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Homomorphisms of Generalized Verma Modules, BGG Parabolic Category $\mathcal{O}^{\mathfrak{p}}$ and Juhl's Conjecture

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Abstract. Let $\mathcal{M}_\lambda(\mathfrak{g},\mathfrak{p}), \mathcal{M}_\mu(\mathfrak{g}',\mathfrak{p}')$ be the generalized Verma modules for $\mathfrak{g} = so(p+1,q+1), \mathfrak{g}' = so(p,q+1)$ induced from characters λ, μ of the standard maximal parabolic (conformal) subalgebras $\mathfrak{p}, \mathfrak{p}' = \mathfrak{g}' \cap \mathfrak{p}$. Motivated by questions about the existence of invariant differential operators in conformal geometry, we explain, reformulate and prove an extended version of Juhl's conjecture on the structure of $\mathcal{U}(\mathfrak{g}')$-homomorphisms of generalized Verma modules from $\mathcal{M}_\lambda(\mathfrak{g}',\mathfrak{p}')$ to $\mathcal{M}_\mu(\mathfrak{g},\mathfrak{p})$. The answer has a natural formulation as a branching problem in the BGG parabolic category $\mathcal{O}^{\mathfrak{p}'}$ rather than the set of generalized Verma modules alone.
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1. Introduction and Motivation

The question we are going to answer in the article has its original motivation in geometry. Let (M^n, g) be a Riemannian manifold, $i_\Sigma : \Sigma^{n-1} \hookrightarrow M^n$ embedded codimension one (i.e. $(n-1)$-dimensional) submanifold and $i^\star_\Sigma(g)$ the induced metric on Σ^{n-1}. One of the basic problems in geometrical analysis on Riemannian or conformal manifolds defined by these data is the existence, uniqueness and properties of natural differential (scalar) operators

$$D_N(M^n, \Sigma^{n-1}, g, \lambda) : C^\infty(M^n) \to C^\infty(\Sigma^{n-1})$$

of order $N \in \mathbb{N}$ and depending polynomially on $\lambda \in \mathbb{C}$, which are conformally invariant in the sense that

$$e^{-(\lambda-N)i^\star_\Sigma(\varphi)} D_N(M^n, \Sigma^{n-1}, e^{2\varphi}g, \lambda) e^{\lambda\varphi} = D_N(M^n, \Sigma^{n-1}, g, \lambda)$$

for each $\varphi \in C^\infty(M^n)$.

It is difficult to handle this problem for a general metric g, but the situation simplifies considerably in the case of a homogeneous flat domain realized on an

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open orbit of the partial flag manifold. The case of our interest in this article corresponds to $M^n = S^n$ resp. $M^n = \mathbb{R}^n$ and $\Sigma^{n-1} = S^{n-1}$ resp. $\Sigma^{n-1} = \mathbb{R}^{n-1}$ in the compact resp. non-compact models of induced representations of (conformal) Lie algebra $\mathfrak{g}(n+1,1) = so(n+1,1)$, and its signature generalizations.

For any simple Lie algebra $\mathfrak{g}$ and its parabolic subalgebra $\mathfrak{p}$ we have the Langlands decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{l} \oplus \mathfrak{n}_+$ and the Iwasawa decomposition $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{n}_+$. Here $\mathfrak{l}$ denotes the Levi factor of $\mathfrak{p}$, $\mathfrak{n}_+$ its nilradical and $\mathfrak{n}_-$ the opposite nilradical. In this article we focus on the maximal parabolic subalgebra with abelian nilradical given by omitting the first simple root, which is the data for the flat Cartan model of conformal geometry.

There is a well-known equivalence between invariant differential operators acting on induced representations and homomorphisms of generalized Verma modules, realized by the pairing

$$Ind_P^G(\mathbb{V}_\lambda) \times \mathcal{M}(\mathfrak{g}, \mathfrak{p}, \mathbb{V}_\lambda^\star) \to \mathbb{C} \tag{1}$$

for any finite dimensional irreducible $\mathfrak{p}$-module $\mathbb{V}_\lambda$ and its dual $\mathbb{V}_\lambda^\star$. This allows to turn the former motivating problem into the question of $\mathfrak{g}(p,q+1) = so(p,q+1)$-homomorphisms of generalized Verma modules

$$\mathcal{M}(\mathfrak{g}(p,q+1), \mathfrak{p}(p,q+1), \mathbb{V}_{\lambda_1}) \to \mathcal{M}(\mathfrak{g}(p+1,q+1), \mathfrak{p}(p+1,q+1), \mathbb{V}_{\lambda_2}), \tag{2}$$

where $\mathbb{V}_{\lambda_i}, i = 1,2$ denote finite dimensional irreducible inducing representations of $\mathfrak{p}(p,q+1)$ resp. $\mathfrak{p}(p+1,q+1)$.

Let us denote the standard inclusion

$$i : \mathfrak{g}(n,1) \hookrightarrow \mathfrak{g}(n+1,1),$$

characterized by the fact that the highest weight vector Y_n of $\mathfrak{l}$-module $\mathfrak{n}$ is preserved by $i(\mathfrak{l}')$, the image of the Levi factor of $\mathfrak{g}(n,1)$. Recently in [9], A. Juhl constructed a collection of elements $D_N(\lambda)$ in

$$Hom_{\mathcal{U}(\mathfrak{g}(n,1))}(\mathcal{M}(\mathfrak{g}(n,1), \mathfrak{p}(n,1), \mathbb{C}_{\lambda-N}), \mathcal{M}(\mathfrak{g}(n+1,1), \mathfrak{p}(n+1,1), \mathbb{C}_\lambda)),$$

numbered by $N \in \mathbb{N}$ and polynomially dependend on the character $\lambda \in \mathbb{C}$ of $\mathfrak{p}(n,1)$-module $\mathbb{C}_{\lambda-N}$, such that $D_N(\lambda) \in \mathcal{U}(\mathfrak{n}_-(n+1,1))$ is induced by

$$\begin{aligned} \mathcal{U}(\mathfrak{g}(n,1)) \otimes \mathbb{C}_{\lambda-N} &\to \mathcal{U}(\mathfrak{g}(n+1,1)) \otimes \mathbb{C}_\lambda, \\ V \otimes 1 &\mapsto i(V) D_N(\lambda). \end{aligned} \tag{3}$$

Then he formulated the following conjectures (see [9] for the case $\mathfrak{g}(n,2)$; the cases of remaining signatures have, according to A. Juhl, an analogous formulation):

Conjecture 1.1. The set of families $\{D_N(\lambda)\}_{N\in\mathbb{N}}$ generates the space

$$\mathcal{M}(\mathfrak{g}(p,q+1), \mathfrak{p}(p,q+1), \mathbb{C}_{\lambda-N}) \to \mathcal{M}(\mathfrak{g}(p+1,q+1), \mathfrak{p}(p+1,q+1), \mathbb{C}_\lambda),\ N \in \mathbb{N}$$

of all homomorphisms of $\mathcal{U}(\mathfrak{g}(p,q+1))$-modules.

This construction was then subsequently considered also in the Lorentzian signature, [1].

The first aim of the article is to prove these conjectures for a generic value of the inducing character λ in the case of any signature. The tool used to complete this task is based on the analysis of character formulas for corresponding parabolic subalgebras. The second aim is a direct analysis of the space of homomorphisms or, equivalently, the space of singular vectors, for certain discrete subset of the values of inducing character. The result is that in these special cases some of the $\mathfrak{g}(p,q+1)$-generalized Verma modules which decompose a given $\mathfrak{g}(p+1,q+1)$-generalized Verma module form non-trivial extensions, i.e. they represent objects (nontrivial extension classes) in the BGG parabolic category $\mathcal{O}^{\mathfrak{p}}$ of the pair $\mathfrak{g}(p,q+1)$, $\mathfrak{p}(p,q+1)$. We give the complete list of modules appearing in the branching problem, thus completing the decomposition as a task in the BGG parabolic category $\mathcal{O}^{\mathfrak{p}}$ rather then the set of Verma modules alone, see e.g. [7], [3].

Notice that in the special case corresponding to the domain of unitarity D_U for generalized Verma modules $\mathcal{M}(\mathfrak{g}(p+1,q+1),\mathfrak{p}(p+1,q+1),\mathbb{C}_\lambda)$ of scalar type, this decomposition was treated by geometrical techniques and orbit methods in [11]. In the recent article [12], the author develops branching problem for generalized Verma modules even for non-standard embeddings of smaller Lie algebra in the bigger one. However, the decomposition is considered only if the induction parameters belong to a certain region.

In addition, the technique of distributive Fourier transformation allows to treat a panorama of examples of parabolic subalgebras with abelian nilradical and their finite dimensional representations in a rather explicit way, based on the reformulation of the action of $\mathfrak{n}'_+$ as the system of hypergeometric differential equations for $\mathfrak{l}'$-invariants in $\mathcal{U}(\mathfrak{n}_-)$. This line of development forms the content of the forthcoming article, [10].

2. Basic Properties of Generalized Verma Modules, their Homomorphisms and BGG Parabolic Category $\mathcal{O}^{\mathfrak{p}}$

Let $\mathfrak{g}(p+1,q+1)$, $p+q=n$, denote the Lie algebra $so(p+1,q+1)$ and $\mathfrak{p}(p,q+1)=\mathfrak{p}_{so(p+1,q+1)}$ its maximal parabolic subalgebra generated by all simple roots except the first one. The parabolic Lie subalgebra $\mathfrak{p}(p+1,q+1)$ has the Iwasawa decomposition $\mathfrak{p}(p+1,q+1)=\mathfrak{m}(p,q)\oplus\mathfrak{a}\oplus\mathfrak{n}_+(p,q)$, where $\mathfrak{m}(p,q)=so(p,q)$, $\mathfrak{a}=\mathbb{R}$ and $\mathfrak{n}_+(p,q)=\mathbb{R}^{p,q}$. We say that two couples $(so(p+1,q+1)$, $\mathfrak{p}(p+1,q+1))$ and $(so(p,q+1),\mathfrak{p}(p,q+1))$ are compatible (or standard) if $\mathfrak{p}(p+1,q+1)\cap so(p,q+1)=\mathfrak{p}(p,q+1)$. The compatibility of two couples implies that the standard embedding $i:so(p,q+1)\hookrightarrow so(p+1,q+1)$ induces embedding $\tilde{i}:\mathfrak{n}_+(p-1,q)\hookrightarrow\mathfrak{n}(p,q)$ corresponding to the isomorphism $\mathfrak{n}_+(p,q)\simeq\mathfrak{n}_+(p-1,q)\oplus\mathbb{R}$ of $so(p-1,q)$-modules. We denote by big latin letters the Lie groups of corresponding Lie algebras.

Let $I_\lambda(\mathfrak{g})$ be the $\mathcal{U}(\mathfrak{g})$-submodule of $\mathcal{U}(\mathfrak{g})\otimes\mathbb{V}_\lambda$ generated by

$$(X\otimes 1-1\otimes X\cdot v)\in\mathcal{U}(\mathfrak{g})\otimes\mathbb{V}_\lambda,\ X\in\mathfrak{p},v\in\mathbb{V}_\lambda. \tag{4}$$

The quotient module $\mathcal{M}(\mathfrak{g},\mathfrak{p},\mathbb{V}_\lambda):=\mathcal{U}(\mathfrak{g})/I_\lambda(\mathfrak{g})$ is called the generalized Verma module of $\mathfrak{g}$ induced from the finite dimensional $\mathfrak{p}$-module $\mathbb{V}_\lambda$ with highest weight λ.

Let $\mathfrak{g}$ be a simple Lie algebra. The general scheme in the classification of homomorphisms of generalized Verma $\mathcal{U}(\mathfrak{g})$-modules looks as follows. The highest weight vector of $\mathcal{M}(\mathfrak{g},\mathfrak{p},\mathbb{V}_\lambda)=\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{p})}\mathbb{V}_\lambda$ is $1\otimes v_\lambda$, with v_λ the highest weight vector of $\mathbb{V}_\lambda$. The generalized Verma modules decompose on direct sums of their weight spaces with finite multiplicity. By a homomorphism of generalized Verma modules we mean left $\mathcal{U}(\mathfrak{g})$-module homomorphism

$$\mathcal{M}_\mu(\mathfrak{g},\mathfrak{p})=\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{p})}\mathbb{V}_\mu\to\mathcal{M}_\nu(\mathfrak{g},\mathfrak{p})=\mathcal{U}(\mathfrak{g})\otimes_{\mathcal{U}(\mathfrak{p})}\mathbb{V}_\nu. \tag{5}$$

Neccessary conditions for the existence of such a homomorphism acting between $\mathcal{M}_\mu(\mathfrak{g},\mathfrak{p})$ and $\mathcal{M}_\nu(\mathfrak{g},\mathfrak{p})$ are

1. μ and ν are linked by an affine action of the Weyl group $W(\mathfrak{g})$ of $\mathfrak{g}$, i.e. $\nu=w\cdot\mu$ for some $w\in W(\mathfrak{g})$.

2. The weights μ,ν are integral, i.e. $(\mu+\delta)(H_\alpha)\in\mathbb{Z}$ for all coroots H_α of $\mathfrak{g}$.

The homomorphisms of generalized Verma modules are generally not injective as in the case of homomorphisms of Verma modules, and

$$dim(Hom_{\mathcal{U}(\mathfrak{g})}(\mathcal{M}_\mu(\mathfrak{g},\mathfrak{p}),\mathcal{M}_\nu(\mathfrak{g},\mathfrak{p}))$$

may be strictly bigger than one. For example, the standard homomorphisms of generalized Verma modules (i.e. those induced from lifts of homomorphisms of Verma modules) are easy to classify, see [13].

For a standard parabolic subalgebra $\mathfrak{b}\subset\mathfrak{p}\subset\mathfrak{g}$ (corresponding to the choice of the subset of simple roots of $\mathfrak{g}$) the BGG parabolic category $\mathcal{O}^\mathfrak{p}$ is defined as the full category $Mod(\mathcal{U}(\mathfrak{g}))$ of $\mathcal{U}(\mathfrak{g})$-modules whose objects $M\in Mod(\mathcal{U}(\mathfrak{g}))$ satisfy:

1. M is finitely generated $\mathcal{U}(\mathfrak{g})$-module.

2. As an $\mathcal{U}(\mathfrak{l})$-module, M is direct sum of finite dimensional simple modules.

3. M is locally $\mathfrak{n}_+$-finite.

Recall that there are three types of basic objects in the parabolic category $\mathcal{O}^\mathfrak{p}$ - the Verma modules $\mathcal{M}_\lambda(\mathfrak{g},\mathfrak{p})$, the simple modules $\mathcal{L}_\lambda(\mathfrak{g},\mathfrak{p})$ and the projective modules $\mathcal{P}_\lambda(\mathfrak{g},\mathfrak{p})$. The object P in abelian category $\mathcal{O}^\mathfrak{p}$ is called projective if the left exact functor $Hom_{\mathcal{O}^\mathfrak{p}}(P,-)$ is also right exact. The parabolic category $\mathcal{O}^\mathfrak{p}$ has enough projectives: for each $M\in\mathcal{O}^\mathfrak{p}$ there is a projective object $P\in\mathcal{O}^\mathfrak{p}$ and an epimorphism $P\to M$. Another characterization of projectivity: $P\in\mathcal{O}^\mathfrak{p}$ is projective if for an epimorphism $\pi:M\to N$ and a morphism $\psi:P\to N$, there is a morphism $\phi:P\to M$ such that $\pi\circ\phi=\psi$. Similarly, parabolic category $\mathcal{O}^\mathfrak{p}$ has enough injective objects Q characterized by universal monomorphism property: for each $M\in\mathcal{O}^\mathfrak{p}$ there is an injective object Q and a monomorphism $M\to Q$.

3. $so(p, q+1)$-homomorphisms and $so(p, q+1)$-singular Vectors for $so(p+1, q+1)$-generalized Verma Modules Induced from Characters

In this section we construct discrete family of 1-dimensional continuous families of $\mathcal{U}(so(p, q+1))$-homomorphisms between generalized Verma $so(p, q+1)$- resp. $so(p+1, q+1)$-modules induced from character. This also amounts to the construction of $so(p, q+1)$-singular vectors in the target generalized $so(p+1, q+1)$-Verma module.

For the Lie algebra $so(p+1, q+1)$ of signature $(p+1, q+1)$ $(p+q=n)$, let J be the diagonal matrix with the number of $(p+1)$ 1's and $(q+1)$ -1's. The set of matrices $(i, j=1, \ldots, n)$

$$M_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & e_i^T \otimes e_j - Je_j^T \otimes e_i J & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; H_0 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$Y_i^- = \sqrt{2}\begin{pmatrix} 0 & e_i & 0 \\ 0 & 0 & -Je_i^T \\ 0 & 0 & 0 \end{pmatrix}, \; Y_i^+ = \sqrt{2}\begin{pmatrix} 0 & 0 & 0 \\ e_i^T & 0 & 0 \\ 0 & -e_i J & 0 \end{pmatrix} \tag{6}$$

gives the matrix realization of Iwasawa decomposition of $so(p+1, q+1)$. Here $\{e_i\}_i$ is the basis of $\mathbb{R}^{p,q}$. The following commutation relations for $so(p+1, q+1)$ will be useful:

$$\begin{aligned} &[Y_i^+, Y_j^-] = 2(\delta_{ij} H_0 + M_{ij}), \\ &[H_0, Y_i^{\pm}] = \pm Y_i^{\pm}, \\ &[M_{ij}, Y_k^{\pm}] = \delta_{jk} Y_i^{\pm} - \delta_{ik} Y_j^{\pm}. \end{aligned} \tag{7}$$

It is now elementary to extend the results in [9] (for the signature $(n+1, 1)$) resp. [1] (for the signature $(n, 2)$) to any signature. The following identities will be helpful:

$$\begin{aligned} &[Y_1^+, (Y_1^-)^2] = -2Y_1^- + 4Y_1^- H_0, \\ &[Y_1^+, (Y_i^-)^2] = 2Y_1^- + 4Y_1^- M_{1i}, \; i = 2, \ldots, n. \end{aligned}$$

The next Lemma is a key step to construct continuous families of homomorphisms of generalized Verma modules. Its proof differs from [9] and is based on suitable inductive procedure. We use the obvious shorthand notation for Lie subalgebras appearing in the Iwasawa decomposition of
$so(p+1, q+1)$, e.g. $\mathfrak{m}_n = \mathfrak{m}(so(p+1, q+1)) = so(p, q)$ and
$\mathfrak{n}_{n-} = \mathfrak{n}_-(so(p+1, q+1)) \simeq \mathbb{R}^{p,q}$, etc.
As we shall see all the results are independent of signature and depend on $n = p+q$ only. The subscript by $\triangle^-$ denotes the underlying dimension.

Lemma 3.1. *For any signature $(p+1, q+1)$ $(p+q=n)$ and $j \in \mathbb{N}$, we have*

$$\begin{aligned} &[Y_1^+, (\triangle_{n-1}^-)^j] - 2j(p+q-1-2j) Y_1^- (\triangle_{n-1}^-)^{j-1} - 4j Y_1^- (\triangle_{n-1}^-)^{j-1} H_0 \\ &\in \mathcal{U}(\mathfrak{n}_{n-}) \mathfrak{m}_n. \end{aligned} \tag{8}$$

Proof. The proof goes by induction on j. Let us recall the conventional notation $\triangle_{n-1}^- := \sum_{j=1}^{n-1}(Y_j^-)^2$. The case $j=1$ amounts to

$$[Y_1^+, \sum_{j=1}^{p}(Y_j^-)^2 + \sum_{j=p+1}^{n-1}(Y_j^-)^2] = (-2Y_1^- + 4Y_1^- H_0) +$$
$$(2Y_1^- + 4Y_2^- M_{1,2}) + \cdots + (2Y_1^- + 4Y_p^- M_{1,p}) +$$
$$(2Y_1^- + 4Y_{p+1}^- M_{1,p+1}) + \cdots + (2Y_1^- + 4Y_{n-1}^- M_{1,n-1}) =$$
$$2(p+q-3)Y_1^- + 4Y_1^- H_0 \; mod \; \mathcal{U}(\mathfrak{n}_{n-})\mathfrak{m}_n \tag{9}$$

and the claim is proved.

Let us now assume that the claim is true for $j \in \mathbb{N}$, i.e.

$$[Y_1^+, (\triangle_{n-1}^-)^j] = 2j(p+q-1-2j)Y_1^-(\triangle_{n-1}^-)^{j-1} +$$
$$4jY_1^-(\triangle_{n-1}^-)^{j-1}H_0 \; mod \; \mathcal{U}(\mathfrak{n}_{n-})\mathfrak{m}_n. \tag{10}$$

Then

$$[Y_1^+, (\triangle_{n-1}^-)^{j+1}] = \triangle_{n-1}^-[Y_1^+, (\triangle_{n-1}^-)^j] + [Y_1^+, \triangle_{n-1}^-](\triangle_{n-1}^-)^j =$$
$$\triangle_{n-1}^-(2j(p+q-1-2j)Y_1^-(\triangle_{n-1}^-)^{j-1} + 4jY_1^-(\triangle_{n-1}^-)^{j-1}H_0) +$$
$$(2(p+q-3)Y_1^- + 4Y_1^- H_0)(\triangle_{n-1}^-)^j$$
$$= 2(j+1)(p+q-1-2(j+1))Y_1^-(\triangle_{n-1}^-)^j +$$
$$4(j+1)Y_1^-(\triangle_{n-1}^-)^j H_0 \; mod \; \mathcal{U}(\mathfrak{n}_{n-})\mathfrak{m}_n \tag{11}$$

and the claim follows. ∎

A direct consequence of the previous Lemma yields the explicit form of homomorphisms or, when evaluated, singular vectors in the target generalized Verma module. The Theorem is divided into two parts according to the homogeneity of the homomorphism.

Theorem 3.2. *1. (Families of even order) Let $(so(p+1,q+1), so(p,q+1))$ $(p+q=n)$ be the couple of orthogonal Lie algebras. For any $p, q, N \in \mathbb{N}$, $(p+q) \geq 3$ and $\lambda \in \mathbb{C}$, the element*

$$D_{2N}(\lambda) = \sum_{j=0}^{N} a_j(\lambda)(\triangle_{n-1}^-)^{j-1}(Y_n^-)^{2N-2j} \in \mathcal{U}(\mathfrak{n}_{n+1-}) \tag{12}$$

satisfies

$$[Y_i^+, \sum_{j=0}^{N} a_j(\lambda)(\triangle_{n-1}^-)^{j-1}(Y_n^-)^{2N-2j}] \in \mathcal{U}(\mathfrak{n}_{n+1-})(\mathfrak{m}_{n+1} \oplus \mathbb{C}(H_0 - \lambda)) \tag{13}$$

for $i = 1, \ldots, n-1$ iff the coeficients $\{a_j\}_{j=0}^N$ fulfill the recursive relations

$$(N-j+1)(2N-2j+1)a_{j-1} + j(p+q-1+2\lambda-4N+2j)a_j = 0, \tag{14}$$

$j = 1, \ldots, N$. In effect, the left multiplication by this element induces $\mathcal{U}(so(p,q+1))$-homomorphism

$$\mathcal{M}_{\lambda-N}(\mathfrak{g}(p,q+1), \mathfrak{p}(p,q+1)) \to \mathcal{M}_\lambda(\mathfrak{g}(p+1,q+1), \mathfrak{p}(p+1,q+1)) \tag{15}$$

2. *(Families of odd order) For any $p, q, N \in \mathbb{N}$, $p+q=n$ $((p+q)\geq 3)$ and $\lambda \in \mathbb{C}$ the element*

$$D_{2N+1}(\lambda)=\sum_{j=0}^{N} b_j(\lambda)(\triangle^-_{n-1})^{j-1}(Y^-_n)^{2N-2j+1}\in \mathcal{U}(\mathfrak{n}_{n+1-}) \tag{16}$$

satisfies

$$[Y^+_i,\sum_{j=0}^{N} a_j(\lambda)(\triangle^-_{n-1})^{j-1}(Y^-_n)^{2N-2j+1}]\in \mathcal{U}(\mathfrak{n}_{n+1-})(\mathfrak{m}_{n+1}\oplus\mathbb{C}(H_0-\lambda)), \tag{17}$$

$i=1,\dots,n-1$ *iff the coefficients* $\{b_j\}_{j=0}^N$ *fulfill the recursive relations*

$$(N-j+1)(2N-2j+3)b_{j-1}+j(p+q-3+2\lambda-4N+2j)b_j=0, \tag{18}$$

$j=1,\dots,N$.

As we shall prove in the next section, this set of singular vectors (enumerated by $N\in\mathbb{N}$) is complete and sufficient to decompose a given generalized Verma module with respect to a rank one less orthogonal Lie subalgebra.

4. The Composition Series for Branching Problem of Generalized Verma Modules

In the previous section we produced a collection of $so(p,q+1)$-homomorphism from $\mathfrak{g}'=so(p,q+1)$-generalized Verma modules to a fixed $\mathfrak{g}=so(p+1,q+1)$-generalized Verma module (regarded as $so(p,q+1)$-module via standard embedding $so(p,q+1)\hookrightarrow so(p+1,q+1)$) or, when evaluated, the collection of $so(p,q+1)$-singular vectors in the $so(p+1,q+1)$-generalized Verma module. The remaining question is whether the construction in the previous section produced complete (exhausting) family of singular vectors.

One way to analyze this question is based on character identities for the restriction of generalized Verma modules with respect to a reductive subalgebra $\mathfrak{g}'$ for which the parabolic subalgebra $\mathfrak{p}':=\mathfrak{g}'\cap\mathfrak{p}$ is standard ($\mathfrak{p}=\mathfrak{l}+\mathfrak{n}_+, \mathfrak{p}'=\mathfrak{l}'+\mathfrak{n}'_+$), see e.g. [12]. Let $\mathbb{V}_\lambda$ be a finite dimensional $\mathfrak{l}$-module with highest weight $\lambda\in\Lambda^+(\mathfrak{l})$ and likewise $\mathbb{V}_{\lambda'}$ be a finite dimensional representation of $\mathfrak{l}'$, $\lambda'\in\Lambda^+(\mathfrak{l}')$. Given a vector space V we denote $S(V)=\oplus_{l=0}^\infty S_l(V)$ the symmetric tensor algebra on V. Let us extend the adjoint action of $\mathfrak{l}'$ on $\mathfrak{n}_+/(\mathfrak{n}_+\cap\mathfrak{g}')$ to $S(\mathfrak{n}_+/(\mathfrak{n}_+\cap\mathfrak{g}'))$. We set

$$m(\lambda',\lambda)=Hom_{\mathfrak{l}'}(\mathbb{V}_{\lambda'},\mathbb{V}_\lambda|_{\mathfrak{l}'}\otimes S(\mathfrak{n}_+/(\mathfrak{n}_+\cap\mathfrak{g}'))). \tag{19}$$

Theorem 4.1. *([12], Theorem 3.9) Suppose $\mathfrak{p}$ is $\mathfrak{g}'$-compatible standard parabolic subalgebra of $\mathfrak{g}$, $\lambda\in\Lambda^+(\mathfrak{l})$. Then*

1. *$m(\lambda',\lambda)<\infty$ for all $\lambda'\in\Lambda^+(\mathfrak{l}')$.*

2. In the Grothendieck group of $\mathcal{O}^{\mathfrak{p}'}$ *there is* $\mathfrak{g}'$*-isomorphism*

$$\mathcal{M}_\lambda(\mathfrak{g},\mathfrak{p})|_{\mathfrak{g}'} \simeq \bigoplus_{\lambda'\in\Lambda^+(\mathfrak{l}')} m(\lambda',\lambda)\mathcal{M}_{\lambda'}(\mathfrak{g}',\mathfrak{p}').$$

A consequence of this Theorem is that in the case of multiplicity free $\mathfrak{l}'$-module $\mathfrak{n}_+/(\mathfrak{n}_+\cap\mathfrak{g}'))$ and for generic character λ the decomposition of generalized Verma module $\mathcal{M}_\lambda(\mathfrak{g},\mathfrak{p})$ (induced from character λ) with respect to $\mathfrak{g}'$ is multiplicity free. Moreover, for any value of the character λ the following relation holds true in the Grothendieck group of $\mathcal{O}^{\mathfrak{p}'}$:

Corollary 4.2. *For* $\mathfrak{g} \equiv so(p+1,q+1), \mathfrak{g}' \equiv so(p,q+1)$ *with standard maximal parabolic subalgebras* $\mathfrak{p},\mathfrak{p}'$ *given by omitting the first simple root, we have* $\mathfrak{n}_+ \simeq \mathbb{R}^{p,q}, \mathfrak{n}_+\cap\mathfrak{g}' \simeq \mathbb{R}^{p-1,q}$ *and* $\mathfrak{n}_+/(\mathfrak{n}_+\cap\mathfrak{g}') \simeq \mathbb{R}$ *transforms as the character of the Levi subalgebra of* $\mathfrak{g}'$*. Then* $m(\lambda,\lambda')=1$ *if and only if* $\lambda'=\lambda-j, j\in\mathbb{N}$ *and* $m(\lambda,\lambda')=0$ *otherwise. In the Grothendieck group of* $\mathcal{O}^{\mathfrak{p}'}$ *holds*

$$\mathcal{M}_\lambda(\mathfrak{g},\mathfrak{p}) \simeq \bigoplus_{j\in\mathbb{N}} \mathcal{M}_{\lambda-j}(\mathfrak{g}',\mathfrak{p}').$$

4.1. Branching rules for the generic value λ of the inducing character of generalized Verma modules.

In this subsection we prove that the previous observation on the relation in the Grothendieck group corresponds, in case of a generic inducing character, to the actual branching rule for the couple $(\mathfrak{g},\mathfrak{g}')\equiv(so(p+1,q+1),so(p,q+1))$ and a generalized Verma $\mathcal{U}(\mathfrak{g})$-module induced from the character $\lambda\in\mathbb{C}$ of $\mathfrak{p}\subset\mathfrak{g}$.

First of all we note that for any λ there is always a direct sum decomposition of $\mathfrak{g}'$-module $\mathcal{M}_\lambda(\mathfrak{g},\mathfrak{p})$ into the even and odd part,

$$\mathcal{M}_\lambda(\mathfrak{g},\mathfrak{p}) = \left(\sum_{k=0}^{\infty}\langle w_{2k}\rangle\right)\oplus\left(\sum_{k=0}^{\infty}\langle w_{2k+1}\rangle\right) = U^{even}\oplus U^{odd},$$

according to the homogeneity of an element in the polynomial algebra $\mathcal{U}(\mathfrak{n}_-)$. Here w_l denotes the singular vector in $\mathcal{M}_\lambda(\mathfrak{g},\mathfrak{p})$ corresponding to the image of the homomorphism $D_l(\lambda)$, $l\in\mathbb{N}$, and $< w_l >$ denotes its $\mathcal{U}(\mathfrak{g}')$-span. The spaces $\langle w_j\rangle = \mathcal{U}(\mathfrak{g}')w_j$ are invariant under the action of $\mathfrak{n}'_-$, while under the action of $\mathfrak{n}'$ the spaces $\langle w_j\rangle$ are mapped to the sum $\sum_{\ell\in\mathbb{N}}\langle w_{j-2\ell}\rangle$. Hence

$$W_{2k} = \sum_{\ell=0}^{k}\langle w_{2l}\rangle,\ k\in\mathbb{N}$$

form a $\mathfrak{g}'$-filtration of U^{even} by invariant subspaces. Analogous result is true for the odd part and consequently the spaces U^{even} and U^{odd} have invariant filtrations under the $\mathfrak{g}'$-action.

We shall now discuss decomposition problem of U^{even}, U^{odd} for $\lambda\neq k-\frac{n}{2}$. The values of $\lambda\in\mathbb{C}, \lambda\neq k-\frac{n}{2}$ are henceforth termed generic.

Theorem 4.3. *Let $\lambda \in \mathbb{C}, \lambda \neq k - \frac{n}{2}$ for $k \in \mathbb{N}$, i.e. let λ be generic. Then*

$$U^{even} = \bigoplus_{j=0}^{\infty} \mathcal{M}_{\lambda-2j}(\mathfrak{g}', \mathfrak{p}'),$$
$$U^{odd} = \bigoplus_{i=0}^{\infty} \mathcal{M}_{\lambda-2i-1}(\mathfrak{g}', \mathfrak{p}') \tag{20}$$

gives the direct sum decomposition of the left hand side into irreducible submodules under the restriction from $\mathfrak{g}$ to $\mathfrak{g}'$. The embedding of $\mathcal{M}_{\lambda-2j}(\mathfrak{g}', \mathfrak{p}') \hookrightarrow \mathcal{M}_{\lambda}(\mathfrak{g}, \mathfrak{p})$ resp. $\mathcal{M}_{\lambda-2i-1}(\mathfrak{g}', \mathfrak{p}') \hookrightarrow \mathcal{M}_{\lambda}(\mathfrak{g}, \mathfrak{p})$ is induced by the singular vector $w_{2j} \in \mathcal{M}_{\lambda}(\mathfrak{g}, \mathfrak{p})$ resp. $w_{2i+1} \in \mathcal{M}_{\lambda}(\mathfrak{g}, \mathfrak{p})$, $i, j \in \mathbb{N}$.

Proof. The singular vector w_l generates a cyclic $\mathfrak{g}'$-submodule $\langle w_l \rangle$ in $\mathcal{M}_{\lambda-l}(\mathfrak{g}, \mathfrak{p})$ with the highest weight $\lambda_l = (\lambda - l|0, \dots, 0)$. The vectors have mutually different infinitesimal characters, because the difference of the quadratic Casimir for w_j resp. w_i is

$$|\lambda_j + \delta|^2 - |\lambda_i + \delta|^2 = (i-j)(2\lambda + n - (i+j)),$$

which is nonzero by assumptions of Theorem. Here δ denotes the half of the sum of simple roots of $\mathfrak{g}'$. This conclusion implies direct sum decomposition in Corollary 4.2 and the result follows. ∎

4.2. Branching rules for the non-generic value λ of inducing character of generalized Verma module.

The remaining task is the analysis of the composition series for non-generic values $\lambda \in \mathbb{C}$ of induced character in a given block of $\mathcal{O}^{\mathfrak{p}'}$, characterized as a locus given by special values of quadratic Casimir operator. Recall that these values correspond to the known classification of homomorphisms of generalized Verma $so(p, q+1)$-modules induced from character, [5]. As we shall see, parabolic $\mathcal{O}^{\mathfrak{p}'}$ category naturally appears in our decomposition problem.

Because the weights used to induce generalized Verma modules are characters of reductive Levi factor of $\mathfrak{g}$, we are basically left with $sl(2)$-theory (generated by the first simple root of $\mathfrak{g}$). It is then natural to remind as a motivation the structure of $\mathcal{O}^{\mathfrak{b}}$ category for $sl(2)$ and then return back to our former problem.

Example 4.4. Throughout this example we use the notation

$$\mathcal{M}_{\lambda} = \mathcal{M}_{\lambda}(sl(2,\mathbb{C}), \mathfrak{b}), \mathcal{L}_{\lambda} = \mathcal{L}_{\lambda}(sl(2,\mathbb{C}), \mathfrak{b}), \mathcal{P}_{\lambda} = \mathcal{P}_{\lambda}(sl(2,\mathbb{C}), \mathfrak{b}).$$

As for $\mathfrak{g} = sl(2, \mathbb{C})$, the dual of Cartan subalgebra $\mathfrak{h}^{\star}$ is isomorphic to $\mathbb{C}$. The non-integral weights are linked by action of the Weyl group to no comparable weights (in the standard ordering), and so the only interesting subcategories (blocks) $\mathcal{O}_{\lambda}$ of the Borel category $\mathcal{O}^{\mathfrak{b}}$ are given by $\lambda \in \mathbb{Z}$. Let us consider the orbit of the Weyl group for $\lambda, \mu := w \cdot \lambda = -\lambda - 2$, $\lambda \in \mathbb{N}$. There is no lower weight associated to μ, consequently $\mathcal{L}_\mu = \mathcal{M}_\mu$ $(dim(\mathcal{L}_\mu) = \infty)$. For $\lambda \in \mathbb{N}$ we get $dim(\mathcal{L}_\lambda) < \infty$ and there is a short exact sequence

$$0 \to \mathcal{L}_\mu \to \mathcal{M}_\lambda \to \mathcal{L}_\lambda \to 0. \tag{21}$$

For the dominant integral weight λ we have $\mathcal{P}_\lambda = \mathcal{M}_\lambda$. Its dual in $\mathcal{O}^{\mathfrak{b}}$-category $\mathcal{Q}(\lambda) := \tilde{\mathcal{P}}_\lambda$ is the injective module whose socle is $\mathcal{L}_\lambda$ and its head is $\mathcal{L}_\mu$. The top quotient of $\mathcal{P}_\mu$ is $\mathcal{L}_\mu = \mathcal{M}_\mu$, and it follows from the BGG reciprocity $[\mathcal{P}_\mu : \mathcal{M}_\lambda] = [\mathcal{M}_\lambda : \mathcal{L}_\mu]$, $\lambda, \mu \in \mathfrak{h}^\star$ that there is a non-split short exact sequence

$$0 \to \mathcal{M}_\lambda \to \mathcal{P}_\mu \to \mathcal{M}_\mu \to 0. \tag{22}$$

The dual of projective module $\tilde{\mathcal{P}}_\mu$ is $\mathcal{Q}_\mu \simeq \mathcal{P}_\mu$. The (quadratic) Casimir operator $z \in \mathcal{U}(\mathfrak{g})$ acts by scalar $\lambda^2 + 2\lambda$ on both $\mathcal{M}_\lambda$ and $\mathcal{M}_\mu$. The element $z - (\lambda^2 + 2\lambda)$ is nonzero when acting on $\mathcal{P}_\mu$, but $(z - (\lambda^2 + 2\lambda))^2$ is trivial on $\mathcal{P}_\mu$.

In conclusion, there are five isomorphism classes of indecomposable objects in $\mathcal{O}^{\mathfrak{b}}_\lambda$:

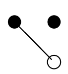
$$\mathcal{L}_\lambda, \mathcal{L}_\mu = \mathcal{M}_\mu, \mathcal{M}_\lambda = \mathcal{P}_\lambda, \mathcal{Q}_\lambda = \tilde{\mathcal{M}}_\lambda, \mathcal{P}_\mu = \mathcal{Q}_\mu.$$

We shall now analyze explicitly the first few cases when the nontrivial composition series emerges. We focus on the nontrivial part of the decomposition, which means that the vector complement in the decomposition consists of generalized Verma modules with mutually different infinitesimal characters, hence direct summands in the decomposition. We discuss the even case only, the discussion of odd case goes along the same lines. Recall the convention $\triangle' := \triangle_{n-1}$ $(n = p+q.)$

The first non-trivial case corresponds to the value λ for which $2\lambda+n-3 = 0$, $N = 1$ and $D_2(\lambda) = (2\lambda + n - 3)Y_n^{-2} + \triangle'$. Hence for this value of λ the homomorphism reduces to $\triangle'$ and so with respect to the homogeneity of Y_n^-, the first row $\mathcal{M}_\lambda(\mathfrak{g}', \mathfrak{p}')$ given by $\mathcal{U}(\mathfrak{g}')$-span of highest weight vector of $\mathcal{M}_\lambda(\mathfrak{g}, \mathfrak{p})$ contains the nontrivial submodule (its singular vector is generated by the image of $\triangle'$). Taken together with the third row $\mathcal{M}_{\lambda-2}(\mathfrak{g}, \mathfrak{p})$ form the nontrivial (nonsplit) extension class

$$0 \to \mathcal{M}_\lambda(\mathfrak{g}', \mathfrak{p}') \to \mathcal{P}_{\lambda-2}(\mathfrak{g}', \mathfrak{p}') \to \mathcal{M}_{\lambda-2}(\mathfrak{g}', \mathfrak{p}') \to 0, \tag{23}$$

where $\mathcal{P}_{\lambda-2}(\mathfrak{g}', \mathfrak{p}')$ is an object in the block of the parabolic category $\mathcal{O}^{\mathfrak{p}'}$. The picture representing such a situation is



where (anti)diagonals represent the singular vectors and the degeneration of particular singular vector for the previously mentioned value of λ is pictured in such a way that the missing monomials (in $Y_n^{-2}, \triangle'$) correspond to white circles and the nontrivial present monomials to black circles. The first resp. the third rows correspond to $\mathcal{U}(\mathfrak{g}')$-span of v_λ resp. $Y_n^{-2} v_\lambda$, where v_λ is the highest weight vector of $\mathcal{M}_\lambda(\mathfrak{g}, \mathfrak{p})$.

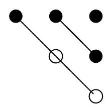
The next case is related to the appearance of a non-trivial composition series, whose source is the fourth order operator ($N=2$)

$$D_4(\lambda)=(2\lambda+n-7)(2\lambda+n-5)Y_n^{-4}+(2\lambda+n-5)Y_n^{-2}\triangle'+(\triangle')^2.$$

The computation of infinitesimal character implies that this happens for λ fulfilling $2\lambda+n-5=0$. For such λ, the generalized Verma $\mathcal{U}(\mathfrak{g}')$-modules $\mathcal{M}_\lambda(\mathfrak{g}',\mathfrak{p}'),\mathcal{M}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')$ form nontrivial extension

$$0\to\mathcal{M}_\lambda(\mathfrak{g}',\mathfrak{p}')\to\mathcal{P}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')\to\mathcal{M}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')\to 0, \tag{24}$$

realizing an object $\mathcal{P}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')$. The generalized Verma module $\mathcal{M}_\lambda(\mathfrak{g}',\mathfrak{p}')$ has a nontrivial composition structure in itself - its nontrivial submodule $\mathcal{M}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')\subset\mathcal{M}_\lambda(\mathfrak{g}',\mathfrak{p}')$ is generated by the image of $\triangle'^2$. The picture in which the first and the third row represent $\mathcal{P}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')$ and the second row has a different infinitesimal character is drawn on the following picture:



The last explicit case we mention corresponds to λ fulfilling $2\lambda+n-7=0$. The sixth order operator (here $N=3$) generating the family of singular vector is

$$\begin{aligned} D_6(\lambda) &= (2\lambda+n-11)(2\lambda+n-9)(2\lambda+n-7)Y_n^{-6} \\ &\quad +(2\lambda+n-9)(2\lambda+n-7)Y_n^{-4}\triangle' \\ &\quad +(2\lambda+n-7)Y_n^{-2}(\triangle')^2+(\triangle')^3. \end{aligned} \tag{25}$$

In this case we observe the emergence of two objects in the parabolic category $\mathcal{O}^{\mathfrak{p}'}$. The first comes from the nontrivial extension

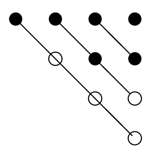
$$0\to\mathcal{M}_\lambda(\mathfrak{g}',\mathfrak{p}')\to\mathcal{P}_{\lambda-6}(\mathfrak{g}',\mathfrak{p}')\to\mathcal{M}_{\lambda-6}(\mathfrak{g}',\mathfrak{p}')\to 0, \tag{26}$$

while the second from

$$0\to\mathcal{M}_{\lambda-2}(\mathfrak{g}',\mathfrak{p}')\to\mathcal{P}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')\to\mathcal{M}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')\to 0. \tag{27}$$

Note that $\mathcal{M}_\lambda(\mathfrak{g}',\mathfrak{p}')$ has nontrivial filtered structure - its submodule is a generalized Verma module generated by the image of $(\triangle')^3$. Similarly, $\mathcal{M}_{\lambda-2}(\mathfrak{g}',\mathfrak{p}')$ has nontrivial composition series - its maximal generalized Verma submodule $\mathcal{M}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')$ is generated by the image of $\triangle'$. The modules $\mathcal{P}_{\lambda-6}(\mathfrak{g}',\mathfrak{p}'),\mathcal{P}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')$ have different infinitesimal character.

The picture in which the first and the fourth resp. the second and the third row represent $\mathcal{P}_{\lambda-6}(\mathfrak{g}',\mathfrak{p}')$ resp. $\mathcal{P}_{\lambda-4}(\mathfrak{g}',\mathfrak{p}')$ is



Theorem 4.5. *Let $\mathcal{M}_\lambda(\mathfrak{g},\mathfrak{p})$ be a family of generalized Verma $\mathcal{U}(\mathfrak{g})$-modules induced from character λ, where $\mathfrak{g}=so(p+1,q+1)$ $(p+q=n)$ and $\mathfrak{p}\subset\mathfrak{g}$ its standard maximal parabolic subalgebra given by omitting the first simple root. Let $\mathfrak{g}'=so(p,q+1)$ be the reductive subalgebra of $\mathfrak{g}$ and $\mathfrak{p}'=\mathfrak{g}'\cap\mathfrak{p}$.*

As an $\mathcal{U}(\mathfrak{g}')$-module, $\mathcal{M}_\lambda(\mathfrak{g},\mathfrak{p})$ has a contribution to the non-trivial composition structure from both the even and odd homogeneity homomorphisms:

1. *The case of even homogeneity homomorphisms corresponds to $\lambda\in\mathbb{C}$ fulfilling $2\lambda+n=2N+1$, $N\in\mathbb{N}_+$. In the decomposition there are $[\frac{N+1}{2}]$ modules*

$$\mathcal{P}_{\lambda-2N}(\mathfrak{g}',\mathfrak{p}'),\mathcal{P}_{\lambda-2N+2}(\mathfrak{g}',\mathfrak{p}'),\dots,\mathcal{P}_{\lambda-2N+2[\frac{N-1}{2}]}(\mathfrak{g}',\mathfrak{p}').$$

These modules appear as nontrivial extensions in short exact sequences

$$\begin{aligned}&0\to\mathcal{M}_{\lambda-2[\frac{N-1}{2}]}(\mathfrak{g}',\mathfrak{p}')\to\mathcal{P}_{\lambda-2N+2[\frac{N-1}{2}]}(\mathfrak{g}',\mathfrak{p}')\\&\quad\to\mathcal{M}_{\lambda-2N+2[\frac{N-1}{2}]}(\mathfrak{g}',\mathfrak{p}')\to 0,\\&\dots\\&0\to\mathcal{M}_{\lambda-2j}(\mathfrak{g}',\mathfrak{p}')\to\mathcal{P}_{\lambda-2N+2j}(\mathfrak{g}',\mathfrak{p}')\to\mathcal{M}_{\lambda-2N+2j}(\mathfrak{g}',\mathfrak{p}')\to 0,\\&\dots\\&0\to\mathcal{M}_{\lambda}(\mathfrak{g}',\mathfrak{p}')\to\mathcal{P}_{\lambda-2N}(\mathfrak{g}',\mathfrak{p}')\to\mathcal{M}_{\lambda-2N}(\mathfrak{g}',\mathfrak{p}')\to 0,\end{aligned}\qquad(28)$$

where $j=0,1,\dots,[\frac{N-1}{2}]$. The j-th module $\mathcal{M}_{\lambda-2j}(\mathfrak{g}',\mathfrak{p}')$, $j=0,1,\dots,[\frac{N-1}{2}]$ has a nontrivial composition series - its maximal submodule $\mathcal{M}_{\lambda-2N+2j}(\mathfrak{g}',\mathfrak{p}')$ is generated by the image of ${\triangle'}^{N-2j}:={\triangle_{n-1}}^{N-2j}$ and the quotient

$$\mathcal{P}_{\lambda-2N+2j}(\mathfrak{g}',\mathfrak{p}')/\mathcal{M}_{\lambda-2j}(\mathfrak{g}',\mathfrak{p}')$$

is simple module. The module $\mathcal{P}_{\lambda-2N+2j}(\mathfrak{g}',\mathfrak{p}')$ is realized in the generalized Verma $so(p+1,q+1)$-module by $\mathcal{U}(\mathfrak{g}')$-span of singular vectors w_{2j},w_{2N-2j}. Let us introduce the finite set $S:=\{\lambda-2j,\lambda-2N+2j|j=0,1,\dots,[\frac{N-1}{2}]\}$, so $S':=\{\{\lambda-2\mathbb{N}\}\setminus S|N\in\mathbb{N}\}$ is infinite. Then we have the branching rule

$$\mathcal{M}^{even}_\lambda(\mathfrak{g},\mathfrak{p})\simeq\bigoplus_{j=0,1,\dots,[\frac{N-1}{2}]}\mathcal{P}_{\lambda-2N+2j}(\mathfrak{g}',\mathfrak{p}')\bigoplus_{\lambda'\in S'}\mathcal{M}_{\lambda'}(\mathfrak{g}',\mathfrak{p}').\qquad(29)$$

2. *The case of odd homogeneity homomorphisms corresponds to $\lambda\in\mathbb{C}$ fulfilling $2(\lambda-1)+n=2N+1$, $N\in\mathbb{N}_+$. In the decomposition there are $[\frac{N+1}{2}]$ modules*

$$\mathcal{P}_{\lambda-2N-1}(\mathfrak{g}',\mathfrak{p}'),\mathcal{P}_{\lambda-2N+1}(\mathfrak{g}',\mathfrak{p}'),\dots,\mathcal{P}_{\lambda-2N-1+2[\frac{N-1}{2}]}(\mathfrak{g}',\mathfrak{p}').$$

These modules appear as nontrivial extensions in short exact sequences

$$\begin{aligned}&0\to\mathcal{M}_{\lambda-1-2[\frac{N-1}{2}]}(\mathfrak{g}',\mathfrak{p}')\to\mathcal{P}_{\lambda-2N-1+2[\frac{N-1}{2}]}(\mathfrak{g}',\mathfrak{p}')\\&\quad\to\mathcal{M}_{\lambda-2N-1+2[\frac{N-1}{2}]}(\mathfrak{g}',\mathfrak{p}')\to 0,\\&\dots\\&0\to\mathcal{M}_{\lambda-1-2j}(\mathfrak{g}',\mathfrak{p}')\to\mathcal{P}_{\lambda-2N-1+2j}(\mathfrak{g}',\mathfrak{p}')\to\mathcal{M}_{\lambda-2N-1+2j}(\mathfrak{g}',\mathfrak{p}')\to 0,\\&\dots\\&0\to\mathcal{M}_{\lambda-1}(\mathfrak{g}',\mathfrak{p}')\to\mathcal{P}_{\lambda-2N-1}(\mathfrak{g}',\mathfrak{p}')\to\mathcal{M}_{\lambda-2N-1}(\mathfrak{g}',\mathfrak{p}')\to 0,\end{aligned}\qquad(30)$$

where $j = 0, 1, \ldots, [\frac{N-1}{2}]$. *The* j*-th module*

$$\mathcal{M}_{\lambda-2j-1}(\mathfrak{g}', \mathfrak{p}'),\ j = 0, 1, \ldots, [\tfrac{N-1}{2}]$$

has nontrivial composition series - its maximal submodule $\mathcal{M}_{\lambda-2N-1+2j}(\mathfrak{g}', \mathfrak{p}')$ *is generated by the image of* $Y_n^- {\triangle'}^{N-2j}$ *and the quotient*

$$\mathcal{P}_{\lambda-2N-1+2j}(\mathfrak{g}', \mathfrak{p}')/\mathcal{M}_{\lambda-1-2j}(\mathfrak{g}', \mathfrak{p}')$$

is simple module. The module $\mathcal{P}_{\lambda-2N-1+2j}(\mathfrak{g}', \mathfrak{p}')$ *is realized in the generalized Verma* $so(p+1, q+1)$*-module by* $\mathcal{U}(\mathfrak{g}')$*-span of singular vectors* $w_{2j+1}, w_{2N+1-2j}$.

Let us introduce the finite set $\tilde{S} := \{\lambda - 1 - 2j, \lambda - 2N - 1 + 2j | j = 0, 1, \ldots, [\frac{N-1}{2}]\}$, *so* $\tilde{S}' := \{\{\lambda - 2\mathbb{N} - 1\} \setminus \tilde{S} | N \in \mathbb{N}\}$ *is infinite. Then we have the branching rule*

$$\mathcal{M}^{odd}_{\lambda}(\mathfrak{g}, \mathfrak{p}) \simeq \bigoplus_{j=0,1,\ldots,[\frac{N-1}{2}]} \mathcal{P}_{\lambda-2N-1+2j}(\mathfrak{g}', \mathfrak{p}') \bigoplus_{\lambda' \in \tilde{S}'} \mathcal{M}_{\lambda'}(\mathfrak{g}', \mathfrak{p}'). \tag{31}$$

Finally, we have direct sum decomposition of $\mathfrak{g}'$*-modules (which is even true for any* λ*):*

$$\mathcal{M}_{\lambda}(\mathfrak{g}, \mathfrak{p}) \simeq \mathcal{M}^{even}_{\lambda}(\mathfrak{g}, \mathfrak{p}) \bigoplus \mathcal{M}^{odd}_{\lambda}(\mathfrak{g}, \mathfrak{p}). \tag{32}$$

Proof. The general case follows the scheme indicated in the discussion of the structure of singular vectors preceded this Theorem. Corollary 4.2 implies that the elements constructed in Theorem 3.2 cover all singular vectors. Moreover, for any non-generic value of the inducing character λ (determined in Theorem 4.3) there is a finite number of couples of singular vectors with equal infinitesimal character, as follows again from Theorem 4.3. These couples are enumerated in Equation 28 for even homogeneity resp. Equation 30 for odd homogeneity case. The non-triviality of each extension class is an elementary direct check applied to the singular vector (evaluated at the corresponding non-generic value λ) based on Equation 7, Equation 8. ■

The techniques used in the article do not allow further analysis of constructed extension classes of generalized Verma modules. In [4], there are certain partial results describing (non-recursive) scheme to compute Kazhdan-Lusztig polynomials associated to Hermitian symmetric spaces. In particular, Kazhdan-Lusztig polynomials are determined in the case of regular block of zero weight in Proposition 5.1., p. 288, [4], with the following result (in the even dimensional orthogonal case): $P_{w_i,w_j}(u)$ is trivial for incomparable w_i, w_j, $P_{w_i,w_j}(u) = 1 + u^{j-n-1}$ for $n+2 \le j \le 2n-1$, $1 \le i \le 2n-j$ and $P_{w_i,w_j}(u) = 1$ otherwise. In the odd orthogonal case, the structure of Kazhdan-Lusztig polynomials is even simpler. As the extension classes $Ext^{\star}_{\mathcal{U}(\mathfrak{g})}(\mathcal{M}_{w_i}, \mathcal{L}_{w_j})$ are the coefficients of Kazhdan-Lusztig polynomials, they are at most one dimensional. In a basic example, taking into account the relationship between extension classes and Lie algebra cohomology classes for parabolic subalgebras with commutative nilradicals ([7]), one can directly compare the extension class produced in the branching rule with its geometrical realization based on the Lie algebra cohomology method, see [2]. However, in many cases are our results realized in singular infinitesimal character and so the structure of Kazhdan-Lusztig polynomials is to our best knowledge not known.

Another remark closely to the last paragraph is that in many cases the extension classes appearing in the main Theorem are projective objects of parabolic BGG category $\mathcal{O}^{\mathfrak{p}'}$.

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