Orbits of Distal Actions on Locally Compact Groups

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Abstract. We discuss properties of orbits of (semi)group actions on locally compact groups. In particular, we show that if a compactly generated locally compact abelian group acts distally on $G$ then the closure of each of its orbits is a minimal closed invariant set (i.e. the action has [MOC]). We also show that for such an action distality is preserved if we go modulo any closed normal invariant subgroup and hence [MOC] is also preserved. We also show that any semigroup action on $G$ has [MOC] if and only if the corresponding actions on a compact invariant metrizable subgroup $K$ and on the quotient space $G/K$ have [MOC].

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1. Introduction

Let $X$ be a Hausdorff space and let $\Gamma$ be a (topological) semigroup acting continuously on $X$ by continuous self-maps. The action of $\Gamma$ on $X$ is said to be distal if for any two distinct points $x, y \in X$, the closure of $\{(\gamma(x), \gamma(y)) \mid \gamma \in \Gamma\}$ does not intersect the diagonal $\{(a, a) \mid a \in X\}$. It is said to be pointwise distal if for each $\gamma \in \Gamma$, the action of $\{\gamma^n\}_{n \in \mathbb{N}}$ on $X$ is distal. The $\Gamma$-action on $X$ is said to have [MOC] (minimal orbit closures) if the closure of every $\Gamma$-orbit is a minimal closed $\Gamma$-invariant set, i.e. for $x, y \in X$, if $y \in \Gamma(x)$ then $\Gamma(y) = \Gamma(x)$. The notion of distality was introduced by Hilbert (cf. Ellis [7], Moore [13]) and studied by many in different contexts, (see Abels [1]-[2], Furstenberg [8], Raja-Shah [17] and the references cited therein).

Let $G$ be a locally compact (Hausdorff) group and let $e$ denote the identity of $G$. Let $\Gamma$ be a semigroup acting continuously on $G$ by endomorphisms. Then the $\Gamma$-action on $G$ is distal if and only if $e \not\in \Gamma x$ for all $x \in G \setminus \{e\}$. Note that if the $\Gamma$-action on $G$ has [MOC], then it is distal; for if $e \in \Gamma x$, then $\{e\} = \Gamma e = \Gamma x$ and hence $x = e$. What we are interested in is the converse: If the $\Gamma$-action on

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$G$ is distal, does it have [MOC]? The answer is known to be affirmative in any of the following cases: (1) $G$ is compact (2) $\Gamma$ is compact, (3) $G$ is a connected Lie group and $\Gamma$ is a subgroup of $\text{Aut}(G)$ (4) $\Gamma$ is a group and $G$ is discrete, or more generally, all $\Gamma$-orbits are closed. If $\Gamma$ is a group and if $\Gamma'$ is a closed co-compact normal subgroup, then the $\Gamma$-action on $G$ has [MOC] if and only if the $\Gamma'$-action on $G$ has [MOC] (cf. [13]); it is easy to see that the same equivalence is true for distality. For a general locally compact group $G$ and a group $\Gamma$ which acts on $G$ by automorphisms, the answer to the above question is not known. But in case of a certain kind of $\Gamma$, we get the following:

**Theorem 1.1.** Let $G$ be a locally compact group and let $\Gamma$ be a compactly generated locally compact abelian group such that $\Gamma$ acts on $G$ by automorphisms. Then the following are equivalent:

1. The $\Gamma$-action on $G$ is distal
2. The $\Gamma$-action on $G$ has [MOC].

Let us now discuss general actions on compact spaces. For a compact space $K$, let $\Gamma$ be a semigroup of continuous bijective self-maps of $K$. Then $\Gamma$ is a subsemigroup of $C(K)$, the group of all continuous bijective self-maps on $K$. Let $[\Gamma]$ be the group generated by $\Gamma$ in $C(K)$. We know that $\Gamma$ acts distally on $K$ if and only if $E(\Gamma)$, the closure of $\Gamma$ in $K^K$ with weak topology, is a group (see [7], Theorem 1 which is for group actions and it can easily be seen that the same proof works for semigroup actions). It is obvious that $E(\Gamma)$ is compact since $K^K$ is so. When $E(\Gamma)$ is a group, we have $E(\Gamma) = E([\Gamma])$; moreover, for any $x \in K$, $\Gamma(x) = E(\Gamma)(x) = E([\Gamma])(x)$. In particular if $K$ is a compact group and $\Gamma$ acts on $K$ by automorphisms and $[\Gamma]$ is as above, then the following are equivalent:

1. The $\Gamma$-action on $K$ is distal.
2. The $[\Gamma]$-action on $K$ is distal.
3. The $\Gamma$-action on $K$ has [MOC].
4. The $[\Gamma]$-action on $K$ has [MOC].

Moreover, for a closed subgroup $H$ of the compact group $K$ which is $\Gamma$-invariant (i.e. $\gamma(H) = H$ for all $\gamma \in \Gamma$), the above equivalence is also true for the actions of $\Gamma$ and $[\Gamma]$ on $K/H$. Note that for such an $H$, the corresponding $\Gamma$-action on the homogeneous space $K/H = \{xH \mid x \in K\}$ is canonically defined as $\gamma(xH) = \gamma(x)H$ for all $\gamma \in \Gamma$; it is clearly well-defined.

In [17], it is shown that distality of a semigroup action is preserved by factor actions modulo compact invariant subgroups. We show that a similar result holds for [MOC], (see also Remark 2.2).

**Theorem 1.2.** Let $G$ be a locally compact group and let $\Gamma$ be a subsemigroup of $\text{Aut}(G)$. Let $K$ be a compact metrizable $\Gamma$-invariant subgroup of $G$. Then the $\Gamma$-action on $G$ has [MOC] if and only if $\Gamma$-actions on both $K$ and $G/K$ have [MOC].
The following result is about factor actions modulo closed normal invariant subgroups.

**Theorem 1.3.** Let $G$ and $\Gamma$ be as in Theorem 1.1. Let $H$ be a closed normal $\Gamma$-invariant subgroup of $G$. Then the $\Gamma$-action on $G$ has [MOC] if and only if $\Gamma$-actions on both $H$ and $G/H$ have [MOC].

We will later show that a similar result holds for distality for a larger class of $\Gamma$.

A locally compact group $G$ is said to be **distal** (resp. **pointwise distal**) if the conjugacy action of $G$ on $G$ is distal (resp. pointwise distal). A distal group is obviously pointwise distal. Abelian groups, discrete groups and compact groups are obviously distal. Nilpotent groups, connected groups of polynomial growth are distal (cf. [19]) and p-adic Lie groups of type $R$ and p-adic Lie groups of polynomial growth are pointwise distal (cf. Raja [14] and [15]).

In [17], jointly with C. R. E. Raja, we have shown that any locally compact group is pointwise distal if and only if it has shifted convolution property; i.e. for any probability measure $\mu$ on $G$, whose concentration functions do not converge to zero, there exists $x \in \text{supp} \mu$, the support of $\mu$, such that $\mu^n x^{-n} \to \omega_H$, the Haar measure of some compact group $H$ which is normalised by $\text{supp} \mu$. For a probability measure $\mu$ on $G$, the $n$-th convolution function of $\mu$ is defined as $f_n(\mu, C) = \sup_{g \in G} \mu^n(Cg)$, for any compact subset $C$ of $G$. We say that the concentration functions of $\mu$ do not converge to zero if there exists a compact set $C$ such that $f_n(\mu, C) \not\to 0$ as $n \to \infty$, (see [17] for more details). The following corollary is a consequence of Theorem 6.1 of [17] and Theorem 1.1.

**Corollary 1.4.** Let $G$ be a locally compact group. Then the following are equivalent:

1. $G$ is pointwise distal.
2. $G$ has shifted convolution property.
3. For every $g \in G$, the conjugation action of $\{g^n\}_{n \in \mathbb{Z}}$ on $G$ has [MOC].

A locally compact group $G$ is said to be a **generalised FC**-group (resp. **FC-**-nilpotent) if $G$ has closed normal subgroups $\{G = G_0, \ldots, G_n = \{e\}\}$ such that $G_{i+1} \subset G_i$ and $G_i/G_{i+1}$ is a compactly generated group with relatively compact conjugacy classes (resp. every orbit of the conjugacy action of $G$ on $G_i/G_{i+1}$ is relatively compact) for all $i = 0, 1, \ldots, n-1$. Any compactly generated abelian group (resp. any polycyclic group) is a generalised FC-group. Any compactly generated group $G$ has polynomial growth if and only if it is FC-nilpotent; and it is a generalised FC-group (cf. [12]). Note that generalised FC-groups are compactly generated (cf. [12], Proposition 2).

Recall that a subgroup $\Gamma$ of Aut($G$) is said to be equicontinuous (at $e$) if and only if there exists a neighbourhood base at $e$ consisting of $\Gamma$-invariant neighbourhoods; in case of totally disconnected groups, this is equivalent to the
existence of a neighbourhood base at \( e \) consisting of compact open \( \Gamma \)-invariant subgroups. If \( \Gamma \) is compact, then it is easy to see that \( \Gamma \) is equicontinuous. If \( G \) is a totally disconnected group and if \( \Gamma \) has a polycyclic subgroup of finite index and it acts distally on \( G \), then \( \Gamma \) is equicontinuous (cf. [11], Corollary 2.4). If any group \( \Gamma \) acts on \( G \) by automorphisms and its image in \( \text{Aut}(G) \) is equicontinuous then we say that the \( \Gamma \)-action on \( G \) is equicontinuous.

For a totally disconnected locally compact group \( G \), we have the following:

**Proposition 1.5.** Let \( G \) be a totally disconnected locally compact group and let \( \Gamma \) be a generalised \( FC^- \)-group which acts on \( G \) by automorphisms. Then the following are equivalent.

1. The \( \Gamma \)-action on \( G \) is distal.
2. The \( \Gamma \)-action on \( G \) has [MOC].
3. The \( \Gamma \)-action on \( G \) is equicontinuous.

In Section 2, we discuss factor actions modulo compact (resp. closed normal) invariant groups and prove Theorem 1.2, Proposition 1.5 and an analogue of Theorem 1.3 for distal actions of a more general class of groups. In Section 3, we prove the equivalence of distality and [MOC] of certain actions, namely, Theorem 1.1. Note that if \( \Gamma \) acts on \( G \) by automorphisms, for convenience, \( \Gamma \) is often equated with its image in \( \text{Aut}(G) \), whenever there is no loss of any generality.

### 2. Orbits of Factor Actions

In this section we discuss [MOC] of factor actions modulo compact invariant groups and modulo closed normal invariant groups. We first show that [MOC] is preserved if we go modulo a compact invariant subgroup by proving Theorem 1.2. Before that we prove a proposition which proves a special case of the theorem in case the compact subgroup is a Lie group.

**Proposition 2.1.** Let \( G \) be a locally compact group and let \( \Gamma \) be a subsemigroup of \( \text{Aut}(G) \). Let \( K \) and \( L \) be compact \( \Gamma \)-invariant subgroups of \( G \) such that \( L \) is a normal subgroup of \( K \) and \( K/L \) is a Lie group. Then the \( \Gamma \)-action on \( G/L \) has [MOC] if and only if \( \Gamma \)-actions on both \( G/K \) and \( K/L \) have [MOC].

**Proof.** Step 1. Let \( G, \Gamma, K \) and \( L \) be as in the hypothesis. One way implication “only if” is easy to prove. Suppose the \( \Gamma \)-action on \( G/L \) has [MOC]. Then clearly the \( \Gamma \)-action on \( K/L \) also has [MOC], as \( K \) is closed and \( \Gamma \)-invariant. Now we want to show that the \( \Gamma \)-action on \( G/K \) has [MOC]. Let \( x \in G \) and let \( yK \in \overline{\Gamma(xK)} \) in \( G/K \) for some \( y \in G \). Then \( yK \subset \overline{\Gamma(x)K} = \overline{\Gamma(x)K} \) and hence \( yk \in \overline{\Gamma(x)} \) for some \( k \in K \). In particular, we get that \( ykL \subset \overline{\Gamma(x)L} = \overline{\Gamma(x)L} \) as \( L \) is compact. Hence \( ykL \in \overline{\Gamma(xL)} \) in \( G/L \). Since the \( \Gamma \)-action on \( G/L \) has [MOC], we get that \( \overline{\Gamma(xL)} = \overline{\Gamma(ykL)} \) and hence \( x \in \overline{\Gamma(y)K} \) as \( k \in K, L \subset K \) and both \( L \) and \( K \) are \( \Gamma \)-invariant groups. This implies that \( xK \in \overline{\Gamma(yK)} \) in \( G/K \) and
hence the $\Gamma$-action on $G/K$ has [MOC]. Note that the condition that $K/L$ is a Lie group is not used in the proof of the “only if” statement.

**Step 2.** Now we prove the “if” statement. Suppose $\Gamma$-actions on both $G/K$ and $K/L$ have [MOC]. This implies that the $\Gamma$-action on $K/L$ is distal as $K/L$ is a group. We will first show, using compactness of $K$, that since the $\Gamma$-action on $G/K$ has [MOC], it is distal. This, together with the previous assertion, would imply that the $\Gamma$-action on $G/L$ is distal. Let $x,y,a \in G$ be such that $\gamma_d(xK) \to aK$ and $\gamma_d(yK) \to aK$ in $G/K$ for some $\{\gamma_d\} \subset \Gamma$. We need to show that $xK = yK$. Since $K$ is compact, it is easy to show that $\gamma(y^{-1}xK) \to eK$. Since $\{eK\}$ is $\Gamma$-invariant in $G/K$, [MOC] of the $\Gamma$-action on $G/K$ implies that $y^{-1}xK = eK$, and hence, $xK = yK$.

For any $g \in G$, let $g' = gL$. The map $g \mapsto g'$ is a continuous proper map from $G$ to $G/L$. Let $x \in G$ and let $y' \in \overline{\Gamma(x')}$ for some $y \in G$. We want to show that $x' \in \overline{\Gamma(y')}$, then $yK \in \overline{\Gamma(xK)}$, and as the $\Gamma$-action on $G/K$ has [MOC], $xK \in \overline{\Gamma(yK)}$. This implies that $xk \in \overline{\Gamma(y)}$ for some $k \in K$, and hence, $x'k' \in \overline{\Gamma(y')}$. Let $\{\gamma_d\}$ and $\{\beta_d\}$ be nets in $\Gamma$ such that $\gamma_d(x') \to y'$ and $\beta_d(y') \to x'k'$. 

**Step 3.** Let $\Gamma_0$ be the closure of the image of $\Gamma$ in $\text{Aut}(K/L)$. Suppose $\Gamma_0$ is compact. Then $\Gamma_0$, being a compact semigroup, is a group. Let $\beta$ and $\gamma$ be limit points of images of $\{\beta_d\}$ and $\{\gamma_d\}$ in $\Gamma_0$ respectively. Then

$$\gamma_d(x'k') \to y'\gamma(k') \in \overline{\Gamma(y')}, \quad \beta_d(y'\gamma(k')) \to x'k'\alpha(k') \in \overline{\Gamma(y')}$$

where $\alpha = \beta\gamma \in \text{Aut}(K/L)$. Similarly we get that for

$$k_n = k'\alpha(k')\cdots\alpha^{n-1}(k') \in K/L, \quad x_n = x'k_n \in \overline{\Gamma(y')}, \quad \text{for all } n \in \mathbb{N}.$$ 

As $\Gamma_0$ is a compact group, there exists a sequence $\{n_j\} \subset \mathbb{N}$ such that $\alpha^{n_j} \to I$, the identity of $\text{Aut}(K/L)$. Passing to a subsequence if necessary, we may assume that $k_{n_j} \to c' = cL \in K/L$, for some $c \in K$. Hence $x'c' \in \overline{\Gamma(y')}$. Now as $\alpha^{n_j} \to I$,

$$k_{2n_j} = k_{n_j}\alpha^{n_j}(k_{n_j}) \to (cL)^2 = c^2L.$$ 

Similarly, for all $m \in \mathbb{N}$,

$$k_{mn_j} = k_{n_j}\alpha^{n_j}(k_{n_j})\cdots\alpha^{(m-1)n_j}(k_{n_j}) \to c^mL \in K/L$$

and $xc^mL \in \overline{\Gamma(yL)}$. Since $K/L$ is a compact (Lie) group, $c' = eL$ is in the closure of $\{c^mL\}_{m \in \mathbb{N}}$ in $K/L$ and hence $x' \in \overline{\Gamma(y')}$, i.e. $\overline{\Gamma(x')} = \overline{\Gamma(y')}$. Hence the $\Gamma$-action on $G/L$ has [MOC].

Since $K/L$ is a Lie group, $K/K^0L$ is finite, and hence, $\text{Aut}(K/K^0L)$ is finite. Arguing as above for $K^0L$ in place of $L$, we get that the $\Gamma$-action on $G/K^0L$ has [MOC] and we may assume that $K = K^0L$, i.e. $K/L$ is connected.

**Step 4.** Let $Z$ be the subgroup of $K$ such that $L \subset Z$ and $Z/L$ is the center of $K/L$. Then $Z$ and $Z^0L$ are closed and $\Gamma$-invariant. Moreover, $K/Z$ is a connected semisimple Lie group and hence its automorphism group is compact. Therefore arguing as in Step 3 for $Z$ in place of $L$, we get that the $\Gamma$-action on $G/Z$ has...
[MOC], and since \( Z/Z^0L \) is finite, the \( \Gamma \)-action on \( G/Z^0L \) also has [MOC]. Now replacing \( K \) by \( Z^0L \), we may assume that \( K/L \) is a connected abelian Lie group.

Let \([\Gamma]\) be the group generated by \( \Gamma \) in \( \text{Aut}(K/L) \). Then \([\Gamma]\) also acts distally on \( K/L \). By Lemma 2.5 of [2], there exists a finite set of compact (normal) \( [\Gamma] \)-invariant subgroups \( \{K_0, \ldots, K_n\} \) in \( K \) such that \( K = K_0 \supset K_1 \supset \cdots \supset K_n = L \) and the image of \([\Gamma]\) in \( \text{Aut}(K_i/K_{i+1}) \) is finite for each \( i \in \{0, \ldots, n-1\} \).

Arguing as in Step 3 for \( K_1 \) in place of \( L \), we get that the \( \Gamma \)-action on \( G/K_1 \) has [MOC]. Since the image of \( \Gamma \) in \( \text{Aut}(K_i/K_{i+1}) \) is finite, using the above argument repeatedly for \( K_i/K_{i+1} \) in place of \( K/L \), we get that the \( \Gamma \)-action on \( G/K_{i+1} \) has [MOC], \( 1 \leq i \leq n-1 \). Since \( K_n = L \), the \( \Gamma \)-action on \( G/L \) has [MOC].


**Proof of Theorem 1.2.** Let \( G \), \( \Gamma \) and \( K \) be as in the hypothesis. The “only if” statement follows as in Step 1 of the proof of Proposition 2.1. Now we prove the “if” statement. Suppose that \( \Gamma \)-actions on both \( G/K \) and \( K \) have [MOC]. Hence \( \Gamma \)-actions on \( G/K \), \( K \) and \( G \) are distal, (see Step 2 of the proof of Proposition 2.1). Let \( K \) consist of closed (compact) \( \Gamma \)-invariant subgroups \( C \) of \( K \) such that the \( \Gamma \)-action on \( G/C \) has [MOC]. Then \( K \) is nonempty as \( K \) belongs to \( K \). We put an order on \( K \) by set inclusion. Let \( A = \{K_d\} \) be a totally ordered subset of \( K \). We show that \( K' = \cap K_d \in K \).

For any \( x \in G \) and \( y \in \Gamma(x)K' \), we show that \( \Gamma(x)K' = \Gamma(y)K' \). We know that \( \Gamma(x)K_d = \Gamma(y)K_d \) for each \( d \). First we show that \( \cap_{d} \Gamma(x)K_d = \Gamma(x)K' \). One way inclusion is obvious. Let \( a \in \cap_{d} \Gamma(x)K_d \). Then \( C_d = \Gamma(x) \cap a K_d \neq \emptyset \) for all \( d \).

\( A' = \{C_d\} \) is a collection of compact sets and intersection of finitely many subsets in \( A' \) is nonempty since \( A \) is totally ordered. Hence \( \cap_{d} C_d \) is nonempty. But

\[
\cap_{d} C_d = \cap_{d} (\Gamma(x) \cap a K_d) = \Gamma(x) \cap (\cap_{d} a K_d) = \Gamma(x) \cap a K' \neq \emptyset.
\]

Hence \( a \in \Gamma(x)K' \). Therefore, \( \cap_{d} \Gamma(x)K_d = \Gamma(x)K' \). Similarly, \( \cap_{d} \Gamma(y)K_d = \Gamma(y)K' \). This implies that \( \Gamma(x)K' = \Gamma(y)K' \) and hence the \( \Gamma \)-action on \( G/K' \) has [MOC], i.e. \( K' \in K \).

By Zorn’s Lemma, there exists a minimal element in \( K \), say \( M \). Here, \( M \) is a compact \( \Gamma \)-invariant subgroup of \( K \) such that the \( \Gamma \)-action on \( G/M \) has [MOC] and there is no proper subgroup of \( M \) in \( K \). We show that \( M = \{e\} \). If possible suppose \( M \) is nontrivial. Since \( M \subset K \) is compact and metrizable and since the \( \Gamma \)-action on \( M \) is distal, it is not ergodic and there exists a (nontrivial) irreducible unitary representation \( \chi \) of \( M \) such that \( \chi \Gamma \) is finite upto equivalence classes (cf. [3], Theorem 2.1, see also [16] as the action of the group \([\Gamma]\) generated by \( \Gamma \) is also distal). Let \( L = \cap_{\gamma \in \Gamma} \ker(\chi \gamma) \). Then \( L \) is a proper closed (compact) normal \( \Gamma \)-invariant subgroup of \( M \) and since \( \chi \Gamma \) is finite upto equivalence classes, \( M/L \) is a (compact) Lie group. Moreover, the \( \Gamma \)-action on \( M/L \) is distal (cf. [17], Theorem 3.1) and hence it has [MOC]. By Proposition 2.1, we get that the \( \Gamma \)-action on \( G/L \) has [MOC]. Hence \( L \in K \), a contradiction to the minimality of \( M \) in \( K \). Hence \( M = \{e\} \) and the \( \Gamma \)-action on \( G \) has [MOC]. This completes the proof.


**Remark 2.2.** 1. In Theorem 1.2, if \( G \) is first countable then, \( K \) is also first countable, and hence, it is metrizable.
2. Theorem 1.2 holds in case $\Gamma$ is a locally compact $\sigma$-compact group, (for e.g. $\Gamma = \mathbb{Z}$) and $K$ is not (necessarily) metrizable. As in this case, the group $M$ as above is not necessarily metrizable. Here, $\Gamma \ltimes M$ is locally compact and $\sigma$-compact and hence $M$ has arbitrarily small compact normal $\Gamma$-invariant subgroups $M_d$ such that $\bigcap_d M_d = \{e\}$ and $M/M_d$ is second countable and hence metrizable (cf. [9], Theorem 8.7). Now from Theorem 3.1 of [17], if the $\Gamma$-action on $M$ is distal then the corresponding $\Gamma$-action on $M/M_d$ is also distal and hence not ergodic and we get a proper closed normal $\Gamma$-invariant subgroup (of $M/M_d$, and hence,) of $M$, denote it by $L$ again, such that $M/L$ is a Lie group. Now the assertion is obvious from the above proof. Note that any compactly generated locally compact group is $\sigma$-compact.

The following corollary follows from Theorem 3.1 in [17], Theorem 1.1 in [2] and Theorem 1.2 above since every connected locally compact group has a unique maximal compact normal (characteristic) subgroup such that the quotient is a connected Lie group.

**Corollary 2.3.** Let $G$ be a connected locally compact first countable group. Let $\Gamma$ be a subgroup of $\text{Aut}(G)$. Then the $\Gamma$-action on $G$ is distal if and only if it has [MOC].

We now show that [MOC] is preserved by factors modulo closed normal invariant groups. Before that we prove Proposition 1.5 and a Lemma which will be useful in proving Theorem 2.5 below and also Theorem 1.1.

**Proof of Proposition 1.5.** Let $G$ be a locally compact totally disconnected group and let $\Gamma$ be a generalised $FC^-$-group acting on $G$ by automorphisms. Let $\Gamma_0 = \{\gamma \in \Gamma \mid \gamma(x) = x \text{ for all } x \in G\}$. Then $\Gamma_0$ is a closed normal subgroup of $\Gamma$, $\Gamma/\Gamma_0$ is isomorphic to a subgroup of $\text{Aut}(G)$. Also, $\Gamma/\Gamma_0$ is a generalised $FC^-$-group. It is easy to see that we can replace $\Gamma$ by $\Gamma/\Gamma_0$ and assume that $\Gamma \subset \text{Aut}(G)$. We prove that $(1) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$.

Suppose $\Gamma$ acts distally on $G$. As $\Gamma$ is totally disconnected, it has a compact open normal subgroup $C$ such that $\Gamma/C$ has a polycyclic subgroup of finite index (cf. [12]). Since $C$ is compact, by Lemma 2.3 of [11], the $\Gamma$-action on $G$ is also equicontinuous, (see also ’Note added in Proof’ in [11] for non-metrizable groups). Now $G$ has a neighbourhood base at $e$ consisting of open compact subgroups $K_d$ which are $\Gamma$-invariant and $\bigcap_d K_d = \{e\}$. For each $d$, since $G/K_d$ is discrete, the $\Gamma$-action on $G/K_d$ has [MOC]. Let $x \in G$ and let $y \in \overline{\Gamma(x)}$. Then $\Gamma(x)K_d = \Gamma(x) = \Gamma(y)K_d = \overline{\Gamma(y)}K_d$ as $K_d$ is open for all $d$. $\Gamma(x) = \bigcap_d \Gamma(x)K_d = \bigcap_d \Gamma(y)K_d = \Gamma(y)$. This proves that the $\Gamma$-action on $G$ has [MOC]. We know that [MOC] implies distality. □

**Lemma 2.4.** Let $G$ be a locally compact group and let $\Gamma$ be a group acting on $G$ by automorphisms. Suppose that the $\Gamma$-action on $G/G^0$ is equicontinuous. Then there exist open (resp. compact) $\Gamma$-invariant subgroups $H_d$ (resp. $K_d$) such that $H_d = K_d G^0$, $K_d$ is the maximal compact normal subgroup of $H_d$, $K_d \cap G^0 = \bigcap_d K_d$ is the maximal compact normal $\Gamma$-invariant subgroup of $G^0$. In particular, if $G_0$
has no nontrivial compact normal subgroup, then $K_d$ is totally disconnected and $H_d = K_d \times G^0$, a direct product, for all $d$.

Proof. Since the $\Gamma$-action on $G/G^0$ is equicontinuous, there exist open almost connected $\Gamma$-invariant subgroups $H_d$ such that $\{H_d/G^0\}$ form a neighbourhood base at the identity in $G/G^0$ consisting of compact open subgroups.

Choose $H = H_d$ for some fixed $d$. Since $H$ is almost connected, it is Lie projective, and hence, it has a compact normal subgroup $C$ (say) such that $H/C$ is a Lie group with finitely many connected components. Therefore, $H/C$, and hence, $H$ has a maximal compact normal subgroup; we denote it by $C$ again. Then $C$ is characteristic in $H$, and in particular, it is $\Gamma$-invariant. Let $H' = CG^0$. Then $H'/C$ is the connected component of the identity in the Lie group $H/C$. Therefore, $H'$ is an open $\Gamma$-invariant subgroup in $G$ and $K = C \cap G^0$ is the maximal compact normal subgroup of $G^0$. Since $H'/G^0$ is compact and open in $G/G^0$, passing to a subnet, we may assume that $H_d \subset H'$ for all $d$. Let $K_d = C \cap H_d$. Then $K_d$ is a compact normal $\Gamma$-invariant subgroup in $H_d$ and $H_d = K_dG^0$ as $G^0 \subset H_d$. Since $K = C \cap G^0 \subset H_d$, $K = K_d \cap G^0$ and $K_d$ is the maximal compact normal subgroup in $H_d$ for every $d$. Also, since $\cap_d H_d = G^0$, we get that $\cap_d K_d = K$. Moreover, if $K_d \cap G^0 = K$ is trivial, then $K_d$ is totally disconnected and $H_d = K_d \times G^0$, a direct product, as both $K_d$ and $G^0$ are normal in $H_d$, for all $d$.

To prove Theorem 1.3, in view of Theorem 1.1, it is enough if we prove the same statement for distal actions. Here, we prove the following for distal actions of a more general class of groups.

**Theorem 2.5.** Let $G$ be a locally compact group and let $\Gamma$ be a generalised $FC^-$-group which acts on $G$ by automorphisms. Let $H$ be a closed normal $\Gamma$-invariant subgroup. Then the $\Gamma$-action on $G$ is distal if and only if $\Gamma$-actions on both $H$ and $G/H$ are distal.

Proof. Let $G$, $H$ and $\Gamma$ be as in the hypothesis. Suppose $\Gamma$-actions on $G/H$ and $H$ are distal. Then it is easy to see that the $\Gamma$-action on $G$ is distal.

Now we prove the converse. Suppose the $\Gamma$-action on $G$ is distal. Then the $\Gamma$-action on $H$ is also distal. As in the proof of Theorem 1.5, we may assume that $\Gamma \subset \text{Aut}(G)$. We prove that the $\Gamma$-action on $G/H$ is distal. By Theorem 3.3 of [17], the $\Gamma$-action on $G/G^0$ is distal and hence equicontinuous (by Proposition 1.5). By Lemma 2.4, there exists an open $\Gamma$-invariant subgroup $L$ in $G$ such that $L = KG^0$, where $K$ is the maximal compact normal $\Gamma$-invariant subgroup of $L$. We know that $G/L$ is discrete, and hence, so is $G/HL$, where $HL$ is an open $\Gamma$-invariant subgroup. Therefore, it is enough to prove that $\Gamma$ acts distally on $HL/H$. Since $HL/H$ is isomorphic to $L/(L \cap H)$, without loss of any generality, we may assume that $G = L = KG^0$ and $K$ is the maximal compact normal $\Gamma$-invariant subgroup in $G$. In particular, $G/K$ is a connected Lie group.

Here, $HK$ and $K \cap H$ are closed, normal and $\Gamma$-invariant subgroups. By Theorem 3.1 of [17] we know that $\Gamma$ acts distally on $G/K$, $HK/K$ and on $K/(K \cap H)$; the latter is isomorphic to $HK/H$. Hence it is enough to prove that
Γ acts distally on the group $G/HK$ which is isomorphic to $(G/K)/(HK/K)$.

Replacing $G$ by $G/K$ and $H$ by $HK/K$, we may assume that $G$ is a connected Lie group and $H$ is a closed normal Lie subgroup. Let $\mathcal{G}$ be the Lie algebra of $G$. Since the $\Gamma$-action on $G$ is distal, so is the corresponding action of $\{d\gamma \mid \gamma \in \Gamma\}$ on $\mathcal{G}$ (cf. [2], Theorem 1.1). Equivalently, the eigenvalues of $d\gamma$ are of absolute value 1, for all $\gamma \in \Gamma$ (cf. [1], Theorem 1′). Since $H$ is normal and $\Gamma$-invariant, the Lie algebra $\mathcal{H}$ of $H^0$ is a Lie subalgebra which is an ideal invariant under $d\gamma$, for all $\gamma \in \Gamma$, and the Lie algebra of $G/H$ is isomorphic to $G/\mathcal{H}$. Then the eigenvalues of $d\gamma$ on $G/H$ are also of absolute value 1 for all $\gamma \in \Gamma$. Hence $\Gamma$ acts distally on $G/H$ (cf. [1], [2]). This completes the proof. ■

3. Distality and [MOC]

In this section we show that if $\Gamma$ is a locally compact, compactly generated abelian (resp. Moore) group acting on a locally compact group by automorphisms, then distality and [MOC] of the $\Gamma$-action are equivalent. We first prove a proposition which will be useful in proving Theorem 1.1.

Proposition 3.1. Let $G$ and $\Gamma$ be as in Theorem 1.1. Suppose that the $\Gamma$-action on $G$ is distal. Given a net $\{\gamma_d\}$ in $\Gamma$, let

$$M = \{g \in G \mid \{\gamma_d(g)\}_d \text{ is relatively compact}\}.$$ 

Then $M$ is a closed $\Gamma$-invariant subgroup.

Proof. It is obvious that $M$ is a subgroup and it is $\Gamma$-invariant since $\Gamma$ is abelian. Therefore $\overline{M}$ is also a $\Gamma$-invariant subgroup. If $M$ is trivial, then $M = \overline{M}$. Suppose $M$ is a nontrivial subgroup of $G$. Without loss of any generality, we may assume that $G = \overline{M}$, i.e. $M$ is dense in $G$.

Step 1. By Theorem 3.3 of [17], the $\Gamma$-action on $G/G^0$ is distal. Since $\Gamma$ is a compactly generated locally compact abelian group, it is a generalised $FC^-$-group. By Proposition 1.5, the $\Gamma$-action on $G/G^0$ has [MOC] and the $\Gamma$-action on $G/G^0$ is equicontinuous. By Lemma 2.4, there exists an open (resp. compact) $\Gamma$-invariant subgroup $H$ (resp. $K$) such that $H = KG^0$, where $K$ is the maximal compact normal subgroup of $H$. Since $H$ is open and $\Gamma$-invariant, it is enough to show that $H \subset M$ and hence, we may assume that $G = H$. Here, since $K$ is a maximal compact normal $\Gamma$-invariant subgroup, $K \subset M$ and $G/K$ is a connected Lie group without any nontrivial compact subgroup. Moreover, the $\Gamma$-action on $G/K$ is distal (cf. [17], Theorem 3.1). Let $\pi : G \rightarrow G/K$ be the natural projection. Since $K$ is compact, $\pi(M) = \{gK \in G/K \mid \{\gamma_d(gK)\}_d \text{ is relatively compact in } G/K\}$ and $M$ is closed if and only if $\pi(M)$ is closed. Moreover, $\pi(M)$ is dense in $G/K$.

Now, we may replace $G$ by $G/K$ and assume that $G$ is a connected Lie group without any nontrivial compact normal subgroup and $\Gamma \subset \text{Aut}(G)$.

Step 2. Since $G$ has no nontrivial compact central subgroup, $\text{Aut}(G)$ is almost algebraic (as a subgroup of $GL(\mathcal{G})$) (cf. [4]), where $\mathcal{G}$ is the Lie algebra of $G$. Let $\Gamma'$ be the smallest almost algebraic subgroup containing $\Gamma$ in $\text{Aut}(G)$. Here $\Gamma'$ is an open subgroup of finite index in the Zariski closure $\overline{\Gamma}$ of $\Gamma$ in $GL(\mathcal{G})$. 

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By Proposition 3.1, \( M \) belongs to \( \gamma \) on \( G \), a compactly generated locally compact abelian group. Suppose that the \( \Gamma \)-action for all \( \{ \text{subsequence} \}. \) We know that from Proposition 1.5, we get that the \( \Gamma \)-action on \( \text{FC} \) without any nontrivial compact normal subgroup. Note that \( \Gamma \) is a generalised subgroup of \( G \) acting distally on a group without any nontrivial compact central subgroup, \( \Gamma \) is a subgroup of \( \text{Aut}(\cdot) \). From the above proof it is clear that if \( \{ \text{subsequence} \} \). We have that \( \{ \text{compact for all} \} \. Therefore the support of \( \{ \text{compact for all} \} \mid \alpha \in \Gamma \} \) is closed in \( P(G) \), i.e. \( \{ \alpha(\mu) \mid \alpha \in \Gamma \} \) is closed in \( P(G) \).

**Step 3.** We now prove that \( \{ \gamma_d \}_d \) is relatively compact in \( \text{Aut}(G) \). Suppose \( \{ \gamma_d \}_d \) is not relatively compact in \( \text{Aut}(G) \). Since \( \text{Aut}(G) \) is a Lie group, there exists a divergent sequence \( \{ \gamma'_n \} \) in the set \( \{ \gamma_d \}_d \), i.e. \( \{ \gamma'_n \} \) has no convergent subsequence. We know that \( \{ \gamma'_n(g) \} \) is relatively compact for all \( g \) in a dense subgroup \( M \). There exists a countable subgroup \( M_1 \subset M \) which is dense in \( G \). Let \( M_1 = \{ g_i \mid i \in \mathbb{N} \} \). Passing to a subsequence if necessary, we may assume that \( \{ \gamma'_n(g_i) \}_n \) converges for all \( i \). Let \( x_i \in G \) be such that \( \gamma'_n(g_i) \to x_i, \ i \in \mathbb{N} \).

Let \( \mu = \sum_{i=1}^{\infty} (1/2^i) \delta_{x_i} \) and let \( \lambda = \sum_{i=1}^{\infty} (1/2^i) \delta_{x_i} \), where for any \( g \in G \), \( \delta_g \) denotes the Dirac measure at \( g \). Then \( \mu, \lambda \in P(G) \) and it is easy to see that \( \{ \gamma_n(\mu) \} \) converges to \( \lambda \). Now from Step 2, there exists \( \gamma \in \Gamma \) such that \( \gamma'_n(\mu) \to \gamma(\mu) = \lambda \). Since \( M_1 \) is dense in \( G \), the support of \( \mu \) is whole of \( G \). Therefore the support of \( \gamma(\mu) \) is also whole of \( G \). Now by Theorem 1.6 of [5], we get that \( \{ \gamma'_n \} \) is relatively compact and for any limit point \( \beta \) of it, \( \beta(\mu) = \gamma(\mu) \). This implies that \( \beta(g) = \gamma(g) \) for all \( g \in M_1 \) and hence \( \beta = \gamma \). Therefore, \( \gamma'_n \to \gamma, \) i.e. \( \{ \gamma'_n \} \) is convergent.

This contradicts the above assumption that \( \{ \gamma'_n \} \) is divergent. Hence we have that \( \{ \gamma_d \}_d \) is relatively compact in \( \text{Aut}(G) \). Therefore, \( \{ \gamma_d(x) \}_d \) is relatively compact for all \( x \in G \) and \( G = M \), i.e. \( M \) is closed.

**Remark 3.2.** From the above proof it is clear that if \( G \) is a connected Lie group without any nontrivial compact central subgroup, \( \Gamma \) is a subgroup of \( \text{Aut}(G) \) acting distally on \( G \) and if \( \{ \gamma_d \} \subset \Gamma \) is such that \( \{ \gamma_d(g) \}_d \) is relatively compact for all \( g \) in a dense subgroup of \( G \), then \( \{ \gamma_d \} \) is relatively compact in \( \text{Aut}(G) \).

**Proof of Theorem 1.1.** Let \( G \) be a locally compact group and let \( \Gamma \) be a compactly generated locally compact abelian group. Suppose that the \( \Gamma \)-action on \( G \) has \([\text{MOC}]\), then we know that the \( \Gamma \)-action on \( G \) is distal.

Now suppose that the \( \Gamma \)-action on \( G \) is distal. We show that it has \([\text{MOC}]\).

Let \( x \in G \) and let \( y \in \Gamma(x) \). We need to show that \( x \in \Gamma(y) \). We have that \( \gamma_d(x) \to y \) for some \( \{ \gamma_d \} \subset \Gamma \). Let \( M = \{ g \in G \mid \{ \gamma_d(g) \}_d \) is relatively compact \}. By Proposition 3.1, \( M \) is a closed \( \Gamma \)-invariant subgroup and \( x \), and hence, \( y \) belongs to \( M \). Without loss of any generality we may assume that \( M = G \). In view of Theorem 1.2 and Remark 2.2, we can go modulo the maximal compact normal subgroup of \( G^0 \) which is characteristic in \( G \) and assume that \( G^0 \) is a Lie group without any nontrivial compact normal subgroup. Note that \( \Gamma \) is a generalised \( FC^{-} \)-group and the \( \Gamma \)-action on \( G/G^0 \) is distal (by Theorem 3.3 of [17]). Hence from Proposition 1.5, we get that the \( \Gamma \)-action on \( G/G^0 \) is equicontinuous. Let \( H_d = K_d \times G^0 \) be open \( \Gamma \)-invariant subgroups, where \( K_d \) are totally disconnected.
compact $\Gamma$-invariant subgroups such that $\cap_d K_d = \{e\}$ in $G$, (see Lemma 2.4). Then passing to a subnet if necessary, we may assume that $\gamma_d(x) = y k_d g_d = y g_d k_d$, where $k_d \in K_d$ and $g_d \in G^0$, $k_d \to e$, $g_d \to e$. In particular, we get that $\gamma_d^{-1}(y) = x \gamma_d^{-1}(k_d^{-1}) \gamma_d^{-1}(g_d^{-1})$. We know that $\{\gamma_d|_{G^0}\}$ is relatively compact, (see Remark 3.2). Let $\gamma$ be a limit point of $\{\gamma_d|_{G^0}\}$ in Aut($G^0$). Then $\gamma^{-1}$ is a limit point of $\{\gamma_d^{-1}|_{G^0}\}$ in Aut($G^0$). Therefore, passing to a subnet if necessary, we get that $\gamma_d^{-1}(g_d^{-1}) \to \gamma^{-1}(e) = e$ and $\gamma_d^{-1}(y) = x k_d' \gamma_d^{-1}(g_d^{-1}) \to x$

where $k_d' = \gamma_d^{-1}(k_d^{-1}) \in K_d$ and $k_d' \to e$ as $K_d$ are $\Gamma$-invariant and $\cap_d K_d = \{e\}$. In particular, $x \in \Gamma(y)$. Since this is true for any $x \in G$ and any $y \in \Gamma(x)$, the $\Gamma$-action on $G$ has [MOC].

A locally compact group $G$ is said to be a central group or a $Z$-group if $G/Z(G)$ is compact, where $Z(G)$ is the center of $G$. It is said to be a Moore group if all its irreducible unitary representations are finite dimensional. All abelian groups and all compact groups are $Z$-groups and $Z$-groups are also Moore groups. A Moore group has a normal subgroup $H$ of finite index such that $[H,H]$ is compact (cf. [18]). It is easy to see from this, that any Moore group $G$ is $FC^-$-nilpotent as $G_0 = G$, $G_1 = H$, $G_2 = [H,H]$ and $G_3 = \{e\}$. Since $G_0/G_1$ is finite, and $G_1/G_2$ is abelian and $G_2/G_3$ is compact, we have that the conjugacy action of $G$ on $G_i/G_{i+1}$ has relatively compact orbits for all $i = 0, 1, 2$. Hence any compactly generated Moore group has polynomial growth and it is a generalised $FC^-$-group (cf. [12], Theorem 1, Lemma 1).

**Corollary 3.3.** Let $G$ be a locally compact group and let $\Gamma$ be a compactly generated Moore group acting on $G$ by automorphisms. Then the $\Gamma$-action on $G$ is distal if and only if it has [MOC].

The proof of the above corollary is essentially the same as that of Theorem 1.1. As $\Gamma$ is a Moore group, it has a closed normal subgroup $\Gamma_1$ of finite index whose commutator group is relatively compact. (cf. [18], Theorem 1). Then by Lemma 4.1 of [13], it is enough to show that the $\Gamma_1$-action on $G$ has [MOC]. Without loss of any generality, we may assume that $[\Gamma,\Gamma]$ is relatively compact and hence it is easy to see that the group $M$ defined in the above proof is $\Gamma$-invariant. We will not repeat the proof here.

**Remark 3.4.** From above, it is obvious that Theorem 1.1 holds for any compactly generated locally compact group $\Gamma$ such that its commutator subgroup is relatively compact. Moreover from Lemma 4.1 in [13] we know that the action of a group $\Gamma$ on $G$ has [MOC] if the action of any co-compact subgroup of $\Gamma$ on $G$ has [MOC]. Hence Theorems 1.1 and 1.3 hold for compact extensions of such a group $\Gamma$ mentioned above, and in particular, for compact extensions of compactly generated abelian, or more generally, of Moore groups.

We conjecture that Theorem 1.1 holds for an action of any generalised $FC^-$-group. It already holds for the action of such a group on totally disconnected groups, compact groups and connected groups.
References


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