we consider a generating series involving a modified Bessel function and a Gegenbauer polynomial, that generalizes a known series already considered by L. Gegenbauer. We actually use inversion formulas for Fourier and Radon transforms to derive a closed formula for this series when the parameter of the Gegenbauer polynomial is a positive integer. As a by-product, we get a relatively simple integral representation for the generalized Bessel function associated with dihedral groups D_n, $n \geq 2$ when both multiplicities sum to an integer. In particular, we recover a previous result obtained for D_4 and we give a special interest to D_6. Finally, we derive similar results for odd dihedral groups.
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1. Motivation

The dihedral group D_n of order $n \geq 2$ is defined as the group of regular n-gone preserving-symmetries ([8]). It figures among reflection groups associated with irreducible root systems yet ceases to be crystallographic unless $n = 2, 3, 4, 6$. Nevertheless the theory of rational Dunkl operators introduced in the late eighties associates to reduced non necessarily crystallographic root systems generalized Bessel functions that extend spherical functions on symmetric spaces of Euclidean type from a discrete to a continuous range of multiplicities (see Ch.I in [3]). In fact, the radial part of the Laplace-Beltrami operator on these symmetric spaces fits the reflection group-invariant part of the Dunkl Laplacian with special multiplicities (this fact holds for radial parts of a more general class of differential operators). However, they are not easy to handle unlike spherical functions, except possibly in lower ranks. In fact, they are expressed for the four infinite series of irreducible root systems as multivariate hypergeometric series defined via Jack polynomials ([4]). Nonetheless, Jack polynomials may be expressed by means of Gegenbauer polynomials in ranks one and two ([13]). Moreover, probabilistic considerations led to the following expression for the generalized Bessel function

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associated with dihedral systems ([5]). Let $n = 2p, p \geq 1$ and let D_k^W denote the generalized Bessel function depending in this case on the two real variables $x = \rho e^{i\phi}, y = re^{i\theta}, \rho, r \geq 0, \phi, \theta \in [0, \pi/2p]$. Then

$$D_k^W(\rho, \phi, r, \theta) = c_{p,k}\left(\frac{2}{r\rho}\right)^{\gamma} \sum_{j\geq 0} I_{2jp+\gamma}(\rho r) p_j^{l_1,l_0}(\cos(2p\phi)) p_j^{l_1,l_0}(\cos(2p\theta)) \tag{1}$$

where

- $k = (k_0, k_1)$ is a positive-valued multiplicity function, $l_i = k_i - 1/2, i \in \{1, 2\}$, $\gamma = p(k_0 + k_1)$.
- $I_{2jp+\gamma}, p_j^{l_1,l_0}$ are the modified Bessel function of index $2jp + \gamma$ and the j-th orthonormal Jacobi polynomial of parameters l_1, l_0 respectively (the orthogonality (Beta) measure need not to be normalized here. In fact, the normalization only alters the constant $c_{p,k}$ below).
- The constant $c_{p,k}$ depends on p, k and is such that $D_k^W(0, y) = 1$ for all $y = (r, \theta) \in [0, \infty) \times [0, \pi/2p]$ (see [6])
$$c_{p,k} = 2^{k_0+k_1} \frac{\Gamma(p(k_1 + k_0) + 1)\Gamma(k_1 + 1/2)\Gamma(k_0 + 1/2)}{\Gamma(k_0 + k_1 + 1)}.$$
- A similar formula holds for odd dihedral systems (see the fourth section).

Once this relatively simple formula was obtained, the special case $p = 2$ was the main object of a subsequent paper ([6]), aiming to work out the series displayed in (1). The main achievement was then realized when $\gamma = 2(k_0 + k_1)$ is an even integer and according to Corollary 1.2 in [6]

$$D_k^W(\rho, \phi, r, \theta) = \int\int i_{(\gamma-1)/2}\left(\rho r\sqrt{\frac{1 + z_{2\phi,2\theta}(u, v)}{2}}\right) \mu^{l_1}(du)\mu^{l_0}(dv)$$

where

$$i_\alpha(x) := \sum_{m=0}^{\infty} \frac{1}{(\alpha + 1)_m m!}\left(\frac{x}{2}\right)^{2m}, \ \alpha > -1,$$

is the normalized modified Bessel function ([8]). In this paper, we shall see that this achievement is not specific to the value $p = 2$ but rather extends to all $p \geq 1$ provided that $k_0 + k_1$ is a positive integer and is even related to geometrical considerations on spheres that considerably avoid tedious computations performed in [6]. This is seen as follows: start with Dijksma-Koornwinder product formula for Jacobi polynomials ([7]) which may be written in the following way ([6]):

$$c(\alpha, \beta)p_j^{\alpha,\beta}(\cos 2\phi)p_j^{\alpha,\beta}(\cos 2\theta) = (2j + \alpha + \beta + 1)\int\int C_{2j}^{\alpha+\beta+1}(z_{\phi,\theta}(u, v))\mu^\alpha(du)\mu^\beta(dv)$$

where $\alpha, \beta > -1/2$,

$$c(\alpha, \beta) = 2^{\alpha+\beta+1} \frac{\Gamma(\alpha + 1)\Gamma(\beta + 1)}{\Gamma(\alpha + \beta + 1)},$$

$$z_{\phi,\theta}(u,v) = u\cos\theta\cos\phi + v\sin\theta\sin\phi,$$

and μ^α is the symmetric Beta probability measure whose density is given by

$$\mu^\alpha(du) = \frac{\Gamma(\alpha+1)}{\sqrt{\pi}\Gamma(\alpha+1/2)}(1-u^2)^{\alpha-1/2}\mathbf{1}_{[-1,1]}(u)du, \quad \alpha > -1/2.$$

Next, invert the order of integration in (1) to see that

$$D_k^W(\rho,\phi,r,\theta) \propto \int\int \left(\frac{2}{r\rho}\right)^\gamma \sum_{j\geq 0}(2j+k_0+k_1)I_{2jp+\gamma}(\rho r)C_{2j}^{k_0+k_1}(z_{p\phi,p\theta}(u,v)) \\ \mu^\alpha(du)\mu^\beta(dv) \tag{2}$$

where the notation $\propto$ means that equality holds up to a constant factor. Now, the integrand in (2) is obviously the sum of the following series

$$f_{\nu,p}^{\pm}(R,\cos\zeta) := \left(\frac{2}{R}\right)^{p\nu}\sum_{j\geq 0}(\pm 1)^j(j+\nu)I_{p(j+\nu)}(R)C_j^\nu(\cos\zeta) \tag{3}$$

where we set $\nu := k_0+k_1, R := \rho r$ and $\cos\zeta := \cos\zeta(u,v) = z_{p\phi,p\theta}(u,v)$.

Note actually that closed formulas for $f_{\nu,1}^{\pm}$ are due to L. Gegenbauer (equations (4), (5), p.369 in [14])

$$\left(\frac{2}{r\rho}\right)^\gamma \sum_{j\geq 0}(\pm 1)^j(j+\gamma)I_{j+\gamma}(\rho r)C_j^\gamma(\cos\zeta) = \frac{1}{\Gamma(\gamma)}e^{\pm\rho r\cos\zeta}$$

and were used in [6]. For general $p\geq 1$, we shall interpret the sequence

$$(\pm 1)^j I_{p(j+\nu)}(R),\ j\geq 0$$

for fixed R as the Gegenbauer-Fourier coefficients of $\zeta\mapsto f_{\nu,p}^{\pm}(R,\cos\zeta)$ corresponding to the Gegenbauer-Fourier transform studied in [1]. When $\nu = (d-1)/2$ for some integer $d\geq 1$, this is in fact the Fourier transform on the sphere S^d considered as a homogenous space $SO(d+1)/SO(d)$ since spherical functions are expressed through Gegenbauer polynomials ([1] p.356). Thus, deriving closed formulas for $f_{\nu,p}^{\pm}$ when ν is a positive integer amounts to appropriately use inversion formulas for Fourier and Radon transforms. Our main result is then stated as

Theorem 1.1. *Assume ν is an integer and $\nu\geq 1$, then*

$$\left(\frac{R}{2}\right)^{p\nu} f_{\nu,p}^{\pm}(R,\cos\zeta) = \frac{1}{2^\nu(\nu-1)!}\left[-\frac{1}{\sin\zeta}\frac{d}{d\zeta}\right]^\nu \frac{1}{p}\sum_{s=1}^{p} e^{\pm R\cos[(\zeta+2\pi s)/p]}.$$

When $p=2$, we shall use

$$\cos(\zeta/2) = \sqrt{\frac{1+\cos\zeta}{2}}, \quad \zeta\in[0,\pi].$$

together with appropriate formulas for modified Bessel functions in order to recover Corollary 1.2. in [6], while when $p=3$ we shall solve a special cubic equation relying on results from analytic function theory rather than Cardan formulas. The required solution is then expressed by means of Gauss hypergeometric functions ([11]) yielding therefore a somehow explicit formula for the series (2), though much more complicated than the one derived for $p = 2$. For general $p \geq 3$, one can use the expansion proved in Lemma 2.1 below which plays a key role in the proof of our main result. The paper is closed with adapting our method to odd dihedral groups, in particular to D_3 thereby exhausting the list of dihedral groups that are Weyl groups ($p = 1$ corresponds to the product group $\mathbb{Z}_2^2$ for which D_k^W is namely a product of normalized modified Bessel functions).

Remark 1.2. We have a strong belief that the assumption $\nu = k_0 + k_1$ is an integer is an optimal restriction. Indeed, Proposition 1.1 in [6] shows that even for the simplest case corresponding to $p = 2$, one gets a highly complicated expression for D_k^W as soon as ν fails to be an integer.

2. Proof of the main result

Recall the orthogonality relation for Gegenbauer polynomials ([8]):

$$\begin{aligned}\int_0^\pi C_j^\nu(\cos\zeta)C_m^\nu(\cos\zeta)(\sin\zeta)^{2\nu}d\zeta &= \delta_{jm}\frac{\pi\Gamma(j+2\nu)2^{1-2\nu}}{\Gamma^2(\nu)(j+\nu)j!}\\ &= \delta_{jm}\frac{\pi 2^{1-2\nu}\Gamma(2\nu)}{(j+\nu)\Gamma^2(\nu)}C_j^\nu(1)\\ &= \delta_{jm}\nu\frac{\sqrt{\pi}\Gamma(\nu+1/2)}{\Gamma(\nu+1)}\frac{C_j^\nu(1)}{(j+\nu)}\end{aligned}$$

where we used Gauss duplication formula ([8])

$$\sqrt{\pi}\Gamma(2\nu) = 2^{2\nu-1}\Gamma(\nu)\Gamma(\nu+1/2),$$

and the special value ([8])

$$C_j^\nu(1) = \frac{(2\nu)_j}{j!}.$$

Equivalently, if $\mu^\nu(d\cos\zeta)$ is the image of $\mu^\nu(d\zeta)$ under the map $\zeta \mapsto \cos\zeta$, then

$$(j+\nu)\int C_j^\nu(\cos\zeta)C_m^\nu(\cos\zeta)\mu^\nu(d\cos\zeta) = \nu C_j^\nu(1)\delta_{jm}$$

so that (3) yields

$$\nu(\pm 1)^j\left(\frac{2}{R}\right)^{p\nu} I_{p(j+\nu)}(R) = \int P_j^\nu(\cos\zeta)f_{\nu,p}^{\pm}(R,\cos\zeta)\mu^\nu(d\cos\zeta) \tag{4}$$

where

$$P_j^\nu(\cos\zeta) := C_j^\nu(\cos\zeta)/C_j^\nu(1)$$

is the j-th normalized Gegenbauer polynomial. Thus, the j-th Gegenbauer-Fourier coefficients of $\zeta \mapsto f_{\nu,p}^{\pm}(R,\cos\zeta)$ are given by

$$\nu(\pm 1)^j\left(\frac{2}{R}\right)^{p\nu} I_{p(j+\nu)}(R), \quad p \geq 1.$$

Following [1] p.356, the Mehler integral ([9], p.177)

$$P_j^{\nu}(\cos\zeta) = 2^{\nu}\frac{\Gamma(\nu+1/2)}{\Gamma(\nu)\sqrt{\pi}}(\sin\zeta)^{1-2\nu}\int_0^{\zeta}\cos[(j+\nu)t](\cos t - \cos\zeta)^{\nu-1}dt$$

valid for real $\nu > 0$, transforms (4) to $\left(\frac{2}{R}\right)^{p\nu}(\pm 1)^j I_{p(j+\nu)}(R)$

$$\begin{aligned} &= \frac{2^{\nu}}{\pi}\int_0^{\pi} f_{\nu,p}^{\pm}(R,\cos\zeta)\sin\zeta\int_0^{\zeta}\cos[(j+\nu)t](\cos t-\cos\zeta)^{\nu-1}dtd\zeta \\ &= \frac{2^{\nu}}{\pi}\int_0^{\pi}\cos[(j+\nu)t]\int_t^{\pi} f_{\nu,p}^{\pm}(R,\cos\zeta)\sin\zeta(\cos t-\cos\zeta)^{\nu-1}d\zeta dt. \end{aligned} \tag{5}$$

The second integral displayed in the RHS of the second equality is known as the Radon transform of $\zeta \mapsto f_{\nu,p}^{\pm}(R,\cos\zeta)$ and inversion formulas already exist ([1]). As a matter of fact, we firstly need to express $(\pm 1)^{j+\nu} I_{p(j+\nu)}$, when $\nu \geq 1$ is an integer, as the Fourier-cosine coefficient of order $j+\nu$ of some function. This is a consequence of the Lemma below. Secondly, we shall use the appropriate inversion formula for the Radon transform derived in [1]).

Lemma 2.1. *For any integer $p \geq 1$ and any $t \in [0,\pi]$:*

$$2\sum_{j\geq 0}(\pm 1)^j I_{pj}(R)\cos(jt) = I_0(R) + \frac{1}{p}\sum_{s=1}^{p} e^{\pm R\cos[(t+2\pi s)/p]}.$$

Proof. We will prove the $(+)$ part, the proof of the $(-)$ part follows the same lines with minor modifications. Write

$$\begin{aligned} 2\sum_{j\geq 0} I_{pj}(R)\cos(jt) &= \sum_{j\geq 0} I_{pj}(R)[e^{ijt}+e^{-ijt}] \\ &= I_0(R) + \sum_{j\in\mathbb{Z}} I_{pj}(R)e^{ijt} \end{aligned}$$

where we used the fact that $I_j(r) = I_{-j}(r), j \geq 0$. Using the identity

$$\frac{1}{m}\sum_{s=1}^{m} e^{2i\pi sj/m} = \begin{cases} 1 & \text{if} \quad j \equiv 0[m], \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

valid for any integer $m \geq 1$, one obviously gets

$$\sum_{j\in\mathbb{Z}} I_{pj}(R)e^{ijt} = \frac{1}{p}\sum_{s=1}^{p}\sum_{j\in\mathbb{Z}} I_j(R)e^{ij(t+2\pi s)/p}.$$

The $(+)$ part of the Lemma then follows from the generating series for modified Bessel functions ([14]):

$$e^{(z+1/z)R/2} = \sum_{j\in\mathbb{Z}} I_j(R)z^j,\ z\in\mathbb{C}. \qquad \blacksquare$$

Now, we use the Lemma to get

$$I_{pj}(R) = I_0(R)\delta_{j0} + \frac{1}{\pi}\int_0^{\pi} \cos(jt)\frac{1}{p}\sum_{s=1}^{p} e^{\pm R\cos[(t+2\pi s)/p]}dt$$

for any integer $j \geq 0$. Assuming that ν is a stricltly positive integer, one then recovers

$$I_{p(j+\nu)}(R) = \frac{1}{\pi}\int_0^{\pi} \cos[(j+\nu)t]\frac{1}{p}\sum_{s=1}^{p} e^{\pm R\cos[(t+2\pi s)/p]}dt. \tag{7}$$

Note that

$$t \mapsto \int_t^{\pi} f(R, \cos\zeta)\sin\zeta(\cos t - \cos\zeta)^{\nu-1}d\zeta$$

as well as

$$t \mapsto \frac{1}{p}\sum_{s=1}^{p} e^{\pm R\cos[(t+2\pi s)/p]}$$

are even functions. This is true since

$$\zeta \mapsto f(R, \cos\zeta)(\sin\zeta)(\cos t - \cos\zeta)^{\nu-1}$$

is an odd function so that

$$\int_{-t}^{t} f(R, \cos\zeta)\sin\zeta(\cos t - \cos\zeta)^{\nu-1}d\zeta = 0,$$

and since

$$\cos[(-t + 2s\pi)/p] = \cos[(t + 2(p-s)\pi)/p]$$

so that one performs the index change $s \to p - s$ and notes that the terms corresponding to $s = 0$ and $s = p$ are equal. Similar arguments yield the 2π-periodicity of these functions, therefore the Fourier-cosine transforms of their restrictions on $(-\pi, \pi)$ coincide with their Fourier transforms on that interval. As a matter of fact,

$$\left(\frac{R}{2}\right)^{p\nu}\int_t^{\pi} f_{\nu,p}(R, \cos\zeta)\sin\zeta(\cos t - \cos\zeta)^{\nu-1}d\zeta = \frac{1}{2^{\nu}p}\sum_{s=1}^{p} e^{\pm R\cos[(t+2\pi s)/p]}$$

for all t since both functions are continuous and our main result follows from Theorem 3.1. p.363 in [1].

Corollary 2.2. *For any integer $\nu \geq 1$ we have $\sum_{j\geq 0}(2j+\nu)I_{p(2j+\nu)}(R)C_{2j}^{\nu}(\cos\zeta)$*

$$= \frac{1}{2^{\nu}\Gamma(\nu)}\left[-\frac{1}{\sin\zeta}\frac{d}{d\zeta}\right]^{\nu}\frac{1}{p}\sum_{s=1}^{p}\cosh\left(R\cos[(\zeta + 2\pi s)/p]\right).$$

3. Dihdral groups D_4, D_6

3.1. p=2. Letting $p = 2$ and using the fact that cosh is an even function, our main result yields $\left(\frac{4}{R^2}\right)^\nu \sum_{j\geq 0}(2j+\nu)I_{2(2j+\nu)}(R)C_{2j}^\nu(\cos\zeta)$

$$= \frac{1}{2^\nu\Gamma(\nu)}\left[-\frac{4}{R^2\sin\zeta}\frac{d}{d\zeta}\right]^\nu \cosh\left(R\cos(\cdot/2)\right)(\zeta).$$

Noting that for a function f

$$-\frac{4}{R^2\sin\zeta}\frac{d}{d\zeta}f\left(R\cos\frac{\zeta}{2}\right)(\zeta) = \left[\frac{1}{u}\frac{d}{du}f(u)\right]_{|u=R\cos(\zeta/2)},$$

and using the classical formula (see for instance (5.8.3) in [12])

$$\left(\frac{1}{z}\frac{d}{dz}\right)^{\nu-1}\frac{\sin z}{z} = (-1)^{\nu-1}\sqrt{\frac{\pi}{2}}\frac{1}{z^{\nu-1/2}}J_{\nu-1/2}(z),$$

one obtains

$$\left(\frac{4}{R^2}\right)^\nu \sum_{j\geq 0}(2j+\nu)I_{2(2j+\nu)}(R)C_{2j}^\nu(\cos\zeta) = \frac{1}{2\Gamma(2\nu)}i_{\nu-1/2}\left(R\cos\frac{\zeta}{2}\right),$$

and finally recovers Corollary 1.2 in [6] since $c_{2,k}/c(k_1-1/2, k_0-1/2) = \Gamma(2\nu+1)/\nu$.

3.2. p=3. The corresponding dihedral group D_6 is a two dimensional representation of the exceptional Weyl group G_2 ([2]). Let $\zeta \in]0,\pi[$ and start with the linearization formula:

$$4\cos^3(\zeta/3) = \cos\zeta + 3\cos(\zeta/3).$$

Thus, we are led to find a root lying in $[-1,1]$ of the cubic equation

$$Z^3 - (3/4)Z - (\cos\zeta)/4 = 0$$

for $|Z| < 1$. Set $Z = (\sqrt{-1}/2)T, |T| < 2$, the above cubic equation transforms to

$$T^3 + 3T - 2\sqrt{-1}\cos\zeta = 0.$$

The obtained cubic equation already showed up in analytic function theory in relation to the local inversion Theorem ([11] p.265-266). Amazingly (compared to Cardan formulas), its real and both complex roots are expressed through the Gauss hypergeometric function ${}_2F_1$. Since we are looking for real $Z = (\sqrt{-1}/2)T$, we shall only consider the complex roots (see the bottom of p. 266 in [11]):

$$T^\pm = \pm\sqrt{-1}\left[\sqrt{3}\,{}_2F_1\left(-\frac{1}{6},\frac{1}{6},\frac{1}{2};\cos^2\zeta\right) - \frac{1}{3}\cos\zeta\,{}_2F_1\left(\frac{1}{3},\frac{2}{3},\frac{3}{2};\cos^2\zeta\right)\right]$$

so that

$$Z^\pm = \pm\left[\frac{\sqrt{3}}{2}\,{}_2F_1\left(-\frac{1}{6},\frac{1}{6},\frac{1}{2};\cos^2\zeta\right) - \frac{1}{6}\cos\zeta\,{}_2F_1\left(\frac{1}{3},\frac{2}{3},\frac{3}{2};\cos^2\zeta\right)\right].$$

Since for $\zeta = \pi/2$, $\cos\zeta/3 = \cos\pi/6 = \sqrt{3}/2$, it follows that

$$\cos(\zeta/3) = \left[\frac{\sqrt{3}}{2}\,{}_2F_1\left(-\frac{1}{6},\frac{1}{6},\frac{1}{2};\cos^2\zeta\right) - \frac{1}{6}\cos\zeta\,{}_2F_1\left(\frac{1}{3},\frac{2}{3},\frac{3}{2};\cos^2\zeta\right)\right]$$

for all $\zeta \in (0,\pi)$. Now, write $Z = Z(\cos\zeta)$ so that

$$\begin{aligned}\cos[(\zeta+2s\pi)/3] &= \cos(2s\pi/3)\cos(\zeta/3) - \sin(2s\pi/3)\sqrt{1-\cos^2(\zeta/3)}\\ &= \cos(2s\pi/3)Z(\cos\zeta) - \sin(2s\pi/3)\sqrt{1-Z^2(\cos\zeta)}\end{aligned}$$

for any $1 \leq s \leq 3$. It follows that

$$f_{\nu,3}(R,\cos\zeta) = \frac{1}{3\Gamma(\nu)}\left[-\frac{4}{R^3\sin\zeta}\frac{d}{d\zeta}\right]^{\nu}\sum_{s=1}^{3} g_s(RZ(\cos\zeta))$$

where

$$g_s(u) = \cosh\left[\left(\cos(2s\pi/3)u - \sin(2s\pi/3)\sqrt{R^2-u^2}\right)\right], u \in (-1,1).$$

Finally,

$$f_{\nu,3}(R,\cos\zeta) = \frac{1}{3\Gamma(\nu)}\left[\frac{4}{R^3}\frac{d}{du}\right]^{\nu}\sum_{s=1}^{3} h_s(u)_{|u=\cos\zeta}$$

where $h_s(u) := g_s(RZ(u)), 1 \leq s \leq 3$. For instance, let $\nu = 1$, then it is not difficult to see that

$$\frac{d}{du}h_s(u)_{|u=\cos\zeta} = \frac{R}{\sin\zeta/3}\frac{dZ}{du}_{|u=\cos\zeta}\sin\left(\frac{\xi+2\pi s}{3}\right)\sinh\left[\sin\left(\frac{\xi+2\pi s}{3}\right)\right]$$

for any $s \in \{1,2,3\}$ and the derivative of $u \mapsto Z(u)$ is computed using the differentiation formula for ${}_2F_1$:

$$\frac{d}{du}{}_2F_1(a,b,c;u) = \frac{ab}{c}{}_2F_1(a+1,b+1,c+1;u),\ |u|<1, c\neq 0.$$

As the reader may conclude, formulas are cumbersome compared to the ones derived for $p = 2$.

4. Odd Dihedral groups

Let $n \geq 3$ be an odd integer and consider odd dihedral groups D_n, then ([5] p.157)

$$D_k^W(\rho,\phi,r,\theta) = c_{n,k}\left(\frac{2}{r\rho}\right)^{nk}\sum_{j\geq 0} I_{n(2j+k)}(\rho r)p_j^{-1/2,l_0}(\cos(2n\phi))p_j^{-1/2,l_0}(\cos(2n\theta)) \tag{8}$$

where $k \geq 0, \rho, r \geq 0, \theta, \phi \in [0,\pi/n]$, and

$$c_{n,k} = 2^k\Gamma(nk+1)\frac{\sqrt{\pi}\Gamma(k+1/2)}{\Gamma(k+1)}.$$

In order to adapt our method to these groups, we need to write down the product formula for orthonormal Jacobi polynomials in the limiting case $\alpha = -1/2$ or equivalently $k = 0$. This task was achieved in [7] p.194 using implicitly the fact that the Beta distribution μ^α converges weakly to the Dirac mass δ_1 as $\alpha \to -1/2$. In order to fit it into our normalizations, we proceed as follows: use the well-known quadratic transformation ([8]):

$$\begin{aligned} P_j^{-1/2,k-1/2}(1 - 2\sin^2(n\theta)) &= (-1)^j P_j^{k-1/2,-1/2}(2\sin^2(n\theta) - 1) \\ &= (-1)^j \frac{(1/2)_j}{(k)_j} C_{2j}^k(\sin(n\theta)) \end{aligned}$$

where $P_j^{\alpha,\beta}$ is the (non orthonormal) j-th Jacobi polynomial, together with $\cos(2n\theta) = 1 - 2\sin^2(n\theta)$ to obtain

$$P_j^{-1/2,k-1/2}(\cos(2n\theta))P_j^{-1/2,k-1/2}(\cos(2n\phi)) = \left[\frac{(1/2)_j}{(k)_j}\right]^2 C_{2j}^k(\sin(n\theta))C_{2j}^k(\sin(n\phi)).$$

Now, let $k > 0$ and recall that the squared L^2-norm of $P_j^{-1/2,k-1/2}$ is given by ([8])

$$\frac{2^k}{2j+k}\frac{\Gamma(j+1/2)\Gamma(j+k+1/2)}{j!\Gamma(j+k)} = \frac{2^k\sqrt{\pi}\Gamma(k+1/2)}{\Gamma(k)}\frac{(1/2)_j}{(k)_j}\frac{(k+1/2)_j}{(2j+k)j!}.$$

Recall also the special value

$$C_{2j}^k(1) = \frac{(2k)_{2j}}{(2j)!} = 2\frac{(k)_j(k+1/2)_j}{(1/2)_j j!}.$$

It follows that $c(k)p_j^{-1/2,k-1/2}(\cos(2n\theta))p_j^{-1/2,k-1/2}(\cos(2n\phi))$

$$\begin{aligned} &= \frac{(1/2)_j}{(k)_j}\frac{(2j+k)j!}{(k+1/2)_j}C_{2j}^k(\sin(n\theta))C_{2j}^k(\sin(n\phi)) \\ &= \frac{(2j+k)}{C_{2j}^k(1)}C_{2j}^k(\sin(n\theta))C_{2j}^k(\sin(n\phi)) \\ &= (2j+k)\int C_{2j}^k\left(z_{n\phi,n\theta}(u,1)\right)\mu^k(du), \end{aligned}$$

according to [7] p.194, where

$$c(k) := \frac{2^{k+1}\sqrt{\pi}\Gamma(k+1/2)}{\Gamma(k)}.$$

As a matter of fact, we are led again to

$$\left(\frac{2}{R}\right)^{nk}\sum_{j\geq 0}(2j+k)I_{n(2j+k)}(R)C_{2j}^k(\cos\zeta) = \frac{1}{2}[f_{k,n}^+ + f_{k,n}^-](R,\cos\zeta).$$

5. A Concluding Remark

The occurrence of the Radon transform on spheres in higher dimensions in relation to generalized Bessel functions associated with dihedral groups is somehow intriguing and unexplained only analytically. Below is another yet similar occurrence: recall that $D_k^W(\cdot, y)$ is an eigenfunction of the Dunkl Laplacian Δ_k associated with the eigenvalue $|y|^2$ ([3], Ch. I). When acting on D_n-invariant functions, this operator, denoted Δ_k^W, reads ([5]):

$$\Delta_k^W = \partial_r^2 + \frac{2nk+1}{r}\partial_r + \frac{1}{r^2}\left[\partial_\theta^2 + 2nk\cot(n\theta)\partial_\theta\right]$$

for odd integers n while

$$\Delta_k^W = \partial_r^2 + \frac{2p(k_0+k_1)+1}{r}\partial_r + \frac{1}{r^2}\left[\partial_\theta^2 + 2p(k_0\cot(p\theta) - k_1\tan(p\theta))\partial_\theta\right]$$

for even $n = 2p$. But for odd values of n, Proposition 2.3 p.197 in [10] shows that Δ_k^W coincides with the Euclidean Laplacian in dimension $2nk+2$ acting on $SO(2nk+1)$-invariant functions. Besides, a similar interpretation holds for even values of n and equal multiplicities $k_0 = k_1$ after one notices that

$$\cot(\theta) - \tan(\theta) = 2\cot(2\theta).$$

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