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Structure of Root Graded Lie Algebras

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Abstract. We give a complete description of the structure of Lie algebras graded by an infinite irreducible locally finite root system.
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1. Introduction

In 1992, S. Berman and R. Moody [7] introduced the notion of a *Lie algebra graded by an irreducible reduced finite root system.* Their definition was motivated by a construction appearing in the classification of finite dimensional simple Lie algebras containing nonzero toral subalgebras [12]. The classification of root graded Lie algebras in the sense of S. Berman and R. Moody was given, in part, by S. Bermen and R. Moody themselves and was completed by G. Benkart and E. Zelmanov [5] in 1996. They classified Lie algebras graded by an irreducible reduced finite root system using a type-by-type approach; for each type X, the authors give a recognition theorem for centerless Lie algebras graded by a root system of type X. In 1996, E. Neher [11] generalized the notion of root graded Lie algebras by switching from fields of characteristic zero to rings containing $1/6$ and working with locally finite root systems instead of finite root systems. Roughly speaking, according to him, a Lie algebra $\mathcal{L}$ over a ring containing $1/6$ is graded by a reduced locally finite root system R if $\mathcal{L}$ is a $\mathrm{span}_{\mathbb{Q}}R$-graded Lie algebra generated by homogenous submodules of nonzero degrees and that for any nonzero root $\alpha \in R$, there are homogenous elements e and f of degrees α and $-\alpha$ respectively such that $[e, f]$ acts diagonally on $\mathcal{L}$. He realized root graded Lie algebras for reduced types other than F_4, G_2 and E_8 as central extensions of Tits-Kantor-Koecher algebras of certain Jordan pairs. Finally in 2002, B. Allison, G. Benkart and Y. Gao [3] defined a Lie algebra graded by an irreducible finite root system of type BC and studied root graded Lie algebras of type BC_n for $n \geq 2$. In 2003, G. Benkart and O. Smirnov [6] studied Lie algebras graded by a finite root system of

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type BC_1 and finalized the classification of Lie algebras graded by an irreducible finite root system.

A Lie algebra $\mathcal{L}$ graded by an irreducible finite root system R has a weight space decomposition with respect to a splitting Cartan subalgebra of a finite dimensional split simple Lie subalgebra $\mathfrak{g}$ of $\mathcal{L}$, whose set of weights is contained in R. This feature allows us to decompose $\mathcal{L}$ as $\mathcal{L} = \mathcal{M}_1 \oplus \mathcal{M}_2$ in which $\mathcal{M}_1$ is a direct sum of finite dimensional irreducible nontrivial $\mathfrak{g}$-submodules and $\mathcal{M}_2$ is a trivial $\mathfrak{g}$-submodule of $\mathcal{L}$. One can derive a specific vector space $\mathfrak{b}$ from the $\mathfrak{g}$-module structure of $\mathcal{M}_1$. This vector space is equipped with an algebraic structure which is induced by the Lie algebraic structure of $\mathcal{L}$. Moreover, the Lie algebra $\mathcal{L}$ can be reconstructed from the algebra $\mathfrak{b}$ in a prescribed way; see [2] and [3]. This construction led to finding a finite presentation for the universal central extension of a *Lie torus* [13] of some finite types; see [14] and [4]. This motivates us to generalize this construction for Lie algebras graded by infinite root systems.

We give a complete description of the structure of root graded Lie algebras. We fix an infinite irreducible locally finite root system R and show that a Lie algebra $\mathcal{L}$ graded by R can be completely described in terms of the structure of some known algebraic features derived from the structure of $\mathcal{L}$. More precisely, depending on the type of R, we consider a quadruple $\mathfrak{c}$ so called a *coordinate quadruple*. We next correspond to $\mathfrak{c}$, a specific algebra $\mathfrak{b}_\mathfrak{c}$ and a specific Lie algebra $\{\mathfrak{b}_\mathfrak{c}, \mathfrak{b}_\mathfrak{c}\}$. Then for each subspace $\mathcal{K}$ of the center of $\{\mathfrak{b}_\mathfrak{c}, \mathfrak{b}_\mathfrak{c}\}$ satisfying a certain property called *the uniform property*, we define a Lie algebra $\mathcal{L}(\mathfrak{b}_\mathfrak{c}, \mathcal{K})$ and show that it is a Lie algebra graded by R. Conversely, given a Lie algebra $\mathcal{L}$ graded by R, we prove that $\mathcal{L}$ can be decomposed as $\mathcal{M}_1 \oplus \mathcal{M}_2$ where $\mathcal{M}_1$ is a direct sum of certain irreducible nontrivial $\mathcal{G}$-submodules for a locally finite spilt simple Lie subalgebra $\mathcal{G}$ of $\mathcal{L}$ and $\mathcal{M}_2$ is a specific subalgebra of $\mathcal{L}$. We derive a quadruple $\mathfrak{c}$ from the $\mathcal{G}$-module structure of $\mathcal{M}_1$ and show that it is a coordinate quadruple. We also prove that there is a subspace $\mathcal{K}$ of the center of $\{\mathfrak{b}_\mathfrak{c}, \mathfrak{b}_\mathfrak{c}\}$ satisfying the uniform property such that $\mathcal{M}_2$ is isomorphic to the quotient algebra $\{\mathfrak{b}_\mathfrak{c}, \mathfrak{b}_\mathfrak{c}\}/\mathcal{K}$ and moreover $\mathcal{L}$ is isomorphic to $\mathcal{L}(\mathfrak{b}_\mathfrak{c}, \mathcal{K})$.

In the case that the root system R is reduced, our method also suggests another approach to characterize Lie algebras graded by R compared with what offered by E. Neher [11].

The outline of this work is as follows: In the first section, we gather some preliminaries which fill in the background for the readers. In the second section, we compare the structure of a Lie algebra graded by a finite root system with the structure of its root graded subalgebras. Regarding the results of the second section, we devote the third section to the main result of the work, namely, we study the structure of Lie algebras graded by infinite locally finite root systems.

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2. Preliminaries

Throughout this work, $\mathbb{N}$ denotes the set of nonnegative integers and $\mathbb{F}$ is a field of characteristic zero. Unless otherwise mentioned, all vector spaces are considered over $\mathbb{F}$. We denote the dual space of a vector space V by V^*. We also denote by $\mathrm{End}(V)$, the vector space of all linear endomorphisms on V. For a linear transformation $T \in \mathrm{End}(V)$, if the trace of T is defined, we denote it by $tr(T)$. Also for a nonempty set S, by $id_{_S}$ (or id if there is no confusion), we mean the identity map on S and by $|S|$, we mean the cardinality of S. Finally for an index set I, by a conventional notation, we take $\bar{I} := \{\bar{i} \mid i \in I\}$ to be a disjoint copy of I and for each subset J of I, by $\bar{J}$, we mean the subset of $\bar{I}$ corresponding to J.

2.1. Locally Finite Split Simple Lie Algebras. In this subsection, we recall the structure of infinite dimensional locally finite split simple Lie algebras from [10] and state some facts which play key roles in this work. Let us start with the following definition.

Definition 2.1. Let $\mathcal{H}$ be a Lie algebra. We say an $\mathcal{H}$-module $\mathcal{M}$ has a *weight space decomposition with respect to* $\mathcal{H}$, if

$$\mathcal{M} = \oplus_{\alpha\in\mathcal{H}^*}\mathcal{M}_\alpha \text{ where } \mathcal{M}_\alpha := \{x \in \mathcal{M} \mid h \cdot x = \alpha(h)x;\ \ \forall\, h \in \mathcal{H}\}$$

for all $\alpha \in \mathcal{H}^*$. The set $R := \{\alpha \in \mathcal{H}^* \mid \mathcal{M}_\alpha \neq \{0\}\}$ is called the *set of weights* of $\mathcal{M}$ (with respect to $\mathcal{H}$). For $\alpha \in R$, $\mathcal{M}_\alpha$ is called a *weight space*, and any element of $\mathcal{M}_\alpha$ is called a *weight vector* of *weight* α. If a Lie algebra $\mathcal{L}$ has a weight space decomposition with respect to a nontrivial subalgebra H of $\mathcal{L}$ via the adjoint representation, H is called a *split toral subalgebra*. The set of weights of $\mathcal{L}$ is called the *root system* of $\mathcal{L}$ with respect to H, and the corresponding weight spaces are called *root spaces of* $\mathcal{L}$. A Lie algebra $\mathcal{L}$ is called *split* if it contains a *splitting Cartan subalgebra*, that is a split toral subalgebra H of $\mathcal{L}$ with $\mathcal{L}_0 = H$.

The root system of a locally finite split simple Lie algebra with respect to a splitting Cartan subalgebra is a reduced irreducible locally finite root system in the following sense (see [8] and [10]):

Definition 2.2. [9] Let $\mathcal{U}$ be a nontrivial vector space and R be a subset of $\mathcal{U}$. The subset R is said to be a *locally finite root system in* $\mathcal{U}$ of *rank dim*$(\mathcal{U})$ if the followings are satisfied:

(i) R is locally finite, contains zero and spans $\mathcal{U}$.

(ii) For every $\alpha \in R^\times := R \setminus \{0\}$, there exists $\check{\alpha} \in \mathcal{U}^*$ such that $\check{\alpha}(\alpha) = 2$ and $s_\alpha(\beta) \in R$ for $\alpha, \beta \in R$ where $s_\alpha : \mathcal{U} \longrightarrow \mathcal{U}$ maps $u \in \mathcal{U}$ to $u - \check{\alpha}(u)\alpha$. We set by convention $\check{0}$ to be zero.

(iii) $\check{\alpha}(\beta) \in \mathbb{Z}$, for $\alpha, \beta \in R$.

Set $R_{sdiv} := (R \setminus \{\alpha \in R \mid 2\alpha \in R\}) \cup \{0\}$ and call it the *semi-divisible subsystem* of R. The root system R is called *reduced* if $R = R_{sdiv}$.

Suppose that R is a locally finite root system. A nonempty subset S of R is said to be a *subsystem* of R if S contains zero and $s_\alpha(\beta) \in S$ for $\alpha, \beta \in S \setminus \{0\}$.

A subsystem S of R is called *full* if $\mathrm{span}_{\mathbb{F}}S \cap R = S$. Following [9, §2.6], we say two nonzero roots α, β of a subset S of R are *connected* in S if there exist finitely many roots $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_n = \beta$ in S such that $\check{\alpha}_{i+1}(\alpha_i) \neq 0$, $1 \leq i \leq n-1$. Connectedness in $R^\times$ defines an equivalence relation on $R^\times$ and so $R^\times$ is the disjoint union of its equivalence classes called *connected components* of R. A nonempty subset X of R is called *irreducible,* if each two nonzero elements $x, y \in X$ are connected in X and it is called *closed* if $(X+X) \cap R \subseteq X$. It is easy to see that if X is a connected component of a locally finite root system R, then $X \cup \{0\}$ is a closed subsystem of R.

For the locally finite root system R, take $\{R_\lambda \mid \lambda \in \Gamma\}$ to be the class of all finite subsystems of R, and say $\lambda \preccurlyeq \mu$ $(\lambda, \mu \in \Gamma)$ if R_λ is a subsystem of R_μ, then $(\Gamma, \preccurlyeq)$ is a directed set and R is the direct union of $\{R_\lambda \mid \lambda \in \Gamma\}$. Furthermore, if R is irreducible, it is the direct union of its irreducible finite subsystems; see [9].

Two locally finite root systems $(R, \mathcal{U})$ and $(S, \mathcal{V})$ are said to be *isomorphic* if there is a linear transformation $f : \mathcal{U} \longrightarrow \mathcal{V}$ such that $f(R) = S$.

Suppose that I is a nonempty index set and $\mathcal{U} := \oplus_{i \in I} \mathbb{F}\epsilon_i$ is the free $\mathbb{F}$-module over the set I. Define the form

$$\begin{gathered}(\cdot,\cdot) : \mathcal{U} \times \mathcal{U} \longrightarrow \mathbb{F}\\ (\epsilon_i, \epsilon_j) = \delta_{i,j}, \text{ for } i, j \in I\end{gathered}$$

and set

$$\begin{array}{l}\dot{A}_I := \{\epsilon_i - \epsilon_j \mid i, j \in I\},\\ D_I := \dot{A}_I \cup \{\pm(\epsilon_i + \epsilon_j) \mid i, j \in I,\ i \neq j\},\\ B_I := D_I \cup \{\pm\epsilon_i \mid i \in I\},\\ C_I := D_I \cup \{\pm 2\epsilon_i \mid i \in I\},\\ BC_I := B_I \cup C_I.\end{array} \tag{1}$$

One can see that these are irreducible locally finite root systems in their $\mathbb{F}-$span's which we refer to as *type* A, D, B, C and BC respectively. Moreover every irreducible locally finite root system of infinite rank is isomorphic to one of these root systems (see [9, §4.14, §8]). Now we suppose R is an irreducible locally finite root system as above and note that $(\alpha, \alpha) \in \mathbb{N}$ for all $\alpha \in R$. This allows us to define

$$\begin{array}{l}R_{sh} := \{\alpha \in R^\times \mid (\alpha, \alpha) \leq (\beta, \beta);\ \text{ for all } \beta \in R\},\\ R_{ex} := R \cap 2R_{sh},\\ R_{lg} := R^\times \setminus (R_{sh} \cup R_{ex}).\end{array}$$

The elements of R_{sh} (resp. R_{lg}, R_{ex}) are called *short roots* (resp. *long roots, extra-long roots*) of R.

A locally finite split simple Lie algebra is said to be of *type* A, B, C or D if its corresponding root system with respect to a splitting Cartan subalgebra is of type A, B, C or D respectively. In what follows, we recall from [10] the classification of infinite dimensional locally finite split simple Lie algebras. Suppose that J is an index set and $\mathcal{V} := \mathcal{V}_J$ is a vector space with a fixed basis $\{v_j \mid j \in J\}$. One knows that $\mathfrak{gl}(\mathcal{V}) := \mathrm{End}(\mathcal{V})$ together with

$$[\cdot,\cdot] : \mathfrak{gl}(\mathcal{V}) \times \mathfrak{gl}(\mathcal{V}) \longrightarrow \mathfrak{gl}(\mathcal{V});\ (X, Y) \mapsto XY - YX;\ \ X, Y \in \mathfrak{gl}(\mathcal{V})$$

is a Lie algebra. Now for $j,k \in J$, define

$$e_{j,k} : \mathcal{V} \longrightarrow \mathcal{V};\ \ v_i \mapsto \delta_{k,i} v_j, \quad (i \in J), \tag{2}$$

then $\mathfrak{gl}(J) := \mathrm{span}_{\mathbb{F}}\{e_{j,k} \mid j,k \in J\}$ is a Lie subalgebra of $\mathfrak{gl}(\mathcal{V})$.

Lemma 2.3 (Classical Lie algebras of type A). *Suppose that I is a nonempty index set of cardinality greater than 1, I_0 is a fixed subset of I with $|I_0| > 1$ and $\mathcal{V}$ is a vector space with a basis $\{v_i \mid i \in I\}$. Take Λ to be an index set containing 0 and $\{I_\lambda \mid \lambda \in \Lambda\}$ to be the class of all finite subsets of I containing I_0. Set*

$$\mathcal{G} := \mathfrak{sl}(I) := \{\phi \in \mathfrak{gl}(I) \mid tr(\phi) = 0\},$$

and for $\lambda \in \Lambda$, take

$$\mathcal{G}_{I_\lambda} := \mathcal{G}^\lambda := \mathcal{G} \cap span_{\mathbb{F}}\{e_{r,s} \mid r,s \in I_\lambda\}.$$

Then $\mathfrak{sl}(I)$ is a locally finite split simple Lie subalgebra of $\mathfrak{gl}(I)$ with splitting Cartan subalgebra $\mathcal{H} := span\{e_{i,i} - e_{j,j} \mid i,j \in I\}$ and corresponding root system isomorphic to $\dot{A}_I$. Moreover, for $i,j \in I$ with $i \neq j$, we have

$$\mathcal{G}_{\epsilon_i - \epsilon_j} = \mathbb{F} e_{i,j}.$$

Also for each $\lambda \in \Lambda$, $\mathcal{G}^\lambda$ is a finite dimensional split simple Lie subalgebra of $\mathcal{G}$ with splitting Cartan subalgebra $\mathcal{H}^\lambda := \mathcal{H} \cap \mathcal{G}^\lambda$ and $\mathcal{G}$ is the direct union of $\{\mathcal{G}^\lambda \mid \lambda \in \Lambda\}$.

In the following lemma, we see that locally finite split simple Lie algebras of type B can be described in terms of derivations of *Clifford Jordan algebras* which are defined as following:

Definition 2.4 ([13]). Suppose that A is a unital commutative associative algebra over $\mathbb{F}$ and $\mathcal{W}$ is a unitary A-module. Suppose that $g : \mathcal{W} \times \mathcal{W} \longrightarrow A$ is a symmetric A-bilinear form and set $\mathcal{J} := \mathcal{J}(g, \mathcal{W}) := A \oplus \mathcal{W}$. The vector space $\mathcal{J}$ together with the following multiplication

$$(a_1 + w_1)(a_2 + w_2) = a_1 a_2 + g(w_1, w_2) + a_1 w_2 + a_2 w_1$$

for $a_1, a_2 \in A$ and $w_1, w_2 \in \mathcal{W}$ is a Jordan algebra called a *Clifford Jordan algebra.* For $a, b \in \mathcal{J}$, define $D_{a,b} := -[\mathbf{L}_a, \mathbf{L}_b] := \mathbf{L}_b \mathbf{L}_a - \mathbf{L}_a \mathbf{L}_b$ where $\mathbf{L}_a, \mathbf{L}_b$ are left multiplications by a and b respectively. For a subspace V of $\mathcal{J}$, set $D_{V,V}$ to be the subspace of $\mathrm{End}(\mathcal{J})$ spanned by $D_{a,b}$ for $a, b \in V$. One can see that for $w_1, w_2 \in \mathcal{W}$, D_{w_1,w_2} can be identified with $D_{w_1,w_2}|_{\mathcal{W}}$, the restriction of D_{w_1,w_2} to $\mathcal{W}$. This allows us to consider $D_{\mathcal{W},\mathcal{W}}$ as a subalgebra of $\mathfrak{gl}(\mathcal{W})$.

Lemma 2.5 (Classical Lie algebras of type B). *Suppose that I is a nonempty index set. Take $J := \{0\} \uplus I \uplus \bar{I}$ and consider the vector space $\mathcal{V} := \mathcal{V}_J$ as before. Define the bilinear form $(\cdot,\cdot)$ on $\mathcal{V}$ by*

$$\begin{gathered}(v_j, v_{\bar{k}}) := (v_{\bar{k}}, v_j) := 2\delta_{j,k},\ (v_0, v_0) := 2,\\ (v_j, v_k) := (v_j, v_0) := (v_0, v_j) := (v_0, v_{\bar{j}}) := (v_{\bar{j}}, v_0) := (v_{\bar{j}}, v_{\bar{k}}) := 0\end{gathered} \tag{3}$$

for $j,k\in I$ *and set*

$$\mathcal{G}:=\mathfrak{o}_B(I):=\{\phi\in\mathfrak{gl}(J)\mid(\phi(v),w)=-(v,\phi(w)),\ \textit{for all } v,w\in\mathcal{V}\}.$$

Then we have the followings:

(i) $\mathcal{G}$ *is a locally finite split simple Lie subalgebra of* $\mathfrak{gl}(J)$ *with splitting Cartan subalgebra* $\mathcal{H}:=span_{\mathbb{F}}\{h_i:=e_{i,i}-e_{\bar{i},\bar{i}}\mid i\in I\}$ *and corresponding root system isomorphic to* B_I. *Moreover, for* $i,j\in J$ *with* $i\neq j$, *we have*

$$\mathcal{G}_{\epsilon_i-\epsilon_j}=\mathbb{F}(e_{i,j}-e_{\bar{j},\bar{i}}),\ \mathcal{G}_{\epsilon_i+\epsilon_j}=\mathbb{F}(e_{i,\bar{j}}-e_{j,\bar{i}}),\ \mathcal{G}_{-\epsilon_i-\epsilon_j}=\mathbb{F}(e_{\bar{i},j}-e_{\bar{j},i}),$$
$$\mathcal{G}_{\epsilon_i}=\mathbb{F}(e_{i,0}-e_{0,\bar{i}}),\ \mathcal{G}_{-\epsilon_i}=\mathbb{F}(e_{\bar{i},0}-e_{0,i}).$$

(ii) For the Clifford Jordan algebra $\mathcal{J}((\cdot,\cdot),\mathcal{V})$, *we have* $\mathcal{G}=D_{\mathcal{V},\mathcal{V}}$.

(iii) For a fixed subset I_0 *of* I, *take* Λ *to be an index set containing* 0 *such that* $\{I_\lambda\mid\lambda\in\Lambda\}$ *is the class of all finite subsets of* I *containing* I_0. *For each* $\lambda\in\Lambda$, *set*

$$\mathcal{G}_{I_\lambda}:=\mathcal{G}^\lambda:=\mathcal{G}\cap span_{\mathbb{F}}\{e_{r,s}\mid r,s\in\{0\}\cup I_\lambda\cup\bar{I}_\lambda\}.\tag{4}$$

Then $\mathcal{G}^\lambda$ $(\lambda\in\Lambda)$ *is a finite dimensional split simple Lie subalgebra of* $\mathcal{G}$ *of type* B, *with splitting Cartan subalgebra* $\mathcal{H}^\lambda:=\mathcal{H}\cap\mathcal{G}^\lambda$ *and* $\mathcal{G}$ *is the direct union of* $\{\mathcal{G}^\lambda\mid\lambda\in\Lambda\}$.

Lemma 2.6 (Classical Lie algebras of type D)**.** *Suppose that* I *is a nonempty index set and* I_0 *is a fixed subset of* I. *Set* $J:=I\uplus\bar{I}$ *and take* $\{I_\lambda\mid\lambda\in\Lambda\}$, *where* Λ *is an index set containing* 0, *to be the class of all finite subsets of* I *containing* I_0. *Define the bilinear form* $(\cdot,\cdot)$ *on* $\mathcal{V}:=\mathcal{V}_J$ *by*

$$(v_j,v_{\bar{k}}):=(v_{\bar{k}},v_j):=2\delta_{j,k},\ (v_j,v_k):=(v_{\bar{j}},v_{\bar{k}}):=0;\ (j,k\in I),\tag{5}$$

and set

$$\mathcal{G}:=\mathfrak{o}_D(I):=\{\phi\in\mathfrak{gl}(J)\mid(\phi(v),w)=-(v,\phi(w)),\ \textit{for all } v,w\in\mathcal{V}\},$$
$$\mathcal{H}:=span_{\mathbb{F}}\{h_i:=e_{i,i}-e_{\bar{i},\bar{i}}\mid i\in I\}.$$

Also for $\lambda\in\Lambda$, *take*

$$\mathcal{G}_{I_\lambda}:=\mathcal{G}^\lambda:=\mathcal{G}\cap span_{\mathbb{F}}\{e_{r,s}\mid r,s\in I_\lambda\cup\bar{I}_\lambda\}.$$

Then $\mathcal{G}$ *is a locally finite split simple Lie subalgebra of* $\mathfrak{gl}(J)$ *with splitting Cartan subalgebra* $\mathcal{H}$ *and corresponding root system isomorphic to* D_I. *Moreover for* $i,j\in J$ *with* $i\neq j$, *we have*

$$\mathcal{G}_{\epsilon_i-\epsilon_j}=\mathbb{F}(e_{i,j}-e_{\bar{j},\bar{i}}),\ \mathcal{G}_{\epsilon_i+\epsilon_j}=\mathbb{F}(e_{i,\bar{j}}-e_{j,\bar{i}}),\ \mathcal{G}_{-\epsilon_i-\epsilon_j}=\mathbb{F}(e_{\bar{i},j}-e_{\bar{j},i}).$$

Also for each $\lambda\in\Lambda$, $\mathcal{G}^\lambda$ *is a finite dimensional split simple Lie subalgebra of* $\mathcal{G}$, *of type* D, *with splitting Cartan subalgebra* $\mathcal{H}^\lambda:=\mathcal{H}\cap\mathcal{G}^\lambda$, *and* $\mathcal{G}$ *is the direct union of* $\{\mathcal{G}^\lambda\mid\lambda\in\Lambda\}$.

Lemma 2.7 (Classical Lie algebras of type C)**.** *Suppose that* I *is a nonempty index set and* $J:=I\uplus\bar{I}$. *Consider the bilinear form* $(\cdot,\cdot)$ *on* $\mathcal{V}:=\mathcal{V}_J$ *defined by*

$$(v_j,v_{\bar{k}}):=-(v_{\bar{k}},v_j):=2\delta_{j,k},\ (v_j,v_k):=0,\ (v_{\bar{j}},v_{\bar{k}}):=0;\quad(j,k\in I),\tag{6}$$

and set

$$\mathcal{G} := \mathfrak{sp}(I) := \{\phi \in \mathfrak{gl}(J) \mid (\phi(v), w) = -(v, \phi(w)),\ \textit{for all}\ v, w \in \mathcal{V}\}.$$

Also for a fixed subset I_0 of I, take $\{I_\lambda \mid \lambda \in \Lambda\}$ to be the class of all finite subsets of I containing I_0, in which Λ is an index set containing 0, and for each $\lambda \in \Lambda$, set

$$\mathcal{G}_{I_\lambda} := \mathcal{G}^\lambda := \mathcal{G} \cap \mathit{span}\{e_{r,s} \mid r, s \in I_\lambda \cup \bar{I}_\lambda\}. \tag{7}$$

Then $\mathcal{G}$ is a locally finite split simple Lie subalgebra of $\mathfrak{gl}(J)$ with splitting Cartan subalgebra $\mathcal{H} := \mathit{span}_{\mathbb{F}}\{h_i := e_{i,i} - e_{\bar{i},\bar{i}} \mid i \in I\}$. Moreover, for $i, j \in I$ with $i \neq j$, we have

$$\mathcal{G}_{\epsilon_i - \epsilon_j} = \mathbb{F}(e_{i,j} - e_{\bar{j},\bar{i}}),\ \mathcal{G}_{\epsilon_i + \epsilon_j} = \mathbb{F}(e_{i,\bar{j}} + e_{j,\bar{i}}),\ \mathcal{G}_{-\epsilon_i - \epsilon_j} = \mathbb{F}(e_{\bar{i},j} + e_{\bar{j},i}),$$
$$\mathcal{G}_{2\epsilon_i} = \mathbb{F}e_{i,\bar{i}},\ \mathcal{G}_{-2\epsilon_i} = \mathbb{F}e_{\bar{i},i}.$$

Also for $\lambda \in \Lambda$, $\mathcal{G}^\lambda$ is a finite dimensional split simple Lie subalgebra of type C, with splitting Cartan subalgebra $\mathcal{H}^\lambda := \mathcal{H} \cap \mathcal{G}^\lambda$, and $\mathcal{G}$ is the direct union of $\{\mathcal{G}^\lambda \mid \lambda \in \Lambda\}$.

Proposition 2.8. *(i)[10, Theorem VI.7] Suppose that I is an infinite index set, then $\mathfrak{o}_B(I)$ is isomorphic to $\mathfrak{o}_D(I)$. Moreover, if $\mathcal{G}$ is an infinite dimensional locally finite split simple Lie algebra, then $\mathcal{G}$ is isomorphic to exactly one of the Lie algebras $\mathfrak{sl}(I)$, $\mathfrak{o}_B(I)$ or $\mathfrak{sp}(I)$ for some infinite index set I.*

(ii)[10, Corollary VI.8] Suppose that I is an infinite index set, then the Lie algebras $\mathfrak{sl}(I)$ and $\mathfrak{sp}(I)$ have one and $\mathfrak{o}_B(I)$ has two conjugacy classes of splitting Cartan subalgebras under the group of automorphisms of the Lie algebra.

Lemma 2.9. *Suppose $\mathcal{G}$ is a locally finite split simple Lie algebra with a splitting Cartan subalgebra $\mathcal{H}$. Assume $R \subseteq \mathcal{H}^*$ is an irreducible locally finite root system and $\mathcal{G} = \sum_{\alpha \in R_{sdiv}} \mathcal{G}_\alpha$ is the root space decomposition of $\mathcal{G}$ with respect to $\mathcal{H}$. Suppose that S is an irreducible closed subsystem of R and set $\mathfrak{g} := \sum_{\alpha \in S^\times_{sdiv}} \mathcal{G}_\alpha \oplus \sum_{\alpha \in S^\times_{sdiv}} [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]$ as well as $\mathfrak{h} := \mathcal{H} \cap \mathfrak{g}$, then the restriction of*

$$\pi : \mathcal{H}^* \longrightarrow \mathfrak{h}^*;\ \ f \mapsto f|_{\mathfrak{h}},\ \ f \in \mathcal{H}^*$$

to S is injective. Identify $\alpha \in S$ with $\pi(\alpha)$ via π, then $\mathfrak{g}$ is a locally finite split simple Lie subalgebra of $\mathcal{G}$ with splitting Cartan subalgebra $\mathfrak{h}$ and corresponding root system S_{sdiv}.

Proof. We first claim that

$$\begin{array}{l} \textit{if } \alpha, \beta \in S \textit{ and } \alpha - \beta \notin R, \textit{ then there is} \\ h \in \mathfrak{h} \textit{ such that } \alpha(h) > 0 \textit{ and } \beta(h) \leq 0. \end{array} \tag{8}$$

To prove this, we note that since $\alpha - \beta \notin R$, we have $\alpha \neq 0$ and $\beta \neq 0$. Moreover, it follows from the theory of locally finite root systems that $\beta - 2\alpha \notin R$ and $\alpha - 2\beta \notin R$, also if $2\alpha \in R$ or $2\beta \in R$, then $2\alpha - 2\beta \notin R$. Therefore setting

$$\alpha' := \begin{cases} \alpha & \text{if } 2\alpha \notin R \\ 2\alpha & \text{if } 2\alpha \in R, \end{cases} \quad \text{and} \quad \beta' := \begin{cases} \beta & \text{if } 2\beta \notin R \\ 2\beta & \text{if } 2\beta \in R, \end{cases}$$

we have $\alpha', \beta' \in S_{sdiv}^{\times}$ and $\alpha' - \beta' \notin R$. Next we fix $e \in \mathcal{G}_{\alpha'}$ and $f \in \mathcal{G}_{-\alpha'}$ such that $(e, h := [e, f], f)$ is an $\mathfrak{sl}_2$-triple. Since $\alpha' - \beta' \notin R_{sdiv}$, one knows from $\mathfrak{sl}_2$-module theory that $\beta'(h) \leq 0$ while $\alpha'(h) = 2 > 0$. Therefore $h \in [\mathcal{G}_{\alpha'}, \mathcal{G}_{-\alpha'}] \subseteq \mathfrak{h}$, $\alpha(h) > 0$ and $\beta(h) \leq 0$. This completes the proof of the claim. Now suppose $\alpha, \beta \in S$ with $\pi(\alpha) = \pi(\beta)$. We must show $\alpha = \beta$. We prove this through the following three cases:

Case 1. $\alpha, \beta \in S_{sdiv}$: If $\gamma := \alpha - \beta \in R^{\times}$, then since S is a closed subsystem of R and S_{sdiv} is a closed subsystem of S, we get $\gamma \in S_{sdiv}^{\times}$. Thus there is $t \in [\mathcal{G}_{\gamma}, \mathcal{G}_{-\gamma}] \subseteq \mathfrak{h}$ with $\gamma(t) = 2$, so $(\alpha - \beta)(t) = 2$ which contradicts the fact that $\alpha \mid_{\mathfrak{h}} = \beta \mid_{\mathfrak{h}}$. Therefore $\alpha - \beta \notin R^{\times}$. Now if $\alpha - \beta \neq 0$, then $\alpha - \beta \notin R$ and so using (8), one finds $h \in \mathfrak{h}$ with $\alpha(h) > 0$ and $\beta(h) \leq 0$. This is again a contradiction. Therefore $\alpha = \beta$.

Case 2. $\alpha, \beta \notin S_{sdiv}$: In this case $2\alpha, 2\beta \in S_{sdiv}$ and so by Case 1, $2\alpha = 2\beta$ which in turn implies that $\alpha = \beta$.

Case 3. $\alpha \in S_{sdiv}, \beta \notin S_{sdiv}$: If $\alpha - \beta \notin R$, then by (8), there is $h \in \mathfrak{h}$ such that $\alpha(h) > 0$ and $\beta(h) \leq 0$ which contradicts the fact that $\pi(\alpha) = \pi(\beta)$. Also if $\alpha = 2\beta$, then since $\alpha \in S_{sdiv}^{\times}$, there is $h \in [\mathcal{G}_{\alpha}, \mathcal{G}_{-\alpha}] \subseteq \mathfrak{h}$ with $\alpha(h) = 2$. Thus $\alpha(h) \neq \beta(h)$ which is again a contradiction. Therefore $\alpha - \beta \in R$ and $\alpha \neq 2\beta$. Now if $\alpha - \beta \neq 0$, we get that $\alpha - \beta \in R_{sh}$, $\alpha \in R_{lg}$, $\beta \in R_{sh}$, $\gamma := \alpha - 2\beta \in R_{lg}$, $\alpha + \gamma, \alpha - \gamma \in R_{ex}$ and $\alpha + 2\gamma, \alpha - 2\gamma \notin R$. Now since $\gamma \in S_{sdiv}^{\times}$, there is $h \in [\mathcal{G}_{\gamma}, \mathcal{G}_{-\gamma}] \subseteq \mathfrak{h}$ with $\gamma(h) = 2$. Also since $\alpha + 2\gamma, \alpha - 2\gamma \notin R$, one concludes form $\mathfrak{sl}_2$-module theory that $\alpha(h) = 0$. So we have $\beta(h) = \alpha(h) = 0$. But this gives that $2 = \gamma(h) = (\alpha - 2\beta)(h) = 0$, a contradiction. Thus we have $\alpha = \beta$. This completes the proof of the first assertion.

For the last assertion, we note that S is a closed subsystem of R and S_{sdiv} is a closed subsystem of S, so it is easily seen that $\mathfrak{g}$ is a subalgebra of $\mathcal{G}$. This together with the fact that $\pi|_S$ is injective completes the proof. ∎

Definition 2.10. Take $\mathcal{G}$, Λ, $\mathcal{G}^{\lambda}$, and $\mathcal{H}^{\lambda}$ $(\lambda \in \Lambda)$ to be as in one of Lemmas 2.3, 2.5, 2.6 and 2.7. For $\lambda, \mu \in \Lambda$, we say $\lambda \preccurlyeq \mu$ if $\mathcal{G}^{\lambda}$ is a subalgebra of $\mathcal{G}^{\mu}$. Let χ be a representation of $\mathcal{G}$ in a vector space $\mathcal{M}$. We say $\mathcal{M}$ is a *direct limit $\mathcal{G}$-module with directed system* $\{\mathcal{M}^{\lambda} \mid \lambda \in \Lambda\}$ if

- for $\lambda \in \Lambda$, $\mathcal{M}^{\lambda}$ is a finite dimensional subspace of $\mathcal{M}$ and for all $x \in \mathcal{G}^{\lambda}$, $\mathcal{M}^{\lambda}$ is invariant under $\chi(x)$,
- for $\lambda \in \Lambda$, $\chi \mid_{\mathcal{G}^{\lambda}}$ defines a nontrivial finite dimensional irreducible $\mathcal{G}^{\lambda}$-module in $\mathcal{M}^{\lambda}$ having a weight space decomposition with respect to $\mathcal{H}^{\lambda}$ whose weight spaces corresponding to nonzero weights are one dimensional,
- for $\lambda, \mu \in \Lambda$ with $\lambda \preccurlyeq \mu$, $\mathcal{M}^{\lambda} \subseteq \mathcal{M}^{\mu} \subseteq \mathcal{M}$ and as a vector space, $\mathcal{M}$ is the direct union of $\{\mathcal{M}^{\lambda} \mid \lambda \in \Lambda\}$,
- for $\lambda, \mu \in \Lambda$ with $\lambda \preccurlyeq \mu$, the set of weights of $\mathcal{G}^{\lambda}$-module $\mathcal{M}^{\lambda}$ is contained in the set of weights of $\mathcal{G}^{\mu}$-module $\mathcal{M}^{\mu}$ restricted to $\mathcal{H}^{\lambda}$ and $(\mathcal{M}^{\lambda})_{p|_{\mathcal{H}^{\lambda}}} = (\mathcal{M}^{\mu})_p$ for each weight p of $\mathcal{M}^{\mu}$ for which $p|_{\mathcal{H}^{\lambda}}$ is a nonzero weight of $\mathcal{M}^{\lambda}$.

Using standard techniques, one can verify the following propositions:

Proposition 2.11. *Consider $\mathcal{G}$, Λ, $\mathcal{G}^\lambda$ and $\mathcal{H}^\lambda$ $(\lambda \in \Lambda)$ as in Definition 2.10. Suppose that $\mathcal{M}$ is a direct limit $\mathcal{G}$-module with directed system $\{\mathcal{M}^\lambda \mid \lambda \in \Lambda\}$. Then we have the followings:*

(i) $\mathcal{M}$ is an irreducible $\mathcal{G}$-module.

(ii) If $\mathcal{W}$ is another direct limit $\mathcal{G}$-module with directed system $\{\mathcal{W}^\lambda \mid \lambda \in \Lambda\}$, and for each $\lambda \in \Lambda$, $\mathcal{G}^\lambda$-modules $\mathcal{M}^\lambda$ and $\mathcal{W}^\lambda$ are isomorphic, then as two $\mathcal{G}$-modules, $\mathcal{M}$ and $\mathcal{W}$ are isomorphic.

Proposition 2.12. *Suppose that I is a nonempty index set.*

(a) Take $\mathcal{G} := \mathfrak{o}_B(I)$ and use the same notations as in Lemma 2.5. Define

$$\pi : \mathcal{G} \longrightarrow End(\mathcal{V});\ \pi(\phi)(v) = \phi(v);\ \ \phi \in \mathcal{G},\ v \in \mathcal{V},$$

then

(i) π is an irreducible representation of $\mathcal{G}$ in $\mathcal{V}$ equipped with a weight space decomposition with respect to $\mathcal{H}$ whose set of weights is $\{0, \pm\epsilon_i \mid i \in I\}$ with $\mathcal{V}_0 = \mathbb{F}v_0$, $\mathcal{V}_{\epsilon_i} = \mathbb{F}v_i$ and $\mathcal{V}_{-\epsilon_i} = \mathbb{F}v_{\bar{i}}$ for $i \in I$,

(ii) for each $\lambda \in \Lambda$, set

$$\mathcal{V}_{I_\lambda} := \mathcal{V}^\lambda := span_{\mathbb{F}}\{v_r \mid r \in \{0\} \cup I_\lambda \cup \bar{I}_\lambda\}, \tag{9}$$

then $\mathcal{V}$ is a direct limit $\mathcal{G}$-module with directed system $\{\mathcal{V}^\lambda \mid \lambda \in \Lambda\}$.

(b) Use the same notations as in Lemma 2.7 and take $\mathcal{G} := \mathfrak{sp}(I)$.

(i) Define

$$\pi_1 : \mathcal{G} \longrightarrow End(\mathcal{V});\ \pi(\phi)(v) := \phi(v);\ \ \phi \in \mathcal{G},\ v \in \mathcal{V}.$$

Then π_1 is an irreducible representation of $\mathcal{G}$ in $\mathcal{V}$ equipped with a weight space decomposition with respect to $\mathcal{H}$ whose set of weights is $\{\pm\epsilon_i \mid i \in I\}$ with $\mathcal{V}_{\epsilon_i} = \mathbb{F}v_i$ and $\mathcal{V}_{-\epsilon_i} = \mathbb{F}v_{\bar{i}}$ for $i \in I$. Also for $J := I \cup \bar{I}$ and

$$\mathcal{S} := \{\phi \in \mathfrak{gl}(J) \mid tr(\phi) = 0, (\phi(v), w) = (v, \phi(w)),\ for\ all\ v, w \in \mathcal{V}\}, \tag{10}$$

we have that

$$\pi_2 : \mathcal{G} \longrightarrow End(\mathcal{S});\ \pi_2(X)(Y) := [X, Y];\ \ X \in \mathcal{G},\ Y \in \mathcal{S}$$

is an irreducible representation of $\mathcal{G}$ in $\mathcal{S}$ equipped with a weight space decomposition with respect to $\mathcal{H}$ whose set of weights is $\{0, \pm(\epsilon_i \pm \epsilon_j) \mid i, j \in I,\ i \neq j\}$ with $\mathcal{S}_0 = span_{\mathbb{F}}\{e_{r,r} + e_{\bar{r},\bar{r}} - \frac{1}{|I_\lambda|}\sum_{i \in I_\lambda}(e_{i,i} + e_{\bar{i},\bar{i}}) \mid \lambda \in \Lambda, r \in I_\lambda\}$, $\mathcal{S}_{\epsilon_i+\epsilon_j} = \mathbb{F}(e_{i,\bar{j}} - e_{j,\bar{i}})$, $\mathcal{S}_{-\epsilon_i-\epsilon_j} = \mathbb{F}(e_{\bar{i},j} - e_{\bar{j},i})$ and $\mathcal{S}_{\epsilon_i-\epsilon_j} = \mathbb{F}(e_{i,j} + e_{\bar{j},\bar{i}})$ $(i, j \in I, i \neq j)$.

(ii) For $\lambda \in \Lambda$, set

$$\begin{aligned} \mathcal{V}_{I_\lambda} &:= \mathcal{V}^\lambda := span_{\mathbb{F}}\{v_r \mid r \in I_\lambda \cup \bar{I}_\lambda\}, \\ \mathcal{S}_{I_\lambda} &:= \mathcal{S}^\lambda := \mathcal{S} \cap span_{\mathbb{F}}\{e_{r,s} \mid r, s \in I_\lambda \cup \bar{I}_\lambda\}. \end{aligned} \tag{11}$$

Then $\mathcal{V}$ and $\mathcal{S}$ are direct limit $\mathcal{G}$-modules with directed systems $\{\mathcal{V}^\lambda \mid \lambda \in \Lambda\}$ and $\{\mathcal{S}^\lambda \mid \lambda \in \Lambda\}$ respectively.

2.2. Finite Dimensional Case. In this subsection, we state a proposition on representation theory of finite dimensional split simple Lie algebras. This proposition is an essential tool for the proof of our results in the next section. We start with an elementary but important fact about finite dimensional representations of a finite dimensional split semisimple Lie algebra.

Lemma 2.13. *Suppose that $\mathcal{G}$ is a finite dimensional split semisimple Lie algebra with a splitting Cartan subalgebra $\mathcal{H}$ and the root system R. Let $\mathcal{V}$ be a finite dimensional $\mathcal{G}$-module equipped with a weight space decomposition with respect to $\mathcal{H}$. Take Π to be the set of weights of $\mathcal{V}$. If $\alpha \in R^\times$ and $\lambda \in \Pi$ are such that $\alpha+\lambda \in \Pi$, then $\mathcal{G}_\alpha \cdot \mathcal{V}_\lambda \neq \{0\}$. In particular if $\mathcal{V}_{\lambda+\alpha}$ is one dimensional, then $\mathcal{G}_\alpha \cdot \mathcal{V}_\lambda = \mathcal{V}_{\lambda+\alpha}$.*

Proof. Take $e \in \mathcal{G}_\alpha$ and $f \in \mathcal{G}_{-\alpha}$ to be such that $(e, h := [e,f], f)$ is an $\mathfrak{sl}_2$-triple and define $\mathfrak{s} := \mathrm{span}_{\mathbb{F}}\{e,h,f\}$. Set $\mathcal{W} := \sum_{k=-\infty}^{\infty} \mathcal{V}_{\lambda+k\alpha}$, then $\mathcal{W}$ is a finite dimensional $\mathfrak{s}$-submodule and so by Weyl theorem, it is decomposed into finite dimensional irreducible $\mathfrak{s}$-submodules, say $\mathcal{W} = \oplus_{i=1}^{n} \mathcal{W}_i$ where n is a positive integer and $\mathcal{W}_i$, $1 \leq i \leq n$, is a finite dimensional irreducible $\mathfrak{s}$-module. We note that the set of weights of $\mathcal{W}$ with respect to $\mathbb{F}h$ is

$$\Pi' = \{\lambda(h) + 2k \mid k \in \mathbb{Z} \quad \text{and} \quad \lambda + k\alpha \in \Pi\}$$

and that for $k \in \mathbb{Z}$ with $\lambda + k\alpha \in \Pi$, $\mathcal{W}_{\lambda(h)+2k} = \mathcal{V}_{\lambda+k\alpha}$. Now as $\lambda, \lambda+\alpha \in \Pi$, we have $\lambda(h), \lambda(h)+2 \in \Pi'$ and so by $\mathfrak{sl}_2$-module theory, there is $1 \leq i \leq n$ such that $\lambda(h), \lambda(h)+2$ are weights for $\mathcal{W}_i$. Now again using $\mathfrak{sl}_2$-module theory, we get that

$$0 \neq e \cdot (\mathcal{W}_i)_{\lambda(h)} \subseteq e \cdot \mathcal{W}_{\lambda(h)} = e \cdot \mathcal{V}_\lambda \subseteq \mathcal{G}_\alpha \cdot \mathcal{V}_\lambda$$

showing that $\mathcal{G}_\alpha \cdot \mathcal{V}_\lambda \neq \{0\}$. This completes the proof. ∎

Lemma 2.14. *Suppose that $\{e_i, f_i, h_i \mid 1 \leq i \leq n\}$ is a set of Chevalley generators for a finite dimensional split simple Lie algebra $\mathcal{G}$ and $\mathcal{V}$ is a $\mathcal{G}$-module equipped with a weight space decomposition with respect to the Cartan subalgebra $\mathcal{H} := span\{h_i \mid 1 \leq i \leq n\}$. Let v be a weight vector, m be a positive integer and $1 \leq i, j_1, \ldots, j_m \leq n$. Let the set $\{k \in \{1,\ldots,m\} \mid j_k = i\}$ be a nonempty set and $k_1 < \cdots < k_p$ be such that $\{k \in \{1,\ldots,m\} \mid j_k = i\} = \{k_1,\ldots,k_p\}$. Then if $f_i \cdot v = 0$, we have*

$$f_i \cdot e_{j_m} \cdot \cdots \cdot e_{j_1} \cdot v \in \sum_{t=1}^{p} \mathbb{F} e_{j_m} \cdot \cdots e_{j_{k_p}} \cdot \cdots \cdot \widehat{e_{j_{k_t}}} \cdot \cdots \cdot e_{j_{k_1}} \cdot \cdots \cdot e_{j_1} \cdot v,$$

in which "$\widehat{\ }$" means omission.

Proof. Using induction on p, we are done. ∎

Proposition 2.15. *Let $\mathcal{G}_1$ be a finite dimensional split simple Lie algebra with a splitting Cartan subalgebra $\mathcal{H}_1$. Assume $R \subseteq \mathcal{H}_1^*$ is an irreducible finite root system and $\mathcal{G}_1 = \sum_{\alpha \in (R_1)_{sdiv}} (\mathcal{G}_1)_\alpha$ is the root space decomposition of $\mathcal{G}_1$ with respect to $\mathcal{H}_1$. Suppose that R_2 is an irreducible full subsystem of R_1 of rank greater that 1 and* *set* $\mathcal{G}_2 := \sum_{\alpha\in(R_2)^\times_{sdiv}}(\mathcal{G}_1)_\alpha \oplus \sum_{\alpha\in(R_2)^\times_{sdiv}}[(\mathcal{G}_1)_\alpha,(\mathcal{G}_1)_{-\alpha}]$ *as well as* $\mathcal{H}_2 := \mathcal{H}_1\cap\mathcal{G}_2$. *For* $i=1,2$, *assume* $\mathcal{V}_i$ *is a* $\mathcal{G}_i$*-module equipped with a weight space decomposition with respect to* $\mathcal{H}_i$ *and take* Λ_i $(i=1,2)$ *to be the set of wights of* $\mathcal{V}_i$ *with respect to* $\mathcal{H}_i$. *Suppose that*

(i) R_1 *and* R_2 *are of the same type* $X\neq G_2,F_4,E_{6,7,8}$,

(ii) $\Lambda_1\subseteq R_1$ *and* $\Lambda_2\subseteq\{\alpha|_{\mathcal{H}_2}\mid\alpha\in R_2\}$,

(iii) $\mathcal{V}_2\subseteq\mathcal{V}_1$ *with* $(\mathcal{V}_2)_{\alpha|_{\mathcal{H}_2}}\subseteq(\mathcal{V}_1)_\alpha$, *for* $\alpha\in\{\beta\in R_2\mid\beta|_{\mathcal{H}_2}\in\Lambda_2\}\setminus\{0\}$.

Let $\mathcal{W}$ *be a nontrivial finite dimensional irreducible* $\mathcal{G}_2$*-submodule of* $\mathcal{V}_2$ *and take* $\mathcal{U}$ *to be the* $\mathcal{G}_1$*-submodule of* $\mathcal{V}_1$ *generated by* $\mathcal{W}$, *then* $\mathcal{U}$ *is a finite dimensional irreducible* $\mathcal{G}_1$*-module equipped with a weight space decomposition with respect to* $\mathcal{H}_1$ *whose set of nonzero weights is* $(R_1)_{sh}$, *(resp.* $(R_1)^\times_{sdiv}$, *or* $((R_1)_{sdiv})_{sh}$*) if the set of nonzero weights of* $\mathcal{W}$ *is the set of elements of* $(R_2)_{sh}$ *(resp.* $(R_2)^\times_{sdiv}$, *or* $((R_2)_{sdiv})_{sh}$*) restricted to* $\mathcal{H}_2$.

Proof. Take $n:=n_1$ and $\ell:=n_2$ to be the rank of R_1 and R_2 respectively. Using Lemma 2.9, we identify $\beta\in R_2$ with $\beta|_{\mathcal{H}_2}$. Also without loss of generality, we assume R_1,R_2 and bases Δ_1,Δ_2 for $(R_1)_{sdiv}$ and $(R_2)_{sdiv}$ respectively, are as in the following tables:

Type	$R_k (k=1,2)$
A	$\{\pm(\varepsilon_i-\varepsilon_j)\mid 1\leq i<j\leq n_k+1\}\cup\{0\}$, $n_k\geq 2$
B	$\{\pm\varepsilon_i,\pm(\varepsilon_i\pm\varepsilon_j)\mid 1\leq i<j\leq n_k\}\cup\{0\}$, $n_k\geq 2$
C	$\{\pm 2\varepsilon_i,\pm(\varepsilon_i\pm\varepsilon_j)\mid 1\leq i<j\leq n_k\}\cup\{0\}$, $n_k\geq 3$
D	$\{\pm(\varepsilon_i\pm\varepsilon_j)\mid 1\leq i<j\leq n_k\}\cup\{0\}$, $n_k\geq 4$
BC	$\{\pm\varepsilon_i,\pm(\varepsilon_i\pm\varepsilon_j)\mid 1\leq i,j\leq n_k\}$, $n_k\geq 2$

Type	$\Delta_k (k=1,2)$
A	$\{\alpha_i:=\epsilon_{i+1}-\epsilon_i\mid 1\leq i\leq n_k\}$
B	$\{\alpha_1:=\epsilon_1,\alpha_i:=\epsilon_i-\epsilon_{i-1}\mid 2\leq i\leq n_k\}$
C	$\{\alpha_1:=2\epsilon_1,\alpha_i:=\epsilon_i-\epsilon_{i-1}\mid 2\leq i\leq n_k\}$
D	$\{\alpha_1:=\epsilon_1+\epsilon_2,\alpha_i:=\epsilon_i-\epsilon_{i-1}\mid 2\leq i\leq n_k\}$
BC	$\{\alpha_1:=2\epsilon_1,\alpha_i:=\epsilon_i-\epsilon_{i-1}\mid 2\leq i\leq n_k\}$

Suppose that $\mathcal{W}$ is of highest pair (v,α) with respect to Δ_2. Since the set of weights of $\mathcal{W}$ is permuted by the Weyl group of R_2, one gets that $\alpha=\alpha^2_*$ for $*\in\{sh,lg,ex\}$ where for $i=1,2$, α^i_{sh} (resp. α^i_{lg}, or α^i_{ex}) denotes the highest short (resp. long, or extra long) root of R_i with respect to Δ_i. Next suppose that $\{e_i,f_i,h_i\mid 1\leq i\leq n\}$ is a set of Chevalley generators for $\mathcal{G}_1$ with respect to Δ_1, then $\{e_i,f_i,h_i\mid 1\leq i\leq\ell\}$ is a set of Chevalley generators for $\mathcal{G}_2$ with respect to Δ_2. Now as $\mathcal{U}$ is a $\mathcal{G}_1$-submodule of $\mathcal{V}_1$ generated by v, we have

$$\mathcal{U}=\sum_{t,s\in\mathbb{N}}\mathbb{F}(f_{i_t}\cdot\dots\cdot f_{i_1}\cdot e_{j_s}\cdot\dots\cdot e_{j_1}\cdot v) \tag{12}$$

where $i_1,\ldots,i_t,j_1,\ldots,j_s\in\{1,\ldots,n\}$. This implies that $\mathcal{U}$ is finite dimensional as Λ_1 is a subset of the finite root system R_1. So there are a positive integer p and irreducible finite dimensional $\mathcal{G}_1$-submodules $\mathcal{U}_j$ $(1\leq j\leq p)$ of $\mathcal{U}$ such that $\mathcal{U}=\oplus_{j=1}^p\mathcal{U}_j$. But we know that v generates the $\mathcal{G}_1$-submodule $\mathcal{U}$, and that $v\in\mathcal{U}\cap(\mathcal{V}_2)_\alpha\subseteq\mathcal{U}\cap(\mathcal{V}_1)_\alpha=\mathcal{U}_\alpha=\oplus_{j=1}^p(\mathcal{U}_j)_\alpha$, so

$$\text{for any } 1 \le j \le p, \text{ there is a nonzero element } u_j \in (\mathcal{U}_j)_\alpha \text{ such that } v = \sum_{j=1}^p u_j. \tag{13}$$

This in particular implies that each $\mathcal{U}_j$ $(1 \le j \le p)$ is a nontrivial irreducible $\mathcal{G}_1$-module. But we know that for $1 \le j \le p$, the set of weights of $\mathcal{U}_j$ is a subset of R_1, and that it is permuted by the Weyl group of R_1, so the highest weight of $\mathcal{U}_j$ is α_*^1 for $* = sh, lg, ex$. Therefore using the finite dimensional theory, one knows that

$$\textit{the weight spaces of } \mathcal{U}_j \ (1 \le j \le p\ ) \textit{ with respect to } \mathcal{H}_1 \textit{ corresponding to nonzero weights are one dimensional.} \tag{14}$$

Now we are ready to proceed with the proof in the following three steps:

- Step 1. $\mathcal{U}_t$ $(1 \le t \le p)$ is a finite dimensional irreducible $\mathcal{G}_1$-module of highest weight α_*^1 if $\alpha = \alpha_*^2$ for $* = sh, lg, ex$.
- Step 2. $\dim(\mathcal{U}_\alpha) = 1$,
- Step 3. $p = 1$.

Step 1: We use a case-by-case argument to prove the desired point. We note that there is nothing to show if R_1 is of type A or D, and continue as following:

Type B : One can see that in this case, $\alpha_{sh}^1 = \epsilon_n$, $\alpha_{sh}^2 = \epsilon_\ell$, $\alpha_{lg}^1 = \epsilon_n + \epsilon_{n-1}$ and $\alpha_{lg}^2 = \epsilon_\ell + \epsilon_{\ell-1}$. We first assume $\alpha = \alpha_{sh}^2$ and show that the highest weight of $\mathcal{U}_t$ is the highest short root of R_1. For this, it is enough to show that no long root is a weight for $\mathcal{U}_t$. Suppose to the contrary that the set of weights of $\mathcal{U}_t$ contains a long root or equivalently contains all long roots. Setting $\beta := \epsilon_{\ell-1}$, we get that $\alpha + \beta$ is a long root of R_1 and so $\alpha + \beta$ is a weight for $\mathcal{U}_t$. Now fix $x \in (\mathcal{G}_2)_\beta = (\mathcal{G}_1)_\beta$ and note that $\alpha + \beta$ is a weight for $\mathcal{U}_t$. Applying Lemma 2.13 together with (13) and (14), we have $x \cdot u_t \ne 0$. This gives that $0 \ne \sum_{j=1}^p x \cdot u_j = x \cdot v \in (\mathcal{G}_2)_\beta \cdot \mathcal{W}_\alpha \subseteq \mathcal{W}_{\alpha+\beta}$ which is a contradiction as $\alpha + \beta$ is a long root and cannot be a weight for $\mathcal{W}$. Therefore $\mathcal{U}_t$ has no long root as a weight and so we are done in the case that $\alpha = \alpha_{sh}^2$. Next suppose $\alpha = \alpha_{lg}^2$ and note that by (13), u_t is a weight vector for $\mathcal{U}_t$ of weight α. Since α is a long root, the set of weights of $\mathcal{U}_t$ contains all long roots and so the highest long root is the highest weight of $\mathcal{U}_t$.

Type C : In this case, we have $\alpha_{sh}^1 = \epsilon_{n-1} + \epsilon_n$, $\alpha_{sh}^2 = \epsilon_{\ell-1} + \epsilon_\ell$, $\alpha_{lg}^1 = 2\epsilon_n$ and $\alpha_{lg}^2 = 2\epsilon_\ell$. Setting $\beta := \epsilon_{\ell-1} - \epsilon_\ell$, we get $\alpha_{sh}^2 + \beta \in (R_1)_{lg}$. Now using the same argument as in Type B, we are done.

Type BC : In this case, we have $\alpha_{sh}^1 = \epsilon_n$, $\alpha_{sh}^2 = \epsilon_\ell$, $\alpha_{lg}^1 = \epsilon_{n-1} + \epsilon_n$, $\alpha_{lg}^2 = \epsilon_{\ell-1} + \epsilon_\ell$, $\alpha_{ex}^1 = 2\epsilon_n$ and $\alpha_{ex}^2 = 2\epsilon_\ell$. One can also easily see that

$$\{\gamma + \beta \mid \gamma \in (R_1)_{sh}, \beta \in (R_1)_{lg} \cup (R_1)_{ex}\} \cap R_1 \subseteq (R_1)_{sh}.$$

This together with (12), (13) and the fact that $\Delta_1 \subseteq (R_1)_{ex} \cup (R_1)_{lg}$ proves the claim stated in Step 1 in the case that $\alpha = \alpha_{sh}^2$. Now suppose that $\alpha = \alpha_{lg}^2$. Setting

$\beta := \epsilon_{\ell-1} - \epsilon_\ell$, we get that $\alpha+\beta$ is an extra long root. Now we are done using the same argument as in Type B. Next suppose $\alpha = \alpha_{ex}^2$, then by (13), u_t is a weight vector of weight α which is an extra long root. Therefore any extra long root is a weight for $\mathcal{U}_t$ and so the highest weight of $\mathcal{U}_t$ is α_{ex}^1. This completes the proof of Step 1.

Step 2: We first note that depending on the type of R_2, α is one of $\epsilon_\ell, 2\epsilon_\ell, \epsilon_\ell+\epsilon_{\ell-1}$ or $\epsilon_{\ell+1}-\epsilon_1$. If $\alpha = \epsilon_\ell+\epsilon_{\ell-1}$, then either R_2 is of type B or D and by Step 1, the set of nonzero weights of $\mathcal{U}$ coincides with $R^\times$, or R is of type C or BC and the set of nonzero weights of $\mathcal{U}$ coincides with $(R_{sdiv})_{sh}$. In both cases, using induction on $r \in \mathbb{N} \setminus \{0\}$, one can see that

$$\text{if } 1 \le m_1, \dots, m_r \le n \text{ and for each } 1 \le p \le r,\ \alpha_{m_p} + \cdots + \alpha_{m_1} + \alpha \text{ is a weight for } \mathcal{U}, \text{ then } \{m_1, \dots, m_r\} \subseteq \{\ell, \dots, n\} \text{ and } \alpha_{m_r} + \cdots + \alpha_{m_1} + \alpha = \epsilon_q + \epsilon_{q'} \text{ for some } \ell - 1 \le q \ne q' \le n. \tag{15}$$

Also for $\alpha = \epsilon_{\ell+1} - \epsilon_1, \epsilon_\ell, 2\epsilon_\ell$, one can see that

$$\text{if } r \text{ is a positive integer and } 1 \le m_1, \dots, m_r \le n \text{ are such that for each } 1 \le p \le r,\ \alpha_{m_p} + \cdots + \alpha_{m_1} + \alpha \text{ is a weight for } \mathcal{U}, \text{ then } \{m_1, \dots, m_r\} \subseteq \{\ell+1, \dots, n\}. \tag{16}$$

Now suppose that $0 \ne u \in \mathcal{U}_\alpha$, we shall show that u is a scalar multiple of v. Since $u \in \mathcal{U}$, by (12), u is written as a linear combination of weight vectors of the form $f_{i_t} \cdot \cdots \cdot f_{i_1} \cdot e_{j_s} \cdot \cdots \cdot e_{j_1} \cdot v$, $t, s \in \mathbb{N}$, $1 \le i_1, \dots, i_t, j_1, \dots, j_s \le n$. So without loss of generality, we suppose

$$u = f_{i_t} \cdot \cdots \cdot f_{i_1} \cdot e_{j_s} \cdot \cdots \cdot e_{j_1} \cdot v$$

where $t, s \in \mathbb{N}$, and $1 \le i_1, \dots, i_t, j_1, \dots, j_s \le n$. Since u is of weight α, we get that $\alpha + \alpha_{j_1} + \cdots + \alpha_{j_s} - \alpha_{i_1} - \cdots - \alpha_{i_t} = \alpha$. This implies that

$$s = t \quad \text{and} \quad (j_1, \dots, j_s) = (\sigma(i_1), \dots, \sigma(i_t)) \tag{17}$$

for a permutation σ of $\{i_1, \dots, i_t\}$. We note that α is an element of R_2^+ and so it is written as a linear combination of $\{\alpha_i \mid 1 \le i \le \ell\}$ with nonnegative rational coefficients not all equal to zero. Now since $\{\alpha_i \mid 1 \le i \le n\}$ is a base of $(R_1)_{sdiv}$, $\alpha - \alpha_j$ $(\ell+1 \le j \le n)$ is not a root of R_1 and so it is not an element of Λ_1. Therefore

$$f_j \cdot v = 0, \quad \ell+1 \le j \le n. \tag{18}$$

Now this implies that

$$f_j \cdot e_j \cdot v = e_j \cdot f_j \cdot v - h_j \cdot v = 0 - h_j \cdot v \in \mathbb{F}v, \quad (\ell+1 \le j \le n). \tag{19}$$

We also note that as v is a highest vector of $\mathcal{G}_2$-module $\mathcal{W}$, $e_j \cdot v = 0$ for $1 \le j \le \ell$. Therefore one gets that

$$j_1 \in \{\ell+1, \dots, n\} \text{ provided that } s \ne 0. \tag{20}$$

Now we are ready to prove that u is a scalar multiple of v. If $s=0$, there is nothing to prove. So we suppose $s\geq 1$ and use induction on s to prove. If $s=1$, we get the result appealing (17), (20) and (19). Now suppose $s>1$. If $\alpha=\epsilon_\ell+\epsilon_{\ell-1}$, then using (15) together with (17), we get that $i_1\in\{\ell,\ldots,n\}$. This together with (18) and the fact that $2\epsilon_{\ell-1}$ is not a weight for $\mathcal{U}$, implies that $f_{i_1}\cdot v=0$. Next take $1\leq k_1<\ldots<k_r\leq s$ to be the only indices with $j_{k_1}=\cdots=j_{k_r}=i_1$ and use Lemma 2.14 to get

$$f_{i_t}\cdot\cdots\cdot f_{i_1}\cdot e_{j_s}\cdot\cdots\cdot e_{j_1}\cdot v\in\sum_{q=1}^{r}\mathbb{F}f_{i_t}\cdot\cdots\cdot f_{i_2}\cdot e_{j_s}\cdot\cdots\cdot e_{j_{k_r}}\cdot\cdots\cdot\widehat{e_{j_{k_q}}}\cdot\cdots\cdot e_{j_{k_1}}\cdot\cdots\cdot e_{j_1}\cdot v.$$

Now induction hypothesis completes the proof of this step in the case that $\alpha=\epsilon_{\ell-1}+\epsilon_\ell$. Next suppose $\alpha\in\{\epsilon_\ell,2\epsilon_\ell,\epsilon_{\ell+1}-\epsilon_1\}$, then using (16) together with (17), (18), Lemma 2.14 and the same argument as before, we get the result.

Step 3: It is immediate using Step 2 together with (13). ■

3. Lie algebras graded by a finite root system

The structure of Lie algebras graded by an irreducible finite root system has been studied in [7], [5], [2], [11], [3] and [6]. A Lie algebra $\mathcal{L}$ graded by an irreducible finite root system R contains a finite dimensional split simple Lie algebra $\mathcal{G}$ and with respect to a splitting Cartan subalgebra, it is equipped with a weight space decomposition whose set of weights is contained in R. This feature allows us to decompose $\mathcal{L}$ into finite dimensional irreducible $\mathcal{G}$-submodules whose set of nonzero weights is R_{sh}, $R^\times_{sdiv}$ or $(R_{sdiv})_{sh}$. Collecting the components of the same highest weight results in the decomposition

$$\mathcal{L}=(\mathcal{G}\otimes\mathcal{A})\oplus(\mathcal{S}\otimes\mathcal{B})\oplus(\mathcal{V}\otimes\mathcal{C})\oplus\mathcal{D}\tag{21}$$

in which $\mathcal{D}$ is a trivial submodule of $\mathcal{L}$ and $\mathcal{S}$ (resp. $\mathcal{V}$) is the finite dimensional irreducible $\mathcal{G}$-module whose set of nonzero weights is $(R_{sdiv})_{sh}$ (resp. R_{sh}). The Lie algebraic structure on $\mathcal{L}$ induces an algebraic structure on $\mathfrak{b}:=\mathcal{A}\oplus\mathcal{B}\oplus\mathcal{C}$ which we refer to as the *coordinate algebra* of $\mathcal{L}$. The Lie bracket on $\mathcal{L}$ can be recovered using the ingredients involved in describing the product defined on the algebra $\mathfrak{b}$. In this section, we have a comparison view on the coordinate algebras of root graded subalgebras of $\mathcal{L}$. We devote this section to two subsections. In the first subsection, we illustrate the structure of a specific Lie algebra which we shall frequently use in the sequel of the paper. In the second subsection, we consider a Lie algebra $\mathcal{L}$ graded by an irreducible finite root system R and for a full irreducible subsystem S of R which is of the same type as R, we take $\mathcal{L}^S$ to be the Lie subalgebra of $\mathcal{L}$ generated by homogeneous spaces in correspondence to $S^\times$. We show that the coordinate algebra of the Lie subalgebra $\mathcal{L}^S$, which is an algebra graded by S, does not depend on S. In fact, we prove that the coordinate algebra of $\mathcal{L}^S$ coincides with the coordinate algebra of $\mathcal{L}$. Moreover, we describe the Lie bracket on $\mathcal{L}$ in terms of the ingredients involved in describing the Lie bracket on $\mathcal{L}^S$ with respect to its coordinate algebra. Our method is based on a type-by-type approach. Since the proofs for different types are quite similar, we go through the proofs in details for type BC and for other types, we just report the results and leave the proofs to the readers. In the sequel of the paper, if $h : X \longrightarrow Y$ is a map and $x \in X$, we sometimes denote by x^h, the image of x under h.

3.1. A Specific Lie Algebra. By a *star algebra* $(\mathfrak{A}, \star)$, we mean an algebra $\mathfrak{A}$ together with a self-inverting antiautomorphism $\star$ which is referred to as an *involution.*

We call a quadruple $(\mathfrak{a}, *, \mathcal{C}, f)$, a *coordinate quadruple* if one of the followings holds:

- (Type A) $\mathfrak{a}$ is a unital associative algebra, $* = id_{\mathfrak{a}}$, $\mathcal{C} = \{0\}$ and $f\colon \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is the zero map.
- (Type B) $\mathfrak{a} = \mathcal{A} \oplus \mathcal{B}$ where $\mathcal{A}$ is a unital commutative associative algebra and $\mathcal{B}$ is a unital associative $\mathcal{A}$-module equipped with a symmetric bilinear form and $\mathfrak{a}$ is the corresponding Clifford Jordan algebra, $*$ is a linear transformation fixing the elements of $\mathcal{A}$ and skew fixing the elements of $\mathcal{B}$, $\mathcal{C} = \{0\}$ and $f : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is the zero map.
- (Type C) $\mathfrak{a}$ is a unital associative algebra, $*$ is an involution on $\mathfrak{a}$, $\mathcal{C} = \{0\}$ and $f : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is the zero map.
- (Type D) $\mathfrak{a}$ is a unital commutative associative algebra $* = id_{\mathfrak{a}}$, $\mathcal{C} = \{0\}$ and $f : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is the zero map.
- (Type BC) $\mathfrak{a}$ is a unital associative algebra, $*$ is an involution on $\mathfrak{a}$, $\mathcal{C}$ is a unital associative $\mathfrak{a}$-module and $f : \mathcal{C} \times \mathcal{C} \longrightarrow \mathfrak{a}$ is a skew-hermitian form.

Suppose that $\mathfrak{q} := (\mathfrak{a}, *, \mathcal{C}, f)$ is a coordinate quadruple. Denote by $\mathcal{A}$ and $\mathcal{B}$, the fixed and the skew fixed points of $\mathfrak{a}$ under $*$, respectively. Set $\mathfrak{b} := \mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f) := \mathfrak{a} \oplus \mathcal{C}$ and define

$$\begin{gathered} \cdot : \mathfrak{b} \times \mathfrak{b} \longrightarrow \mathfrak{b} \\ (\alpha_1 + c_1, \alpha_2 + c_2) \mapsto (\alpha_1 \cdot \alpha_2) + f(c_1, c_2) + \alpha_1 \cdot c_2 + \alpha_2^* \cdot c_1, \end{gathered} \tag{22}$$

for $\alpha_1, \alpha_2 \in \mathfrak{a}$ and $c_1, c_2 \in \mathcal{C}$. Also for $\beta, \beta' \in \mathfrak{b}$, set

$$\beta \circ \beta' := \beta \cdot \beta' + \beta' \cdot \beta \quad \text{and} \quad [\beta, \beta'] := \beta \cdot \beta' - \beta' \cdot \beta, \tag{23}$$

and for $c, c' \in \mathcal{C}$, define

$$\begin{aligned} &\diamond : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{A}, \quad (c, c') \mapsto \tfrac{f(c,c') - f(c',c)}{2}; \; c, c' \in \mathcal{C}, \\ &\heartsuit : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{B}, \quad (c, c') \mapsto \tfrac{f(c,c') + f(c',c)}{2}; \; c, c' \in \mathcal{C}. \end{aligned} \tag{24}$$

Now suppose that ℓ is a positive integer and for $\alpha, \alpha' \in \mathfrak{a}$ and $c, c' \in \mathcal{C}$, consider the following endomorphisms

$$
\begin{aligned}
&d^{\ell,\mathfrak{b}}_{\alpha,\alpha'}:\mathfrak{b}\longrightarrow\mathfrak{b},\\
&\beta\mapsto\begin{cases}\frac{1}{\ell+1}[[\alpha,\alpha'],\beta] & \mathfrak{q}\text{ is of type }A,\ \beta\in\mathfrak{b},\\ \alpha'(\alpha\beta)-\alpha(\alpha'\beta) & \mathfrak{q}\text{ is of type }B,\ \beta\in\mathfrak{b},\\ \frac{1}{4\ell}[[\alpha,\alpha']+[\alpha^*,\alpha'^*],\beta] & \mathfrak{q}\text{ is of type }C\text{ or }BC,\ \beta\in\mathfrak{a},\\ \frac{1}{4\ell}([\alpha,\alpha']+[\alpha^*,\alpha'^*])\cdot\beta & \mathfrak{q}\text{ is of type }C\text{ or }BC,\ \beta\in\mathcal{C},\\ 0 & \mathfrak{q}\text{ is of type }D,\ \beta\in\mathfrak{b},\end{cases}\\
&d^{\ell,\mathfrak{b}}_{c,c'}:\mathfrak{b}\longrightarrow\mathfrak{b},\\
&\beta\mapsto\begin{cases}\frac{-1}{2\ell}[c\heartsuit c',\beta] & \mathfrak{q}\text{ is of type }BC,\ \beta\in\mathfrak{a},\\ \frac{-1}{2\ell}(c\heartsuit c')\cdot\beta-\frac{1}{2}(f(\beta,c')\cdot c+f(\beta,c)\cdot c') & \mathfrak{q}\text{ is of type }BC,\ \beta\in\mathcal{C},\\ 0 & \text{otherwise},\end{cases}\\
&d^{\ell,\mathfrak{b}}_{\alpha,c}:=d^{\ell,\mathfrak{b}}_{c,\alpha}:=0,\\
&d^{\ell,\mathfrak{b}}_{\alpha+c,\alpha'+c'}:=d^{\ell,\mathfrak{b}}_{\alpha,\alpha'}+d^{\ell,\mathfrak{b}}_{c,c'}.
\end{aligned}
\tag{25}
$$

One can see that for $\beta,\beta'\in\mathfrak{b}$, $d^{\ell,\mathfrak{b}}_{\beta,\beta'}\in Der(\mathfrak{b})$. Next take K to be a subspace of $\mathfrak{b}\otimes\mathfrak{b}$ spanned by

$$
\begin{aligned}
&\alpha\otimes c,\quad c\otimes\alpha,\quad a\otimes b,\\
&\alpha\otimes\alpha'+\alpha'\otimes\alpha,\quad c\otimes c'-c'\otimes c,\\
&(\alpha\cdot\alpha')\otimes\alpha''+(\alpha''\cdot\alpha)\otimes\alpha'+(\alpha'\cdot\alpha'')\otimes\alpha,\\
&f(c,c')\otimes\alpha+(\alpha^*\cdot c')\otimes c-(\alpha\cdot c)\otimes c'
\end{aligned}
$$

for $\alpha,\alpha',\alpha''\in\mathfrak{a}$, $a\in\mathcal{A}$, $b\in\mathcal{B}$, and $c,c'\in\mathcal{C}$. Then $(\mathfrak{b}\otimes\mathfrak{b})/K$ is a Lie algebra under the following Lie bracket

$$[(\beta_1\otimes\beta_2)+K,(\beta_1'\otimes\beta_2')+K]_\ell:=((d^{\ell,\mathfrak{b}}_{\beta_1,\beta_2}(\beta_1')\otimes\beta_2')+K)+(\beta_1'\otimes d^{\ell,\mathfrak{b}}_{\beta_1,\beta_2}(\beta_2'))+K)\tag{26}$$

for $\beta_1,\beta_2,\beta_1',\beta_2'\in\mathfrak{b}$ (see [3, Proposition 5.23] and [2]). We denote this Lie algebra by $\{\mathfrak{b},\mathfrak{b}\}_\ell$ (or $\{\mathfrak{b},\mathfrak{b}\}$ if there is no confusion) and for $\beta_1,\beta_2\in\mathfrak{b}$, we denote $(\beta_1\otimes\beta_2)+K$ by $\{\beta_1,\beta_2\}_\ell$ (or $\{\beta_1,\beta_2\}$ if there is no confusion). We recall the *full skew-dihedral homology group*

$$HF(\mathfrak{b}):=\{\sum_{i=1}^n\{\beta_i,\beta_i'\}_\ell\in\{\mathfrak{b},\mathfrak{b}\}_\ell\mid\sum_{i=1}^n d^{\ell,\mathfrak{b}}_{\beta_i,\beta_i'}=0\}$$

of $\mathfrak{b}$ (with respect to ℓ) from [3] and [2] and note that it is a subset of the center of $\{\mathfrak{b},\mathfrak{b}\}_\ell$. For $\beta_1=a_1+b_1+c_1\in\mathfrak{b}$ and $\beta_2=a_2+b_2+c_2\in\mathfrak{b}$ with $a_1,a_2\in\mathcal{A}$, $b_1,b_2\in\mathcal{B}$ and $c_1,c_2\in\mathcal{C}$, set

$$\beta^*_{\beta_1,\beta_2}:=[a_1,a_2]+[b_1,b_2]-c_1\heartsuit c_2.\tag{27}$$

We say a subset $\mathcal{K}$ of the full skew-dihedral homology group of $\mathfrak{b}$ satisfies the "*uniform property on* $\mathfrak{b}$" if for $\beta_1,\beta_1',\ldots,\beta_n,\beta_n'\in\mathfrak{b}$, $\sum_{i=1}^n\{\beta_i,\beta_i'\}_\ell\in\mathcal{K}$ implies that $\sum_{i=1}^n\beta^*_{\beta_i,\beta_i'}=0$.

Remark 3.1. Suppose that ℓ, ℓ' are two positive integers. If $\mathcal{K}$ is a subset of the full skew-dihedral homology group of $\mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f)$ with respect to ℓ satisfying the uniform property on $\mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f)$, then it is a subset of the full skew-dihedral homology group of $\mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f)$ with respect to ℓ' satisfying the uniform property on $\mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f)$. In other words, the uniform property on $\mathfrak{b}(\mathfrak{a}, *, \mathcal{C}, f)$ dose not depend on ℓ.

3.2. Root Graded Lie Algebras. In this work, we study root graded Lie algebras in the following sense:

Definition 3.2. Suppose that R is an irreducible locally finite root system. We say a Lie algebra $\mathcal{L}$ is *R-graded* or *(graded by R)* with *grading pair* $(\mathcal{G}, \mathcal{H})$ if the followings are satisfied:

i) $\mathcal{G}$ is a locally finite split simple Lie subalgebra of $\mathcal{L}$ with splitting Cartan subalgebra $\mathcal{H}$ and corresponding root system R_{sdiv}.

ii) $\mathcal{L}$ has a weight space decomposition $\mathcal{L} = \oplus_{\alpha \in R} \mathcal{L}_\alpha$ with respect to $\mathcal{H}$ via the adjoint representation.

iii) $\mathcal{L}_0 = \sum_{\alpha \in R^\times} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}]$.

The following lemma easily follows from Lemma 2.9.

Lemma 3.3. *Suppose that R is an irreducible locally finite root system and $\mathcal{L}$ is a Lie algebra graded by R with grading pair $(\mathcal{G}, \mathcal{H})$. Let S be an irreducible full subsystem of R and set*

$$\begin{aligned} \mathcal{L}^{S} &:= \sum_{\alpha \in S^\times} \mathcal{L}_\alpha \oplus \sum_{\alpha \in S^\times} [\mathcal{L}_\alpha, \mathcal{L}_{-\alpha}], \\ \mathcal{G}^{S} &:= \sum_{\alpha \in S^\times_{sdiv}} \mathcal{G}_\alpha \oplus \sum_{\alpha \in S^\times_{sdiv}} [\mathcal{G}_\alpha, \mathcal{G}_{-\alpha}]. \end{aligned} \tag{28}$$

Then $\mathcal{L}^{S}$ is an S-graded Lie subalgebra of $\mathcal{L}$ with grading pair $(\mathcal{G}^{S}, \mathcal{H}^{S} := \mathcal{H} \cap \mathcal{G}^{S})$.

Before going through the main body of this subsection, we want to fix a notation. If X is a subspace of a vector space V_1 and Y is a subspace of a vector space V_2, by a conventional notation, we take $X \dot{\otimes} Y$ to be the vector subspace of $V_1 \otimes V_2$ spanned by $x \otimes y$ for $x \in X$ and $y \in Y$.

3.0.1. Type BC

We first mention that in what follows, we introduce various notations which we use freely throughout the text without specific mention. Suppose that I is a nonempty index set of cardinality $m_n := n > 3$ and I_0 is a nonempty subset of I of cardinality $m_\ell := \ell > 3$. Take $\mathcal{V} := \mathcal{V}^n$ to be a vector space with a basis $\{v_i \mid i \in I \cup \bar{I}\}$ equipped with a nondegenerate symmetric bilinear form $(\cdot, \cdot)$ as in (6). Set $\mathcal{G} := \mathcal{G}^n := \mathfrak{sp}(I)$ and take $\mathcal{S} := \mathcal{S}^n$ to be as in (10). Consider (11), (7) and set

$$\mathcal{V}^\ell := \mathcal{V}_{I_0}, \;\; \mathcal{G}^\ell := \mathcal{G}_{I_0}, \;\; \mathcal{S}^\ell := \mathcal{S}_{I_0}.$$

We take $Id_{\mathcal{V}}$ to be the identity map on $\mathcal{V}$ and define the linear endomorphism $Id_{\mathcal{V}^\ell}$ on $\mathcal{V}$ by

$$\begin{gathered} Id_{\mathcal{V}^\ell}:\mathcal{V}\longrightarrow\mathcal{V} \\ v_i\mapsto v_i,\ v_{\bar{i}}\mapsto v_{\bar{i}},\ v_j\mapsto 0,\ v_{\bar{j}}\mapsto 0;\ \ (i\in I_0,\ j\in I\setminus I_0). \end{gathered}$$

Also for $\lambda=\ell,n$ and $x,y\in\mathcal{G}^\lambda\cup\mathcal{S}^\lambda$, set

$$x\circ_\lambda y:=xy+yx-(1/m_\lambda)tr(xy)Id_{\mathcal{V}^\lambda}. \tag{29}$$

Next for $u,v\in\mathcal{V}$, define

$$\begin{aligned} &[u,v]:\mathcal{V}\longrightarrow\mathcal{V};\ w\mapsto \tfrac{1}{2}((v,w)u+(w,u)v)+\tfrac{1}{2\ell}(u,v)Id_{\mathcal{V}^\ell}(w);\ \ w\in\mathcal{V},\\ &u\circ v:\mathcal{V}\longrightarrow\mathcal{V};\ w\mapsto \tfrac{1}{2}((v,w)u+(u,w)v);\ \ w\in\mathcal{V},\\ &[u,v]_n:\mathcal{V}\longrightarrow\mathcal{V};\ w\mapsto \tfrac{1}{2}((v,w)u+(w,u)v)+\tfrac{1}{2n}(u,v)Id_{\mathcal{V}}(w);\ \ w\in\mathcal{V}. \end{aligned} \tag{30}$$

One can easily see that up to isomorphism

$$\begin{aligned} &\mathcal{G}^\ell=\mathrm{span}\{u\circ v\mid u,v\in\mathcal{V}^\ell\},\quad \mathcal{S}^\ell=\mathrm{span}\{[u,v]\mid u,v\in\mathcal{V}^\ell\},\\ &\mathcal{G}^n=\mathrm{span}\{u\circ v\mid u,v\in\mathcal{V}^n\},\quad \mathcal{S}^n=\mathrm{span}\{[u,v]_n\mid u,v\in\mathcal{V}^n\}. \end{aligned}$$

Suppose that R is an irreducible finite root system of type BC_I and S is the irreducible full subsystem of R of type BC_{I_0}. Suppose that $\mathcal{L}$ is an R-graded Lie algebra with grading pair $(\mathcal{G},\mathcal{H})$ and take $\mathcal{L}^{^S}$, $\mathcal{G}^{^S}$ and $\mathcal{H}^{^S}$ to be as in Lemma 3.3. To simplicity of the notations, we set

$$\mathcal{L}^n:=\mathcal{L},\ \ \mathcal{L}^\ell:=\mathcal{L}^{^S},\ \ [u,v]_\ell:=[u,v];\ \ (u,v\in\mathcal{V}) \tag{31}$$

and note that $\mathcal{G}^{^S}=\mathcal{G}^\ell$. One knows that as a $\mathcal{G}^\ell$-module, $\mathcal{L}^\ell$ can be decomposed into finite dimensional $\mathcal{G}^\ell$-submodules, each of which is an irreducible $\mathcal{G}^\ell$-module with highest weight contained in S. Take

$$\mathcal{L}^\ell=\mathcal{L}^{^S}=\bigoplus_{i\in\mathcal{I}_0}\mathfrak{g}_i\oplus\bigoplus_{j\in\mathcal{J}_0}\mathfrak{s}_j\oplus\bigoplus_{t\in\mathcal{T}_0}V_t\oplus E \tag{32}$$

to be the decomposition of $\mathcal{L}^\ell$ into finite dimensional irreducible $\mathcal{G}^\ell$-modules in which $\mathcal{I}_0,\mathcal{J}_0,\mathcal{T}_0$ are (possibly empty) index sets and for $i\in\mathcal{I}_0$, $j\in\mathcal{J}_0$, and $t\in\mathcal{T}_0$, $\mathfrak{g}_i$ is isomorphic to $\mathcal{G}^\ell$, $\mathfrak{s}_j$ is isomorphic to $\mathcal{S}^\ell$, V_t is isomorphic to $\mathcal{V}^\ell$ and E is a trivial $\mathcal{G}^\ell$-submodule.

Lemma 3.4. *Use the notation as in the text and consider $\mathcal{L}=\mathcal{L}^n$ as a $\mathcal{G}$-module. Then there exist index sets $\mathcal{I}$, $\mathcal{J}$, $\mathcal{T}$ with $\mathcal{I}_0\subseteq\mathcal{I},\mathcal{J}_0\subseteq\mathcal{J},\mathcal{T}_0\subseteq\mathcal{T}$, and a class $\{\mathcal{D}_n,\mathcal{G}_i,\mathcal{S}_j,\mathcal{V}_t\mid i\in\mathcal{I},j\in\mathcal{J},t\in\mathcal{T}\}$ of finite dimensional $\mathcal{G}$-submodules of $\mathcal{L}$ such that*

- *$\mathcal{D}_n$ is a trivial $\mathcal{G}$-module, $\mathcal{G}_i$ is isomorphic to $\mathcal{G}$, $\mathcal{S}_j$ is isomorphic to $\mathcal{S}$, and $\mathcal{V}_t$ is isomorphic to $\mathcal{V}$, for $i\in\mathcal{I},j\in\mathcal{J},\ t\in\mathcal{T}$,*

- $\mathfrak{g}_i \subseteq \mathcal{G}_i,\ \mathfrak{s}_j \subseteq \mathcal{S}_j,\ V_t \subseteq \mathcal{V}_t\ (i \in \mathcal{I}_0,\ j \in \mathcal{J}_0,\ t \in \mathcal{T}_0),$

- $\mathcal{L}^n = \mathcal{L} = \bigoplus_{i\in\mathcal{I}} \mathcal{G}_i \oplus \bigoplus_{j\in\mathcal{J}} \mathcal{S}_j \oplus \bigoplus_{t\in\mathcal{T}} \mathcal{V}_t \oplus \mathcal{D}_n.$

We make a convention that we refer to

$$(\mathcal{I}, \mathcal{J}, \mathcal{T}, \{\mathfrak{g}_i\}, \{\mathcal{G}_i\}, \{\mathfrak{s}_j\}, \{\mathcal{S}_j\}, \{V_t\}, \{\mathcal{V}_t\}, E, \mathcal{D}_n) \tag{33}$$

as an $(R, S)-$ datum for the pair $(\mathcal{L}^n, \mathcal{L}^\ell)$.

Proof. For $i \in \mathcal{I}_0$, by Proposition 2.15, the $\mathcal{G}$-submodule $\mathcal{G}_i$ of $\mathcal{L}$ generated by $\mathfrak{g}_i$ is a finite dimensional $\mathcal{G}$-module isomorphic to $\mathcal{G}$. For $\alpha \in S^\times_{sdiv}$ and $0 \neq x \in (\mathfrak{g}_i)_\alpha \subseteq \mathcal{L}_\alpha$, we have $x \in \mathcal{L}_\alpha \cap \mathfrak{g}_i \subseteq \mathcal{L}_\alpha \cap \mathcal{G}_i = (\mathcal{G}_i)_\alpha$. Now as $\dim(\mathcal{G}_i)_\alpha = \dim(\mathfrak{g}_i)_\alpha = 1$, we get

$$(\mathfrak{g}_i)_\alpha = (\mathcal{G}_i)_\alpha; \quad \alpha \in S^\times_{sdiv}. \tag{34}$$

Claim 1. The sum $\sum_{i\in\mathcal{I}_0} \mathcal{G}_i$ is a direct sum: Suppose that $i_0, i_1, \dots, i_n$ are distinct elements of $\mathcal{I}_0$ and $0 \neq x \in \mathcal{G}_{i_0} \cap \sum_{t=1}^n \mathcal{G}_{i_t}$. Then as $\mathcal{G}_{i_0}$ is an irreducible $\mathcal{G}$-module, we get that

$$\mathcal{G}_{i_0} \subseteq \sum_{t=1}^n \mathcal{G}_{i_t}.$$

This together with (34) implies that for $\alpha \in S^\times_{sdiv} \subseteq R^\times_{sdiv}$,

$$(\mathfrak{g}_{i_0})_\alpha = (\mathcal{G}_{i_0})_\alpha \subseteq \sum_{t=1}^n (\mathcal{G}_{i_t})_\alpha = \sum_{t=1}^n (\mathfrak{g}_{i_t})_\alpha$$

which contradicts the fact that $\sum_{i\in\mathcal{I}_0} \mathfrak{g}_i$ is direct. This completes the proof of Claim 1.

Now for $j \in \mathcal{J}_0$ and $t \in \mathcal{T}_0$, take $\mathcal{S}_j$ and $\mathcal{V}_t$ to be the finite dimensional irreducible $\mathcal{G}$-submodules of $\mathcal{L}$ generated by $\mathfrak{s}_j$ and V_t respectively. Using the same argument as above, one can see that the summations $\sum_{j\in\mathcal{J}_0} \mathcal{S}_j$ and $\sum_{t\in\mathcal{T}_0} \mathcal{V}_t$ are direct. Set

$$\mathcal{G}(n) := \oplus_{i\in\mathcal{I}_0}\mathcal{G}_i, \ \ \mathcal{S}(n) := \oplus_{j\in\mathcal{J}_0}\mathcal{S}_j, \ \ \mathcal{V}(n) := \oplus_{t\in\mathcal{T}_0}\mathcal{V}_t.$$

We note that $\mathcal{G}(n)$ (resp. $\mathcal{S}(n)$ and $\mathcal{V}(n)$) is a $\mathcal{G}$-submodule of $\mathcal{L}$ whose set of weights is R_{sdiv} (resp. $R_{lg} \cup \{0\}$ and R_{sh}).

Claim 2. For $\alpha \in R_{lg},\ x \in \mathcal{G}(n)_\alpha$, and $y \in \mathcal{S}(n)_\alpha$, we have $x + y = 0$ if and only if $x = y = 0$: Suppose that $x + y = 0$. Since $x \in \mathcal{G}(n)_\alpha = \sum_{i\in\mathcal{I}_0}(\mathcal{G}_i)_\alpha$, we get $x = \sum_{i\in\mathcal{I}_0} x_i$ with finitely many nonzero terms $x_i \in (\mathcal{G}_i)_\alpha$, for $i \in \mathcal{I}_0$. Similarly $y = \sum_{j\in\mathcal{J}_0} y_j$ with finitely many nonzero terms $y_j \in (\mathcal{S}_j)_\alpha$, for $j \in \mathcal{J}_0$. Now we recall that $\ell, n > 3$ and R, S are root systems of type BC_n and BC_ℓ respectively. This allows us to pick $\beta_1, \beta_2 \in R_{lg}$ such that $\beta := \alpha + \beta_1 + \beta_2 \in S_{lg}$ and $\alpha + \beta_1 \in R_{lg}$. Fix $a_1 \in \mathcal{G}_{\beta_1}$ and $a_2 \in \mathcal{G}_{\beta_2}$. Using Lemma 2.13, we get that

if x_i $(i \in \mathcal{I}_0)$ is nonzero, then $a_2 \cdot a_1 \cdot x_i$ is a nonzero element of $(\mathcal{G}_i)_\beta = (\mathfrak{g}_i)_\beta$ and similarly if $j \in \mathcal{J}_0$ and $y_j \neq 0$, $a_2 \cdot a_1 \cdot y_j$ is a nonzero element of $(\mathcal{S}_j)_\beta = (\mathfrak{s}_j)_\beta$. (35)

Now since $x+y=0$, we get that $\sum_{i\in\mathcal{I}_0} x_i = -\sum_{j\in\mathcal{J}_0} y_j$ which in turn implies that $\sum_{i\in\mathcal{I}_0} a_2\cdot a_1\cdot x_i = -\sum_{j\in\mathcal{J}_0} a_2\cdot a_1\cdot y_j$. But the right hand side is an element of $\oplus_{j\in\mathcal{J}_0}(\mathfrak{s}_j)_\beta$ and the left hand side is an element of $\oplus_{i\in\mathcal{I}_0}(\mathfrak{g}_i)_\beta$. Therefore $a_2\cdot a_1\cdot x_i = 0$ and $a_2\cdot a_1\cdot y_j = 0$ for $i\in\mathcal{I}_0$ and $j\in\mathcal{J}_0$. This together with (35) implies that for $i\in\mathcal{I}_0$ and $j\in\mathcal{J}_0$, $x_i=0$ and $y_j=0$. This completes the proof of Claim 2.

Claim 3. For $x\in\mathcal{G}(n)_0$ and $y\in\mathcal{S}(n)_0$, $x+y=0$ if and only if $x=y=0$: Suppose that $x+y=0$ and $x\neq 0$. Since $\mathcal{G}(n)_0=\sum_{i\in\mathcal{I}_0}(\mathcal{G}_i)_0$, we have $x=\sum_{i\in\mathcal{I}_0} x_i$ with finitely many nonzero terms $x_i\in(\mathcal{G}_i)_0$, $i\in\mathcal{I}_0$. Fix $t\in\mathcal{I}_0$ such that $x_t\neq 0$. Since x_t is a nonzero element of the irreducible nontrivial $\mathcal{G}$-module $\mathcal{G}_t$, there is $\alpha\in R^\times$ and $0\neq a\in\mathcal{G}_\alpha$ such that $a\cdot x_t\neq 0$. We note that $x\in\mathcal{G}(n)_0$ and $y\in\mathcal{S}(n)_0$, therefore we have $a\cdot x\in\mathcal{G}(n)_\alpha$ and $a\cdot y\in\mathcal{S}(n)_\alpha$. Now as $0=a\cdot x+a\cdot y$, Claim 2 together with the fact that the set of weights of $\mathcal{S}(n)$ is $R_{lg}\cup\{0\}$ implies that $a\cdot x=0$ and $a\cdot y=0$. So $\sum_{i\in\mathcal{I}_0} a\cdot x_i=0$. But by Claim 1, $\sum_{i\in\mathcal{I}_0}\mathcal{G}_i$ is a direct sum, so $a\cdot x_t=0$ which is a contradiction. Therefore $x=0$ and so $y=0$ as well. This completes the proof of Claim 3.

Claim 4. The sum $\mathcal{G}(n)+\mathcal{S}(n)+\mathcal{V}(n)$ is a direct sum: Suppose that $x\in\mathcal{G}(n)$, $y\in\mathcal{S}(n)$ and $z\in\mathcal{V}(n)$ are such that $x+y+z=0$. We have $x=\displaystyle\sum_{\alpha\in R_{sdiv}} x_\alpha$ with $x_\alpha\in\mathcal{G}(n)_\alpha\subseteq\mathcal{L}_\alpha$ for $\alpha\in R_{sdiv}$, $y=\displaystyle\sum_{\alpha\in R_{lg}\cup\{0\}} y_\alpha$ with $y_\alpha\in\mathcal{S}(n)_\alpha\subseteq\mathcal{L}_\alpha$ for $\alpha\in R_{lg}\cup\{0\}$, and $z=\displaystyle\sum_{\alpha\in R_{sh}} z_\alpha$ with $z_\alpha\in\mathcal{V}(n)_\alpha\subseteq\mathcal{L}_\alpha$ for $\alpha\in R_{sh}$. Therefore one gets that

$$\begin{gathered} x_0+y_0=0,\ z_\alpha=0,\ x_\beta+y_\beta=0,\ x_\gamma=0;\\ (\alpha\in R_{sh},\ \beta\in R_{lg},\ \gamma\in R_{ex}). \end{gathered}$$

Now using Claims 2,3, we are done

To complete the proof, we note that as a $\mathcal{G}$-module, $\mathcal{L}$ can be decomposed into finite dimensional irreducible $\mathcal{G}$-submodules with the set of weights contained in R. Now as $\bigoplus_{i\in\mathcal{I}_0}\mathcal{G}_i\oplus\bigoplus_{j\in\mathcal{J}_0}\mathcal{S}_j\oplus\bigoplus_{t\in\mathcal{T}_0}\mathcal{V}_t$ is a submodule of $\mathcal{L}$, one can find index sets $\mathcal{I},\mathcal{J},\mathcal{T}$ with

$$\mathcal{I}_0\subseteq\mathcal{I},\ \mathcal{J}_0\subseteq\mathcal{J},\ \mathcal{T}_0\subseteq\mathcal{T}$$

and a class $\{\mathcal{D}_n,\mathcal{G}_i,\mathcal{S}_j,\mathcal{V}_t\mid i\in\mathcal{I}\setminus\mathcal{I}_0, j\in\mathcal{J}\setminus\mathcal{J}_0, t\in\mathcal{T}\setminus\mathcal{T}_0\}$ of finite dimensional $\mathcal{G}$-submodules such that $\mathcal{D}_n$ is a trivial $\mathcal{G}$-module, $\mathcal{G}_i$ is isomorphic to $\mathcal{G}$ $(i\in\mathcal{I}\setminus\mathcal{I}_0)$, $\mathcal{S}_j$ is isomorphic to $\mathcal{S}$ $(j\in\mathcal{J}\setminus\mathcal{J}_0)$, $\mathcal{V}_t$ is isomorphic to $\mathcal{V}$ $(t\in\mathcal{T}\setminus\mathcal{T}_0)$ and

$$\begin{aligned}\mathcal{L} &= (\bigoplus_{i\in\mathcal{I}_0}\mathcal{G}_i\oplus\bigoplus_{j\in\mathcal{J}_0}\mathcal{S}_j\oplus\bigoplus_{t\in\mathcal{T}_0}\mathcal{V}_t)\oplus(\bigoplus_{i\in\mathcal{I}\setminus\mathcal{I}_0}\mathcal{G}_i\oplus\bigoplus_{j\in\mathcal{J}\setminus\mathcal{J}_0}\mathcal{S}_j\oplus\bigoplus_{t\in\mathcal{T}\setminus\mathcal{T}_0}\mathcal{V}_t\oplus\mathcal{D}_n)\\ &= \bigoplus_{i\in\mathcal{I}}\mathcal{G}_i\oplus\bigoplus_{j\in\mathcal{J}}\mathcal{S}_j\oplus\bigoplus_{t\in\mathcal{T}}\mathcal{V}_t\oplus\mathcal{D}_n.\end{aligned}$$

This completes the proof. ■

From now on, we use the data appearing in the $(R,S)-$datum (33). We take $\mathcal{A}_n$ to be a vector space with a basis $\{a_i\mid i\in\mathcal{I}\}$, $\mathcal{B}_n$ to be a vector space with a basis $\{b_j \mid j \in \mathcal{J}\}$ and $\mathcal{C}_n$ to be a vector space with a basis $\{c_t \mid t \in \mathcal{T}\}$. Then as a $(\mathcal{G} =)\mathcal{G}^n$-module, $\mathcal{L}^n = \mathcal{L}$ can be identified with

$$(\mathcal{G}^n \otimes \mathcal{A}_n) \oplus (\mathcal{S}^n \otimes \mathcal{B}_n) \oplus (\mathcal{V}^n \otimes \mathcal{C}_n) \oplus \mathcal{D}_n. \tag{36}$$

Take

$$\varphi : \mathcal{L} \longrightarrow (\mathcal{G}^n \otimes \mathcal{A}_n) \oplus (\mathcal{S}^n \otimes \mathcal{B}_n) \oplus (\mathcal{V}^n \otimes \mathcal{C}_n) \oplus \mathcal{D}_n \tag{37}$$

to be the canonical identification. Next define $\mathcal{A}_\ell$ to be the vector subspace of $\mathcal{A}_n$ spanned by $\{a_i \mid i \in \mathcal{I}_0\}$, $\mathcal{B}_\ell$ to be the vector subspace of $\mathcal{B}_n$ spanned by $\{b_j \mid j \in \mathcal{J}_0\}$ and $\mathcal{C}_\ell$ to be the vector subspace of $\mathcal{C}_n$ spanned by $\{c_t \mid t \in \mathcal{T}_0\}$. Then it follows from (32) that as a $\mathcal{G}^\ell$-module, $\mathcal{L}^\ell = \mathcal{L}^{^S}$ can be identified with

$$(\mathcal{G}^\ell \dot{\otimes} \mathcal{A}_\ell) \oplus (\mathcal{S}^\ell \dot{\otimes} \mathcal{B}_\ell) \oplus (\mathcal{V}^\ell \dot{\otimes} \mathcal{C}_\ell) \oplus \mathcal{D}_\ell \tag{38}$$

where $\mathcal{D}_\ell := \varphi(E)$. In what follows using [3, Thm. 2.48], for $\mu = \ell, n$, we give the algebraical structure of $\mathcal{L}^{^\mu}$ in terms of the ingredients involved in the decomposition of $\mathcal{L}^\mu$ into finite dimensional irreducible $\mathcal{G}^\mu$-modules. Set $\mathfrak{a}_\mu := \mathcal{A}_\mu \oplus \mathcal{B}_\mu$. Then there are a bilinear map $\cdot_\mu : \mathfrak{a}_\mu \times \mathfrak{a}_\mu \longrightarrow \mathfrak{a}_\mu$ and a linear map $*_\mu : \mathfrak{a}_\mu \longrightarrow \mathfrak{a}_\mu$ such that $(\mathfrak{a}_\mu, \cdot_\mu)$ is a unital associative algebra and $*_\mu$ is an involution on $\mathfrak{a}_\mu$ with $*_\mu$-fixed points $\mathcal{A}_\mu$ and $*_\mu$-skew fixed points $\mathcal{B}_\mu$. Also there is a bilinear map $\cdot_\mu : \mathfrak{a}_\mu \times \mathcal{C}_\mu \longrightarrow \mathcal{C}_\mu$ such that $(\mathcal{C}_\mu, \cdot_\mu)$ is a left unital associative $\mathfrak{a}_\mu$-module equipped with a skew-hermitian form $f_\mu : \mathcal{C}_\mu \times \mathcal{C}_\mu \longrightarrow \mathfrak{a}_\mu$. Take $\mathfrak{b}_\mu := \mathfrak{b}(\mathfrak{a}_\mu, *_\mu, \mathcal{C}_\mu, f_\mu)$ to be defined as in Subsection 3 and set $\cdot_\mu$, $\circ_\mu$, $[\cdot,\cdot]_\mu$, $\diamond_\mu$ and $\heartsuit_\mu$ to be the corresponding features as $\cdot$, $\circ$, $[\cdot,\cdot]$, $\diamond$ and $\heartsuit$ defined in Subsection 3. Also for $\beta, \beta' \in \mathfrak{b}_\mu$, set $d^\mu_{\beta,\beta'} := d^{\mu,\mathfrak{b}_\mu}_{\beta,\beta'}$. By [3, Theorems 2.48, 5.34], $\mathcal{D}_\mu$ is a subalgebra of $\mathcal{L}^\mu$ and there is a subspace $\mathcal{K}_\mu$ of the full skew-dihedral homology group

$$HF(\mathfrak{b}_\mu) = \{\sum_i \{\beta_i, \beta'_i\}_\mu \mid \sum_i d^\mu_{\beta_i,\beta'_i} = 0\}$$

of $\mathfrak{b}_\mu$ with respect to μ such that $\mathcal{D}_\mu$ is isomorphic to the quotient algebra $\{\mathfrak{b}_\mu, \mathfrak{b}_\mu\}_\mu / \mathcal{K}_\mu$. For β_1, β_2, take $\langle \beta_1, \beta_2 \rangle_\mu$ to be the element of $\mathcal{D}_\mu$ corresponding to $\{\beta_1, \beta_2\}_\mu + \mathcal{K}_\mu$, then one has $\langle \mathcal{A}_\mu, \mathcal{B}_\mu \rangle_\mu = \langle \mathcal{A}_\mu, \mathcal{C}_\mu \rangle_\mu = \langle \mathcal{B}_\mu, \mathcal{C}_\mu \rangle_\mu = \{0\}$ and $\mathcal{D}_\mu = \langle \mathcal{A}_\mu, \mathcal{A}_\mu \rangle_\mu + \langle \mathcal{B}_\mu, \mathcal{B}_\mu \rangle_\mu + \langle \mathcal{C}_\mu, \mathcal{C}_\mu \rangle_\mu$. Moreover, the Lie bracket on $\mathcal{L}^\mu$ is given by

$$\begin{aligned}
&[x \otimes a, y \otimes a'] = [x, y] \otimes \tfrac{1}{2}(a \circ_\mu a') + (x \circ_\mu y) \otimes \tfrac{1}{2}[a, a']_\mu + tr(xy)\langle a, a' \rangle_\mu,\\
&[x \otimes a, s \otimes b] = (x \circ_\mu s) \otimes \tfrac{1}{2}[a, b]_\mu + [x, s] \otimes \tfrac{1}{2}(a \circ_\mu b) = -[s \otimes b, x \otimes a],\\
&[s \otimes b, t \otimes b'] = [s, t] \otimes \tfrac{1}{2}(b \circ_\mu b') + (s \circ_\mu t) \otimes \tfrac{1}{2}[b, b']_\mu + tr(st)\langle b, b' \rangle_\mu,\\
&[x \otimes a, u \otimes c] = xu \otimes a \cdot_\mu c = -[u \otimes c, x \otimes a],\\
&[s \otimes b, u \otimes c] = su \otimes b \cdot_\mu c = -[u \otimes c, s \otimes b],\\
&[u \otimes c, v \otimes c'] = (u \circ v) \otimes (c \diamond_\mu c') + [u, v]_\mu \otimes (c \heartsuit_\mu c') + (u, v)\langle c, c' \rangle_\mu,\\
&[\langle \beta, \beta' \rangle_\mu, x \otimes a] = x \otimes d^\mu_{\beta,\beta'}(a) = -[x \otimes a, \langle \beta, \beta' \rangle_\mu],\\
&[\langle \beta, \beta' \rangle_\mu, s \otimes b] = s \otimes d^\mu_{\beta,\beta'}(b) = -[s \otimes b, \langle \beta, \beta' \rangle_\mu],\\
&[\langle \beta, \beta' \rangle_\mu, u \otimes c] = u \otimes d^\mu_{\beta,\beta'}(c) = -[u \otimes c, \langle \beta, \beta' \rangle_\mu],\\
&[\langle \beta_1, \beta_2 \rangle_\mu, \langle \beta'_1, \beta'_2 \rangle_\mu] = \langle d^\mu_{\beta_1,\beta_2}(\beta'_1), \beta'_2 \rangle_\mu + \langle \beta'_1, d^\mu_{\beta_1,\beta_2}(\beta'_2) \rangle_\mu,
\end{aligned} \tag{39}$$

for $x, y \in \mathcal{G}^\mu,\ s, t \in \mathcal{S}^\mu,\ u, v \in \mathcal{V}^\mu,\ a, a' \in \mathcal{A}_\mu,\ b, b' \in \mathcal{B}_\mu,\ c, c' \in \mathcal{C}_\mu$ and $\beta, \beta' \in \mathfrak{b}_\mu$.

Lemma 3.5. *We have $\mathcal{I}=\mathcal{I}_0$, $\mathcal{J}=\mathcal{J}_0$ and $\mathcal{T}=\mathcal{T}_0$.*

Proof. It follows from (39), (36) and (38) that

$$(\mathcal{L}^n)_\alpha=\mathcal{L}_\alpha=\begin{cases}\mathcal{V}^n_\alpha\dot{\otimes}\mathcal{C}_n & \text{if } \alpha\in R_{sh}\\ (\mathcal{G}^n_\alpha\dot{\otimes}\mathcal{A}_n)\oplus(\mathcal{S}^n_\alpha\dot{\otimes}\mathcal{B}_n) & \text{if } \alpha\in R_{lg}\\ \mathcal{G}^n_\alpha\dot{\otimes}\mathcal{A}_n & \text{if } \alpha\in R_{ex}\end{cases}$$

and

$$(\mathcal{L}^\ell)_\alpha=(\mathcal{L}^{^S})_\alpha=\begin{cases}(\mathcal{V}^\ell)_\alpha\dot{\otimes}\mathcal{C}_\ell & \text{if } \alpha\in S_{sh}\\ (\mathcal{G}^\ell_\alpha\dot{\otimes}\mathcal{A}_\ell)\oplus(\mathcal{S}^\ell_\alpha\dot{\otimes}\mathcal{B}_\ell) & \text{if } \alpha\in S_{lg}\\ \mathcal{G}^\ell_\alpha\dot{\otimes}\mathcal{A}_\ell & \text{if } \alpha\in S_{ex}.\end{cases}$$

Now fix $\alpha\in S_{ex}$, then

$$(\mathcal{G}^\ell)_\alpha\dot{\otimes}\mathcal{A}_\ell=(\mathcal{L}^{^S})_\alpha=\mathcal{L}_\alpha=\mathcal{G}^n_\alpha\dot{\otimes}\mathcal{A}_n.$$

This together with the fact that $\mathcal{G}^\ell_\alpha=\mathcal{G}^n_\alpha$ is a one dimensional vector space, implies that the vector space $\mathcal{A}_\ell$ equals the vector space $\mathcal{A}_n$. In particular, we get $\mathcal{I}=\mathcal{I}_0$. Next fix $\alpha\in S_{sh}$, then we have

$$\mathcal{V}^\ell_\alpha\dot{\otimes}\mathcal{C}_\ell=\mathcal{L}^{^S}_\alpha=\mathcal{L}_\alpha=\mathcal{V}^n_\alpha\dot{\otimes}\mathcal{C}_n.$$

This as above, implies that $\mathcal{T}=\mathcal{T}_0$. Finally fix $\alpha\in S_{lg}$, then

$$(\mathcal{G}^\ell_\alpha\dot{\otimes}\mathcal{A}_\ell)\oplus(\mathcal{S}^\ell_\alpha\dot{\otimes}\mathcal{B}_\ell)=\mathcal{L}^{^S}_\alpha=\mathcal{L}_\alpha=(\mathcal{G}^n_\alpha\dot{\otimes}\mathcal{A}_n)\oplus(\mathcal{S}^n_\alpha\dot{\otimes}\mathcal{B}_n).$$

Now as $\mathcal{S}^\ell_\alpha=\mathcal{S}^n_\alpha$ is a one dimensional vector space, $\mathcal{G}^\ell_\alpha=\mathcal{G}^n_\alpha$, $\mathcal{B}_\ell\subseteq\mathcal{B}_n$ and $\mathcal{A}_\ell=\mathcal{A}_n$, we get that the two vector spaces $\mathcal{B}_\ell$ and $\mathcal{B}_n$ are equal and so $\mathcal{J}=\mathcal{J}_0$. ∎

As we have already seen, on the vector space level, we have

$$\mathcal{A}_\ell=\mathcal{A}_n,\ \ \mathcal{B}_\ell=\mathcal{B}_n,\ \ \mathcal{C}_\ell=\mathcal{C}_n$$

which in turn implies that the vector space $\mathfrak{b}_\ell$ equals the vector space $\mathfrak{b}_n$. In the following lemma, we show, in addition, that the algebras $\mathfrak{b}_\ell$ and $\mathfrak{b}_n$ have the same algebraic structure.

Lemma 3.6. *The algebraic structure of $\mathfrak{b}_n$ coincides with the algebraic structure of $\mathfrak{b}_\ell$.*

Proof. Using Lemma 3.5, we set

$$\mathcal{A}:=\mathcal{A}_\ell=\mathcal{A}_n,\ \ \mathcal{B}:=\mathcal{B}_\ell=\mathcal{B}_n,\ \ \mathcal{C}:=\mathcal{C}_\ell=\mathcal{C}_n. \tag{40}$$

Suppose that i,j,k are distinct elements of I_0. Take $x:=e_{i,j}-e_{\bar{j},\bar{i}}\in\mathcal{G}^\ell$, $y:=e_{j,k}-e_{\bar{k},\bar{j}}\in\mathcal{G}^\ell$ and assume $a,a'\in\mathcal{A}$, then by (39), one has

$$\begin{aligned}[x,y]\otimes\frac{1}{2}(a\circ_n a')+(x\circ_n y)\otimes\frac{1}{2}[a,a']_n &= [x\otimes a,y\otimes a']\\ &= [x,y]\otimes\frac{1}{2}(a\circ_\ell a')+(x\circ_\ell y)\otimes\frac{1}{2}[a,a']_\ell.\end{aligned}$$

This in turn implies that

$$[x,y]\otimes(\frac{1}{2}(a\circ_n a')-\frac{1}{2}(a\circ_\ell a'))=(x\circ_n y)\otimes(\frac{1}{2}[a,a']_\ell-\frac{1}{2}[a,a']_n),$$

but the left hand side is an element of $\mathcal{G}\otimes\mathcal{A}$ and the right hand side is an element of $\mathcal{S}\otimes\mathcal{B}$. Therefore as $[x,y]\neq 0$ and $x\circ_n y\neq 0$, we get that

$$\frac{1}{2}[a,a']_\ell-\frac{1}{2}[a,a']_n=0\quad\text{and}\quad\frac{1}{2}(a\circ_n a')-\frac{1}{2}(a\circ_\ell a')=0.$$

This now implies that

$$a\cdot_\ell a'=a\cdot_n a';\;\; a,a'\in\mathcal{A}. \tag{41}$$

Next take i and j to be two distinct elements of I_0. Set $s:=e_{i,j}+e_{\bar{j},\bar{i}}\in\mathcal{S}^\ell$ and $x:=e_{j,\bar{j}}\in\mathcal{G}^\ell$, then using the same argument as above, we have

$$\frac{1}{2}[a,b]_\ell-\frac{1}{2}[a,b]_n=0\quad\text{and}\quad\frac{1}{2}(a\circ_n b)-\frac{1}{2}(a\circ_\ell b)=0$$

for $a\in\mathcal{A}$ and $b\in\mathcal{B}$. This in particular gives that

$$a\cdot_\ell b=a\cdot_n b\quad\text{and}\quad b\cdot_\ell a=b\cdot_n a;\qquad (a\in\mathcal{A},\; b\in\mathcal{B}). \tag{42}$$

Finally, for distinct fixed elements i,j,k of I_0, set $s:=e_{i,\bar{j}}-e_{j,\bar{i}}, t=e_{\bar{i},k}-e_{\bar{k},i}\in\mathcal{S}^\ell$. Then using an analogous argument as before, we have $b\cdot_\ell b'=b\cdot_n b'$ for $b,b'\in\mathcal{B}$. This together with (41) and (42) implies that

$$\mathfrak{a}_\ell=\mathfrak{a}_n\ \text{(as two algebras)}. \tag{43}$$

Now take $x\in\mathcal{G}^\ell$, $s\in\mathcal{S}^\ell$ and $u,v\in\mathcal{V}^\ell$ to be such that $xu\neq 0$ and $sv\neq 0$. Then for $a\in\mathcal{A}$, $b\in\mathcal{B}$ and $c\in\mathcal{C}$, we get using (39) that

$$\begin{aligned} xu\otimes a\cdot_\ell c &= [x\otimes a,u\otimes c]=xu\otimes a\cdot_n c,\\ sv\otimes b\cdot_\ell c &= [s\otimes b,v\otimes c]=sv\otimes b\cdot_n c.\end{aligned}$$

This implies that

$$a\cdot_\ell c=a\cdot_n c\quad\text{and}\quad b\cdot_\ell c=b\cdot_n c$$

for $a\in\mathcal{A}$, $b\in\mathcal{B}$ and $c\in\mathcal{C}$. Therefore we have

$$\mathcal{C}_\ell=\mathcal{C}_n\ \text{(as two } \mathfrak{a}_n\text{-modules)}. \tag{44}$$

Now we are done thanks to (43) and (44). ∎

To continue, regarding Lemma 3.6, we set

$$\mathfrak{a}:=\mathfrak{a}_n=\mathfrak{a}_\ell\quad\text{and}\quad\mathfrak{b}:=\mathfrak{b}_\ell=\mathfrak{b}_n.$$

Also for $\beta,\beta'\in\mathfrak{b}$, we take

$$\begin{aligned}&\beta\cdot\beta':=\beta\cdot_n\beta'=\beta\cdot_\ell\beta',\\ &[\beta,\beta']:=\beta\cdot\beta'-\beta'\cdot\beta,\\ &\beta\circ\beta':=\beta\cdot\beta'+\beta'\cdot\beta.\end{aligned} \tag{45}$$

Lemma 3.7. *For $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$, we have*

$$\langle a, a'\rangle_n = (\tfrac{-1}{\ell} Id_{\mathcal{V}^\ell} + \tfrac{1}{n} Id_{\mathcal{V}}) \otimes \tfrac{1}{2}[a, a']) + \langle a, a'\rangle_\ell,$$
$$\langle b, b'\rangle_n = (\tfrac{-1}{\ell} Id_{\mathcal{V}^\ell} + \tfrac{1}{n} Id_{\mathcal{V}}) \otimes \tfrac{1}{2}[b, b']) + \langle b, b'\rangle_\ell.$$

Also for $c, c' \in \mathcal{C}$, $f_\ell(c, c') = f_n(c, c')$, $c \diamond_\ell c' = c \diamond_n c'$ and $c \heartsuit_\ell c' = c \heartsuit_n c'$. Moreover, we have

$$\begin{aligned} \langle c, c'\rangle_n &= ((\frac{1}{\ell} Id_{\mathcal{V}^\ell} - \frac{1}{n} Id_{\mathcal{V}}) \otimes \frac{1}{2}(c \heartsuit_n c')) + \langle c, c'\rangle_\ell \\ &= ((\frac{1}{\ell} Id_{\mathcal{V}^\ell} - \frac{1}{n} Id_{\mathcal{V}}) \otimes \frac{1}{2}(c \heartsuit_\ell c')) + \langle c, c'\rangle_\ell. \end{aligned}$$

Proof. Fix $x, y \in \mathcal{G}^\ell$ such that $tr(xy) \neq 0$. For $a, a' \in \mathcal{A}$, consider (45) and use (39) to get

$$\begin{aligned} & ([x, y] \otimes \frac{1}{2}(a \circ a')) + ((x \circ_n y) \otimes \frac{1}{2}[a, a']) + tr(xy)\langle a, a'\rangle_n \\ = \; & [x \otimes a, y \otimes a'] \\ = \; & ([x, y] \otimes \frac{1}{2}(a \circ a')) + ((x \circ_\ell y) \otimes \frac{1}{2}[a, a']) + tr(xy)\langle a, a'\rangle_\ell. \end{aligned}$$

This gives that

$$(\frac{tr(xy)}{n} Id_{\mathcal{V}} \otimes \frac{1}{2}[a, a']) - tr(xy)\langle a, a'\rangle_n = (\frac{tr(xy)}{\ell} Id_{\mathcal{V}^\ell} \otimes \frac{1}{2}[a, a']) - tr(xy)\langle a, a'\rangle_\ell.$$

Therefore we have

$$\langle a, a'\rangle_n = (\frac{-1}{\ell} Id_{\mathcal{V}^\ell} + \frac{1}{n} Id_{\mathcal{V}}) \otimes \frac{1}{2}[a, a']) + \langle a, a'\rangle_\ell; \qquad (a, a' \in \mathcal{A}). \tag{46}$$

Similarly, one can get the second equality in the statement. Now suppose that i and j are two distinct elements of I_0. Take $u := v_i$ and $v := v_{\bar{j}}$, then $(u, v) = 0$ and so $[u, v]_n = [u, v]$. Therefore for all $c, c' \in \mathcal{C}$, by (39), we have

$$\begin{aligned} (u \circ v) \otimes (c \diamond_n c') + [u, v] \otimes (c \heartsuit_n c') &= [u \otimes c, v \otimes c'] \\ &= (u \circ v) \otimes (c \diamond_\ell c') + [u, v] \otimes c \heartsuit_\ell c'. \end{aligned}$$

But one knows that $u \circ v \in \mathcal{G}$, $[u, v]$ is a nonzero element of $\mathcal{S}$, $c \diamond_\ell c', c \diamond_n c' \in \mathcal{A}$ and $c \heartsuit_\ell c', c \heartsuit_n c' \in \mathcal{B}$, so we get that

$$c \diamond c' := c \diamond_\ell c' = c \diamond_n c' \quad \text{and} \quad c \heartsuit c' := c \heartsuit_\ell c' = c \heartsuit_n c'; \qquad (c, c' \in \mathcal{C}). \tag{47}$$

This implies that

$$f(c, c') := f_\ell(c, c') = f_n(c, c'); \qquad (c, c' \in \mathcal{C}). \tag{48}$$

Now using the same argument as before, we get the last equality.

Corollary 3.8. *Let $\ell < n$ and suppose that t is a positive integer, $a_i, a'_i \in \mathcal{A}$, $b_i, b'_i \in \mathcal{B}$ and $c_i, c'_i \in \mathcal{C}$ for $1 \leq i \leq t$. Then $\sum_{i=1}^{t}(\langle a_i, a'_i\rangle_\ell + \langle b_i, b'_i\rangle_\ell + \langle c_i, c'_i\rangle_\ell) = 0$ if and only if*

$$\sum_{i=1}^{t}([a_i, a'_i] + [b_i, b'_i] - c_i \diamond c'_i) = 0 \quad \textit{and} \quad \sum_{i=1}^{t}(\langle a_i, a'_i\rangle_n + \langle b_i, b'_i\rangle_n + \langle c_i, c'_i\rangle_n) = 0.$$

Proof. By Lemma 3.7, we have

$$\begin{aligned}&\sum_{i=1}^{t}(\langle a_i, a'_i\rangle_\ell + \langle b_i, b'_i\rangle_\ell + \langle c_i, c'_i\rangle_\ell) = \\ &\sum_{i=1}^{t}(\langle a_i, a'_i\rangle_n + \langle b_i, b'_i\rangle_n + \langle c_i, c'_i\rangle_n) - \\ &(\frac{-1}{\ell} Id_{_{\mathcal{V}^\ell}} + \frac{1}{n} Id_{\mathcal{V}}) \otimes \frac{1}{2}\sum_{i=1}^{t}([a_i, a'_i] + [b_i, b'_i] - c_i \diamond c'_i).\end{aligned}$$

Now as $\sum_{i=1}^{t}(\langle a_i, a'_i\rangle_n + \langle b_i, b'_i\rangle_n + \langle c_i, c'_i\rangle_n) \in \mathcal{D}_n$ and $(\frac{-1}{\ell} Id_{_{\mathcal{V}^\ell}} + \frac{1}{n} Id_{\mathcal{V}}) \otimes \frac{1}{2}\sum_{i=1}^{t}([a_i, a'_i] + [b_i, b'_i] - c_i \diamond c'_i) \in \mathcal{S} \otimes \mathcal{B}$, we are done. ■

Remark 3.9. Consider the decomposition (32) for $\mathcal{L}^{^S} = \mathcal{L}^\ell$ into finite dimensional irreducible $\mathcal{G}^\ell$-submodules and the decomposition of $\mathcal{L} = \mathcal{L}^n$ into finite dimensional irreducible $\mathcal{G}$-submodules as in Lemma 3.4, then contemplating the identification (37), we have, using Lemma 3.7, that

$$\mathcal{L} = (\bigoplus_{i \in \mathcal{I}} \mathcal{G}_i \oplus \bigoplus_{j \in \mathcal{J}} \mathcal{S}_j \oplus \bigoplus_{t \in \mathcal{T}} \mathcal{V}_t) + E.$$

Moreover, setting $\langle \beta, \beta'\rangle^n := \varphi^{-1}(\langle \beta, \beta'\rangle_n)$ and $\langle \beta, \beta'\rangle^\ell := \varphi^{-1}(\langle \beta, \beta'\rangle_\ell)$ for $\beta, \beta' \in \mathfrak{b}$, we get that $\{\langle \beta, \beta'\rangle^n \mid \beta, \beta' \in \mathfrak{b}\}$ spans $\mathcal{D}_n$ and that $\{\langle \beta, \beta'\rangle^\ell \mid \beta, \beta' \in \mathfrak{b}\}$ spans E. Furthermore, thanks to Corollary 3.8, for $t \in \mathbb{N} \setminus \{0\}$, $a_i, a'_i \in \mathcal{A}$, $b_i, b'_i \in \mathcal{B}$ and $c_i, c'_i \in \mathcal{C}$ $(1 \leq i \leq t)$, $\sum_{i=1}^{t}(\langle a_i, a'_i\rangle^\ell + \langle b_i, b'_i\rangle^\ell + \langle c_i, c'_i\rangle^\ell) = 0$ if and only if $\sum_{i=1}^{t}([a_i, a'_i] + [b_i, b'_i] - c_i \diamond c'_i) = 0$ and $\sum_{i=1}^{t}(\langle a_i, a'_i\rangle^n + \langle b_i, b'_i\rangle^n + \langle c_i, c'_i\rangle^n) = 0$.

Recalling (27) and (47), we now ready to state the main result of this part.

Proposition 3.10. *For $e, f \in \mathcal{G} \cup \mathcal{S}$, set*

$$e \circ f := ef + fe - \frac{tr(ef)}{\ell} Id_{_{\mathcal{V}^\ell}}.$$

Also for $\beta_1 = a_1 + b_1 + c_1 \in \mathfrak{b}$ and $\beta_2 = a_2 + b_2 + c_2 \in \mathfrak{b}$ with $a_1, a_2 \in \mathcal{A}$, $b_1, b_2 \in \mathcal{B}$ and $c_1, c_2 \in \mathcal{C}$, set

$$\langle \beta_1, \beta_2\rangle := \langle \beta_1, \beta_2\rangle_\ell, \quad \beta_1^* = c_1, \quad \beta_2^* = c_2$$

and take

$$\mathcal{D} := span\{\langle a, a'\rangle, \langle b, b'\rangle, \langle c, c'\rangle \mid a, a' \in \mathcal{A},\ b, b' \in \mathcal{B}, c, c' \in \mathcal{C}\},$$

then we have

$$\begin{aligned}\mathcal{L} &= (\mathcal{G}\otimes\mathcal{A})\oplus(\mathcal{S}\otimes\mathcal{B})\oplus(\mathcal{V}\otimes\mathcal{C})\oplus\langle\mathfrak{b},\mathfrak{b}\rangle_n\\ &= ((\mathcal{G}\otimes\mathcal{A})\oplus(\mathcal{S}\otimes\mathcal{B})\oplus(\mathcal{V}\otimes\mathcal{C}))+\mathcal{D}\end{aligned}$$

with the Lie bracket given by

$$\begin{aligned}
&[x\otimes a, y\otimes a'] = [x,y]\otimes\tfrac{1}{2}(a\circ a') + (x\circ y)\otimes\tfrac{1}{2}[a,a'] + tr(xy)\langle a,a'\rangle,\\
&[x\otimes a, s\otimes b] = (x\circ s)\otimes\tfrac{1}{2}[a,b] + [x,s]\otimes\tfrac{1}{2}(a\circ b) = -[s\otimes b, x\otimes a],\\
&[s\otimes b, t\otimes b'] = [s,t]\otimes\tfrac{1}{2}(b\circ b') + (s\circ t)\otimes\tfrac{1}{2}[b,b'] + tr(st)\langle b,b'\rangle,\\
&[x\otimes a, u\otimes c] = xu\otimes a\cdot c = -[u\otimes c, x\otimes a],\\
&[s\otimes b, u\otimes c] = su\otimes b\cdot c = -[u\otimes c, s\otimes b],\\
&[u\otimes c, v\otimes c'] = (u\circ v)\otimes(c\diamond c') + [u,v]\otimes(c\heartsuit c') + (u,v)\langle c,c'\rangle,\\
&[\langle\beta_1,\beta_2\rangle, x\otimes a] = \tfrac{-1}{4\ell}(x\circ Id_{\mathcal{V}^\ell}\otimes[a,\beta^*_{\beta_1,\beta_2}] + [x,Id_{\mathcal{V}^\ell}]\otimes a\circ\beta^*_{\beta_1,\beta_2}),\\
&[\langle\beta_1,\beta_2\rangle, s\otimes b] = \tfrac{-1}{4\ell}([s,Id_{\mathcal{V}^\ell}]\otimes(b\circ\beta^*_{\beta_1,\beta_2}) + (s\circ Id_{\mathcal{V}^\ell})\otimes[b,\beta^*_{\beta_1,\beta_2}] + 2tr(sId_{\mathcal{V}^\ell})\langle b,\beta^*_{\beta_1,\beta_2}\rangle),\\
&[\langle\beta_1,\beta_2\rangle, v\otimes c] = \tfrac{1}{2\ell}Id_{\mathcal{V}^\ell}v\otimes(\beta^*_{\beta_1,\beta_2}\cdot c) - \tfrac{1}{2}v\otimes(f(c,\beta_2^*)\cdot\beta_1^* + f(c,\beta_1^*)\cdot\beta_2^*)\\
&[\langle\beta_1,\beta_2\rangle, \langle\beta_1',\beta_2'\rangle] = \langle d^\ell_{\beta_1,\beta_2}(\beta_1'),\beta_2'\rangle + \langle\beta_1', d^\ell_{\beta_1,\beta_2}(\beta_2')\rangle
\end{aligned}\tag{49}$$

for $x,y\in\mathcal{G},\ s,t\in\mathcal{S},\ u,v\in\mathcal{V},\ a,a'\in\mathcal{A},\ b,b'\in\mathcal{B},\ c,c'\in\mathcal{C},\ \beta_1,\beta_2,\beta_1',\beta_2'\in\mathfrak{b}$.

Proof. Suppose that $x,y\in\mathcal{G},\ s,t\in\mathcal{S},\ u,v\in\mathcal{V},\ a,a'\in\mathcal{A},\ b,b'\in\mathcal{B}$ and $c,c'\in\mathcal{C}$, then (39) (for $\mu=n$) together with Lemma 3.7 implies that

$$[x\otimes a, y\otimes a'] = ([x,y]\otimes\frac{1}{2}(a\circ a')) + ((x\circ y)\otimes\frac{1}{2}[a,a']) + tr(xy)\langle a,a'\rangle.$$

Similarly we have

$$[s\otimes b, t\otimes b'] = ([s,t]\otimes\frac{1}{2}(b\circ b')) + ((s\circ t)\otimes\frac{1}{2}[b,b']) + tr(st)\langle b,b'\rangle$$

and

$$[u\otimes c, v\otimes c'] = ((u\circ v)\otimes(c\diamond c')) + ([u,v]\otimes(c\heartsuit c')) + (u,v)\langle c,c'\rangle.$$

Now for $a_1,a_2\in\mathcal{A},\ b_1,b_2\in\mathcal{B}$ and $c_1,c_2\in\mathcal{C}$, set $\beta_1=a_1+b_1+c_1$, $\beta_2=a_2+b_2+c_2$ and take $s_n:=(1/\ell)Id_{\mathcal{V}^\ell}-(1/n)Id_{\mathcal{V}}$. Then recalling $\beta^*_{\beta_1,\beta_2}$ from (27), one can see that for $x\in\mathcal{G},\ s\in\mathcal{S},\ v\in\mathcal{V},\ a\in\mathcal{A},\ b\in\mathcal{B}$ and $c\in\mathcal{C}$,

$$[x\otimes a, s_n\otimes\beta^*_{\beta_1,\beta_2}] = \frac{1}{2\ell}(x\circ Id_{\mathcal{V}^\ell}\otimes[a,\beta^*_{\beta_1,\beta_2}] + [x,Id_{\mathcal{V}^\ell}]\otimes(a\circ\beta^*_{\beta_1,\beta_2})) - \frac{1}{n}x\otimes[a,\beta^*_{\beta_1,\beta_2}],$$

$$\begin{aligned}[s\otimes b, s_n\otimes\beta^*_{\beta_1,\beta_2}] &= \frac{1}{2\ell}([s,Id_{\mathcal{V}^\ell}]\otimes(b\circ\beta^*_{\beta_1,\beta_2})\\ &\quad + (s\circ Id_{\mathcal{V}^\ell})\otimes[b,\beta^*_{\beta_1,\beta_2}]) - \frac{1}{n}s\otimes[b,\beta^*_{\beta_1,\beta_2}] + tr(ss_n)\langle b,\beta^*_{\beta_1,\beta_2}\rangle\end{aligned}$$

and

$$[s_n\otimes\beta^*_{\beta_1,\beta_2}, v\otimes c] = \frac{1}{\ell}Id_{\mathcal{V}^\ell}v\otimes(\beta^*_{\beta_1,\beta_2}\cdot c) - \frac{1}{n}v\otimes(\beta^*_{\beta_1,\beta_2}\cdot c).$$

We next note that

$$\begin{aligned}
[\langle a_1,a_2\rangle_n+\langle b_1,b_2\rangle_n+\langle c_1,c_2\rangle_n, x\otimes a] &= x\otimes(d^n_{a_1,a_2}+d^n_{b_1,b_2}+d^n_{c_1,c_2})(a)\\
&= \frac{1}{2n}x\otimes[\beta^*_{\beta_1,\beta_2},a]\\
[\langle a_1,a_2\rangle_n+\langle b_1,b_2\rangle_n+\langle c_1,c_2\rangle_n, s\otimes b] &= s\otimes(d^n_{a_1,a_2}+d^n_{b_1,b_2}+d^n_{c_1,c_2})(b)\\
&= \frac{1}{2n}s\otimes[\beta^*_{\beta_1,\beta_2},b]\\
[\langle a_1,a_2\rangle_n+\langle b_1,b_2\rangle_n+\langle c_1,c_2\rangle_n, v\otimes c] &= v\otimes(d^n_{a_1,a_2}+d^n_{b_1,b_2}+d^n_{c_1,c_2})(c)\\
&= \frac{1}{2n}v\otimes(\beta^*_{\beta_1,\beta_2}\cdot c)\\
&- v\otimes\frac{1}{2}(f(c,c_2)\cdot c_1+f(c,c_1)\cdot c_2).
\end{aligned}$$

Therefore using Lemma 3.7, an easy verification gives that

$$[\langle a_1,a_2\rangle+\langle b_1,b_2\rangle+\langle c_1,c_2\rangle, x\otimes a]=\frac{-1}{4\ell}((x\circ Id_{\mathcal{V}^\ell})\otimes[a,\beta^*_{\beta_1,\beta_2}]+[x,Id_{\mathcal{V}^\ell}]\otimes(a\circ\beta^*_{\beta_1,\beta_2})),$$

$$\begin{aligned}
[\langle a_1,a_2\rangle+\langle b_1,b_2\rangle+\langle c_1,c_2\rangle, s\otimes b] &= \frac{-1}{4\ell}([s,Id_{\mathcal{V}^\ell}]\otimes(b\circ\beta^*_{\beta_1,\beta_2})+(s\circ Id_{\mathcal{V}^\ell})\otimes[b,\beta^*_{\beta_1,\beta_2}])\\
&- \frac{1}{2\ell}tr(sId_{\mathcal{V}^\ell})\langle b,\beta^*_{\beta_1,\beta_2}\rangle,
\end{aligned}$$

and

$$[\langle a_1,a_2\rangle+\langle b_1,b_2\rangle+\langle c_1,c_2\rangle, v\otimes c]=\frac{1}{2\ell}Id_{\mathcal{V}^\ell}v\otimes(\beta^*_{\beta_1,\beta_2}\cdot c)-\frac{1}{2}v\otimes(f(c,c_2)\cdot c_1+f(c,c_1)\cdot c_2).$$

These together with (39) complete the proof. ■

3.0.2. Types A and D

Suppose that I is an index set of cardinality $n+1>5$ and I_0 is a subset of I of cardinality $\ell+1>5$. Suppose that R is an irreducible finite root system of type $X=\dot{A}_I$ or D_I. Suppose that $\mathcal{V}$ is a vector space with a basis $\{v_i \mid i\in I\}$. Take $\mathcal{G}$ to be either $\mathfrak{sl}(I)$ or $\mathfrak{o}_D(I)$. Set $\mathcal{G}^\ell:=\mathcal{G}_{I_0}$ and suppose that $\mathcal{V}^\ell$ is the subspace of $\mathcal{V}$ spanned by $\{v_i \mid i\in I_0\}$. We take $Id_{\mathcal{V}}$ to be the identity map on $\mathcal{V}$ and define $Id_{\mathcal{V}^\ell}$ as follows:

$$\begin{gathered}
Id_{\mathcal{V}^\ell}:\mathcal{V}\longrightarrow\mathcal{V}\\
v_i\mapsto v_i,\ v_j\mapsto 0;\ \ (i\in I_0,\ j\in I\setminus I_0).
\end{gathered}$$

Using [2] together with the same argument as in §3.0.1, we have the following theorem:

Theorem 3.11. *Suppose that $\mathcal{L}$ is a Lie algebra graded by the irreducible finite root system R of type $X=\dot{A}_I$ or D_I with grading pair $(\mathcal{G},\mathcal{H})$ and let S be the irreducible full subsystem of R of type $\dot{A}_{I_0}$ or D_{I_0} respectively.*

(i) Consider $\mathcal{L}^\ell:=\mathcal{L}^S$ as a $\mathcal{G}^\ell$-module and take

$$\mathcal{L}^\ell=\bigoplus_{i\in\mathcal{I}}\mathfrak{g}_i\oplus E \tag{50}$$

to be the decomposition of $\mathcal{L}^\ell$ into finite dimensional irreducible $\mathcal{G}^\ell$-submodules in which $\mathcal{I}$ is an index set, for $i \in \mathcal{I}$, $\mathfrak{g}_i$ is isomorphic to $\mathcal{G}^\ell$, and E is a trivial $\mathcal{G}^\ell$-submodule. Then regarding $\mathcal{L}$ as a $\mathcal{G}$-module, there exists a class $\{\mathcal{D}_n, \mathcal{G}_i \mid i \in \mathcal{I}\}$ of finite dimensional $\mathcal{G}$-submodules of $\mathcal{L}$ such that

- *$\mathcal{D}_n$ is a trivial $\mathcal{G}$-module and for $i \in \mathcal{I}$, $\mathcal{G}_i$ is isomorphic to $\mathcal{G}$,*
- *$\mathfrak{g}_i \subseteq \mathcal{G}_i$ $(i \in \mathcal{I})$,*
- *$\mathcal{L} = \bigoplus_{i\in\mathcal{I}} \mathcal{G}_i \oplus \mathcal{D}_n$.*

(ii) Take $\mathcal{A}$ to be a vector space with basis $\{a_i \mid i \in \mathcal{I}\}$ and identify $\mathcal{L}$ with $(\mathcal{G} \otimes \mathcal{A}) \oplus \mathcal{D}_n$, say via the natural identification

$$\varphi : \mathcal{L} \longrightarrow (\mathcal{G} \otimes \mathcal{A}) \oplus \mathcal{D}_n.$$

Transfer the Lie algebraic structure of $\mathcal{L}$ to $(\mathcal{G} \otimes \mathcal{A}) \oplus \mathcal{D}_n$. Then $\mathcal{D}_\ell := \varphi(E)$ is a subalgebra of $\varphi(\mathcal{L}^\ell) = (\mathcal{G}^\ell \dot{\otimes} \mathcal{A}) \oplus \mathcal{D}_\ell$ and $\mathcal{D}_n$ is a subalgebras of $(\mathcal{G} \otimes \mathcal{A}) \oplus \mathcal{D}_n$. Moreover, the vector space $\mathcal{A}$ is equipped with an associative algebraic structure if $X = \dot{A}_I$ and with a commutative associative algebraic structure if $X = D_I$.

(iii) There is a subspace $\mathcal{K}_1$ of the full skew-dihedral homology group of $\mathcal{A}$ with respect to n and a subspace $\mathcal{K}_2$ of the full skew-dihedral homology group of $\mathcal{A}$ with respect to ℓ such that $\mathcal{D}_n$ and $\mathcal{D}_\ell$ are isomorphic to the quotient algebras $\{\mathcal{A}, \mathcal{A}\}_n/\mathcal{K}_1$ and $\{\mathcal{A}, \mathcal{A}\}_\ell/\mathcal{K}_2$ respectively, say via

$$\psi_1 : \{\mathcal{A}, \mathcal{A}\}_n/\mathcal{K}_1 \longrightarrow \mathcal{D}_n \quad \text{and} \quad \psi_2 : \{\mathcal{A}, \mathcal{A}\}_\ell/\mathcal{K}_2 \longrightarrow \mathcal{D}_\ell.$$

(iv) For $a, a' \in \mathcal{A}$, take

$$\langle a, a'\rangle_n := \psi_1(\{a, a'\}_n + \mathcal{K}_1) \quad \text{and} \quad \langle a, a'\rangle_\ell := \psi_2(\{a, a'\}_\ell + \mathcal{K}_2).$$

Then for $a, a' \in \mathcal{A}$, we have

$$\langle a, a'\rangle_n = \langle a, a'\rangle_\ell + ((\frac{1}{n+1} Id_{\mathcal{V}} - \frac{1}{\ell+1} Id_{\mathcal{V}^\ell}) \otimes (aa' - a'a).$$

(v) Set

$$\langle a, a'\rangle^n := \varphi^{-1}(\langle a, a'\rangle_n) \quad \text{and} \quad \langle a, a'\rangle^\ell := \varphi^{-1}(\langle a, a'\rangle_\ell); \quad (a, a' \in \mathcal{A}).$$

If $\ell < n$, then for a positive integer t and $a_1, a_1', \ldots, a_t, a_t' \in \mathcal{A}$, we have $\sum_{i=1}^t \langle a_i, a_i'\rangle^\ell = 0$ if and only if $\sum_{i=1}^t \langle a_i, a_i'\rangle^n = 0$ and $\sum_{i=1}^t [a_i, a_i'] = 0$.

(vi) For $x, y \in \mathcal{G}$, set $x \circ y := xy + yx - \frac{2tr(xy)}{\ell+1} Id_{\mathcal{V}^\ell}$ and for $a, a' \in \mathcal{A}$, set $\langle a, a'\rangle := \langle a, a'\rangle_\ell$. Then the Lie bracket on $(\mathcal{G} \otimes \mathcal{A}) \oplus \mathcal{D}_n = (\mathcal{G} \otimes \mathcal{A}) + \mathcal{D}_\ell$ is given

by

$$
\begin{aligned}
&[x\otimes a, y\otimes a'] = \begin{cases} [x,y]\otimes \frac{1}{2}(a\circ a') + (x\circ y)\otimes \frac{1}{2}[a,a'] + tr(xy)\langle a,a'\rangle, & X=\dot{A}_I,\\ [x,y]\otimes aa' + tr(xy)\langle a,a'\rangle, & X=D_I,\end{cases}\\
&[\langle a_1,a_2\rangle, x\otimes a] = \begin{cases} \frac{-1}{2(\ell+1)}(x\circ Id_{\mathcal{V}^\ell}\otimes [a,[a_1,a_2]] & \\ +[x, Id_{\mathcal{V}^\ell}]\otimes a\circ [a_1,a_2] + 2tr(Id_{\mathcal{V}^\ell}x)\langle a,[a_1,a_2]\rangle), & X=\dot{A}_I,\\ 0, & X=D_I,\end{cases}\\
&[\langle a_1,a_2\rangle, \langle a_1',a_2'\rangle] = \begin{cases} \langle d^{\ell,\mathcal{A}}_{a_1,a_2}(a_1'), a_2'\rangle + \langle a_1', d^{\ell,\mathcal{A}}_{a_1,a_2}(a_2')\rangle, & X=\dot{A}_I,\\ 0, & X=D_I,\end{cases}
\end{aligned}
\tag{51}
$$

for $x,y\in\mathcal{G},\ a,a',a_1,a_2,a_1',a_2'\in\mathcal{A}$.

3.0.3. Types B and C

Suppose that I is an index set of cardinality $n>4$ and I_0 is a subset of I of cardinality $\ell>4$. Take $\mathcal{G}$ to be either $\mathfrak{o}_B(I)$ or $\mathfrak{sp}(I)$. Consider (4), (7) and set $\mathcal{G}^\ell := \mathcal{G}_{I_0}$. Suppose that $\mathcal{V}$ is a vector space with a basis $\{v_0, v_i, v_{\bar{i}} \mid i\in I\}$ equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ as in (3) if $\mathcal{G}=\mathfrak{o}_B(I)$ and a vector space with a basis $\{v_i, v_{\bar{i}} \mid i\in I\}$ equipped with a nondegenerate skew-symmetric bilinear form $(\cdot,\cdot)$ as in (6) if $\mathcal{G}:=\mathfrak{sp}(I)$. Consider (9), (11) and set $\mathcal{V}^\ell := \mathcal{V}_{I_0}$. Set

$$T := \begin{cases} I_0\cup \bar{I}_0\cup\{0\} & \text{if } \mathcal{G}=\mathfrak{o}_B(I)\\ I_0\cup\bar{I}_0 & \text{if } \mathcal{G}=\mathfrak{sp}(I)\end{cases}$$

and define $Id_{\mathcal{V}^\ell}:\mathcal{V}\longrightarrow\mathcal{V}$ to be the linear transformation given by

$$v_i\mapsto \begin{cases} v_i & \text{if } i\in T\\ 0 & \text{if } i\in I\cup\bar{I}\setminus T.\end{cases}$$

Finally set $\mathcal{S}:=\mathcal{V}$ and $\mathcal{S}^\ell:=\mathcal{V}^\ell$ if $\mathcal{G}:=\mathfrak{o}_B(I)$ and take $\mathcal{S}$ and $\mathcal{S}^\ell:=\mathcal{S}_{I_0}$ to be as in (10) and (11) respectively if $\mathcal{G}=\mathfrak{sp}(I)$.

One can use [2] together with the same argument as in §3.0.1 to get the following theorem:

Theorem 3.12. *Suppose that $\mathcal{L}$ is a Lie algebra graded by a root system R of type $X=B_I$ or C_I with grading pair $(\mathcal{G},\mathcal{H})$ and let S be the irreducible full subsystem of R of type B_{I_0} or C_{I_0} respectively.*

(i) Consider $\mathcal{L}^\ell := \mathcal{L}^{^S}$ as a $\mathcal{G}^\ell$-module and let

$$\mathcal{L}^\ell = \bigoplus_{i\in\mathcal{I}}\mathfrak{g}_i \oplus \bigoplus_{j\in\mathcal{J}}\mathfrak{s}_j\oplus E \tag{52}$$

be the decomposition of $\mathcal{L}^\ell$ into finite dimensional irreducible $\mathcal{G}^\ell$-submodules in which E is a trivial $\mathcal{G}^\ell$-submodule, $\mathcal{I},\mathcal{J}$ are index sets, $\mathfrak{g}_i$ $(i\in\mathcal{I})$ is isomorphic to $\mathcal{G}^\ell$ and $\mathfrak{s}_j$ $(j\in\mathcal{J})$ is isomorphic to $\mathcal{S}^\ell$. Then there exists a class $\{\mathcal{D}_n,\mathcal{G}_i,\mathcal{S}_j \mid i\in\mathcal{I}, j\in\mathcal{J}\}$ of finite dimensional $\mathcal{G}$-submodules of $\mathcal{L}$ such that

- *$\mathcal{D}_n$ is a trivial $\mathcal{G}$-module, $\mathcal{G}_i$ is isomorphic to $\mathcal{G}$ and $\mathcal{S}_j$ is isomorphic to $\mathcal{S}$, for $i\in\mathcal{I}, j\in\mathcal{J}$,*

- *$\mathfrak{g}_i\subseteq\mathcal{G}_i,\ \mathfrak{s}_j\subseteq\mathcal{S}_j\ (i\in\mathcal{I},\ j\in\mathcal{J})$,*

- *$\mathcal{L}=\bigoplus_{i\in\mathcal{I}}\mathcal{G}_i\oplus\bigoplus_{j\in\mathcal{J}}\mathcal{S}_j\oplus\mathcal{D}_n$.*

(ii) Take $\mathcal{A}$ and $\mathcal{B}$ to be vector spaces with bases $\{a_i\mid i\in\mathcal{I}\}$ and $\{b_j\mid j\in\mathcal{J}\}$ respectively and identify $\mathcal{L}$ with $(\mathcal{G}\otimes\mathcal{A})\oplus(\mathcal{S}\otimes\mathcal{B})\oplus\mathcal{D}_n$, say via the natural identification

$$\varphi:\mathcal{L}\longrightarrow(\mathcal{G}\otimes\mathcal{A})\oplus(\mathcal{S}\otimes\mathcal{B})\oplus\mathcal{D}_n.$$

Transfer the Lie algebraic structure of $\mathcal{L}$ to $(\mathcal{G}\otimes\mathcal{A})\oplus(\mathcal{S}\otimes\mathcal{B})\oplus\mathcal{D}_n$. Then $\mathcal{D}_\ell:=\varphi(E)$ and $\mathcal{D}_n$ are subalgebras of $(\mathcal{G}\otimes\mathcal{A})\oplus(\mathcal{S}\otimes\mathcal{B})\oplus\mathcal{D}_n$.

(iii) Set $\mathfrak{a}:=\mathcal{A}\oplus\mathcal{B}$. If $\mathcal{G}=\mathfrak{o}_B(I)$, $\mathcal{A}$ is equipped with a unital commutative associative algebraic structure and the vector space $\mathcal{B}$ is equipped with a unital $\mathcal{A}$-module structure. Also there is a symmetric $\mathcal{A}$-bilinear form $f:\mathcal{B}\times\mathcal{B}\longrightarrow\mathcal{A}$ such that $\mathfrak{a}=\mathcal{J}(f,\mathcal{B})$. Also if $\mathcal{G}=\mathfrak{sp}(I)$, $\mathfrak{a}$ is equipped with a star algebraic structure such that $\mathcal{A}$ (resp. $\mathcal{B}$) is the set of fixed (resp. skew fixed) points of $\mathfrak{a}$ under the involution on $\mathfrak{a}$.

(iv) There is a subspace $\mathcal{K}_1$ of the full skew-dihedral homology group of $\mathfrak{a}$ with respect to n and a subspace $\mathcal{K}_2$ of the full skew-dihedral homology group of $\mathfrak{a}$ with respect to ℓ such that $\mathcal{D}_n$ and $\mathcal{D}_\ell$ are isomorphic to the quotient algebras $\{\mathfrak{a},\mathfrak{a}\}_n/\mathcal{K}_1$ and $\{\mathfrak{a},\mathfrak{a}\}_\ell/\mathcal{K}_2$ respectively, with corresponding isomorphisms $\psi_1\colon\{\mathfrak{a},\mathfrak{a}\}_n/\mathcal{K}_1\longrightarrow\mathcal{D}_n$ and $\psi_2\colon\{\mathfrak{a},\mathfrak{a}\}_\ell/\mathcal{K}_2\longrightarrow\mathcal{D}_\ell$.

(v) For $\alpha,\alpha'\in\mathfrak{a}$, take

$$\langle\alpha,\alpha'\rangle_n:=\psi_1(\{\alpha,\alpha'\}_n+\mathcal{K}_1)\quad\text{and}\quad\langle\alpha,\alpha'\rangle_\ell:=\psi_2(\{\alpha,\alpha'\}_\ell+\mathcal{K}_2).$$

Then if $\alpha,\alpha'\in\mathcal{A}$ or $\alpha,\alpha'\in\mathcal{B}$, we have

$$\langle\alpha,\alpha'\rangle_n=\langle\alpha,\alpha'\rangle_\ell+((\frac{1}{n}Id_{\mathcal{V}}-\frac{1}{\ell}Id_{\mathcal{V}^\ell})\otimes(1/2)(\alpha\alpha'-\alpha'\alpha)).$$

(vi) For $\alpha,\alpha'\in\mathfrak{a}$, set

$$\langle\alpha,\alpha'\rangle^n:=\varphi^{-1}(\langle\alpha,\alpha'\rangle_n)\quad\text{and}\quad\langle\alpha,\alpha'\rangle^\ell:=\varphi^{-1}(\langle\alpha,\alpha'\rangle_\ell).$$

If $\ell<n$, then for $t\in\mathbb{N}\setminus\{0\}$, $a_1,a_1',\dots,a_t,a_t'\in\mathcal{A}$ and $b_1,b_1',\dots,b_t,b_t'\in\mathcal{B}$, we have $\sum_{i=1}^t\langle a_i,a_i'\rangle^\ell+\sum_{i=1}^t\langle b_i,b_i'\rangle^\ell=0$ if and only if

$$\sum_{i=1}^t\langle a_i,a_i'\rangle^n+\sum_{i=1}^t\langle b_i,b_i'\rangle^n=0\quad\text{and}\quad\sum_{i=1}^t[a_i,a_i']+\sum_{i=1}^t[b_i,b_i']=0.$$

(vii) For $e,f\in\mathcal{G}\cup\mathcal{S}$, set

$$e\circ f:=ef+fe-\frac{tr(ef)}{\ell}Id_{\mathcal{V}^\ell},$$

and for $\alpha,\alpha'\in\mathfrak{a}$, *set*

$$\langle\alpha,\alpha'\rangle:=\langle\alpha,\alpha'\rangle_\ell,$$

then the Lie bracket on $(\mathcal{G}\otimes\mathcal{A})\oplus(\mathcal{S}\otimes\mathcal{B})\oplus\mathcal{D}_n=((\mathcal{G}\otimes\mathcal{A})\oplus(\mathcal{S}\otimes\mathcal{B}))+\mathcal{D}_\ell$ *is given by*

$$\begin{aligned}
&[x\otimes a,y\otimes a']=[x,y]\otimes aa'+tr(xy)\langle a,a'\rangle,\\
&[x\otimes a,s\otimes b]=xs\otimes ab,\\
&[s\otimes b,t\otimes b']=D_{s,t}\otimes f(b,b')+(s,t)\langle b,b'\rangle,\\
&[\langle\alpha,\alpha'\rangle,x\otimes a]=x\otimes d^{\ell,\mathfrak{a}}_{\alpha,\alpha'}(a),\\
&[\langle\alpha,\alpha'\rangle,s\otimes b]=s\otimes d^{\ell,\mathfrak{a}}_{\alpha,\alpha'}(b),\\
&[\langle\alpha_1,\alpha_2\rangle,\langle\alpha'_1,\alpha'_2\rangle]=\langle d^{\ell,\mathfrak{a}}_{\alpha_1,\alpha_2}(\alpha'_1),\alpha'_2\rangle+\langle\alpha'_1,d^{\ell,\mathfrak{a}}_{\alpha_1,\alpha_2}(\alpha'_2)\rangle,
\end{aligned}\tag{53}$$

(see Definition 2.4) for $x,y\in\mathcal{G},\ s,t\in\mathcal{S},\ a,a'\in\mathcal{A},\ b,b'\in\mathcal{B},$ *and* $\alpha,\alpha',\alpha_1,\alpha_2,\alpha'_1,\alpha'_2\in\mathfrak{a}$ *if* $\mathcal{G}=\mathfrak{o}_B(I)$ *and it is given by*

$$\begin{aligned}
&[x\otimes a,y\otimes a']=[x,y]\otimes\tfrac{1}{2}(a\circ a')+(x\circ y)\otimes\tfrac{1}{2}[a,a']+tr(xy)\langle a,a'\rangle,\\
&[x\otimes a,s\otimes b]=(x\circ s)\otimes\tfrac{1}{2}[a,b]+[x,s]\otimes\tfrac{1}{2}(a\circ b),\\
&[s\otimes b,t\otimes b']=[s,t]\otimes\tfrac{1}{2}(b\circ b')+(s\circ t)\otimes\tfrac{1}{2}[b,b']+tr(st)\langle b,b'\rangle,\\
&[\langle\alpha,\alpha'\rangle,x\otimes a]=\tfrac{-1}{4\ell}((x\circ Id_{\mathcal{V}^\ell})\otimes[a,\beta^*_{\alpha,\alpha'}]+[x,Id_{\mathcal{V}^\ell}]\otimes(a\circ\beta^*_{\alpha,\alpha'})),\\
&[\langle\alpha,\alpha'\rangle,s\otimes b]=\tfrac{-1}{4\ell}([s,Id_{\mathcal{V}^\ell}]\otimes(b\circ\beta^*_{\alpha,\alpha'})+(s\circ Id_{\mathcal{V}^\ell})\otimes[b,\beta^*_{\alpha,\alpha'}]+2tr(sId_{\mathcal{V}^\ell})\langle b,\beta^*_{\alpha,\alpha'}\rangle),\\
&[\langle\alpha_1,\alpha_2\rangle,\langle\alpha'_1,\alpha'_2\rangle]=\langle d^{\ell,\mathfrak{a}}_{\alpha_1,\alpha_2}(\alpha'_1),\alpha'_2\rangle+\langle\alpha'_1,d^{\ell,\mathfrak{a}}_{\alpha_1,\alpha_2}(\alpha'_2)\rangle,
\end{aligned}\tag{54}$$

(see (27)) for $x,y\in\mathcal{G},\ s,t\in\mathcal{S},\ a,a'\in\mathcal{A},\ b,b'\in\mathcal{B},\ \alpha,\alpha',\alpha_1,\alpha_2,\alpha'_1,\alpha'_2\in\mathfrak{a}$ *if* $\mathcal{G}=\mathfrak{sp}(I)$.

4. Root graded Lie algebras - general case

In this section, we give certain recognition theorems which characterize Lie algebras graded by an infinite irreducible locally finite root system. The main target of the present section is to generalize the decomposition (21) for Lie algebras graded by infinite root systems. For a Lie algebra $\mathcal{L}$ graded by an infinite irreducible locally finite root system with grading pair $(\mathfrak{g},\mathfrak{h})$, we first decompose $\mathcal{L}$ as a direct sum of a certain subalgebra of $\mathcal{L}$ and a certain locally finite completely reducible $\mathfrak{g}$-submodule. This in particular results in a generalized decomposition for $\mathcal{L}$ as in (21). We next reconstruct the structure of $\mathcal{L}$ in terms of the ingredients involved in this decomposition. Moreover, we prove that any Lie algebra graded by an irreducible infinite locally finite root system arises in this way. As in the previous section, we concentrate our attention on type BC and for other types, we just report the results.

4.1. Recognition Theorem for Type BC**.** Suppose that I is an infinite index set and ℓ is a positive integer greater than 3. We assume R is an irreducible locally finite root system of type BC_I and take $\mathcal{G}$, $\mathcal{S}$ and $\mathcal{V}$ to be as in Lemmas 2.7 and 2.12. We show that an R-graded Lie algebra $\mathcal{L}$ can be decomposed into

$$(\mathcal{G}\otimes\mathcal{A})\oplus(\mathcal{S}\otimes\mathcal{B})\oplus(\mathcal{V}\otimes\mathcal{C})\oplus\mathcal{D}\tag{55}$$

in which $\mathcal{A},\mathcal{B}$ and $\mathcal{C}$ are vector spaces and $\mathcal{D}$ is a subalgebra of $\mathcal{L}$. We equip $\mathfrak{b} := \mathcal{A}\oplus\mathcal{B}\oplus\mathcal{C}$ with a unital associative star algebraic structure and show that $\mathcal{D}$ can be expressed as a quotient of the algebra $\{\mathfrak{b},\mathfrak{b}\}_\ell$ by a subspace of the full skew-dihedral homology group of $\mathfrak{b}$, with respect to ℓ, satisfying the uniform property on $\mathfrak{b}$. Conversely, for vector spaces A, B, C and D with specific natures, we form the decomposition (55), equip it with a Lie bracket and show that it is an R-graded Lie algebra.

Theorem 4.1. *Suppose that I is an infinite index set and ℓ is an integer greater than 3. Assume R is an irreducible locally finite root system of type BC_I and $\mathcal{V}$ is a vector space with a basis $\{v_i \mid i\in I\cup\bar I\}$. Suppose that $(\cdot,\cdot)$ is a bilinear form as in (6), set $\mathcal{G} := \mathfrak{sp}(I)$ and consider $\mathcal{S}$ as in (10). Fix a subset I_0 of I of cardinality ℓ and take R_0 to be the full irreducible subsystem of R of type BC_{I_0}. Suppose that $\{R_\lambda \mid \lambda\in\Lambda\}$ is the class of all finite irreducible full subsystems of R containing R_0, where Λ is an index set containing zero. For $\lambda\in\Lambda$, take $\mathcal{G}^\lambda$ as in Lemma 7 and $\mathcal{V}^\lambda,\mathcal{S}^\lambda$ as in (11). Next define*

$$\begin{aligned}\mathfrak{I}_\lambda : \mathcal{V} &\longrightarrow \mathcal{V}\\ v_i &\mapsto \begin{cases} v_i & i\in I_\lambda\cup\bar I_\lambda\\ 0 & \textit{otherwise}\end{cases}\end{aligned}$$

and for $e,f\in\mathcal{G}\cup\mathcal{S}$, define

$$e\circ f := ef+fe-\frac{tr(ef)}{l}\mathfrak{I}_0.$$

(i) Suppose that $(\mathfrak{a},*,\mathcal{C},f)$ is a coordinate quadruple of type BC and $\mathcal{A}$, $\mathcal{B}$ are $*$-fixed and $*$-skew fixed points of $\mathfrak{a}$ respectively. Set $\mathfrak{b} := \mathfrak{b}(\mathfrak{a},*,\mathcal{C},f)$ and take $[\cdot,\cdot],\circ,\heartsuit,\diamond$ to be as in Subsection 3. For $\beta_1,\beta_2\in\mathfrak{b}$, consider $d^{\ell,\mathfrak{b}}_{\beta_1,\beta_2}$ as in (25) and take $\beta^*_{\beta_1,\beta_2}, \beta^*_1$ and β^*_2 as in Proposition 3.10 and (27). For a subset $\mathcal{K}$ of $HF(\mathfrak{b})$ satisfying the uniform property on $\mathfrak{b}$, set

$$\mathcal{L}(\mathfrak{b},\mathcal{K}) := (\mathcal{G}\otimes\mathcal{A})\oplus(\mathcal{S}\otimes\mathcal{B})\oplus(\mathcal{V}\otimes\mathcal{C})\oplus(\{\mathfrak{b},\mathfrak{b}\}_\ell/\mathcal{K}).$$

Then setting $\langle\beta,\beta'\rangle := \{\beta,\beta'\}+\mathcal{K}$, $\beta,\beta'\in\mathfrak{b}$, $\mathcal{L}(\mathfrak{b},\mathcal{K})$ together with

$$\begin{aligned}
&[x\otimes a, y\otimes a'] = [x,y]\otimes\tfrac12(a\circ a') + (x\circ y)\otimes\tfrac12[a,a'] + tr(xy)\langle a,a'\rangle,\\
&[x\otimes a, s\otimes b] = (x\circ s)\otimes\tfrac12[a,b] + [x,s]\otimes\tfrac12(a\circ b) = -[s\otimes b, x\otimes a],\\
&[s\otimes b, t\otimes b'] = [s,t]\otimes\tfrac12(b\circ b') + (s\circ t)\otimes\tfrac12[b,b'] + tr(st)\langle b,b'\rangle,\\
&[x\otimes a, u\otimes c] = xu\otimes a\cdot c = -[u\otimes c, x\otimes a],\\
&[s\otimes b, u\otimes c] = su\otimes b\cdot c = -[u\otimes c, s\otimes b],\\
&[u\otimes c, v\otimes c'] = (u\circ v)\otimes(c\diamond c') + [u,v]\otimes(c\heartsuit c') + (u,v)\langle c,c'\rangle,\\
&[\langle\beta_1,\beta_2\rangle, x\otimes a] = \tfrac{-1}{4\ell}((x\circ\mathfrak{I}_0)\otimes[a,\beta^*_{\beta_1,\beta_2}] + [x,\mathfrak{I}_0]\otimes(a\circ\beta^*_{\beta_1,\beta_2})),\\
&[\langle\beta_1,\beta_2\rangle, s\otimes b]=\tfrac{-1}{4\ell}([s,\mathfrak{I}_0]\otimes(b\circ\beta^*_{\beta_1,\beta_2})+(s\circ\mathfrak{I}_0)\otimes[b,\beta^*_{\beta_1,\beta_2}]+2tr(s\mathfrak{I}_0)\langle b,\beta^*_{\beta_1,\beta_2}\rangle),\\
&[\langle\beta_1,\beta_2\rangle, v\otimes c] = \tfrac{1}{2\ell}\mathfrak{I}_0 v\otimes(\beta^*_{\beta_1,\beta_2}\cdot c) - \tfrac12 v\otimes(f(c,\beta^*_2)\cdot\beta^*_1 + f(c,\beta^*_1)\cdot\beta^*_2)\\
&[\langle\beta_1,\beta_2\rangle, \langle\beta'_1,\beta'_2\rangle] = \langle d^\ell_{\beta_1,\beta_2}(\beta'_1),\beta'_2\rangle + \langle\beta'_1, d^\ell_{\beta_1,\beta_2}(\beta'_2)\rangle
\end{aligned}\tag{56}$$

for $x,y\in\mathcal{G}$, $s,t\in\mathcal{S}$, $u,v\in\mathcal{V}$, $a,a'\in\mathcal{A}$, $b,b'\in\mathcal{B}$, $c,c'\in\mathcal{C}$, $\beta_1,\beta_2,\beta'_1,\beta'_2\in\mathfrak{b}$, is an R-graded Lie algebra with grading pair $(\mathcal{G},\mathcal{H})$ where $\mathcal{H}$ is the splitting Cartan subalgebra of $\mathcal{G}$ defined in Lemma 2.7.

(ii) If $\mathcal{L}$ is an R-graded Lie algebra with grading pair $(\mathfrak{g},\mathfrak{h})$, then there is a coordinate quadruple $(\mathfrak{a},*,\mathcal{C},f)$ of type BC and a subspace $\mathcal{K}$ of $\mathfrak{b}:=\mathfrak{b}(\mathfrak{a},*,\mathcal{C},f)$ satisfying the uniform property on $\mathfrak{b}$ such that $\mathcal{L}$ is isomorphic to $\mathcal{L}(\mathfrak{b},\mathcal{K})$.

Remark 4.2. One can check that up to isomorphism the Lie algebra $\mathcal{L}(\mathfrak{b},\mathcal{K})$ does not depend on the choice of ℓ and I_0.

Proof. (i) We prove that $\mathcal{L}(\mathfrak{b},\mathcal{K})$ together with (56) is a Lie algebra. For $\lambda\in\Lambda$ set $n_\lambda:=|I_\lambda|$ and $\mathcal{L}^\lambda:=(\mathcal{G}^\lambda\dot{\otimes}\mathcal{A})\oplus(\mathcal{S}^\lambda\dot{\otimes}\mathcal{B})\oplus(\mathcal{V}^\lambda\dot{\otimes}\mathcal{C})\oplus\langle\mathfrak{b},\mathfrak{b}\rangle$. Also for $a,a'\in\mathcal{A},b,b'\in\mathcal{B}$, and $c,c'\in\mathcal{C}$, set

$$\begin{aligned}
&\langle a,a'\rangle_\lambda:=(((\tfrac{-1}{\ell}\mathfrak{I}_0+\tfrac{1}{n_\lambda}\mathfrak{I}_\lambda)\otimes\tfrac{1}{2}[a,a'])+\langle a,a'\rangle,\\
&\langle b,b'\rangle_\lambda:=((\tfrac{-1}{\ell}\mathfrak{I}_0+\tfrac{1}{n_\lambda}\mathfrak{I}_\lambda)\otimes\tfrac{1}{2}[b,b'])+\langle b,b'\rangle,\\
&\langle c,c'\rangle_\lambda:=((\tfrac{1}{\ell}\mathfrak{I}_0-\tfrac{1}{n_\lambda}\mathfrak{I}_\lambda)\otimes\tfrac{1}{2}(c\heartsuit c'))+\langle c,c'\rangle,\\
&\langle a,b\rangle_\lambda=\langle b,c\rangle_\lambda=\langle a,c\rangle_\lambda:=0.
\end{aligned}$$

Take $\langle\mathfrak{b},\mathfrak{b}\rangle_\lambda:=\operatorname{span}\{\langle a,a'\rangle_\lambda,\langle b,b'\rangle_\lambda,\langle c,c'\rangle_\lambda\mid a,a'\in\mathcal{A},b,b'\in\mathcal{B},c,c'\in\mathcal{C}\}$ and note that as $\mathcal{K}$ satisfies the uniform property on $\mathfrak{b}$, we have

$$\mathcal{L}^\lambda:=(\mathcal{G}^\lambda\dot{\otimes}\mathcal{A})\oplus(\mathcal{S}^\lambda\dot{\otimes}\mathcal{B})\oplus(\mathcal{V}^\lambda\dot{\otimes}\mathcal{C})\oplus\langle\mathfrak{b},\mathfrak{b}\rangle_\lambda.$$

For $\beta=a+b+c,\beta'=a'+b'+c'\in\mathfrak{b}$, set $\langle\beta,\beta'\rangle_\lambda:=\langle a,a'\rangle_\lambda+\langle b,b'\rangle_\lambda+\langle c,c'\rangle_\lambda$.

Now consider the linear transformation $\psi:\mathfrak{b}\otimes\mathfrak{b}\longrightarrow\langle\mathfrak{b},\mathfrak{b}\rangle_\lambda$ mapping $\beta\otimes\beta'$ to $\langle\beta,\beta'\rangle_\lambda$. It is not difficult to see that for the subspace K of $\mathfrak{b}\otimes\mathfrak{b}$ defined in Subsection 3, $\psi(K)=\{0\}$. So ψ induces a linear transformation $\dot{\psi}:\{\mathfrak{b},\mathfrak{b}\}_{n_\lambda}\longrightarrow\langle\mathfrak{b},\mathfrak{b}\rangle_\lambda$ mapping $\{\beta,\beta'\}_{n_\lambda}$ to $\langle\beta,\beta'\rangle_\lambda$, $\beta,\beta'\in\mathfrak{b}$. Take $\mathcal{K}_\lambda$ to be the kernel of $\dot{\psi}$. If $t\in\mathbb{N}\setminus\{0\}$, $a_i,a_i'\in\mathcal{A}$, $b_i,b_i'\in\mathcal{B}$ and $c_i,c_i'\in\mathcal{C}$ $(1\leq i\leq t)$ are such that $\sum_{i=1}^t(\{a_i,a_i'\}_{n_\lambda}+\{b_i,b_i'\}_{n_\lambda}+\{c_i,c_i'\}_{n_\lambda})\in\mathcal{K}_\lambda$, then $\sum_{i=1}^t(\langle a_i,a_i'\rangle_\lambda+\langle b_i,b_i'\rangle_\lambda+\langle c_i,c_i'\rangle_\lambda)=0$. This implies that

$$((\frac{-1}{\ell}\mathfrak{I}_0+\frac{1}{n_\lambda}\mathfrak{I}_\lambda)\otimes\frac{1}{2}\sum_{i=1}^t([a_i,a_i']+[b_i,b_i']-(c_i\heartsuit c_i'))+\sum_{i=1}^t(\langle a_i,a_i'\rangle+\langle b_i,b_i'\rangle+\langle c_i,c_i'\rangle)=0. \quad (57)$$

This in turn implies that $\sum_{i=1}^t(\langle a_i,a_i'\rangle+\langle b_i,b_i'\rangle+\langle c_i,c_i'\rangle)=0$. Therefore we get that $\mathcal{K}_\lambda$ is a subset of the full skew-dihedral homology group of $\mathfrak{b}$ with respect to ℓ. But if $\lambda\neq0$, (57) implies that $\sum_{i=1}^t([a_i,a_i']+[b_i,b_i']-(c_i\heartsuit c_i'))=0$. Now it follows using this together with the fact that $\mathcal{K}_\lambda$ is a subset of the full skew-dihedral homology group of $\mathfrak{b}$ with respect to ℓ, that $\mathcal{K}_\lambda$ is a subset of the full skew-dihedral homology group of $\mathfrak{b}$ with respect to n_λ. Now it follows from [3, Chapter V] that $\mathcal{L}^\lambda$ together with the product introduced in (56) restricted to $\mathcal{L}^\lambda\times\mathcal{L}^\lambda$ defines a Lie algebra. Therefore $\mathcal{L}$ together with $[\cdot,\cdot]$ is a Lie algebra as $\mathcal{L}=\cup_{\lambda\in\Lambda}\mathcal{L}^\lambda$. Now one can easily see that $\mathcal{L}$ has a weight space decomposition $\mathcal{L}=\oplus_{\alpha\in R}\mathcal{L}_\alpha$ with respect to $\mathcal{H}$ in which

$$\mathcal{L}_\alpha=\begin{cases}\mathcal{V}_\alpha\dot{\otimes}\mathcal{C} & \text{if } \alpha\in R_{sh}\\ (\mathcal{G}_\alpha\dot{\otimes}\mathcal{A})\oplus(\mathcal{S}_\alpha\dot{\otimes}\mathcal{B}) & \text{if } \alpha\in R_{lg}\\ \mathcal{G}_\alpha\dot{\otimes}\mathcal{A} & \text{if } \alpha\in R_{ex}\\ (\mathcal{G}_0\dot{\otimes}\mathcal{A})\oplus(\mathcal{S}_0\dot{\otimes}\mathcal{B})\oplus\langle\mathfrak{b},\mathfrak{b}\rangle & \text{if } \alpha=0\end{cases}$$

and that $\mathcal{L}$ is an R-graded Lie algebra with grading pair $(\mathcal{G},\mathcal{H})$.

(ii) For $\lambda\in\Lambda$, set

$$\mathcal{L}^\lambda := \textstyle\sum_{\alpha\in R_\lambda^\times}\mathcal{L}_\alpha\oplus\sum_{\alpha\in R_\lambda^\times}[\mathcal{L}_\alpha,\mathcal{L}_{-\alpha}],$$

$$\mathfrak{g}^\lambda := \textstyle\sum_{\alpha\in(R_\lambda)^\times_{sdiv}}\mathfrak{g}_\alpha\oplus\sum_{\alpha\in(R_\lambda)^\times_{sdiv}}[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]$$

and note that $\mathfrak{g}^\lambda$ is isomorphic to $\mathcal{G}^\lambda$. We know by Lemma 3.3 that $\mathcal{L}^\lambda$ is an R_λ-graded Lie algebra with grading pair $(\mathfrak{g}^\lambda,\mathfrak{h}^\lambda:=\mathfrak{g}^\lambda\cap\mathfrak{h})$. Consider $\mathcal{L}^0$ as a $\mathfrak{g}^0$-module and suppose that $\{\mathcal{G}_i^0,\mathcal{S}_j^0,\mathcal{V}_t^0,\mathcal{D}_0\mid i\in\mathcal{I},j\in\mathcal{J},t\in\mathcal{T}\}$ is a class of finite dimensional $\mathfrak{g}^0$-submodules of $\mathcal{L}^0$ such that

- $\mathcal{L}^0=\sum_{i\in\mathcal{I}}\mathcal{G}_i^0\oplus\sum_{j\in\mathcal{J}}\mathcal{S}_j^0\oplus\sum_{i\in\mathcal{T}}\mathcal{V}_t^0\oplus\mathcal{D}_0$,
- $\mathcal{D}_0$ is a trivial $\mathfrak{g}^0$-submodule of $\mathcal{L}^0$,
- for $i\in\mathcal{I},j\in\mathcal{J}$ and $t\in\mathcal{T}$, $\mathcal{G}_i^0$ is isomorphic to $\mathcal{G}^0$, $\mathcal{S}_j^0$ is isomorphic to $\mathcal{S}^0$, and $\mathcal{V}_t^0$ is isomorphic to $\mathcal{V}^0$.

Now for $\lambda\in\Lambda$, consider $\mathcal{L}^\lambda$ as a $\mathfrak{g}^\lambda$-module via the adjoint representation. Using Lemmas 3.4 and 3.5, one finds finite dimensional irreducible $\mathfrak{g}^\lambda$-submodules $\mathcal{G}_i^\lambda$, $\mathcal{S}_j^\lambda$, $\mathcal{V}_t^\lambda$ $(i\in\mathcal{I},j\in\mathcal{J},t\in\mathcal{T})$ of $\mathcal{L}^\lambda$ and a trivial $\mathfrak{g}^\lambda$-submodule $\mathcal{D}_\lambda$ such that

$$(\mathcal{I},\mathcal{J},\mathcal{T},\{\mathcal{G}_i^0\},\{\mathcal{G}_i^\lambda\},\{\mathcal{S}_j^0\},\{\mathcal{S}_j^\lambda\},\{\mathcal{V}_t^0\},\{\mathcal{V}_t^\lambda\},\mathcal{D}_0,\mathcal{D}_\lambda)$$

is an $(R^\lambda,R^0)-$datum for the pair $(\mathcal{L}^\lambda,\mathcal{L}^0)$ (see (33)). We know from Subsection 3 that there is a coordinate quadruple $(\mathfrak{a},*,\mathcal{C},f)$ of type BC and a subspace $\mathcal{K}_\lambda$ of the full skew-dihedral homology group of $\mathfrak{b}:=\mathfrak{b}(\mathfrak{a},*,\mathcal{C},f)$ with respect to $n_\lambda=|I_\lambda|$ such that $\mathcal{D}_\lambda$ is a subalgebra of $\mathcal{L}^\lambda$ isomorphic to the quotient algebra $\{\mathfrak{b},\mathfrak{b}\}_{n_\lambda}/\mathcal{K}_\lambda$, say via $\phi_\lambda:\{\mathfrak{b},\mathfrak{b}\}_{n_\lambda}/\mathcal{K}_\lambda\longrightarrow\mathcal{D}_\lambda$. Now for $\beta,\beta'\in\mathfrak{b}$, set

$$\langle\beta,\beta'\rangle^\lambda:=\phi_\lambda(\{\beta,\beta'\}_\lambda+\mathcal{K}_\lambda). \tag{58}$$

Take $\mathcal{A}$ and $\mathcal{B}$ to be the $*-$fixed and $*-$skew fixed points of $\mathfrak{a}$ respectively and note that

$$\mathcal{D}_\lambda=\mathrm{span}\{\langle a,a'\rangle^\lambda,\langle b,b'\rangle^\lambda,\langle c,c'\rangle^\lambda\mid a,a'\in\mathcal{A},b,b'\in\mathcal{B},c,c'\in\mathcal{C}\}.$$

We now proceed with the proof in the following steps:

Step 1: For $i\in\mathcal{I}$, $j\in\mathcal{J}$, $t\in\mathcal{T}$ and $\lambda,\mu\in\Lambda$ with $\lambda\preccurlyeq\mu$, $\mathcal{G}_i^\mu$ is the $\mathfrak{g}^\mu$-submodule of $\mathcal{L}^\mu$ generated by $\mathcal{G}_i^\lambda$, $\mathcal{S}_j^\mu$ is the $\mathfrak{g}^\mu$-submodule of $\mathcal{L}^\mu$ generated by $\mathcal{S}_j^\lambda$ and $\mathcal{V}_t^\mu$ is the $\mathfrak{g}^\mu$-submodule of $\mathcal{L}^\mu$ generated by $\mathcal{V}_t^\lambda$. In other words,

$$(\mathcal{I},\mathcal{J},\mathcal{T},\{\mathcal{G}_i^\lambda\},\{\mathcal{G}_i^\mu\},\{\mathcal{S}_j^\lambda\},\{\mathcal{S}_j^\mu\},\{\mathcal{V}_t^\lambda\},\{\mathcal{V}_t^\mu\},\mathcal{D}_\lambda,\mathcal{D}_\mu)$$

is an $(R^\mu,R^\lambda)-$datum for the pair $(\mathcal{L}^\mu,\mathcal{L}^\lambda)$: It is immediate using the fact that $\mathfrak{g}^\lambda$ is a subalgebra of $\mathfrak{g}^\mu$.

Step 2: For $\lambda\in\Lambda$, $\mathcal{L}^\lambda=\sum_{i\in\mathcal{I}}\mathcal{G}_i^\lambda\oplus\sum_{j\in\mathcal{J}}\mathcal{S}_j^\lambda\oplus\sum_{i\in\mathcal{T}}\mathcal{V}_t^\lambda\oplus\mathcal{D}_0$: By Step 1 and Remark 3.9, $\mathcal{L}^\lambda=(\sum_{i\in\mathcal{I}}\mathcal{G}_i^\lambda\oplus\sum_{j\in\mathcal{J}}\mathcal{S}_j^\lambda\oplus\sum_{i\in\mathcal{T}}\mathcal{V}_t^\lambda)+\mathcal{D}_0$. Suppose $d\in\mathcal{D}_0$,

$x \in \sum_{i\in\mathcal{I}}\mathcal{G}_i^\lambda \oplus \sum_{j\in\mathcal{J}}\mathcal{S}_j^\lambda \oplus \sum_{i\in\mathcal{T}}\mathcal{V}_t^\lambda$ and $x+d=0$. Since $d\in\mathcal{D}_0$, there are $t\in\mathbb{N}\setminus\{0\}$, $a_i,a_i'\in\mathcal{A}$, $b_i,b_i'\in\mathcal{B}$ and $c_i,c_i'\in\mathcal{C}$ $(1\le i\le t)$ such that $d=\sum_{i=1}^t\langle a_i,a_i'\rangle^0+\langle b_i,b_i'\rangle^0+\langle c_i,c_i'\rangle^0$. It follows from Step 1 and Lemma 3.7 that there is $y\in\sum_{i\in\mathcal{I}}\mathcal{G}_i^\lambda\oplus\sum_{j\in\mathcal{J}}\mathcal{S}_j^\lambda\oplus\sum_{i\in\mathcal{T}}\mathcal{V}_t^\lambda$ such that $d=y+\sum_{i=1}^t\langle a_i,a_i'\rangle^\lambda+\langle b_i,b_i'\rangle^\lambda+\langle c_i,c_i'\rangle^\lambda$. Now as $0=x+d=x+y+\sum_{i=1}^t\langle a_i,a_i'\rangle^\lambda+\langle b_i,b_i'\rangle^\lambda+\langle c_i,c_i'\rangle^\lambda$, we get that $x+y=0$ and $\sum_{i=1}^t(\langle a_i,a_i'\rangle^\lambda+\langle b_i,b_i'\rangle^\lambda+\langle c_i,c_i'\rangle^\lambda)=0$. Take $\mu\in\Lambda$ to be such that $\lambda\preccurlyeq\mu$, then using Step 1, one gets that the pairs $(\mathcal{L}^\lambda,\mathcal{L}^\mu)$ and $(\mathcal{L}^0,\mathcal{L}^\lambda)$ play the same role as the pair $(\mathcal{L}^\ell,\mathcal{L}^n)$ in Subsection 3. Using Remark 3.9 for the pair $(\mathcal{L}^\lambda,\mathcal{L}^\mu)$, one gets that $\sum_{i=1}^t(\langle a_i,a_i'\rangle^\mu+\langle b_i,b_i'\rangle^\mu+\langle c_i,c_i'\rangle^\mu)=0$ and $\sum_{i=1}^t([a_i,a_i']+[b_i,b_i']-c_i\heartsuit c_i')=0$. Next using Remark 3.9 for the pair $(\mathcal{L}^0,\mathcal{L}^\mu)$, we get that $d=\sum_{i=1}^t(\langle a_i,a_i'\rangle^0+\langle b_i,b_i'\rangle^0+\langle c_i,c_i'\rangle^0)=0$. This completes the proof of this step.

Step 3: $\mathcal{K}_0$ satisfies the uniform property on $\mathfrak{b}$: Suppose that

$$\sum_{i=1}^t(\{a_i,a_i'\}_\ell+\{b_i,b_i'\}_\ell+\{c_i,c_i'\}_\ell)\in\mathcal{K}_0,$$

for $t\in\mathbb{N}\setminus\{0\}$, $a_1,a_1',\ldots,a_t,a_t'\in\mathcal{A}$, $b_1,b_1',\ldots,b_t,b_t'\in\mathcal{B}$ and $c_1,c_1',\ldots,c_n,c_n'\in\mathcal{C}$, so $\sum_{i=1}^n(\langle a_i,a_i'\rangle^0+\langle b_i,b_i'\rangle^0+\langle c_i,c_i'\rangle^0)=0$. Now take $\lambda\in\Lambda\setminus\{0\}$, then by Step 1, $(\mathcal{L}^0,\mathcal{L}^\lambda)$ plays the same role as the pair $(\mathcal{L}^\ell,\mathcal{L}^n)$ in Subsection 3 and so an argument analogous to the proof of Step 2 shows that

$$\sum_{i=1}^t([a_i,a_i']+[b_i,b_i']-c_i\heartsuit c_i')=0.$$

This completes the proof.

Using Steps 1,2, we get the following three steps:

Step 4:

$$\bigcup_{\lambda\in\Lambda}\sum_{i\in\mathcal{I}}\mathcal{G}_i^\lambda=\sum_{i\in\mathcal{I}}\bigcup_{\lambda\in\Lambda}\mathcal{G}_i^\lambda,\ \bigcup_{\lambda\in\Lambda}\sum_{j\in\mathcal{J}}\mathcal{S}_j^\lambda=\sum_{j\in\mathcal{J}}\bigcup_{\lambda\in\Lambda}\mathcal{S}_j^\lambda,\ \bigcup_{\lambda\in\Lambda}\sum_{t\in\mathcal{T}}\mathcal{V}_t^\lambda=\sum_{t\in\mathcal{T}}\bigcup_{\lambda\in\Lambda}\mathcal{V}_t^\lambda.$$

Step 5: We have

$$\bigcup_{\lambda\in\Lambda}\mathcal{L}^\lambda=\bigcup_{\lambda\in\Lambda}(\sum_{i\in\mathcal{I}}\mathcal{G}_i^\lambda)+\bigcup_{\lambda\in\Lambda}(\sum_{j\in\mathcal{J}}\mathcal{S}_j^\lambda)+\bigcup_{\lambda\in\Lambda}\sum_{t\in\mathcal{T}}(\mathcal{V}_t^\lambda)+\mathcal{D}_0.$$

Step 6: $(\sum_{i\in\mathcal{I}}\bigcup_{\lambda\in\Lambda}\mathcal{G}_i^\lambda)+(\sum_{j\in\mathcal{J}}\bigcup_{\lambda\in\Lambda}\mathcal{S}_j^\lambda)+(\sum_{t\in\mathcal{T}}\bigcup_{\lambda\in\Lambda}\mathcal{V}_t^\lambda)+\mathcal{D}_0$ is a direct sum.

Now we are ready to go through the last step.

Step 7: The assertion stated in (ii) is true: Take $\mathcal{A}$ to be a vector space with a basis $\{a_i\mid i\in\mathcal{I}\}$, $\mathcal{B}$ to be a vector space with a basis $\{b_j\mid j\in\mathcal{J}\}$, and $\mathcal{C}$ to be a vector space with a basis $\{c_t\mid t\in\mathcal{T}\}$. Using Steps 1-2,4-6, we get that

$$\begin{aligned}\mathcal{L}=\bigcup_{\lambda\in\Lambda}\mathcal{L}_\lambda &= \bigcup_{\lambda\in\Lambda}(\sum_{i\in\mathcal{I}}\mathcal{G}_i^\lambda)+\bigcup_{\lambda\in\Lambda}(\sum_{j\in\mathcal{J}}\mathcal{S}_j^\lambda)+\bigcup_{\lambda\in\Lambda}(\sum_{t\in\mathcal{T}}\mathcal{V}_t^\lambda)+\mathcal{D}_0\\ &= (\bigoplus_{i\in\mathcal{I}}\bigcup_{\lambda\in\Lambda}\mathcal{G}_i^\lambda)\oplus(\bigoplus_{j\in\mathcal{J}}\bigcup_{\lambda\in\Lambda}\mathcal{S}_j^\lambda)\oplus(\bigoplus_{t\in\mathcal{T}}\bigcup_{\lambda\in\Lambda}\mathcal{V}_t^\lambda)\oplus\mathcal{D}_0.\end{aligned}$$

Now consider $\mathcal{L}$ as a $\mathfrak{g}$-module via the adjoint representation and for $i \in \mathcal{I}$, $j \in \mathcal{J}$ and $t \in \mathcal{T}$, set

$$\mathcal{G}^{(i)} := \bigcup_{\lambda\in\Lambda} \mathcal{G}_i^\lambda,\ \mathcal{S}^{(j)} := \bigcup_{\lambda\in\Lambda} \mathcal{S}_j^\lambda,\ \mathcal{V}^{(t)} := \bigcup_{\lambda\in\Lambda} \mathcal{V}_t^\lambda,$$

then by Propositions 2.11 and 2.12, $\mathcal{G}^{(i)}$ is a $\mathfrak{g}$-submodule of $\mathcal{L}$ isomorphic to $\mathfrak{g} \simeq \mathcal{G}$, $\mathcal{S}^{(j)}$ is a $\mathfrak{g}$-submodule isomorphic to $\mathcal{S}$ and $\mathcal{V}^{(t)}$ is a $\mathfrak{g}$-submodule isomorphic to $\mathcal{V}$. Therefore as a vector space, we can identify $\mathcal{L}$ with

$$(\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus (\mathcal{V} \otimes \mathcal{C}) \oplus \mathcal{D}_0.$$

such that for each $\lambda \in \Lambda$, $\mathcal{L}^\lambda$ is identified with

$$(\mathcal{G}^\lambda \dot{\otimes} \mathcal{A}) \oplus (\mathcal{S}^\lambda \dot{\otimes} \mathcal{B}) \oplus (\mathcal{V}^\lambda \dot{\otimes} \mathcal{C}) \oplus \mathcal{D}_0.$$

Now for $\lambda \in \Lambda$, $(\mathcal{L}^\lambda, \mathcal{L}^0)$ plays the same role as $(\mathcal{L}^n, \mathcal{L}^\ell)$ in §3.0.1 and so we are done using Step 3 together with Proposition 3.10. ∎

4.2. Recognition Theorem for Types A and D.

Theorem 4.3. *Suppose that I is an infinite index set and I_0 is a subset of I of cardinality $\ell > 5$. Let R be an irreducible locally finite root system of type $X = D_I$ or $X = \dot{A}_I$. Suppose that $\mathcal{V}$ is a vector space with a basis $\{v_i \mid i \in I\}$ and take $\mathcal{G}$ to be the infinite dimensional split simple Lie algebra of type X as in Lemmas 2.3 or 2.6 respectively. Define*

$$\mathfrak{I}_0 : \mathcal{V} \longrightarrow \mathcal{V} \quad v_i \mapsto \begin{cases} v_i & i \in I_0 \\ 0 & \textit{otherwise.} \end{cases}$$

Also for $x, y \in \mathcal{G}$, define

$$x \circ y := xy + yx - \frac{2tr(xy)}{l+1}\mathfrak{I}_0.$$

Suppose that $(\mathcal{A}, id_\mathcal{A}, \{0\}, \mathbf{0})$ is a coordinate quadrable of type X and $\mathcal{K}$ is a subset of the full skew-dihedral homology group of $\mathcal{A}$ satisfying the uniform property on $\mathcal{A}$. Set

$$\mathcal{L}(\mathcal{A}, \mathcal{K}) := (\mathcal{G} \otimes \mathcal{A}) \oplus \langle \mathcal{A}, \mathcal{A} \rangle,$$

in which $\langle \mathcal{A}, \mathcal{A} \rangle$ is the quotient space $\{\mathcal{A}, \mathcal{A}\}_\ell / \mathcal{K}$ (see Subsection 3) and for $a, a' \in \mathcal{A}$, take $\langle a, a' \rangle := \{a, a'\}_\ell + \mathcal{K}$, then $\mathcal{L}(\mathcal{A}, \mathcal{K})$ together with

$$\begin{aligned}
&[x \otimes a, y \otimes a'] = \begin{cases} [x, y] \otimes \frac{1}{2}(a \circ a') + (x \circ y) \otimes \frac{1}{2}[a, a'] + tr(xy)\langle a, a' \rangle, & X = \dot{A}_I, \\ [x, y] \otimes aa' + tr(xy)\langle a, a' \rangle, & X = D_I, \end{cases} \\
&[\langle a_1, a_2 \rangle, x \otimes a] = \begin{cases} -\frac{1}{2(\ell+1)}((x \circ Id_{\mathcal{V}^\ell}) \otimes [a, [a_1, a_2]] & \\ +[x, Id_{\mathcal{V}^\ell}] \otimes (a \circ [a_1, a_2]) + 2tr(Id_{\mathcal{V}^\ell} x)\langle a, [a_1, a_2] \rangle), & X = \dot{A}_I, \\ 0 & X = D_I, \end{cases} \\
&[\langle a_1, a_2 \rangle, \langle a_1', a_2' \rangle] = \begin{cases} \langle d^{\ell,\mathcal{A}}_{a_1,a_2}(a_1'), a_2' \rangle + \langle a_1', d^{\ell,\mathcal{A}}_{a_1,a_2}(a_2') \rangle, & X = \dot{A}_I, \\ 0, & X = D_I, \end{cases}
\end{aligned} \tag{59}$$

for $x, y \in \mathcal{G}$, $a, a', a_1, a_2, a_1', a_2' \in \mathcal{A}$, is a Lie algebra graded by R with grading pair $(\mathcal{G}, \mathcal{H})$ where $\mathcal{H}$ is defined as in Lemma 2.3 or Lemma 2.6. Moreover, up to isomorphism any R-graded Lie algebra gives rise in this manner.

4.3. Recognition Theorem for Types B and C.

Theorem 4.4. *Suppose that I is an infinite index set and I_0 is a subset of I of cardinality $\ell > 4$. Take $\mathcal{G}$ to be either $\mathfrak{o}_B(I)$ or $\mathfrak{sp}(I)$. Suppose that $\mathcal{V}$ is a vector space with a basis $\{v_0, v_i, v_{\bar{i}} \mid i \in I\}$ equipped with a nondegenerate symmetric bilinear form $(\cdot,\cdot)$ as in (3) if $\mathcal{G} = \mathfrak{o}_B(I)$ and it is a vector space with a basis $\{v_i, v_{\bar{i}} \mid i \in I\}$ equipped with a nondegenerate skew-symmetric bilinear form $(\cdot,\cdot)$ as in (6) if $\mathcal{G} := \mathfrak{sp}(I)$. Set*

$$T := \begin{cases} I_0 \cup \bar{I}_0 \cup \{0\} & \text{if } \mathcal{G} = \mathfrak{o}_B(I) \\ I_0 \cup \bar{I}_0 & \text{if } \mathcal{G} = \mathfrak{sp}(I) \end{cases}$$

and define $\mathfrak{I}_0 : \mathcal{V} \longrightarrow \mathcal{V}$ to be the linear transformation defined by

$$v_i \mapsto \begin{cases} v_i & \text{if } i \in T \\ 0 & \text{if } i \in I \cup \bar{I} \setminus T. \end{cases}$$

Next set $\mathcal{S} := \mathcal{V}$ if $\mathcal{G} := \mathfrak{o}_B(I)$ and take $\mathcal{S}$ to be as in (10) if $\mathcal{G} = \mathfrak{sp}(I)$. For $e, f \in \mathcal{G} \cup \mathcal{S}$, set

$$e \circ f := ef + fe - \frac{tr(ef)}{\ell}\mathfrak{I}_0.$$

Suppose that R is an irreducible locally finite root system of type $X = B_I$ or $X = C_I$ and $(\mathfrak{a}, *, \mathcal{C}, f)$ is a coordinate quadrable of type X. Take $\mathcal{A}$ and $\mathcal{B}$ to be the set of $*-$fixed and $*-$skew fixed points of $\mathfrak{a}$ respectively. For a subset $\mathcal{K}$ of the full skew-dihedral homology group of $\mathfrak{a}$ satisfying the uniform property on $\mathfrak{a}$, set

$$\mathcal{L}(\mathfrak{a}, \mathcal{K}) := (\mathcal{G} \otimes \mathcal{A}) \oplus (\mathcal{S} \otimes \mathcal{B}) \oplus \langle \mathfrak{a}, \mathfrak{a} \rangle,$$

in which $\langle \mathfrak{a}, \mathfrak{a} \rangle$ is the quotient space $\{\mathfrak{a}, \mathfrak{a}\}_\ell/\mathcal{K}$ (see Subsection 3), and for $\alpha, \alpha' \in \mathfrak{a}$, take $\langle \alpha, \alpha' \rangle := \{\alpha, \alpha'\} + \mathcal{K}$. Then $\mathcal{L}(\mathfrak{a}, \mathcal{K})$ together with

$$\begin{aligned} &[x \otimes a, y \otimes a'] = [x, y] \otimes aa' + tr(xy)\langle a, a' \rangle, \\ &[x \otimes a, s \otimes b] = xs \otimes ab, \\ &[s \otimes b, t \otimes b'] = D_{s,t} \otimes f(b, b') + (s, t)\langle b, b' \rangle \\ &[\langle \alpha_1, \alpha_2 \rangle, x \otimes a] = x \otimes d^{\ell,\mathfrak{a}}_{\alpha_1,\alpha_2}(a), \\ &[\langle \alpha_1, \alpha_2 \rangle, s \otimes b] = s \otimes d^{\ell,\mathfrak{a}}_{\alpha_1,\alpha_2}(b), \\ &[\langle \alpha_1, \alpha_2 \rangle, \langle \alpha'_1, \alpha'_2 \rangle] = \langle d^{\ell,\mathfrak{a}}_{\alpha_1,\alpha_2}(\alpha'_1), \alpha'_2 \rangle + \langle \alpha_1, d^{\ell,\mathfrak{a}}_{\alpha'_1,\alpha'_2}(\alpha_2) \rangle \end{aligned} \tag{60}$$

(see Definition 2.4) for $x, y \in \mathcal{G}$, $s, t \in \mathcal{S}$, $\alpha, \alpha', \alpha_1, \alpha_2, \alpha'_1, \alpha'_2 \in \mathfrak{a}$, $a, a' \in \mathcal{A}$ and $b, b' \in \mathcal{B}$, if $\mathcal{G} = \mathfrak{o}_B(I)$ and

$$\begin{aligned} &[x \otimes a, y \otimes a'] = [x, y] \otimes \tfrac{1}{2}(a \circ a') + (x \circ y) \otimes \tfrac{1}{2}[a, a'] + tr(xy)\langle a, a' \rangle, \\ &[x \otimes a, s \otimes b] = (x \circ s) \otimes \tfrac{1}{2}[a, b] + [x, s] \otimes \tfrac{1}{2}(a \circ b), \\ &[s \otimes b, t \otimes b'] = [s, t] \otimes \tfrac{1}{2} b \circ b' + (s \circ t) \otimes \tfrac{1}{2}[b, b'] + tr(st)\langle b, b' \rangle, \\ &[\langle \alpha, \alpha' \rangle, x \otimes a] = \tfrac{-1}{4\ell}((x \circ \mathfrak{I}_0) \otimes [a, \beta^*_{\alpha,\alpha'}] + [x, \mathfrak{I}_0] \otimes (a \circ \beta^*_{\alpha,\alpha'})), \\ &[\langle \alpha, \alpha' \rangle, s \otimes b] = \tfrac{-1}{4\ell}([s, \mathfrak{I}_0] \otimes (b \circ \beta^*_{\alpha_1,\alpha_2}) + (s \circ \mathfrak{I}_0) \otimes [b, \beta^*_{\alpha_1,\alpha_2}] + 2tr(s\mathfrak{I}_0)\langle b, \beta^*_{\alpha,\alpha'} \rangle), \\ &[\langle \alpha_1, \alpha_2 \rangle, \langle \alpha'_1, \alpha'_2 \rangle] = \langle d^{\ell,\mathfrak{a}}_{\alpha_1,\alpha_2}(\alpha'_1), \alpha'_2 \rangle + \langle \alpha'_1, d^{\ell,\mathfrak{a}}_{\alpha_1,\alpha_2}(\alpha'_2) \rangle \end{aligned} \tag{61}$$

(see (27)) for $x, y \in \mathcal{G}$, $s, t \in \mathcal{S}$, $a, a' \in \mathcal{A}$, $b, b' \in \mathcal{B}$, $\alpha, \alpha', \alpha_1, \alpha_2, \alpha'_1, \alpha'_2 \in \mathfrak{a}$, *if* $\mathcal{G} = \mathfrak{sp}(I)$, *is a Lie algebra graded by* R *with grading pair* $(\mathcal{G}, \mathcal{H})$ *where* $\mathcal{H}$ *is defined as in Lemma 2.5 or Lemma 2.7. Moreover, up to isomorphism any* R*-graded Lie algebra gives rise in this manner.*

References

[1] Allison, B. N., S. Azam, S. Berman, Y. Gao, and A. Pianzola, *Extended affine Lie algebras and their root systems*, Mem. Amer. Math. Soc. **603** (1997), 1–122.

[2] Allison, B. N., G. Benkart, and Y. Gao, *Central extensions of Lie algebras graded by finite root systems*, Math. Ann. **316** (2000), 499–527.

[3] —, *Lie algebras graded by the root systems* BC_r, $r \geq 2$, Mem. Amer. Math. Soc. **158** , no. 751 (2002), x+158 pp.

[4] Azam, S., H. Yamane, and M. Yousofzadeh, *A finite presentation of universal covering of Lie tori*, Publ. Res. Inst. Math. Sci, **46** (2010), 507–548.

[5] Benkart, G., and E. Zelmanov, *Lie algebras graded by finite root systems and intersection matrix algebras*, Invent. Math. **126** (1996), 1–45.

[6] Benkart, G., and O. Smirnov, *Lie algebras graded by the root system* BC_1, J. Lie theory, **13** (2003), 91–132.

[7] Berman, S., and R. Moody, *Lie algebras graded by finite root systems and the intersection matrix algebras of Slodowy*, Invent. Math. **108** (1992), 323–347.

[8] Bourbaki, N., ≪Groupes et algèbres de Lie, Chap. 4-6,≫Hermann, Paris, 1968.

[9] Loos, O., and E. Neher, *Locally finite root systems*, *Locally finite root systems*, Mem. Amer. Math. Soc. **171**, no. 811 (2004), x+214pp.

[10] Neeb, K. H., and N. Stumme, *The classification of locally finite split simple Lie algebras*, J. Reine angew. Math. **533** (2001), 25–53.

[11] Neher, E., *Lie algebras graded by* 3*-graded root systems and Jordan pairs covered by grids*, Amer. J. Math. **118** (1996), 439–491.

[12] Seligman, G. B., "Rational methods in Lie algebras," M. Dekker Lect. Notes in Pure and Appl. Math. **17**, New York, 1976.

[13] Yoshii, Y., *Root systems extended by an abelian group, their Lie algebras and the classification of Lie algebras of type* B, J. Lie Theory **14** (2004), 371–394.

[14] Yousofzadeh, M., *A presentation of Lie tori of type B_ℓ*, Publ. Res. Inst. Math. Sci. **44**, no. 1 (2008), 1–44.

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