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A New Character Formula for Lie Algebras and Lie Groups

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Abstract. The aim of this paper is to present a new character formula for finite-dimensional representations of finite-dimensional complex semisimple Lie algebras and compact semisimple Lie Groups. Some applications of the new formula include the exact determination of the number of weights in a representation, new recursion formulas for multiplicities and, in some cases, closed formulas for the multiplicities themselves.
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1. Introduction

The effective determination of the characters of the compact semisimple Lie groups is a subject that probably goes back to the early 20th century and was mainly carried out by Hermann Weyl. As he narrated in his book [12] on the Classical Groups, he had succeeded in determining the character of all semi-simple continuous groups by means of a combination of the analytical methods developed earlier by Elie Cartan and Issai Schur. Later, in collaboration with Richard Brauer, he pursued the task of deriving the most fundamental results by means of purely algebraic methods, which culminated in the works of Hans Freudental [3], albeit his algebraic results seemed less elegant and less intuitive than the analytic ones. The main tool for most of the character-related works have since been the Weyl Character Formula [4, Theorem 24.3].

Using Weyl's formula one can conceivably compute the character of any semisimple Lie group, at the expense of carrying out a complicated "division", so it is perhaps not the most suitable for practical computations. It is, however, a beautiful and most useful theoretical statement, whose applications extend far beyond that of simply computing the characters.

Here is a brief list of the previously known character formulas other than Weyl's: Freudenthal's [3] formula , Kostant's [6] formula, Littelmann's [8] root and path operators, and Sahi's [9] formula. We shall not be discussing here the strengths and weaknesses of each of these formulas, which seem to be quite powerful

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and elegant on their own. It is important to acknowledge that Freudenthal's formula has shown to be adequate for computer implementation and has been used by many computer algebra systems and in the determination of long character tables [1], although it is not optimal for this task. In our Ph.D. thesis [10] we have shown this fact and that Sahi's formula is optimal in a precise sense.

We do not intend to qualify our formula to belong to this select list, but merely to report its discovery, provide a simple proof, and discuss some of its intriguing consequences and applications. We focus on the following: new recursion formulas for the multiplicities, the counting of the weights in a representation, and obtaining closed formulas for the multiplicities themselves. It is interesting to notice that in our formula, the prevalent role is played not by the simple roots, but by the positive non-simple ones, which suggests that the inherent combinatorics might be simpler than that which uses all of the positive roots.

The rest of this section is used only to establish the notation, most of which appears to be standard, so it may be skipped by an experienced reader, except perhaps for Definitions 1.1 through 1.4, where we introduce the signed character and the girdle.

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank n over $\mathbb{C}$, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, $\mathfrak{h}^*$ its dual space, R the root system of $\mathfrak{g}$ in $\mathfrak{h}^*$, $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ a choice of simple roots in R, R^+ the positive roots, and $R^- = -R^+$ the negative roots. The $\mathbb{R}$-span of Δ in $\mathfrak{h}^*$ is an Euclidean space E where a symmetric non-degenerate bilinear form $\langle\, ,\rangle$ on E can be chosen up to a scalar factor to be the Killing form of $\mathfrak{h}$. The elements of E are called *weights*. As usual, we define the *root lattice* Q of $\mathfrak{g}$ as the $\mathbb{Z}$-span of Δ in E and define the the *co-roots* $\alpha^\vee$ as $2\alpha/\langle\alpha,\alpha\rangle$ for each α in R. The weights $\omega_1, \ldots, \omega_n$ in E defined by $\langle\omega_i, \alpha_j^\vee\rangle = \delta_{ij}$, where δ_{ij} is the Kronecker delta, are called *fundamental weights* of $\mathfrak{g}$ and their $\mathbb{Z}$-span is a lattice P in E called the *weight lattice* of $\mathfrak{g}$. This contains Q as a sublattice of finite index, known as the *index of connection*, which is equal to the determinant d of the *Cartan matrix* $C = [\langle\alpha_i, \alpha_j^\vee\rangle]$. The subset P^+ of P whose weights λ satisfy $\langle\lambda, \alpha^\vee\rangle \geq 0$ for all α in R forms a "convex cone" in P called the cone of *dominant weights*. All weights λ in P are such that $\langle\lambda, \alpha^\vee\rangle \in \mathbb{Z}$ for all α in R and thus are called *integral*. The weight lattice is partially ordered by the dominance partial order: $\mu \leq \lambda$ if and only if $\lambda - \mu$ is a sum (possibly zero) of positive roots. The *Weyl group* W of R is defined as the subgroup of $GL(E)$ generated by the reflections $s_\alpha(\lambda) = \lambda - \langle\lambda, \alpha^\vee\rangle\alpha$ for $\alpha \in R$. It can be shown that W is a Coxeter group generated only by the *simple* reflections s_{α_i}, $i = 1, 2, \ldots, n$, and each element in w in W has as well-defined length, namely the length of a shortest expression of w in terms of the simple generators s_{α_i}. All these definitions can be found for instance in [5] or [4].

For each dominant integral weight λ, the irreducible representation V^λ of highest weight λ is finite dimensional and its weights form a *saturated subset* P^λ of P in the sense of [4, §13.4]. In particular, this means P^λ is finite and invariant under W. Let $\mathbb{C}[P]$ be the group algebra with basis e^μ for all μ in P, where e^μ denotes the function $t \mapsto e^{t\langle\mu,\rho\rangle}$ and ρ is equal to one half the sum of all the positive roots. The action of W on P extends naturally to an action on $\mathbb{C}[P]$ via $w \cdot e^\mu = e^{w(\mu)}$, for all w in W, so $\mathbb{C}[P]$ becomes a W-module.

We define the (formal) *character* of V^λ as the sum

$$\chi_\lambda = \sum_{\mu \in P^\lambda} m_\lambda(\mu) e^\mu,$$

where the coefficient $m_\lambda(\mu)$ is the multiplicity of the weight μ in the representation V^λ, i.e. the dimension of the *weight space*

$$V^\lambda(\mu) = \{v \in V^\lambda \mid h \cdot v = \mu(h)v, \text{for all } h \text{ in } \mathfrak{h}\}.$$

The main purpose of a character formula is providing a means of computing the multiplicities without constructing V^λ explicitly. Our formula will accomplish this in a way that is significantly different from what has been done before. The most important character formula is probably the *Weyl Character Formula* [4, §24.3]:

$$\sum_{w \in W} (-1)^{l(w)} e^{w(\rho)} \chi_\lambda = \sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)}, \tag{1}$$

which is valid for all dominant integral λ. We shall abbreviate this simply by WCF.

In order to express our formula conveniently, we shall extend the usual notions of character and multiplicity and introduce an auxiliary element in $\mathbb{C}[P]$.

Definition 1.1. For each weight λ, let w_λ be the unique shortest Weyl group element such that $w_\lambda(\lambda + \rho)$ is dominant. We define $\overline{\lambda} = w_\lambda(\lambda + \rho) - \rho$.

This means $\overline{\lambda} = \lambda$ when λ is dominant for, in that case, $w_\lambda = 1$. If $\lambda + \rho$ is *regular*, that is, if $\langle \lambda + \rho, \alpha \rangle \neq 0$ for all $\alpha \in R$, then $\overline{\lambda}$ is dominant.

Definition 1.2. For any weight λ let

$$\varepsilon_\lambda = \begin{cases} (-1)^{l(w_\lambda)}, & \text{if } \lambda + \rho \text{ is regular} \\ 0, & \text{otherwise} \end{cases}$$

Definition 1.3. Let the *signed character* be

$$\widetilde{\chi}_\lambda = \varepsilon_\lambda \chi_{\overline{\lambda}}.$$

Here we observe that if λ is dominant, then $\widetilde{\chi}_\lambda = \chi_\lambda$ is just the usual character. Otherwise $\widetilde{\chi}_\lambda$ is equal to plus or minus the character $\chi_{\overline{\lambda}}$ when $\lambda + \rho$ is regular, or zero when $\lambda + \rho$ is not regular.

The coefficients $\widetilde{m}_\lambda(\mu)$ in $\widetilde{\chi}_\lambda$ are equal to $\varepsilon_\lambda m_{\overline{\lambda}}(\mu)$ and also coincide with the usual multiplicities when λ is dominant. In any case, it is clear that $\widetilde{m}_\lambda(\mu) = 0$ if $\mu \notin P^{\overline{\lambda}}$ or $\lambda + \rho$ is not regular.

Definition 1.4. For each $\lambda \in P^+$ we define the *girdle* as the following element in $\mathbb{C}[P]$:

$$\Theta_\lambda = \sum_{\mu \in P^\lambda} e^\mu.$$

The girdle is the characteristic function of the (finite) set P^λ.

It is well known that for a dominant integral λ the set P^λ is finite, invariant under the action of the Weyl group W and each dominant $\mu \in P^\lambda$ satisfies $\mu \leq \lambda$, hence the girdle belongs to the algebra of W-invariant functions $\mathbb{C}[P]^W$ along with the formal characters and can be expressed unquely as a $\mathbb{Z}$-linear combination of the irreducible characters [4, §22.5]. Hence

$$\Theta_\lambda = \sum_{\mu \leq \lambda} c_{\lambda\mu}\chi_\mu, \tag{2}$$

where the sum is over all dominant weights $\mu \in P^\lambda$ and $c_{\lambda\mu} \in \mathbb{Z}$. However, since $m_\lambda(\lambda) = 1$ [4, Theorem 20.2], we can write

$$\Theta_\lambda = \chi_\lambda + \sum_{\mu < \lambda} c_{\lambda\mu}\chi_\mu. \tag{3}$$

Our main result, Theorem 2.1, provides explicit values for the coefficients $c_{\lambda\mu}$, which can be computed from the combinatorics of the positive non-simple roots $R^+ \backslash \Delta$. We will denote these roots simply by Δ'.

The combinatorics of Δ'. For any subset Φ of the vector space E, let $\langle\Phi\rangle$ denote the sum of all the elements in Φ, and for each subset Φ of Δ', let $\mathcal{F}_\Phi$ be the collection of all subsets Ψ of Δ' such that $\langle\Psi\rangle = \langle\Phi\rangle$. We can partition $\mathcal{F}_\Phi$ into $\mathcal{F}^0_\Phi$ and $\mathcal{F}^1_\Phi$ according to whether the subsets in each one contain an even or odd number of elements respectively. The number $c_{\langle\Phi\rangle} = |\mathcal{F}^0_\Phi| - |\mathcal{F}^1_\Phi|$ is the difference between the number of all possible ways of writing $\langle\Phi\rangle$ as an even sum of roots in Δ' and the number of all possible ways of writing $\langle\Phi\rangle$ as an odd sum of roots in Δ' without repetitions. Let $\mathcal{F}$ be the union of all $\langle\Phi\rangle$ for all non-empty subsets Φ of Δ'.

For instance, for Lie the algebra A_2, we have $\Delta' = \mathcal{F} = \{\alpha_1 + \alpha_2\}$ and $c_{\alpha_1+\alpha_2} = -1$. For B_2, we have $\Delta' = \{\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2\}$, $\mathcal{F} = \{\alpha_1 + \alpha_2, \alpha_1 + 2\alpha_2, 2\alpha_1 + 3\alpha_2\}$, $c_{\alpha_1+\alpha_2} = -1$, so $c_{\alpha_1+2\alpha_2} = -1$ and $c_{2\alpha_1+3\alpha_2} = 1$, and so on.

A_2	A_3	A_4	A_5	A_6	A_7	A_8
1	7	45	371	3943	53173	860833
B_2	B_3	B_4	B_5	B_6	B_7	B_8
3	37	531	11147	299253	9985765	395408447
		C_4	C_5	C_6	C_7	C_8
		527	11023	294347	9748655	383324695
		D_4	D_5	D_6	D_7	D_8
		121	2429	65805	2190955	87435085
G_2		F_4		E_6	E_7	
11		4781		272217	86643209	

Table 1: Size of the set $\mathcal{F}$ for lower-rank semisimple Lie algebras

A generating function for the $c_{\langle\Phi\rangle}$. Let us consider the product

$$\prod_{\alpha \in \Delta'} (1 - e^{-\alpha}). \tag{4}$$

By multiplying out its factors and using the above definitions for $\langle\Phi\rangle$ and $c_{\langle\Phi\rangle}$, we can rewrite it as

$$\prod_{\alpha\in\Delta'}(1-e^{-\alpha}) = 1+\sum_{\Phi}(-1)^{|\Phi|}e^{-\langle\Phi\rangle}, \tag{5}$$

where the summation is over all nonempty subsets Φ of Δ'. Then, by factoring out the terms in $e^{\langle\Phi\rangle}$, we get

$$\prod_{\alpha\in\Delta'}(1-e^{-\alpha}) = 1+\sum_{\langle\Phi\rangle\in\mathcal{F}} c_{\langle\Phi\rangle}e^{-\langle\Phi\rangle}. \tag{6}$$

Thus, (4) is a generating function for the numbers $c_{\langle\Phi\rangle}$.

An upper bound for the number of terms on the right-hand side of (6) is obviously $2^{|\Delta'|}$, but by carrying out explicit expansions of (4) we have noticed the actual number of terms is much smaller, as can be seen on Table 1 for all semisimple Lie algebras of rank less than or equal to 8, except for E_8, which is not listed.

2. Main Result

Let ε_λ, $\widetilde{\chi}_\lambda$, $c_{\lambda\mu}$ and $c_{\langle\Phi\rangle}$ be as in the introduction. The following statement is our main result. It provides explicit values for the coefficients $c_{\lambda\mu}$ and states that the matrix $(c_{\lambda\mu})$ is triangular.

Theorem 2.1 (New Character Formula). *If λ is a dominant integral weight then the following identity holds:*

$$\chi_\lambda + \sum_{\langle\Phi\rangle\in\mathcal{F}} c_{\langle\Phi\rangle}\widetilde{\chi}_{\lambda-\langle\Phi\rangle} = \Theta_\lambda. \tag{7}$$

Furthermore, for each $\langle\Phi\rangle\in\mathcal{F}$ such that $\widetilde{\chi}_{\lambda-\langle\Phi\rangle}\neq 0$, we have $\overline{\lambda-\langle\Phi\rangle}<\lambda$.

Equation (7) will follow at once from Theorems 2.3 and 2.4 below. These and the triangularity statement above will be proved in Section 4.

Although the coefficients $c_{\langle\Phi\rangle}$ in (7) can be negative, it is interesting to notice that:

Theorem 2.2. *If λ is dominant integral, then*

$$\chi_\lambda = \Theta_\lambda + \sum_{\mu<\lambda} d_{\lambda\mu}\Theta_\mu \tag{8}$$

where the $d_{\lambda\mu}$ are nonnegative integers and μ runs over the dominant integral weights smaller than λ.

The fact that the $d_{\lambda\mu}$ are nonnegative follows from the fact that the Θ_μ in (8) are distinct and μ occurs with multiplicity equal to 1 in χ_μ, and each coefficient in Θ_λ is equal to 1.

Algebra	Character Formula
A_2	$\chi_\lambda = \Theta_\lambda + \widetilde{\chi}_{\lambda-\rho}$
B_2	$\chi_\lambda = \Theta_\lambda + \widetilde{\chi}_{\lambda-\alpha_1-\alpha_2} + \widetilde{\chi}_{\lambda-\alpha_1-2\alpha_2} - \widetilde{\chi}_{\lambda-2\alpha_1-3\alpha_2}$
G_2	$\chi_\lambda = \Theta_\lambda + \widetilde{\chi}_{\lambda-\alpha_1-\alpha_2} + \widetilde{\chi}_{\lambda-2\alpha_1-\alpha_2} + \widetilde{\chi}_{\lambda-3\alpha_1-\alpha_2}$ $- \widetilde{\chi}_{\lambda-4\alpha_1-2\alpha_2} - \widetilde{\chi}_{\lambda-4\alpha_1-3\alpha_2} - \widetilde{\chi}_{\lambda-5\alpha_1-2\alpha_2}$ $- \widetilde{\chi}_{\lambda-5\alpha_1-3\alpha_2} + \widetilde{\chi}_{\lambda-6\alpha_1-4\alpha_2} + \widetilde{\chi}_{\lambda-7\alpha_1-4\alpha_2}$ $+ \widetilde{\chi}_{\lambda-8\alpha_1-4\alpha_2} - \widetilde{\chi}_{\lambda-9\alpha_1-5\alpha_2}$
A_3	$\chi_\lambda = \Theta_\lambda + \widetilde{\chi}_{\lambda-\alpha_2-\alpha_3} + \widetilde{\chi}_{\lambda-\alpha_1-\alpha_2} + \widetilde{\chi}_{\lambda-\alpha_1-\alpha_2-\alpha_3}$ $- \widetilde{\chi}_{\lambda-\alpha_1-2\alpha_2-\alpha_3} - \widetilde{\chi}_{\lambda-\alpha_1-2\alpha_2-2\alpha_3}$ $- \widetilde{\chi}_{\lambda-2\alpha_1-2\alpha_2-\alpha_3} + \widetilde{\chi}_{\lambda-2\alpha_1-3\alpha_2-2\alpha_3}$
B_3	$\chi_\lambda = \Theta_\lambda + \widetilde{\chi}_{\lambda-\alpha_2-\alpha_3} + \widetilde{\chi}_{\lambda-\alpha_1-\alpha_2} + \widetilde{\chi}_{\lambda-\alpha_2-2\alpha_3} + \widetilde{\chi}_{\lambda-\alpha_1-\alpha_2-\alpha_3}$ $+ \widetilde{\chi}_{\lambda-\alpha_1-\alpha_2-2\alpha_3} - \widetilde{\chi}_{\lambda-\alpha_1-2\alpha_2-\alpha_3} - \widetilde{\chi}_{\lambda-2\alpha_2-3\alpha_3}$ $- \widetilde{\chi}_{\lambda-\alpha_1-2\alpha_2-2\alpha_3} - \widetilde{\chi}_{\lambda-2\alpha_1-2\alpha_2-\alpha_3} - 2\widetilde{\chi}_{\lambda-\alpha_1-2\alpha_2-3\alpha_3}$ $- \widetilde{\chi}_{\lambda-2\alpha_1-2\alpha_2-2\alpha_3} - \widetilde{\chi}_{\lambda-\alpha_1-2\alpha_2-4\alpha_3} - \widetilde{\chi}_{\lambda-2\alpha_1-2\alpha_2-3\alpha_3}$ $+ \widetilde{\chi}_{\lambda-2\alpha_1-3\alpha_2-3\alpha_3} + \widetilde{\chi}_{\lambda-\alpha_1-3\alpha_2-5\alpha_3} + \widetilde{\chi}_{\lambda-2\alpha_1-3\alpha_2-4\alpha_3}$ $+ \widetilde{\chi}_{\lambda-2\alpha_1-4\alpha_2-3\alpha_3} + \widetilde{\chi}_{\lambda-3\alpha_1-3\alpha_2-3\alpha_3} + \widetilde{\chi}_{\lambda-\alpha_1-4\alpha_2-5\alpha_3}$ $+ \widetilde{\chi}_{\lambda-2\alpha_1-3\alpha_2-5\alpha_3} + \widetilde{\chi}_{\lambda-2\alpha_1-4\alpha_2-4\alpha_3} + \widetilde{\chi}_{\lambda-3\alpha_1-4\alpha_2-3\alpha_3}$ $+ \widetilde{\chi}_{\lambda-2\alpha_1-4\alpha_2-5\alpha_3} - \widetilde{\chi}_{\lambda-2\alpha_1-5\alpha_2-5\alpha_3} - \widetilde{\chi}_{\lambda-3\alpha_1-5\alpha_2-4\alpha_3}$ $- \widetilde{\chi}_{\lambda-2\alpha_1-5\alpha_2-6\alpha_3} - 2\widetilde{\chi}_{\lambda-3\alpha_1-5\alpha_2-5\alpha_3} - \widetilde{\chi}_{\lambda-2\alpha_1-5\alpha_2-7\alpha_3}$ $- \widetilde{\chi}_{\lambda-3\alpha_1-5\alpha_2-6\alpha_3} - \widetilde{\chi}_{\lambda-4\alpha_1-5\alpha_2-5\alpha_3} - \widetilde{\chi}_{\lambda-3\alpha_1-5\alpha_2-7\alpha_3}$ $+ \widetilde{\chi}_{\lambda-3\alpha_1-6\alpha_2-6\alpha_3} + \widetilde{\chi}_{\lambda-3\alpha_1-6\alpha_2-7\alpha_3} + \widetilde{\chi}_{\lambda-4\alpha_1-6\alpha_2-6\alpha_3}$ $+ \widetilde{\chi}_{\lambda-3\alpha_1-6\alpha_2-8\alpha_3} + \widetilde{\chi}_{\lambda-4\alpha_1-6\alpha_2-7\alpha_3} - \widetilde{\chi}_{\lambda-4\alpha_1-7\alpha_2-8\alpha_3}$

Table 2: The character formulas for the lower-rank semisimple Lie algebras

Due to the triangularity, we can use (3) to effectively compute any character χ_λ recursively. Table 2 lists the character formulas for the first few lower-rank complex semisimple Lie algebras.

Theorem 2.3. *Let λ be a dominant integral weight. Then, the following identity holds true:*

$$\sum_{w\in W} w \cdot \frac{e^\lambda}{\prod_{\alpha\in\Delta}(1-e^{-\alpha})} = \Theta_\lambda. \tag{9}$$

Theorem 2.4. *If λ is a dominant integral weight, then the following identity holds true:*

$$\sum_{w\in W} w \cdot \frac{e^\lambda}{\prod_{\alpha\in\Delta}(1-e^{-\alpha})} = \chi_\lambda + \sum_{\langle\Phi\rangle\in\mathcal{F}} c_{\langle\Phi\rangle}\widetilde{\chi}_{\lambda-\langle\Phi\rangle}. \tag{10}$$

The exact meaning of equations (9) and (10) depend on a certain expansion convention which will be discussed in Section 4.

3. Applications

Recursions for the multiplicities. It is a straightforward consequence of our main theorem the existence of new recursion formulas for the multiplicities:

Corollary 3.1 (Recursion Formula for the Multiplicities). *If λ is a dominant integral weight and μ is any weight in P^λ, then*

$$m_\lambda(\mu) = 1 - \sum_{\langle\Phi\rangle\in\mathcal{F}} c_{\langle\Phi\rangle}\widetilde{m}_{\lambda-\langle\Phi\rangle}(\mu). \tag{11}$$

This not only provides an effective means for computing the multiplicities $m_\lambda(\mu)$ recursively, but also the possibility of obtaining closed formulas for the multiplicities themselves in certain cases. This possibility will be addressed below.

Counting the weights in a representation. The next corollary is probably the first known closed formula for the number of weights in P^λ.

Corollary 3.2. *If λ is a dominant integral weight, then*

$$|P^\lambda| = \dim V^\lambda + \sum_{\langle\Phi\rangle\in\mathcal{F}} \varepsilon_{\lambda-\langle\Phi\rangle}c_{\langle\Phi\rangle}\dim V^{\overline{\lambda-\langle\Phi\rangle}}. \tag{12}$$

Furthermore, if $\lambda=(m_1,\dots,m_n)$ is expressed in the weight basis then $|P^\lambda|$ is a polynomial in the $m_1,\dots,m_n$.

Here we are adopting the convention that $V^{\overline{\lambda}}=0$ when $\lambda+\rho$ is not regular. Otherwise $\overline{\lambda}$ is dominant integral and $\dim V^{\overline{\lambda}}$ makes sense.

Proof. Recall that if λ is dominant integral, an important consequence of the WCF is that

$$\dim V^\lambda = \prod_{\alpha\in R^+}\frac{\langle\alpha,\lambda+\rho\rangle}{\langle\alpha,\rho\rangle}. \tag{13}$$

This follows from the WCF by regarding e^μ as the complex function $t\mapsto e^{t\langle\mu,\rho\rangle}$ and passing to the limit as $t\to 0$. Applying the same procedure to equation (7) and using the WCF we obtain the result (12). ∎

It is easy to see that (12) is a in fact polynomial in the $m_1,m_2,\dots,m_n$, since (13) is clearly a polynomial expression in these indeterminates. It is perhaps less obvious that the degree of (12) will actually be equal to n. This will follow from the next corollary, which expresses the number of weights in P^λ in terms of the k-th Todd polynomial. This is the homogeneous polynomial $\mathcal{T}_k$ given by the expansion of the generating function

$$\prod_{i\geq 1}\frac{tx_i}{1-e^{-tx_i}} = \sum_{k=0}^{\infty}\mathcal{T}_k(x_1,x_2,\dots)t^k \tag{14}$$

Corollary 3.3. *If λ is a dominant integral weight, then*

$$|P^\lambda| = \sum_{w\in W}\frac{1}{\prod_{i=1}^n\langle\rho,w\alpha_i\rangle}\sum_{j=0}^{n}\frac{\langle\rho,w\lambda\rangle^j}{j!}\mathcal{T}_{n-j}(\langle\rho,w\alpha_1\rangle,\dots,\langle\rho,w\alpha_n\rangle) \tag{15}$$

Furthermore, if $\lambda=(m_1,\dots,m_n)$ is expressed in the weight basis then $|P^\lambda|$ is a polynomial of degree n in the $m_1,\dots,m_n$.

Proof. For complex variables $s, z_1, \dots, z_n$, consider the function

$$F(s, z_1, \dots, z_n) = \prod_{i=1}^{n} \frac{s z_i}{1 - e^{-s z_i}}.$$

It is well-known that this is analytic in a neighborhood of 0 in $\mathbb{C}^{n+1}$ and hence has a power series expansion of the form

$$F(s, z_1, \dots, z_n) = \sum_{k=0}^{\infty} s^k \mathcal{T}_k(z_1, \dots, z_n),$$

where each $\mathcal{T}_k$ is a homogeneous symmetric polynomial of degree k known as the k-th Todd polynomial in $z_1, \dots, z_n$.

Interpreting e^{μ} as the complex function $t \to e^{t\langle \mu, \rho \rangle}$, we can rewrite formula (9) as

$$\sum_{w \in W} \frac{e^{t\langle w\lambda, \rho\rangle}}{\prod_{i=1}^{n}(1 - e^{-t\langle w\alpha_i, \rho\rangle})} = \sum_{\mu \in P^{\lambda}} e^{t\langle \mu, \rho\rangle}. \tag{16}$$

Each term in the left-hand side sum is a meromorphic function with a pole at $t = 0$, so it can be expanded as a Laurent power series:

$$\sum_{w \in W} \frac{e^{t\langle w\lambda, \rho\rangle}}{\prod_{i=1}^{n} t\langle w\alpha_i, \rho\rangle} \sum_{k=0}^{\infty} \mathcal{T}_k(\langle w\alpha_1, \rho\rangle, \dots, \langle w\alpha_n, \rho\rangle) t^k = \sum_{\mu \in P^{\lambda}} e^{t\langle \mu, \rho\rangle}. \tag{17}$$

By comparing the zero-th order term on each side of (17) we obtain equation (15). ∎

The numbers on Table 3 would probably have been difficult to obtain directly by counting the number of terms in each character, but were easily obtained with our formulas using LiE [11] and an optimized program written in C++ by the author.

The calculation of these numbers took advantage of the fact that the size of $P^{k\rho}$ is a polynomial in k whose degree is equal to the rank of the Lie algebra. This is a well known general fact [2, Théorème 32] about the number of lattice points in dilated polyhedra, but it also follows at once from our formulas, for instance by replacing λ with $k\rho$ in (15) and expanding, we get the n-th degree polynomial in k:

$$|P^{k\rho}| = \sum_{j=0}^{n} \left(\sum_{w \in W} \frac{\langle \rho, w\rho\rangle^j}{j! \prod_{i=1}^{n} \langle \rho, w\alpha_i\rangle} \mathcal{T}_{n-j}(\langle \rho, w\alpha_1\rangle, \dots, \langle \rho, \alpha_n\rangle) \right) k^j. \tag{18}$$

These polynomials, known as Ehrhart polynomials, were obtained by means of (18) for all simple Lie algebras of rank less than or equal to 8 and are listed on Table 3.

The meaning of the coefficients in the Erharht polynomial are generally unknown except for its leading term. However, a striking fact emerges at once from Table 3: the coefficient of k in the Ehrhart polynomial of $P^{k\rho}$ is equal to the number of positive roots. This suggests that

$$\sum_{w \in W} \frac{\langle w\rho, \rho\rangle}{\prod_{i=1}^{n} \langle w\alpha_i, \rho\rangle} \mathcal{T}_{n-1}(\langle w\alpha_1, \rho\rangle, \dots, \langle w\alpha_n, \rho\rangle) = |R^+|. \tag{19}$$

λ	ρ	2ρ	3ρ	4ρ	10ρ	100ρ
A_2	7	19	37	61	331	30301
B_2	12	37	76	129	741	70401
G_2	31	109	235	409	2461	240601
A_3	38	201	586	1289	17561	16150601
B_3	136	871	2728	6229	90991	87390901
C_3	135	863	2701	6165	89991	86390901
A_4	291	3081	13531	39801	1364601	12610451001
B_4	2304	30249	143392	438561	16150761	155909261601
C_4	2265	29657	140449	429345	15796761	152405261601
D_4	601	7009	31897	95569	3390721	32004661201
F_4	15145	219529	1075969	3346369	127013641	1248566942401
$\vdots$						
E_6	1.2×10^6	6.8×10^7	7.3×10^8	4.0×10^9	9.3×10^{11}	9.0×10^{17}
E_7	3.1×10^8	3.5×10^{10}	5.8×10^{11}	4.3×10^{12}	2.5×10^{15}	2.5×10^{22}
E_8	4.9×10^{11}	1.2×10^{14}	3.0×10^{15}	2.9×10^{16}	4.4×10^{19}	4.4×10^{27}

Table 3: Number of weights in the representations $V^{k\rho}$

We were unable to find a rational explanation for this fact at the moment.

Another striking fact is that for the Lie algebras of types B and C the respective Ehrhart polynomials seem to coincide up to the term of degree 2 and we were able to confirm this up to rank 10.

The Lie algebras A_2 and A_3. For type A_2, Table 2 says that

$$\chi_\lambda = \Theta_\lambda + \widetilde{\chi}_{\lambda-\rho}$$

whenever λ is dominant integral. The signed character on the right hand side is nonzero as long as $(\lambda-\rho)+\rho = \lambda$ is strictly dominant, hence expressing $\lambda = (p,q)$ in the weight basis, this simply means that the signed character is nonzero as long as $p > 0$ and $q > 0$. Therefore, we can state that

Corollary 3.4. *For type A_2, if λ is a dominant integral weight, then*

$$\chi_\lambda = \sum_{k\geq 0} \Theta_{\lambda-k\rho} \tag{20}$$

where the summation runs over all non-negative integers such that, except when $k=0$, $\lambda - k\rho$ is regular and dominant. If $\lambda = (x,y)$ is expressed in the weight basis, this condition is simply $k = 0, 1, \ldots, \min(x,y)$.

Table 5 lists the first few characters for the Lie algebra A_2 in terms of the girdles. It is interesting to notice that all the coefficients are nonnegative integers.

If follows at once that if λ is dominant integral, then the number of weights in V^λ is given by

$$|P^\lambda| = \dim V^\lambda - \dim V^{\lambda-\rho},$$

Algebra	Ehrhart polynomial $\|P^{k\rho}\|$
A_2	$1 + 3k + 3k^2$
B_2	$1 + 4k + 7k^2$
G_2	$1 + 6k + 24k^2$
A_3	$1 + 6k + 15k^2 + 16k^3$
B_3	$1 + 9k + 39k^2 + 87k^3$
C_3	$1 + 9k + 39k^2 + 86k^3$
A_4	$1 + 10k + 45k^2 + 110k^3 + 125k^4$
B_4	$1 + 16k + 126k^2 + 608k^3 + 1553k^4$
C_4	$1 + 16k + 126k^2 + 604k^3 + 1518k^4$
D_4	$1 + 12k + 66k^2 + 204k^3 + 318k^4$
F_4	$1 + 24k + 294k^2 + 2364k^3 + 12462k^4$
A_5	$1 + 15k + 105k^2 + 435k^3 + 1080k^4 + 1296k^5$
B_5	$1 + 25k + 310k^2 + 2470k^3 + 12985k^4 + 36145k^5$
C_5	$1 + 25k + 310k^2 + 2460k^3 + 12790k^4 + 34988k^5$
D_5	$1 + 20k + 190k^2 + 1100k^3 + 4030k^4 + 7872k^5$
A_6	$1 + 21k + 210k^2 + 1295k^3 + 5250k^4 + 13377k^5 + 16807k^6$
B_6	$1 + 36k + 645k^2 + 7560k^3 + 62595k^4 + 351252k^5 + 1037367k^6$
C_6	$1 + 36k + 645k^2 + 7540k^3 + 61950k^4 + 343020k^5 + 995828k^6$
D_6	$1 + 30k + 435k^2 + 3980k^3 + 24870k^4 + 104004k^5 + 235340k^6$
E_6	$1 + 36k + 630k^2 + 7020k^3 + 54270k^4 + 289440k^5 + 895536k^6$
A_7	$1 + 28k + 378k^2 + 3220k^3 + 18865k^4 + 76608k^5 + 200704k^6 + 262144k^7$
B_7	$1 + 49k + 1197k^2 + 19285k^3 + 225715k^4 + 1946259k^5 + 11481631k^6 + 35402983k^7$
C_7	$1 + 49k + 1197k^2 + 19250k^3 + 224070k^4 + 1912512k^5 + 11123084k^6 + 33742440k^7$
D_7	$1 + 42k + 861k^2 + 11340k^3 + 105630k^4 + 712236k^5 + 3300892k^6 + 8271168k^7$
E_7	$1 + 63k + 1953k^2 + 39375k^3 + 572670k^4 + 6242670k^5 + 50021748k^6 + 248454360k^7$
A_8	$1 + 36k + 630k^2 + 7056k^3 + 55755k^4 + 320544k^5 + 1316574k^6 + 3542940k^7 + 4782969k^8$
B_8	$1 + 64k + 2044k^2 + 43232k^3 + 673190k^4 + 8011136k^5 + 71657404k^6 + 439552864k^7 + 1400424097k^8$
C_8	$1 + 64k + 2044k^2 + 43176k^3 + 669620k^4 + 7906864k^5 + 69909280k^6 + 422739232k^7 + 1326439432k^8$
D_8	$1 + 56k + 1540k^2 + 27496k^3 + 353780k^4 + 3418128k^5 + 24687488k^6 + 123995296k^7 + 334582920k^8$
E_8	$1 + 120k + 7140k^2 + 279720k^3 + 8070300k^4 + 181243440k^5 + 3244615920k^6 + 45648947520k^7 + 438191214480k^8$

Table 4: Ehrhart polynomials for the representations $V^{k\rho}$ for the lower-rank semisimple Lie algebras

$$
\begin{aligned}
\chi_{(1,0)} &= \Theta_{(1,0)}\\
\chi_{(0,1)} &= \Theta_{(0,1)}\\
\chi_{(2,0)} &= \Theta_{(2,0)}\\
\chi_{(0,2)} &= \Theta_{(0,2)}\\
\chi_{(2,1)} &= \Theta_{(2,1)} + \Theta_{(1,0)}\\
&\vdots\\
\chi_{(4,3)} &= \Theta_{(4,3)} + \Theta_{(3,2)} + \Theta_{(2,1)} + \Theta_{(1,0)}\\
&\vdots\\
\chi_{(4,6)} &= \Theta_{(4,6)} + \Theta_{(3,5)} + \Theta_{(2,4)} + \Theta_{(1,3)} + \Theta_{(0,2)}\\
\chi_{(6,5)} &= \Theta_{(6,5)} + \Theta_{(5,4)} + \Theta_{(4,3)} + \Theta_{(3,2)} + \Theta_{(2,1)} + \Theta_{(1,0)}\\
&\vdots
\end{aligned}
$$

Table 5: Characters of A_2 expressed as sums of girdles in the weight basis

as long as λ is regular, otherwise it is equal to just $\dim V^\lambda$. From this, by expressing λ in the weight basis, we obtain the following

Corollary 3.5. *If $\lambda = (x, y)$ is a dominant integral weight for the Lie algebra A_2 expressed in the weight basis, then*

$$|P^\lambda| = \frac{1}{2}x^2 + 2xy + \frac{1}{2}y^2 + \frac{3}{2}x + \frac{3}{2}y + 1.$$

Table 6 shows the evaluation of formula (15) for the simple Lie algebras of rank 2. It is not hard to do the same for the simple Lie algebras of rank 3, but the polynomials get significantly bigger. The next corollary shows this for A_3.

Algebra	$\lvert P^\lambda\rvert$ ($\lambda = (x, y)$ in the weight basis)
A_2	$\frac{1}{2}x^2 + 2xy + \frac{1}{2}y^2 + \frac{3}{2}x + \frac{3}{2}y + 1$
B_2	$2x^2 + 4xy + y^2 + 2x + 2y + 1$
G_2	$3x^2 + 9y^2 + 12xy + 3x + 3y + 1$

Table 6: The polynomials $|P^\lambda|$ for the rank-2 algebras

Corollary 3.6. *If $\lambda = (x, y, z)$ is dominant integral for the Lie algebra A_3 expressed in the weight basis, then*

$$
\begin{aligned}
|P^\lambda| &= \frac{1}{6}x^3 + x^2y + \frac{3}{2}x^2z + 2xy^2 + 6\,xyz + \frac{3}{2}xz^2 + \frac{2}{3}y^3 + 2y^2z + yz^2 + \frac{1}{6}z^3\\
&+ x^2 + 4xy + 3xz + 2y^2 + 4yz + z^2 + \frac{11}{6}x + \frac{7}{3}y + 116z + 1.
\end{aligned}
$$

Exact formulas for the multiplicities. It is sometimes possible to obtain exact formulas for the multiplicities themselves by solving the recurrences. We carry this out for the multiplicity $m_\lambda(0, 0)$ of the representation V^λ of the Lie algebras B_2 and G_2. It is known that these are also the dimensions for certain representations of the Weyl group W. The techniques we used for solving the recurrences can be found for instance in [2, Chapitre II].

For B_2, the zero weight space occurs only when $\lambda = (p, 2q)$ in the weight basis, where p and q are non-negative integers, so for simplicity, let us consider $a_{p,2q} = \tilde{m}_{(p,2q)}(0,0)$. Now, Table 2 says that

$$\chi_\lambda = \Theta_\lambda + \widetilde{\chi}_{\lambda-\alpha_1-\alpha_2} + \widetilde{\chi}_{\lambda-\alpha_1-2\alpha_2} - \widetilde{\chi}_{\lambda-2\alpha_1-3\alpha_2},$$

where $\alpha_1 = (-2, 2)$ and $\alpha_2 = (2, -1)$ in the weight basis, hence by considering $\lambda = (p+1, 2p+2)$ in this formula we get the recurrence relation

$$a_{p+1,2q+2} = 1 - a_{p,2q+2} - a_{p+1,2q} + a_{p,2q},$$

which is valid for all integers $p, q \geq 0$. Multiplying both sides of this identity by $x^p y^{2q}$, summing over all $p, q \geq 0$, and solving for

$$G(x,y) = \sum_{p,q\geq 0} a_{p,2q} x^p y^{2q},$$

we finally arive at the generating function for the $a_{p,2q}$, namely

$$G(x,y) = \frac{1+xy^2}{(1-x)(1-x^2)(1-y^2)^2}.$$

Expanding the right hand side as in [2, Chapitre II, §5], we obtain the following rational decomposition:

$$G(x,y) = \frac{1}{4(1+x)(1-y^2)} + \frac{1}{4(1-x)(1-y^2)} + \frac{1+y^2}{2(1-x)^2(1-y^2)^2}.$$

Now each term in this decomposition can be represented as the product of two geometric series, except for the rightmost one which is the product of two series of the type

$$\frac{1}{(1-z)^k} = \sum_{n\geq 0} \binom{n+k-1}{k-1} z^n, \quad k \in \mathbb{N}, \tag{21}$$

From the decomposition we can easily isolate the coefficient $a_{p,2q}$. Finally, if we multiply the result by a suitable factor, we can lift the restriction on the weights being of the form $(p, 2q)$, obtaining the following result.

Corollary 3.7. *For any integers $p, q \geq 0$, the dimension of the zero weight space of $V^{(p,q)}$ of the Lie algebra B_2 is equal to the quasipolinomial*

$$m_{(p,q)}(0,0) = \left(\frac{1+(-1)^p}{4} + \frac{(p+1)(q+1)}{2}\right)\frac{1+(-1)^q}{2}.$$

Of course this means that whenever q is odd, then $m_{(p,q)}(0,0) = 0$.

For G_2, the zero weight space occurs in every irreducible representation, so we may index these spaces by $\lambda = (p, q)$, where p and q are non-negative integers.

Following a procedure similar to that which was carried out for B_2, and using the information on Table 2, we arrive at the generating function

$$G(x,y) \quad = \quad \frac{g(x,y)}{(1-x)^2(1-x^2)(1-x^3)(1-y)^3(1-y^2)},$$

where

$$\begin{aligned} g(x,y) \quad &= \quad 1-x-y+x^2+3yx+y^2-2yx^2-2y^2x-2yx^4- \\ &- \quad 2y^2x^3+yx^5+3y^2x^4+y^3x^3-y^2x^5-y^3x^4+y^3x^5. \end{aligned}$$

This, in turn, may be decomposed in the following way:

$$\begin{aligned} G(x,y) \quad &= \quad \frac{1}{2(1-x)^3(1-y)^3}+\frac{1}{2(1-x)^2(1-y)^4}+\frac{29}{72(1-x)^2(1-y)^2}- \\ &- \quad \frac{5}{12(1-x)^3(1-y)^2}-\frac{3}{4(1-x)^2(1-y)^3}+\frac{1}{6(1-x)^4(1-y)^2}+ \\ &+ \quad \frac{3}{8(1-x^2)(1-y^2)}+\frac{2-2x}{9(1-x^3)(1-y)^2}. \end{aligned}$$

Now, each term may be expressed as a product of either a geometric series or a series like (21), and the coefficient of x^py^q can be easily identified. This gives us the following result.

Corollary 3.8. *For any integers $p,q\geq 0$, the dimension of the zero weight space of $V^{(p,q)}$ of the Lie algebra G_2 is equal to the quasipolinomial*

$$\begin{aligned} m_{(p,q)}(0,0) \quad &= \quad \frac{1}{2}\binom{p+2}{2}\binom{q+2}{2}+\frac{1}{2}(p+1)\binom{q+3}{3}+\frac{29}{72}(p+1)(q+1)- \\ &- \quad \frac{5}{12}\binom{p+2}{2}(q+1)-\frac{3}{4}(p+1)\binom{q+2}{2}+\frac{1}{6}\binom{p+3}{3}(q+1)+ \\ &+ \quad \frac{4\sqrt{3}}{27}\sin\left(\frac{2\pi(p+1)}{3}\right)+\frac{3}{32}(1+(-1)^p)(1+(-1)^q). \end{aligned}$$

4. Proof of the Main Result

The purpose of this section is to prove our main result. We shall be regarding certain functions as rational functions of the $e^{-\alpha_i}$, i.e, as elements in the field $\mathbb{C}(e^{-\alpha_1},\ldots,e^{-\alpha_n})$ or in the ring of formal Laurent series $\mathbb{C}\langle\langle e^{-\alpha_1},\ldots,e^{-\alpha_n}\rangle\rangle$. In order to simplify the notation, we shall sometimes replace $e^{-\alpha_i}$ with x_i by means of the appropriate homomorphism. Thus, for instance, if $\alpha=a_1\alpha_1+a_2\alpha_2+\cdots+a_n\alpha_n$, where $a_i\in\mathbb{Z}$, then $e^{-\alpha}=(e^{-\alpha_1})^{a_1}(e^{-\alpha_2})^{a_2}\cdots(e^{-\alpha_n})^{a_n}\equiv x_1^{a_1}x_2^{a_2}\cdots x_n^{a_n}$. We shall also be adopting the following expansion convention:

$$(1-e^{-\alpha})^{-1}=1+e^{-\alpha}+e^{-2\alpha}+\cdots\,,\ \text{for}\ \alpha>0, \tag{22}$$

and

$$(1-e^{-\alpha})^{-1}=-e^{\alpha}-e^{2\alpha}-\cdots\,,\ \text{for}\ \alpha<0, \tag{23}$$

which can be regarded as formal power series in the $x_1, x_2, \ldots, x_n$.

As we mentioned, the statements in our main result, Theorem 2.1, will follow by proving that both sides of (7) are actually equal to the following “rational function”:

$$\sum_{w \in W} w \cdot \frac{e^{\lambda}}{\prod_{\alpha \in \Delta}(1 - e^{-\alpha})}. \tag{24}$$

This amounts to proving Theorems 2.3 and 2.4. The statement on the triangularity will require some arguments on the dominance partial order which will be given at the end of this section.

We would like to begin by proving Theorem 2.4, which will follow as a consequence of the usual WCF. We only need to show that the WCF still holds true for our signed characters.

Lemma 4.1 (WCF). *If λ is any integral weight and $\widetilde{\chi}_\lambda$ is a signed character then*

$$a_\rho \widetilde{\chi}_\lambda = a_{\lambda+\rho}, \tag{25}$$

where $a_\mu = \sum_{w \in W} (-1)^{l(w)} e^{w\mu}$.

Here we shall use the fact [4, Lemma 24.3] that

$$a_\rho = e^\rho \prod_{\alpha \in R^+} (1 - e^\alpha).$$

Proof. 4.1 First observe that a_μ is W-alternating, that is $w a_\mu = a_{w\mu} = (-1)^{l(w)} a_\mu$. This follows at once from the fact that the simple reflections in W permute R^+ and send exactly one positive root to its negative. If λ is dominant integral, then (25) is just the usual WCF, and there is nothing to show. Now, if $\lambda + \rho$ is not regular, then $s_\alpha(\lambda+\rho) = \lambda + \rho$ for some positive root α, hence

$$\begin{aligned}
a_{\lambda+\rho} &= a_{s_\alpha(\lambda+\rho)} \\
&= \sum_{w \in W} (-1)^{l(w)} e^{w s_\alpha(\lambda+\rho)} \\
&= \sum_{u \in W} (-1)^{l(u s_\alpha)} e^{u(\lambda+\rho)} \\
&= -\sum_{u \in W} (-1)^{l(u)} e^{u(\lambda+\rho)} = -a_{\lambda+\rho},
\end{aligned}$$

therefore $a_{\lambda+\rho} = 0$. Here it is useful to recall that $\det w = (-1)^{l(w)}$, so $(-1)^{l(u s_\alpha)} = \det(u s_\alpha) = \det(u)\det(s_\alpha) = -(-1)^{l(u)}$. However, in this case, the left-hand side of (25) is also zero by definition, so the equality holds.

Finally, if $\lambda+\rho$ is regular, then $\overline{\lambda}=w_\lambda(\lambda+\rho)-\rho$ is dominant integral and

$$\begin{aligned}
a_{\lambda+\rho} &= \sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)} \\
&= \sum_{u\in W}(-1)^{l(uw_\lambda^{-1})}e^{uw_\lambda(\lambda+\rho)} \\
&= (-1)^{l(w_\lambda)}\sum_{u\in W}(-1)^{l(u)}e^{u(\overline{\lambda}+\rho)} \\
&= (-1)^{l(w_\lambda)}a_{\overline{\lambda}+\rho},
\end{aligned}$$

which, by the usual WCF, is equal to $a_\rho(-1)^{l(w_\lambda)}\chi_{\overline{\lambda}}$, and hence equal to $a_\rho\widetilde{\chi}_\lambda$, by the definition of signed character. ■

Proof. (of 2.4) With the above considerations, we just have to carry out a straightforward calculation:

$$\begin{aligned}
a_\rho\sum_{w\in W} w\cdot\frac{e^\lambda}{\prod_{\alpha\in\Delta}(1-e^{-\alpha})} &= \\
&= \sum_{w\in W}(-1)^{l(w)}w\cdot\left(e^{\lambda+\rho}\prod_{\alpha\in R^+\setminus\Delta}(1-e^{-\alpha})\right) \\
&= \sum_{w\in W}(-1)^{l(w)}w\cdot\left(e^{\lambda+\rho}\left(1+\sum_{\langle\Phi\rangle\in\mathcal{F}}c_{\langle\Phi\rangle}e^{-\langle\Phi\rangle}\right)\right) \\
&= a_{\lambda+\rho}+\sum_{\langle\Phi\rangle\in\mathcal{F}}c_{\langle\Phi\rangle}\sum_{w\in W}(-1)^{l(w)}e^{w(\lambda-\langle\Phi\rangle+\rho)} \\
&= a_{\lambda+\rho}+\sum_{\langle\Phi\rangle\in\mathcal{F}}c_{\langle\Phi\rangle}a_{\lambda-\langle\Phi\rangle+\rho} \\
&= a_\rho\chi_\lambda+a_\rho\sum_{\langle\Phi\rangle\in\mathcal{F}}c_{\langle\Phi\rangle}\widetilde{\chi}_{\lambda-\langle\Phi\rangle},
\end{aligned}$$

where the last step is due to the WCF for signed characters, Lemma 4.1. ■

Now, in order to show Theorem 2.3, we shall need a few technical lemmas and also bit of notation. For each w in W let $\Phi_w=w^{-1}R^-\cap R^+$ be the set of all positive roots that are sent to negative roots by w and set $\Delta_w=\Phi_w\cap\Delta$, the set of simple roots sent to their negatives by w. It will be shown below (Lemma 4.7) that, unless $w=1$, Φ_w contains at least one simple root and hence Δ_w is nonempty. The complement of this set in the set of positive simple roots is $\Delta'_w=\Delta\setminus\Delta_w$.

With this notation, we let $K_{w\lambda}$ be the half-open "cone" generated by the linearly independent vectors $w\alpha$ ($\alpha\in\Delta_w$) and $-w\alpha$ ($\alpha\in\Delta'_w$) with vertex at $w\lambda$, namely the set

$$K_{w\lambda}=\left\{w\lambda+\sum_{\alpha\in\Delta_w}k_\alpha w\alpha-\sum_{\alpha\in\Delta'_w}k_\alpha w\alpha\right\},$$

where the k_α are integers, $k_\alpha > 0$ for $\alpha \in \Delta_w$, and $k_\alpha \geq 0$ for $\alpha \in \Delta'_w$. It is clear that, if λ is dominant integral and $w = 1$, then $P^\lambda \subset K_\lambda$, however, the next lemma will show that P^λ actually lies in the "complement" of all such cones, if $w \neq 1$

Let $\delta_{w\lambda}$ be the characteristic function of $K_{w\lambda}$, ie $\delta_{w\lambda}(\mu) = 1$ when $\mu \in K_{w\lambda}$ and zero otherwise.

Lemma 4.2. *If λ a dominant integral weight, then $P^\lambda \cap K_{w\lambda}$ is empty, unless $w = 1$.*

Proof. In effect, it is well known that $\mu \in P^\lambda$ if and only if $w\mu \leq \lambda$ for all w in W, hence if and only if $\lambda - w\mu = \sum_{\alpha\in\Delta} j_\alpha(w)\alpha$ where the $j_\alpha(w)$ are non-negative integers. Therefore $\mu \in P^\lambda$ if and only if

$$\mu = w\lambda + \sum_{\alpha\in\Delta_w} (-j_\alpha(w^{-1}))w\alpha - \sum_{\alpha\in\Delta'_w} j_\alpha(w^{-1})w\alpha,$$

which clearly does not belong to $K_{w\lambda}$, unless $w = 1$. ∎

Regarding (24) as a formal infinite sum on the weights (by means of the expansion convention above), to prove that (9) holds true, we must only show that the coefficients on its lefthand-side are all equal to zero, except for those corresponding to a weight in P^λ, and show that these nonzero coefficients are actually all equal to 1. This is what we shall prove next.

Proposition 4.3. *Consider the formal series $\sum_{\mu\in P} c_{\lambda\mu}e^\mu$ to be the formal expansion of (24) according to the conventions adopted above. Then the coefficient of e^μ in that expansion is equal to*

$$c_{\lambda\mu} = \sum_{w\in W} (-1)^{|\Delta_w|}\delta_{w\lambda}(\mu).$$

Furthermore, if $\mu \in P^\lambda$ then $c_{\lambda\mu} = 1$.

Proof. Since Δ is linearly independent, given w and μ we can solve

$$w\lambda + \sum_{\alpha\in\Delta_w} (k_\alpha + 1)w\alpha - \sum_{\alpha\in\Delta'_w} k_\alpha w\alpha = \mu$$

uniquely for $k = (k_\alpha)$. Now a straightforward formal calculation shows that:

$$\begin{aligned}
&\sum_{w\in W} w\cdot \frac{e^{\lambda}}{\prod_{\alpha\in\Delta}(1-e^{-\alpha})} = \\
&= \sum_{w\in W} e^{w\lambda} \prod_{\alpha\in\Delta_w} (1-e^{-w\alpha})^{-1} \prod_{\alpha\in\Delta'_w} (1-e^{-w\alpha})^{-1} \\
&= \sum_{w\in W} e^{w\lambda} \prod_{\alpha\in\Delta_w} \left(-e^{w\alpha} \sum_{k_\alpha=0}^{\infty} e^{k_\alpha w\alpha}\right) \prod_{\alpha\in\Delta'_w} \left(\sum_{k_\alpha=0}^{\infty} e^{-k_\alpha w\alpha}\right) \\
&= \sum_{w\in W} (-1)^{|\Delta_w|} \sum_{k\in\mathbb{Z}_+^{|\Delta|}} \exp\left(w\lambda + \sum_{\alpha\in\Delta_w} (k_\alpha+1)w\alpha - \sum_{\alpha\in\Delta'_w} k_\alpha w\alpha\right) \\
&= \sum_{w\in W} (-1)^{|\Delta_w|} \sum_{\mu\in K_{w\lambda}} e^{\mu} \\
&= \sum_{\mu\in P} \left(\sum_{w\in W} (-1)^{|\Delta_w|}\delta_{w\lambda}(\mu)\right) e^{\mu}.
\end{aligned}$$

This proves the $c_{\lambda\mu}$ are of the specified form. The fact that $c_{\lambda\mu} = 1$ when $\mu \in P^\lambda$ follows from the previous lemma. ■

At this point we know for sure that the following formal identity is true:

$$\sum_{w\in W} w\cdot \frac{e^{\lambda}}{\prod_{\alpha\in\Delta}(1-e^{-\alpha})} = \Theta_\lambda + \sum_{\mu\in P\setminus P^\lambda} c_{\lambda\mu} e^{\mu}, \tag{26}$$

It remains to show that the extra term on the right hand side vanishes. To show this, we shall regard the left hand side as a rational function and the right hand side as a Laurent power series as we discussed at the beginning of this section. It might be useful to recall that e^μ can be regarded as a shorthand notation for the function $t \mapsto e^{t\langle\mu,\rho\rangle}$, $t \in \mathbb{C}$. In this sense the following technical lemma will guarantee the existence of a region in which that rational function is represented by the Laurent power series on the right hand side.

Lemma 4.4. *For each w in W and λ in P, the series*

$$(-1)^{|\Delta_w|} \sum_{k\in\mathbb{Z}_+^{|\Delta|}} \exp\left(w\lambda + \sum_{\alpha\in\Delta_w} (k_\alpha+1)w\alpha - \sum_{\alpha\in\Delta'_w} k_\alpha w\alpha\right) \tag{27}$$

converges absolutely to the rational function

$$\frac{e^{w\lambda}}{\prod_{\alpha\in\Delta}(1-e^{-w\alpha})} \tag{28}$$

when we set $x_j = e^{-\alpha_j}$ for $j = 1,\dots,n$ in the polydisc $U = \{x \in \mathbb{C}^n \mid |x_i| < 1\}$, where $n = |\Delta|$. Furthermore in eq. (26) the right hand side converges absolutely to the left hand side in U.

Proof. For $j = 1, \ldots, n$, set $w\alpha_j = w_{1j}\alpha_1 + \cdots + w_{nj}\alpha_n$, where $n = |\Delta|$. For a fixed j, since $\alpha_j > 0$, we must have either all $w_{ij} \leq 0$ ($w\alpha_j < 0$) or all $w_{ij} \geq 0$ ($w\alpha_j > 0$) for $i = 1, \ldots, n$. Hence the terms in (27) and the denominator in (28) involve only non-negative integral powers of $x_j = e^{-\alpha_j}$ for $j = 1, \ldots, n$. Furthermore, in series (27), all terms have $\exp(w\lambda + \sum_{\alpha\in\Delta_w} w\alpha)$ as a common factor and, once this is factored out, the remaining expression is a multigeometric series in the x_j, so it converges absolutely in the polydisc U and uniformly in compacta $K \subset U$ to the rational function

$$\frac{1}{\prod_{\alpha\in\Delta_w}(1 - e^{w\alpha})\prod_{\alpha\in\Delta'_w}(1 - e^{-w\alpha})},$$

hence, (27) converges to the rational function

$$(-1)^{|\Delta_w|}\frac{\exp(w\lambda + \sum_{\alpha\in\Delta_w} w\alpha)}{\prod_{\alpha\in\Delta_w}(1 - e^{w\alpha})\prod_{\alpha\in\Delta'_w}(1 - e^{-w\alpha})}.$$

However, dividing the numerator and the denominator in this expression by $(-1)^{|\Delta_w|}\exp(\sum_{\alpha\in\Delta_w} w\alpha) = \prod_{\alpha\in\Delta_w}(-e^{w\alpha})$, we obtain

$$\frac{e^{w\lambda}}{\prod_{\alpha\in\Delta_w}(-e^{-w\alpha})\prod_{\alpha\in\Delta_w}(1 - e^{w\alpha})\prod_{\alpha\in\Delta'_w}(1 - e^{-w\alpha})} = \frac{e^{w\lambda}}{\prod_{\alpha\in\Delta}(1 - e^{-w\alpha})},$$

which is (28). The second statement is a consequence of the absolute convergence since the series on the right hand side of (26) is obtained by summing (27) over W (a finite sum) and rearranging its terms. ∎

The following proposition completes the proof of Theorem 2.3 in showing that the extra term on the right hand side of (26) is equal to zero.

Proposition 4.5. *If λ is dominant integral and μ is not in P^λ, then*

$$c_{\lambda\mu} = \sum_{w\in W}(-1)^{|\Delta_w|}\delta_{w\lambda}(\mu) = 0.$$

Proof. It is useful to recall once more that we can regard e^μ as a shorthand for the function $t \mapsto e^{t\langle\mu,\rho\rangle}$, which allows us to look at certain expressions, like the one on left hand side of (26), as a rational function, and regard the right hand side of that same equation as a Laurent power series. We shall do the same with both sides of equation (10).

Now, from Lemma (4.4), we know that in (26) the rational function in the lefthand side, namely

$$\sum_{w\in W} w\cdot\frac{e^\lambda}{\prod_{\alpha\in\Delta}(1 - e^\alpha)},$$

is represented in the polydisc U by the power series in the right hand side. On the other hand, we know from Theorem 2.4, that the same expression is formally equal, and thus converges in U, to the Laurent polynomial in the right hand side of (10). From the uniqueness of the expansion, we conclude that series and polynomial are identical, and hence the right hand side of (26) must be supported on P^λ, as required. ∎

To complete the proof of our main result, it only remains to show the triangularity, namely that the weights $\lambda - \langle\Phi\rangle$ in (7) are strictly smaller than λ in the dominance order. For this we shall need a lemma of Kostant and a few auxiliary results. Some of the ideas that follow can be found in greater generality and beauty in [7], but here we preferred to state less general statements and provide elementary proofs.

Lemma 4.6 (Kostant). *Let μ be a weight in P. Then μ belongs to P^ρ if and only if $\mu = \rho - \langle\Phi\rangle$ for some subset Φ of R^+. Moreover, the multiplicity $m_\rho(\mu)$ is equal to the number of such subsets.*

Proof. This is Lemma 5.9 in [7]. ∎

For each w in the Weyl group W, recall the definition of $\Phi_w = w^{-1}R^- \cap R^+$, the set of all positive roots that are sent to negative roots by w, and let Φ'_w be its complement in R^+.

Lemma 4.7. *Let $w \in W$. If Φ_w is nonempty then it contains a simple root.*

Proof. Let w_0 be the longest element in W, namely the unique element in W such that $w_0 R^+ = R^-$. First we observe that

$$\begin{aligned}\Phi'_w = R^+ \backslash (w^{-1}R^- \cap R^+) = w^{-1}R^+ \cap R^+ &= w^{-1}w_0 R^- \cap R^+ \\ &= (w_0 w)^{-1} R^- \cap R^+ = \Phi_{w_0 w}.\end{aligned}$$

Now, if Φ_w contains Δ, then, in particular, the positive system $w^{-1}R^-$ contains the simple system Δ, so $w^{-1}R^- = R^+$. Thus $w^{-1} = w_0$ and Φ'_w is empty.

Repeating the same argument with $w_0 w$ instead of w we see that if Φ'_w contains Δ then Φ_w is empty. ∎

Lemma 4.8. *If Φ is a subset of R^+ then*

$$\langle \rho - \langle\Phi\rangle, \rho - \langle\Phi\rangle \rangle \leq \langle \rho, \rho \rangle,$$

and equality holds if and only if $\Phi = \Phi_w$ for some w in W.

Proof. Let $w \in W$ be the (unique) element such that $\mu = w(\rho - \langle\Phi\rangle)$ is dominant. By Lemma 4.6, $\rho - \langle\Phi\rangle \in P^\rho$, hence $\mu \in P^\rho$. This also means that $\tau = \rho - \mu$ is a sum of positive roots [4, §21.3], so $\langle \tau, \mu \rangle \geq 0$. It follows that

$$\begin{aligned}\langle \rho, \rho \rangle &= \langle \tau + \mu, \tau + \mu \rangle = \langle \tau, \tau \rangle + 2\langle \tau, \mu \rangle + \langle \mu, \mu \rangle \\ &\geq \langle \mu, \mu \rangle = \langle \rho - \langle\Phi\rangle, \rho - \langle\Phi\rangle \rangle,\end{aligned}$$

which shows the inequality.

Now suppose that the equality holds, so we must actually have $\tau = 0$. Therefore $\mu = \rho$, so $w^{-1}\rho = \rho - \langle\Phi\rangle$. On the other hand, if this identity is true for some $w \in W$, then obviously $\langle \rho - \langle\Phi\rangle, \rho - \langle\Phi\rangle \rangle = \langle \rho, \rho \rangle$.

It follows at once from the definition that $w^{-1}\rho = \rho - \langle\Phi_w\rangle$, but since the multiplicity $m_\rho(w^{-1}\rho) = m_\rho(\rho) = 1$, we conclude from Lemma 4.6 that $\Phi = \Phi_w$. ■

We are finally in place to prove our main result, Theorem 2.1.

Proof. (of 2.1) As we mentioned above, Theorems 2.3 and 2.4 together imply that (7) is true.

Now we shall prove that all the weights of the form $\overline{\lambda - \langle\Phi\rangle}$ for which $\lambda - \langle\Phi\rangle$ occur in (7) are strictly smaller than λ in the dominance order. That is to show that when λ is dominant integral and Φ is a nonempty subset of Δ', then $\overline{\lambda - \langle\Phi\rangle} = w(\lambda - \langle\Phi\rangle + \rho) - \rho < \lambda$, where $w \in W$ is the unique element such that $w(\lambda - \langle\Phi\rangle + \rho)$ is dominant. It is useful to notice that Φ contains no simple root, so it cannot be equal to any set of the type Φ_w by Lemma 4.7.

Put $\mu = \overline{\lambda - \langle\Phi\rangle}$. Since $\lambda - \langle\Phi\rangle + \rho \le \lambda + \rho$, then

$$\mu + \rho = w(\lambda - \langle\Phi\rangle + \rho) \le \lambda + \rho,$$

so $\mu \le \lambda$. If we had $\mu = \lambda$, then we would also have

$$\langle\lambda - \langle\Phi\rangle + \rho, \lambda - \langle\Phi\rangle + \rho\rangle = \langle\lambda + \rho, \lambda + \rho\rangle.$$

But, in this case, Lemma 4.8 would imply that

$$0 \le 2\langle\lambda, \langle\Phi\rangle\rangle = \langle\rho - \langle\Phi\rangle, \rho - \langle\Phi\rangle\rangle - \langle\rho, \rho\rangle \le 0,$$

that is $\langle\rho - \langle\Phi\rangle, \rho - \langle\Phi\rangle\rangle = \langle\rho, \rho\rangle$, and thus $\Phi = \Phi_w$ for some $w \in W$. But, as it was remarked, this is impossible, hence $\mu < \lambda$, as required. ■

5. Conclusions

We have discovered a new character formula for complex semisimple Lie algebras and compact Lie Groups which seems to be very different from previously-known formulas, and we have shown a few consequences and applications. We believe that our formula could offer some new, interesting insights into the combinatorics of root systems. The existence of (7) suggests that one could perhaps find a combinatorial formula for the characters. We know that for the Lie algebras of type A this is true, the characters being essentially the Schur polynomials. We intend to investigate this possibility in our future work.

The fact that it might be possible to solve the recursions and obtain exact formulas for the multiplicities also seems to open intriguing possibilities, since formulas of this kind seem to be mostly unknown. It is conceivable that using techniques like those developed by Erhart [2, Chap. II, §5], many of our recursions could be solved exactly.

Since the scalar product is W-invariant, we can rewrite expression (15) as

$$\sum_{w\in W} \frac{1}{\prod_{i=1}^{n}\langle w\rho, \alpha_i\rangle} \sum_{j=0}^{n} \frac{\langle w\rho, \lambda\rangle^j}{j!} \mathcal{T}_{n-j}(\langle w\rho, \alpha_1\rangle, \ldots, \langle w\rho, \alpha_n\rangle) \tag{29}$$

which becomes a sum on the orbit of the regular weight ρ. This formula might be slightly more interesting than (15), for there exist efficient algorithms for generating the weights in a Weyl group orbit sequentially which do not require the storage of a large amount of information.

There seems to be other intriguing possibilities and applications and we intend to be addressing them in future works, for instance the relationship between our coefficients and the Kostant partition function.

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All computations in this work where carried out using the LiE Computer Algebra System [11] and C++ programs written by the author.

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