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Invariant Distributions on Non-Distinguished Nilpotent Orbits with Application to the Gelfand Property of $(GL_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{R}))$

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Abstract. We study invariant distributions on the tangent space to a symmetric space. We prove that an invariant distribution with the property that both its support and the support of its Fourier transform are contained in the set of non-distinguished nilpotent orbits, must vanish. We deduce, using recent developments in the theory of invariant distributions on symmetric spaces, that the symmetric pair $(GL_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{R}))$ is a Gelfand pair. More precisely, we show that for any irreducible smooth admissible Fréchet representation (π, E) of $GL_{2n}(\mathbb{R})$ the space of continuous functionals $Hom_{Sp_{2n}(\mathbb{R})}(E, \mathbb{C})$ is at most one dimensional. Such a result was previously proven for p-adic fields in M. J. Heumos and S. Rallis, Symplectic-Whittaker models for Gl_n, Pacific J. Math. **146** (1990), 247–279, and for $\mathbb{C}$ in E. Sayag, $(GL_{2n}(\mathbb{C}), Sp_{2n}(\mathbb{C}))$ is a Gelfand pair, arXiv:0805.2625 [math.RT].
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1. Introduction

Let (V, ω) be a finite dimensional symplectic vector space over $\mathbb{R}$. Consider the standard imbedding $Sp(V) := Aut(V, \omega) \subset GL(V)$ and the natural action of $Sp(V) \times Sp(V)$ on $GL(V)$. In this paper we prove the following theorem:

Theorem A. Any $Sp(V) \times Sp(V)$ - invariant distribution on $GL(V)$ is invariant with respect to transposition.

It has the following corollary in representation theory:

Theorem B. Let (V, ω) be a symplectic vector space and let E be an irreducible admissible smooth Fréchet representation of $GL(V)$. Then

$$\dim Hom_{Sp(V)}(E, \mathbb{C}) \leq 1$$

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In the language of [AGS], Theorem B means that the pair $(GL(V), Sp(V))$ is a Gelfand pair. In particular, Theorem B implies that the spectral decomposition of the unitary representation $L^2(GL(V)/Sp(V))$ is multiplicity free (see e.g. [Lip]).

Theorem B is deduced from Theorem A using the Gelfand-Kazhdan method (adapted to the archimedean case in [AGS]).

The analogue of Theorem A and Theorem B for non-archimedean fields were proven in [HR] using the method of Gelfand and Kazhdan. A simple argument over finite fields is explained in [GG] and using this a simpler proof of the non-archimedean case was written in [OS3]. Recently, one of us, using the ideas of [AG2] extended the result to the case $F = \mathbb{C}$ (see [Say1]).

Our proof of Theorem A is based on the methods of [AG2]. In that work the notion of regular symmetric pair was introduced and shown to be a useful tool in the verification of the Gelfand property. Thus, the main result of the present work is the *regularity* of the symmetric pair $(GL(V), Sp(V))$. The notion of regularity concerns the action of additional symmetries of the symmetric space on the space of invariant distributions on its tangent space. In previous works the proof of regularity of symmetric pairs was based either on some simple considerations or on a criterion that requires negativity of certain eigenvalues (this was implicit in [JR], [RR] and was explicated in [AG2], [AG3], [Say1]).

The pair $(GL(V), Sp(V))$ does not satisfy the above mentioned criterion and requires new techniques.

1.1. Main ingredients of the proof.
To show regularity we study distributions on the space
$$\mathfrak{q} = \{X \in gl_{2n} : JX^t = XJ\} \text{ where } J = \begin{pmatrix} 0_n & Id_n \\ -Id_n & 0_n \end{pmatrix}.$$
More precisely, we are interested in those distributions that are invariant with respect to the conjugation action of Sp_{2n} and supported on the nilpotent cone. To classify the nilpotent orbits of the action we use the method of [GG] to identify these orbits with nilpotent orbits of the adjoint action of GL_n on its Lie algebra. This allows us to show that there exists a unique *distinguished* (in the sense of e.g. [Sek]) nilpotent orbit $\mathcal{O} \subset \mathfrak{q}$ and that this orbit is open in the nilpotent cone of $\mathfrak{q}$. Next, we use the theory of D-modules, as in [AG4], to prove that there are no distributions supported on non-distinguished orbits whose Fourier transform is also supported on non-distinguished orbits (see Theorem 4.1).

1.2. Related works.
The problem of identifying symmetric pairs that are Gelfand pairs was studied by various authors. In the case of symmetric spaces of rank one this problem was studied extensively in [RR], [vD], [BvD] both in the archimedean and non-archimedean case. Recently, cases of symmetric spaces of high rank were studied in [AGS], [AG2], [AG3], [Say2]. However, as hinted above, all these works could treat a restricted class of symmetric pairs, first introduced in [Sek] that are now commonly called *nice* symmetric pairs.

The pair $(GL(V), Sp(V))$ is *not* a nice symmetric pair and additional methods are needed to study invariant distributions on the corresponding symmetric space. For that, we use the theory of D-modules as in [AG4] and analysis of the nilpotent cone of the pair in question, in order to prove the Gelfand property.

In the non-archimedean case, the pair (GL_{2n}, Sp_{2n}) is a part of a list (GL_{2n}, H_k, ψ_k), $k = 0, 1, ..., n$, of twisted Gelfand pairs that provide a model in the sense of [BGG] to the unitary representations of GL_{2n}. Namely, every irreducible unitarizable representation of GL_{2n} appears exactly once in $\bigoplus_{k=0}^{n} Ind_{H_k}^{GL_{2n}}(\psi_k)$ (see [OS1],[OS2],[OS3]). Considering the strategy taken in those works, a major first step in transferring these results to the archimedean case is taken in the present paper. Indeed, recently, in [AOS] we use the results of the present work to study uniqueness and disjointness for the archimedean analogues of the above pairs.

1.3. Structure of the paper.
In section 2 we give some preliminaries on distributions, symmetric pairs and Gelfand pairs. We introduce the notion of regular symmetric pairs and show that Theorem 7.4.5 of [AG2] and the results of [Say1] allow us to reduce the Gelfand property of the pair in question to proving that the pair is regular. In section 4 we prove the main technical result on distributions, Theorem 4.1. It states that under certain conditions there are no distributions supported on non-distinguished nilpotent orbits. The proof is based on the theory of D-modules. In section 5 we use Theorem 4.1 to prove that the pair $(GL(V), Sp(V))$ is regular.

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2. Preliminaries

2.1. Nash Manifolds, Bundles.
We will use the theory of Schwartz functions and distributions as developed in [AG1]. This theory is developed for Nash manifolds. Nash manifolds are smooth semi-algebraic manifolds but in the present work only smooth real algebraic manifolds are considered. Therefore the reader can safely replace the word *Nash* by *smooth real algebraic.*

We remind that T_X, the tangent bundle of the Nash manifold X, carries a Nash structure. Furthermore, if $Z \subset X$ is a Nash submanifold we denote by $N_Z^X := (T_X|_Z)/T_Z$ the normal bundle to Z in X. We also denote by $CN_Z^X := (N_Z^X)^*$ the co-normal bundle. For a point $z \in Z$ we denote by $N_{Z,z}^X$ the normal space to Z in X at the point z and by $CN_{Z,z}^X$ the co-normal space.

2.2. Schwartz distributions on Nash manifolds.
Schwartz functions are functions that decay, together with all their derivatives, faster than any polynomial. On $\mathbb{R}^n$ it is the usual notion of Schwartz function. For precise definitions of those notions we refer the reader to [AG1].

Let X be a Nash manifold and let $\mathcal{S}(X)$ be the Fréchet space of Schwartz functions on X. We let $\mathcal{S}^*(X) := \mathcal{S}(X)^*$ be the space of Schwartz distributions on

X. More generally, for any Nash vector bundle E over X we denote by $\mathcal{S}(X,E)$ the space of Schwartz sections of E and by $\mathcal{S}^*(X,E)$ its dual space.

For a smooth manifold X and $Z \subset X$ a closed subset we denote by $\mathcal{S}^*_X(Z)$ the space of distributions on X supported in Z. Thus

$$\mathcal{S}^*_X(Z) := \{\xi \in \mathcal{S}^*(X) | \mathrm{Supp}(\xi) \subset Z\}.$$

More generally, for a locally closed subset $Y \subset X$ we denote $\mathcal{S}^*_X(Y) := \mathcal{S}^*_{X \setminus (\overline{Y} \setminus Y)}(Y)$. In the same way, for any bundle E on X we define $\mathcal{S}^*_X(Y,E)$.

Lemma 2.1. *For a Nash manifold X and an open Nash submanifold $U \subset X$, we have the following exact sequence*

$$0 \to \mathcal{S}^*_X(X \setminus U) \to \mathcal{S}^*(X) \to \mathcal{S}^*(U) \to 0.$$

For the proof see e.g. [AG1] Theorem 5.4.3.

Remark 2.2. The above sequence fail to be right exact when usual distributions are considered.

2.3. Invariant distributions: Basic tools.

Given a group G acting on a space X we would like to study distributions on X that are G equivariant. For this we employ a few standard techniques. For the benefit of the reader we state the exact statements below with some commentary.

The first is a technique allowing the study of a equivariant distribution on a space by using a stratification of the space. If X is a Nash manifold we say that $X = \bigcup_{k=0}^{l} X_k$ is a Nash stratification of X if X_i are Nash manifolds and $\bigcup_{k=0}^{r} X_k$ is open for $0 \le r \le l$.

If G is a Nash group acting on X and the strata are G- invariant then we say that the stratification is G-invariant.

Proposition 2.3. *Let a Nash group G act on a Nash manifold X. Let $Z \subset X$ be a closed subset.*

Let $Z = \bigcup_{i=0}^{l} Z_i$ be a Nash G-invariant stratification of Z. Let χ be a character of G. Suppose that for any $k \in \mathbb{Z}_{\geq 0}$ and $0 \le i \le l$ we have $\mathcal{S}^*(Z_i, Sym^k(CN^X_{Z_i}))^{G,\chi} = 0$. Then $\mathcal{S}^*_X(Z)^{G,\chi} = 0$.

This proposition immediately follows from Corollary B.2.4 in [AGS].

The next proposition is an analogue of 2.36 from [BZ]. It is called there Frobenius reciprocity. It allows one to study invariant distributions on a space by means of invariant distributions on a smaller space. More precisely,

Proposition 2.4 (Frobenius reciprocity). *Let a Nash group G act transitively on a Nash manifold Z. Let $\varphi : X \to Z$ be a G-equivariant Nash map. Let $z \in Z$. Let G_z be its stabilizer. Let X_z be the fiber of z. Let χ be a character of G and let $\chi' = \chi \cdot \Delta_G|_{G_z} \cdot \Delta_{G_z}^{-1}$ where Δ_H denotes the modular character of the group H.*

Then $\mathcal{S}^*(X)^{G,\chi}$ is canonically isomorphic to $\mathcal{S}^*(X_z)^{G_z,\chi'}$

For proof see [AG2], Theorem 2.5.7.

The next lemma states the obvious fact that the Fourier transform of an invariant distribution on V is invariant.

Let V be a vector space over $\mathbb{R}$. Let B be a non-degenerate bilinear form on V.

We define $\mathcal{F}_B : S(V) \to S(V)$ by the formula

$$\mathcal{F}_B(f)(v) = \int f(w) e^{2\pi i B(v,w)} dw$$

Here dw is the self dual Haar measure on V with respect to B.

We also denote by $\mathcal{F}_B : \mathcal{S}^*(V) \to \mathcal{S}^*(V)$ the dual map.

For any Nash manifold M we also denote by $\mathcal{F}_B : \mathcal{S}^*(M \times V) \to \mathcal{S}^*(M \times V)$ the partial Fourier transform.

If there is no ambiguity, we will write $\mathcal{F}_V$, and sometimes just $\mathcal{F}$, instead of $\mathcal{F}_B$.

The next simple observation will be very useful so we record it as a lemma.

Lemma 2.5. *Let V be a finite dimensional vector space over $\mathbb{R}$. Let a Nash group G act linearly on V. Let B be a G-invariant non-degenerate symmetric bilinear form on V. Let M be a Nash manifold with an action of G. Let $\xi \in \mathcal{S}^*(V \times M)$ be a G-invariant distribution. Then $\mathcal{F}_B(\xi)$ is also G-invariant.*

2.4. Singular support of distributions.

The *Singular Support* of a distribution ξ is a certain subvariety of the cotangent bundle of X that is related to the support of the distribution and will be used in the sequel. Here we give a brief review and the reader should consult [Bor] or [AG4] Appendix B for more details.

We begin by recalling the definition of a D-module and the singular support of a D-module.

Assume that X is a smooth affine algebraic variety defined over $\mathbb{R}$ and consider the algebra $D(X)$ of polynomial differential operators on X. On $D(X)$ there is a natural filtration by the order of the differential operator and the associated graded algebra is isomorphic to $\mathcal{O}(T^*(X))$ the regular functions on the cotangent bundle of X. Given a $D(X)$-module M certain filtrations F on M are called *good*. One then consider the module $\mathrm{Gr}_F(M)$ over the commutative algebra $\mathrm{Gr}(D(X)) = \mathcal{O}(T^*(X))$. One can show that the support of this module is a subvariety of $T^*(X)$ which is independent of the good filtration F. It is called the *singular support* of the module M and denoted by $SS(M)$.

Let X be a smooth algebraic variety. Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))$. Let M_ξ be the D_X-submodule of $\mathcal{S}^*(X(\mathbb{R}))$ generated by ξ. We denote by $SS(\xi) \subset T^*X$ the singular support of M_ξ (for the definition see [Bor]). We will call it the *singular support* of ξ. In more concrete terms, let $J = Ann(\xi) = \{d \in D(X) : d\xi = 0\}$ be the ideal in $D(X)$ annihilating ξ, and let $I \subset \mathcal{O}(T^*(X))$ be the ideal generated by the symbols of elements of J. Then $SS(\xi)$ is the zero set of I.

Remark 2.6. (i) A similar, but not equivalent notion is sometimes called in the literature a 'wave front set of ξ'.

(ii) In some of the literature different notations are used. Namely, sometimes the expression singular support of a distribution is a subset of X not to be confused with our $SS(\xi)$ which is a subset of T^*X. In those cases our notion is called there characteristic variety.

Let X be a smooth algebraic variety. We denote by $p_X : T^*X \to X$ the standard projection. Furthermore, for $S \subset X$ we denote by $\overline{S}^{Zar}$ its Zariski closure in X. Below is a list of properties of the Singular support. Proofs can be found in [AG4] section 2.3 and Appendix B.

Lemma 2.7. *Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))$. Then $\overline{\mathrm{Supp}(\xi)}^{Zar} = p_X(SS(\xi))$.*

Lemma 2.8. *Let an algebraic group G act on X. Let $\mathfrak{g}$ denote the Lie algebra of G. Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))^{G(\mathbb{R})}$. Then*

$$SS(\xi) \subset \{(x,\phi) \in T^*X \mid \forall \alpha \in \mathfrak{g}\, \phi(\alpha(x)) = 0\}.$$

For the next lemma we need some further notations. Let (V,B) be a quadratic space. Let X be a smooth algebraic variety. Consider B as a map $B : V \to V^*$. Identify $T^*(X \times V)$ with $T^*X \times V \times V^*$. We define $F_V : T^*(X \times V) \to T^*(X \times V)$ by $F_V(\alpha, v, \phi) := (\alpha, -B^{-1}\phi, Bv)$.

Lemma 2.9. *Let (V,B) be a quadratic space. Let $Z \subset X \times V$ be a closed subvariety, invariant with respect to homotheties in V. Let $\xi \in S^*(X \times V)$ and suppose that* $\mathrm{Supp}(\xi) \subset Z(\mathbb{R})$*. Then $SS(\mathcal{F}_V(\xi)) \subset F_V(p^{-1}_{X \times V}(Z))$.*

The integrability Theorem

We recall the notion of co-isotropic subvariety in the context of algebraic varieties.

Definition 2.1. Let M be a smooth algebraic variety and ω be a symplectic form on it. Let $Z \subset M$ be an algebraic subvariety. We call it M**-co-isotropic** if one of the following equivalent conditions holds.

1. The ideal sheaf of regular functions that vanish on $\overline{Z}$ is closed under Poisson bracket.

2. At every smooth point $z \in Z$ we have $T_zZ \supset (T_zZ)^\perp$. Here, $(T_zZ)^\perp$ denotes the orthogonal space to (T_zZ) in (T_zM) with respect to ω.

3. For a generic smooth point $z \in Z$ we have $T_zZ \supset (T_zZ)^\perp$.

If there is no ambiguity, we will call Z a co-isotropic variety.

Note that every non-empty M-co-isotropic variety is of dimension at least $\frac{1}{2}\dim(M)$.

Finally, the following is a corollary of the integrability theorem ([KKS], [Mal], [Gab]):

Theorem 2.10. *Let X be a smooth algebraic variety. Let $\xi \in \mathcal{S}^*(X(\mathbb{R}))$. Then $SS(\xi)$ is co-isotropic with respect to the standard symplectic form.*

3. Gelfand pairs and invariant distributions

In this section we recall a technique due to Gelfand and Kazhdan (see [GK]) which allows to deduce statements in representation theory from statements on invariant distributions. For more detailed description see [AGS], section 2.

Let G be a reductive group. By an *admissible representation of* $G(\mathbb{R})$ we mean an admissible smooth Fréchet representation of moderate growth of $G(\mathbb{R})$ as in [Wa2] 11.5.

We denote by $\mathcal{R}ep(G)$ the category of admissible smooth Fréchet representations of moderate growth of $G(\mathbb{R})$. By the theorem of Casselman-Wallach this category is equivalent to the category of Harish-Chandra modules. The latter category admits a natural duality. Shifting this duality to $\mathcal{R}ep(G)$, we denote for an admissible smooth Fréchet representation $E \in Ob(Rep(G))$ by $\widetilde{E}$ the dual Frechet representation. The next definition for Gelfand Pair was introduced in [AGS]. It follows the spirit of [Gro] who used a similar definition for the p-adic case.

Definition 3.1. Let $H \subset G$ be a pair of reductive groups. We say that (G, H) is a Gelfand Pair if for any irreducible admissible smooth Fréchet representation (π, E) of G we have
$$\dim Hom_{H(\mathbb{R})}(E, \mathbb{C}) \leq 1.$$

Remark 3.1. In the literature different notions for Gelfand pairs were studied both in the real and in the p-adic case, e.g. [vD, BvD]. On some of these and their interconnection see [AGS].

We will use the following theorem from [AGS] which is a version of a classical theorem of Gelfand and Kazhdan.

Theorem 3.2. *Let $H \subset G$ be reductive groups and let τ be an involutive anti-automorphism of G and assume that $\tau(H) = H$. Suppose $\tau(\xi) = \xi$ for all bi $H(\mathbb{R})$-invariant distributions ξ on $G(\mathbb{R})$. Then for any irreducible admissible smooth Fréchet representation (π, E) of G we have*
$$\dim Hom_{H(\mathbb{R})}(E, \mathbb{C}) \cdot \dim Hom_{H(\mathbb{R})}(\widetilde{E}, \mathbb{C}) \leq 1$$
Furthermore, if $G = GL_n$ and $H \subset \mathrm{GL}_n$ is transpose invariant subgroup then (GL_n, H) is a Gelfand pair.

For proof see [AGS], section 2.

Consider now Theorem B, namely that $(GL_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{R}))$ is a Gelfand Pair. Applying 3.2) with $\tau(g) = g^t$ we see that it is a direct consequence of Theorem A regarding invariant distributions.

3.1. Symmetric pairs.
In this subsection we review some tools developed in [AG2] that enable to prove that, granting certain hypothesis, that a symmetric pair is a Gelfand pair.

A **symmetric pair** is a triple (G, H, θ) where $H \subset G$ are reductive groups, and θ is an involution of G such that $H = G^\theta$. In cases when there is no ambiguity we will omit θ

For a symmetric pair (G, H, θ) we define an anti-involution $\sigma : G \to G$ by $\sigma(g) := \theta(g^{-1})$, denote $\mathfrak{g} := \mathrm{Lie}(G)$, $\mathfrak{h} := \mathrm{Lie}(H)$, $\mathfrak{q} := \{X \in \mathfrak{g} | \theta(X) = -X\}$. Note that H acts on $\mathfrak{q}$ by the adjoint action.

Denote also $G^\sigma := \{g \in G | \sigma(g) = g\}$ and define a *symmetrization map* $s : G(\mathbb{R}) \to G^\sigma(\mathbb{R})$ by $s(g) := g\sigma(g)$.

It is well known, and easy to verify, that he symmetrization map $s : G \to G^\sigma$ is submersive and hence open.

Let V be an algebraic finite dimensional representation over $\mathbb{R}$ of a reductive group L. Since L is reductive, the subspace $V^L = \{v \in V : gv = v, \forall g \in L\}$ has a canonical complement

$$V = V^L \oplus V_e.$$

Thus, V_e is the subspace of V where L acts effectively.

Let (G, H, θ) be a symmetric pair. We apply the previous notation for the action of H on $\mathfrak{q}$. Thus the effective piece of $\mathfrak{q}$ will be denoted by $\mathfrak{q}_e$.

We denote by $\mathcal{N}_{G,H}$ the subset of all the nilpotent elements in $\mathfrak{q}_e$. Denote $\mathcal{R}_{G,H} := \mathfrak{q}_e - \mathcal{N}_{G,H}$.

Remark 3.3. Our notion of $\mathcal{R}_{G,H}$ coincides with the notion $R(\mathfrak{q})$ used in [AG2], Notation 2.1.10. This follows from Lemma 7.1.11 in [AG2].

A theorem of Kostant and Rallis

We will also need the following Proposition, whose proof is based on [KR]. We include the proof for the benefit of the readers.

Proposition 3.4. *Let $\pi : \mathfrak{q} \to Spec(\mathcal{O}(\mathfrak{q})^H)$ be the projection, where $\mathcal{O}(\mathfrak{q})$ denote the space of regular functions on the algebraic variety $\mathfrak{q}$.*

1. *Then $Spec(\mathcal{O}(\mathfrak{q})^H)$ is an affine space.*
2. *Let $x \in \mathcal{N}_{G,H}$ be a smooth point. Then $Ker(d_x\pi) = T_x(\mathcal{N}_{G,H})$.*
3. *Let $x \in \mathcal{N}_{G,H}$ be a smooth point. Then π submersive at x.*
4. *Let U be the set of smooth points in $\mathcal{N}_{G,H}$. Let $b = \pi(0)$. Then*

$$N_U^{\mathfrak{q}_e} \cong U \times T_b(Spec(\mathcal{O}(\mathfrak{q})^H))$$

Proof. 1 follows from [Ser] (see p. 21 in [KR]). For 2 let $\mathcal{I} = \{f \in \mathcal{O}(\mathfrak{q})^H : f(0) = 0\}$ and J be the ideal in $\mathcal{O}(\mathfrak{q})$ generated by I. By Theorem 14 of [KR], $\mathcal{J}$ is a radical ideal. We now show, using the Nullstellensatz, that

$$Ker(d_x\pi) = T_x(\mathcal{N}_{G,H}).$$

Indeed,

$$T_x(\mathcal{N}_{G,H}) = \bigcap_{\{f\in\mathcal{O}(\mathfrak{q})|f(\mathcal{N}_{G,H})=0\}} Ker(d_xf) = \bigcap_{f\in rad(\mathcal{J})} Ker(d_xf) =$$
$$\bigcap_{f\in\mathcal{J}} Ker(d_xf) = \bigcap_{f\in\mathcal{I}} Ker(d_xf) = Ker(d_x(\pi)).$$

We now prove 3. By Theorem 3 of [KR], for any such x

$$\dim T_x(\mathcal{N}_{G,H}) = \dim(\mathcal{N}_{G,H}) = \dim(\mathfrak{q}) - \dim(Spec(\mathcal{O}(\mathfrak{q})^H)).$$

Thus,

$$\dim(Im(d_x\pi)) = \dim(Spec(\mathcal{O}(\mathfrak{q})^H)).$$

This proves that π is submersive at x. For 4 note that for any $x \in U$, $d_x\pi$ sets up an isomorphism between $d_x\pi : T_x(\mathfrak{q}_e)/T_x(\mathcal{N}_{G,H}) \to T_b(Spec(\mathcal{O}(\mathfrak{q})^H))$. Thus,

$$N_U^{\mathfrak{q}_e} = T(\mathfrak{q}_e)|_U/T(U) \cong U \times T_b(Spec(\mathcal{O}(\mathfrak{q})^H)). \qquad \blacksquare$$

Gelfand property of symmetric pairs

The Gelfand Kazhdan criterion allow us to verify the Gelfand property of a pair (G, H) provided that we have an anti-automorphism of G satisfying certain properties. In the case of a symmetric pair an obvious choice for such an automorphism is σ and we are led to the following notion.

Definition 3.2. We say that a symmetric pair (G, H, θ) is a *Gelfand-Kazhdan pair* (for short *GK pair*) if any $H(\mathbb{R}) \times H(\mathbb{R})$ - invariant distribution on $G(\mathbb{R})$ is σ-invariant.

An obvious obstruction to a symmetric pair being a GK pair is the existence of closed orbit supporting an invariant distribution which is not σ invariant. Thus the following definition allows us to concentrate on those pairs that have a chance to be GK pairs.

Definition 3.3. We call a symmetric pair (G, H, θ) *good* if for any closed $H(\mathbb{R}) \times H(\mathbb{R})$ orbit $O \subset G(\mathbb{R})$, we have $\sigma(O) = O$.

We define an involution $\theta : GL_{2n} \to GL_{2n}$ by

$$\theta(x) = Jx^{-t}J^{-1} \tag{1}$$

where

$$J = \begin{pmatrix} 0_n & Id_n \\ -Id_n & 0_n \end{pmatrix}. \tag{2}$$

Note that $(GL_{2n}, Sp_{2n}, \theta)$ is a symmetric pair.

Theorem A can be rephrased in the following way:

Theorem A'. The pair (GL_{2n}, Sp_{2n}) defined over $\mathbb{R}$ is a GK pair.

In what follows we recall a technique to verify the GK property of a symmetric pair.

Descendants of symmetric pairs

Proposition 3.5. *Let (G, H, θ) be a symmetric pair. Let $g \in G(\mathbb{R})$ such that HgH is closed. Let $x = s(g)$. Then x is semisimple.*

This result is implicit in [KR]. For this specific statement see [AG2], Proposition 7.2.1.

Definition 3.4. In the notations and assumptions of the previous proposition we will say that the pair $(G_x, H_x, \theta|_{G_x})$ is a descendant of (G, H, θ). Here G_x (and similarly for H) denotes the stabilizer of x in G.

In [Sek] this is called a sub-symmetric pair.

Regular symmetric pairs

An important notion that we use here is that of a *regular* symmetric space. To introduce it we first introduce a class of elements in G, that were called *admissible* in [AG2]. These elements provide additional symmetries of the symmetric space and the regularity condition is satisfied if these elements preserve H-invariant distributions on $\mathfrak{q}$. We now recall the definitions from [AG2].

Definition 3.5. Let (G, H, θ) be a symmetric pair. We call an element $g \in G(\mathbb{R})$ *admissible* if
(i) $Ad(g)$ commutes with θ (or, equivalently, $s(g) \in Z(G)$).
(ii) $Ad(H)|_{\mathfrak{q}} \subset \langle Ad(H), Ad(g) \rangle|_{\mathfrak{q}}$ is of index at most 2. Both are considered as subgroups in $Aut(\mathfrak{q})$
(iii) For any closed $Ad(H)$ orbit $O \subset \mathfrak{q}$ we have $Ad(g)O = O$.

We are now able to introduce the notion of *regularity*

Definition 3.6. We call a symmetric pair (G, H, θ) **regular** if for any admissible $g \in G(\mathbb{R})$ such that every $H(\mathbb{R})$-invariant distribution on $\mathcal{R}_{G,H}$ is also $Ad(g)$-invariant, we have
(*) every $H(\mathbb{R})$-invariant distribution on $\mathfrak{q}_e$ is also $Ad(g)$-invariant.

Clearly, the product of regular pairs is regular (see [AG2], Proposition 7.4.4).

We will deduce Theorem A' (and hence Theorem A) from the following Theorem:

Theorem C. The pair (GL_{2n}, Sp_{2n}) defined over $\mathbb{R}$ is regular.

The deduction is based on the following theorem (see [AG2], Theorem 7.4.5.):

Theorem 3.6. *Let (G, H, θ) be a good symmetric pair such that all its descendants (including itself) are regular. Then it is a GK pair.*

Corollary 3.7. *Theorem C implies Theorem A.*

Proof. The pair (GL_{2n}, Sp_{2n}) is good by Corollary 3.1.3 of [Say1]. In [Say1] it is shown that all the descendance of the pair (GL_{2n}, Sp_{2n}) are products of pairs of the form (GL_{2m}, Sp_{2m}) and $((GL_{2m})_{\mathbb{C}/\mathbb{R}}, (Sp_{2m})_{\mathbb{C}/\mathbb{R}})$, here $G_{\mathbb{C}/\mathbb{R}}$ denotes the restriction of scalars (in particular $G_{\mathbb{C}/\mathbb{R}}(\mathbb{R}) = G(\mathbb{C})$). By Corollary 3.3.1. of [Say1] the pair $((GL_{2m})_{\mathbb{C}/\mathbb{R}}, (Sp_{2m})_{\mathbb{C}/\mathbb{R}})$ is regular. Now clearly Theorem C implies Theorem A' and hence Theorem A. ∎

4. Invariant distributions supported on non-distinguished nilpotent orbits in symmetric pairs

For this section we fix a symmetric pair (G, H, θ).

Definition 4.1. We say that a nilpotent element $x \in \mathfrak{q}$ is distinguished if

$$\mathfrak{g}_x \cap \mathfrak{q}_e \subset \mathcal{N}_{G,H}$$

Theorem 4.1. *Let $A \subset \mathcal{N}_{G,H}$ be an H-invariant closed subset and assume that all elements of A are non-distinguished. Let $W = \mathcal{S}^*_{\mathfrak{q}}(A)^H$. Then $W \cap \mathcal{F}(W) = 0$.*

Remark 4.2. We believe that the methods of [SZ] allow to show the same result without the assumption of H-invariance.

The proof is based on the following proposition:

Proposition 4.3. *Let $A \subset \mathcal{N}_{G,H}$ be an H-invariant closed subset and assume that all elements of A are non-distinguished. Denote by*

$$B = \{(\alpha, \beta) \in A \times A : [\alpha, \beta] = 0\} \subset \mathfrak{q}_e \times \mathfrak{q}_e.$$

Identify $T^*(\mathfrak{q}_e)$ with $\mathfrak{q}_e \times \mathfrak{q}_e$. Then there is no non-empty $T^*(\mathfrak{q}_e)$-co-isotropic subvariety of B.

Proof. It follows from [KR] that H has finitely many orbits on $\mathcal{N}_{G,H}$. Stratify

A by its orbits $\mathcal{O}_1, ..., \mathcal{O}_r$. Namely, $\bigcup_{j=1}^{k} \mathcal{O}_j$ is open in A for any $k = 1, ..., r$. Let $Z_i = \bigcup_{j=i}^{r} \mathcal{O}_j$. Let $C \subset B$ be a co-isotropic subvariety of B.

We will show by induction on i that $C \subset (Z_i \times A) \cap B$. For $i = 1$ this is clear and the inductive step requires to show that $C \cap (\mathcal{O}_i \times A) \cap B = \emptyset$. For this, it is enough to show that $p^{-1}(\mathcal{O}_i) \cap B$ does not include a non empty co-isotropic subvariety.

Let $\mathcal{O} = \mathcal{O}_i$. Let $N_{\mathcal{O}}$ be the conormal bundle to $\mathcal{O}$ considered as a subset of $\mathfrak{q}_e \times \mathfrak{q}_e$. Note that $\dim(N_{\mathcal{O}}) = \dim(\mathfrak{q}_e)$ and

$$N_{\mathcal{O}} = \{(a, b) \in \mathcal{O} \times \mathfrak{q}_e | \forall H \in \mathfrak{h} \text{ we have } \langle Ha, b\rangle = 0\} =$$
$$\{(a, b) \in \mathcal{O} \times \mathfrak{q}_e | \forall H \in \mathfrak{h} \text{ we have } \langle H, [a, b]\rangle = 0\} = \{(a, b) : a \in \mathcal{O}, b \in \mathfrak{q}_e, [a, b] = 0\}.$$

Since $\mathcal{O}$ is not distinguished, for any $x \in \mathcal{O}$ the variety $p^{-1}(x) \cap B \subset p^{-1}(x) \cap N_{\mathcal{O}}$ is a closed proper subvariety. Hence $\dim(p^{-1}(\mathcal{O}) \cap B) < \dim(\mathfrak{q}_e)$

Thus $p^{-1}(\mathcal{O}) \cap B$ can not contain a co-isotropic subvariety. This finishes the proof. ■

Proof. [Proof of Theorem 4.1] For $\xi \in W \cap \mathcal{F}(W)$ we will show that $SS(\xi)$, the singular support of ξ, is contained in B, where B is defined as in proposition 4.3 above. By the integrability theorem [Gab] the singular support is co-isotropic and hence by Proposition 4.3 it is empty. This will finish the proof.

Recall that $SS(\xi) \subset T^*(\mathfrak{q}_e) \cong \mathfrak{q}_e \times \mathfrak{q}_e$.

Consider the projection $pr : \mathfrak{q}_e \times \mathfrak{q}_e \to \mathfrak{q}_e$. Then $\mathrm{pr}(SS(\xi))$ is the Zariski closure of $\mathrm{Supp}(\xi) \subset A$ (see section 2.4). Thus, $SS(\xi) \subset A \times \mathfrak{q}_e$.

Since $\mathrm{Supp}(Four(\xi)) \subset A$ and A is homogenous we obtain $SS(\xi) \subset A \times A$ (see section 2.4). Since ξ is H-invariant we conclude that

$$SS(\xi) \subset \{(a, b) \in T^*(\mathfrak{q}_e) | \langle Xa, b\rangle = 0, \forall X \in \mathfrak{g}\}.$$

However the later set is, under our identification equals to

$$\{(a, b) \in \mathfrak{q}_e \times \mathfrak{q}_e : [a, b] = 0\}$$

We conclude that $SS(\xi) \subset B$ and the proof is complete. ■

5. Regularity of the pair (GL_{2n}, Sp_{2n})

In this section we prove the main result of the paper:

Theorem C. The pair (GL_{2n}, Sp_{2n}) defined over $\mathbb{R}$ is regular.

For the rest of this section we let (G, H) be the symmetric pair $(GL_{2n}(\mathbb{R}), Sp_{2n}(\mathbb{R}))$.

Remark 5.1. In our case there is one non trivial admissible element up to the action of $Z(G)H$, namely, the element $\alpha = \begin{pmatrix} 0_n & Id_n \\ Id_n & 0_n \end{pmatrix}$. Thus the regularity condition needs to be verified only for this element.

Our argument below will not use this fact.

To show regularity we analyze the geometry of nilpotent orbits.

5.1. H orbits on $\mathfrak{q}$.

Proposition 5.2. *There exists a unique distinguished H-orbit in $\mathcal{N}_{G,H}(\mathbb{R})$. This orbit is open in $\mathcal{N}_{G,H}(\mathbb{R})$ and invariant with respect to any admissible $g \in G$.*

For the proof we will use the following Proposition (this is Proposition 2.1 of [GG]):

Proposition 5.3. *Let F be an arbitrary field. For $x \in GL_n(F)$ define*

$$\gamma(x) = \begin{pmatrix} x & 0 \\ 0 & I_n \end{pmatrix}$$

Then γ induces a bijection between the set of conjugacy classes in $GL_n(F)$ and the set of orbits of $Sp_{2n}(F) \times Sp_{2n}(F)$ in $GL_{2n}(F)$.

Corollary 5.4. *Let $d : \mathrm{gl}_n \to \mathfrak{q}$ be defined by*

$$d(X) = \begin{pmatrix} X & 0 \\ 0 & X^t \end{pmatrix}.$$

Then d induces a bijection between nilpotent conjugacy classes in gl_n and H orbits in $\mathcal{N}_{G,H}$.

Proof.

Let $s : GL_{2n} \to GL_{2n}^{\sigma}$ be given by $s(g) = g\sigma(g)$. Let $W = s(GL_{2n}(\mathbb{R}))$. By Proposition 5.3, the map $s \circ \gamma$ induces a bijection between conjugacy classes in $GL_n(\mathbb{R})$ and H orbits on W.

Let $e : \mathcal{N} \to GL_n$ be given by $e(X) = 1 + X$ where $\mathcal{N}$ is the cone of nilpotent elements in gl_n. Let $\ell : W \to \mathfrak{q}$ given by $\ell(w) = w - 1$.

Then, it is easy to see that the map $d|_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}_{G,H}$ coincides with the composition $\ell \circ s \circ \gamma \circ e$. The map $\ell \circ s \circ \gamma \circ e : \mathcal{N} \to \mathfrak{q}$ defines an injection of the set of orbits $\mathcal{N}/Ad(GL_{2n})$ into $\mathfrak{q}/H$.

We first show that $\ell(W) \supset \mathcal{N}_{G,H}$.

Indeed, since the symmetrization map is open the set $W = s(GL_{2n}(\mathbb{R}))$ is open and thus $\ell(W)$ is open. Since $0 \in \ell(W)$ is conjugation invariant it must contains all nilpotent elements. Thus $\ell \circ s \circ \gamma$ surject onto nilpotent orbits. Finally, to show that $\ell \circ s \circ \gamma$ is surjective onto the nilpotent orbits we note that whenever $\ell \circ s \circ \gamma(u) \in \mathcal{N}_{G,H}$ we must have u unipotent. ∎

We are now ready to prove the proposition.

Proof. [Proof of Proposition 5.2] It is easy to see that if X is non regular nilpotent then $d(X)$ is not distinguished. Also, a simple verification shows that if $X = J_n$ is a standard Jordan block then $d(J_n)$ is distinguished. The invariance of $C = Ad(H)d(J_n)$ with respect to admissible elements follows from the uniqueness. Thus we only need to show that C is open in $\mathcal{N}_{G,H}$. For this we will show that

C is dense in $\mathcal{N}_{G,H}$. Indeed, $\overline{C} \supset d(\overline{Ad(GL_n)J_n}) = d(\mathcal{N})$, where $\mathcal{N}$ is the set of nilpotent elements in gl_n. But C is $Ad(H)$-invariant and this implies that $\overline{C} = \mathcal{N}_{G,H}$ ∎

5.2. Proof of Theorem C.

Let A be the union of all non-distinguished elements. Note that A is closed.

We first prove

Proposition 5.5. *Let $g \in G$ be an admissible element. Let ξ be any H-invariant distribution on $\mathfrak{q}_e$ which is anti-invariant with respect to $Ad(g)$. Assume* $\mathrm{Supp}(\xi) \subset \mathcal{N}_{G,H}$, *then* $\mathrm{Supp}(\xi) \subset A$.

Proof. Let $\mathcal{O}_0 \subset \mathcal{N}_{G,H}$ be the distinguished orbit. Let $\widetilde{H} = \langle Ad(H), Ad(g)\rangle$ be the group of automorphisms of $\mathfrak{q}_e$ generated by the adjoint action of H and g. Let χ be the character of $\widetilde{H}$ defined by $\chi(\widetilde{H} - H) = -1$. We need to show

$$\mathcal{S}^*_{\mathfrak{q}_e}(O_0)^{\widetilde{H},\chi} = 0$$

By Proposition 2.3 it is enough to show

$$\mathcal{S}^*(\mathcal{O}_0, Sym^k(CN^{\mathfrak{q}_e}_{\mathcal{O}_0}))^{\widetilde{H},\chi} = 0$$

Notice that $\widetilde{H}$ acts trivially on $Spec(\mathcal{O}(\mathfrak{q}_e)^H)$. Hence, by Proposition 3.4 item 4, the bundle $N^{\mathfrak{q}_e}_{\mathcal{O}_0}$ is trivial as a $\widetilde{H}$ bundle. This completes the proof. ∎

Using the Proposition we can now deduce Theorem C.

Indeed, let $g \in G$ be an admissible element.

Assume that any distribution ξ on $\mathcal{R}_{G,H}$ which is H-invariant is also $Ad(g)$-invariant.

We have to show that any H-invariant distribution on $\mathfrak{q}_e$ that is anti invariant with respect to $Ad(g)$ is zero.

Let η be such a distribution. Since its restriction to $\mathcal{R}_{G,H}$ is still H-invariant it must be, by our assumptions, $Ad(g)$-invariant and anti $Ad(g)$-invariant and hence zero. We conclude that η is supported in $\mathcal{N}_{G,H}$. Thus, by Proposition 5.5 it is supported in A. Similarly, $Four(\eta)$ is also supported in A and η is H-invariant. Thus, by theorem 4.1 we obtain $\eta = 0$. This complete the proof.

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