On Local Structure of Pseudo-Riemannian Poisson Manifolds and Pseudo-Riemannian Lie Algebras

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Abstract. Pseudo-Riemannian Poisson manifolds and pseudo-Riemannian Lie algebras were introduced by M. Boucetta. In this paper, we prove that all pseudo-Riemannian Lie algebras are solvable. Based on our main result and some properties of pseudo-Riemannian Lie algebras, we classify Riemann–Lie algebras of arbitrary dimension and pseudo-Riemannian Lie algebras of dimension at most 3.

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1. Introduction

M. Boucetta introduced the notion of Poisson manifold with compatible pseudo-metric in [1] and a new class of Lie algebras called pseudo-Riemannian Lie algebras in [2]. He proved that a linear Poisson structure on the dual of a Lie algebra has a compatible pseudo-metric if and only if the Lie algebra is a pseudo-Riemannian Lie algebra, and that the Lie algebra obtained by linearizing at a point in a Poisson manifold with compatible pseudo-metric is a pseudo-Riemannian Lie algebra. See [2] for more details of pseudo-Riemannian Poisson manifolds and their relationship with pseudo-Riemannian Lie algebras. Furthermore in [3], Boucetta established five equivalent conditions for \( g \) to be a Riemann–Lie algebra. In this paper we prove that every pseudo-Riemannian Lie algebra is solvable and give a simple proof of Boucetta’s result.

The paper is organized as follows. In Section 2, we collect some basic definitions and properties of pseudo-Riemannian Poisson manifolds and pseudo-Riemannian Lie algebras, and then translate it into our language, which is easier. In Section 3, we prove that no semisimple Lie algebra admits a pseudo-Riemannian Lie algebra structure; via the Levi decomposition, this implies our main result (Theorem 3.1). Boucetta classified Riemann–Lie algebras in [3], and in [2] claimed without proof to classify pseudo-Riemannian Lie algebras of dimension 2 or 3. theorem 1.6 in [2], which classifies 3-dimensional pseudo-Riemannian Lie algebras
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and Riemann–Lie algebras, is not quite correct. In the last section, using our method, we give an explicit classification of Riemann–Lie algebras (Theorem 4.7) and pseudo-Riemannian Lie algebras of dimensions 2 and 3 (Theorem 4.9).

2. Preliminaries

Let $P$ be a Poisson manifold and $\Pi$ be the Poisson bivector field. The Poisson bracket on $P$ is given by

$$\{f_1, f_2\} = \Pi(df_1, df_2) \quad \forall f_1, f_2 \in C^\infty(P).$$

We also have a bundle map $\sharp : T^*P \to TP$ defined by

$$\beta(\sharp(\alpha)) = \Pi(\alpha, \beta) \quad \forall \alpha, \beta \in T^*P.$$

The Poisson tensor induces a Lie bracket on the space of differential 1-forms $\Omega^1(P)$:

$$[\alpha, \beta] = L_{\sharp(\alpha)}\beta - L_{\sharp(\beta)}\alpha - d(\Pi(\alpha, \beta)).$$

Assume that there exists a pseudo-metric of signature $(p,q)$ on the cotangent bundle $T^*P$, that is, a smooth symmetric contravariant 2-form $\langle \cdot, \cdot \rangle$ on $P$ such that $\langle \cdot, \cdot \rangle|_x$ is nondegenerate on $T^*_xP$ with signature $(p,q)$, at each point $x \in P$. According to [4], there is a contravariant connection $D$, called the Levi-Civita contravariant connection associated with the triple $(P, \Pi, \langle \cdot, \cdot \rangle)$, given by

$$2\langle D_\alpha \beta, \gamma \rangle = \sharp(\alpha)\langle \beta, \gamma \rangle + \sharp(\beta)\langle \alpha, \gamma \rangle - \sharp(\gamma)\langle \alpha, \beta \rangle$$

$$+ \langle [\alpha, \beta], \gamma \rangle + \langle [\gamma, \alpha], \beta \rangle + \langle [\gamma, \beta], \alpha \rangle,$$

where $\alpha, \beta, \gamma \in \Omega^1(P)$. Further, $D$ satisfies the following conditions:

$$D_\alpha \beta - D_\beta \alpha = [\alpha, \beta];$$

$$\sharp(\alpha)\langle \beta, \alpha \rangle = \langle D_\alpha \beta, \gamma \rangle + \langle \beta, D_\alpha \gamma \rangle.$$

**Definition 2.1 ([2], Definition 1.1).** The triple $(P, \Pi, \langle \cdot, \cdot \rangle)$ is called a pseudo-Riemannian Poisson manifold if, for all $\alpha, \beta, \gamma \in \Omega^1(P)$,

$$D\Pi(\alpha, \beta, \gamma) = \sharp(\alpha)\Pi(\beta, \gamma) - \Pi(D_\alpha \beta, \gamma) - \Pi(\beta, D_\alpha \gamma) = 0.$$

When $\langle \cdot, \cdot \rangle$ is positive definite, the triple is called a Riemann–Poisson manifold.

For all $x \in P$, taking the linear approximation to the Poisson structure [5], we get a Lie algebra structure on $\text{Ker} \sharp_x$. In order to study the structure of $\text{Ker} \sharp_x$, we need the following definition, due to [2].

Let $g$ be a real Lie algebra and $(\cdot, \cdot)$ be a nondegenerate symmetric bilinear form on $g$. Define a bilinear map $(u,v) \mapsto uv$ on $g$ by

$$2(uv, w) = ([u, v], w) + ([w, u], v) + ([w, v], u) \quad \forall u, v, w \in g. \quad (2.1)$$

This map is called the infinitesimal Levi-Civita connection associated with $(\cdot, \cdot)$. Indeed, if $G$ is a connected Lie group with Lie algebra $g$, then $(\cdot, \cdot)$ defines a
left invariant pseudo-Riemannian metric on \( G \). The Levi-Civita connection \( \nabla \) associated with this metric is given by

\[
\nabla_u v^l = (uv)^l \quad \forall u, v \in g,
\]

where \( v^l \) denotes the left invariant vector field associated with \( u \). One may easily see that the equality (2.1) is equivalent to the following identities:

\[
uv - vu = [u, v] \quad (\text{PR1})
\]
\[
(uv, w) + (v, uw) = 0 \quad (\text{PR2})
\]

**Definition 2.2.** The pair \((g, (\cdot, \cdot))\), or \( g \) for short, is called a (real) pseudo-Riemannian Lie algebra if it satisfies (PR1), (PR2) and

\[
[wv, w] + [u, vw] = 0 \quad \forall u, v, w \in g. \quad (\text{PR3})
\]

If the bilinear form \((\cdot, \cdot)\) is positive definite, then \( g \) is called a Riemann–Lie algebra.

The following theorems of M. Boucetta describe the relationship between pseudo-Riemannian Poisson manifolds and pseudo-Riemannian Lie algebras.

**Theorem 2.3** ([2], Theorem 1.1). Let \((P, \Pi, \langle \cdot, \cdot \rangle)\) be a pseudo-Riemannian Poisson manifold. Then the Lie algebra \( \text{Ker}^{\#} x \) obtained by linearizing the Poisson structure at \( x \) is a pseudo-Riemannian Lie algebra, for every point \( x \in P \) such that the restriction of \( \langle \cdot, \cdot \rangle \) to \( \text{Ker}^{\#} x \) is nondegenerate.

**Theorem 2.4** ([2], Theorem 1.2). Let \( g \) be a real Lie algebra. The dual \( g^* \) endowed with its linear Poisson structure \( \Pi \) has a pseudo-metric \( \langle \cdot, \cdot \rangle \) for which the triple \((g^*, \Pi, \langle \cdot, \cdot \rangle)\) is a pseudo-Riemannian Poisson manifold if and only if \( g \) is a pseudo-Riemannian Lie algebra.

By (PR1), we may write condition (PR3) in another form:

\[
(uv)w - w(uv) + u(wv) - (wv)u = 0. \quad (\text{PR3}')
\]

So we may redefine pseudo-Riemannian Lie algebras, as follows.

**Definition 2.5.** An algebra \( g \) with a nondegenerate symmetric bilinear form \((\cdot, \cdot)\) is called a pseudo-Riemannian Lie algebra if the conditions (PR2) and (PR3’) are satisfied for all \( u, v, w \in g \).

With this definition, \([u, v] = uv - vu\) defines a Lie algebra structure on \( g \), since (PR3’) implies the Jacobi identity, as one may easily see. Thus \( g \) is a Lie algebra. So the two definitions are equivalent. Given \( u \in g \), denote by \( l_u \) and \( r_u \) the left and right multiplications by \( u \). Then (PR2) and (PR3) may be written as

\[
(l_u v, w) + (v, l_u w) = 0 \quad \text{and} \quad [r_u v, w] + [u, r_u w] = 0.
\]

**Remark 2.6.** If \( g \) is an abelian Lie algebra, then the product is trivial, that is, \( xy = 0 \) for all \( x, y \in g \).
3. Main results

In this section, we will prove the main theorem of this paper.

Theorem 3.1. Every Lie algebra over a field of characteristic 0 with a product satisfying (PR1) and (PR3) is solvable. Consequently, every pseudo-Riemannian Lie algebra $(\mathfrak{g},\langle\cdot,\cdot\rangle)$ is solvable.

The following results are immediate consequences of the main theorem.

Corollary 3.2. Let $(P,\Pi,\langle\cdot,\cdot\rangle)$ be a pseudo-Riemannian Poisson manifold. Then the Lie algebra $\text{Ker}_x\Pi$ is solvable, for every point $x \in P$ such that the restriction of $\langle\cdot,\cdot\rangle$ to $\text{Ker}_x\Pi$ is nondegenerate.

Corollary 3.3. Let $\mathfrak{g}$ be a real Lie algebra. If the dual $\mathfrak{g}^\ast$ endowed with its linear Poisson structure $\Pi$ has a pseudo-metric $\langle\cdot,\cdot\rangle$ for which the triple $(\mathfrak{g}^\ast,\Pi,\langle\cdot,\cdot\rangle)$ is a pseudo-Riemannian Poisson manifold, then $\mathfrak{g}$ is solvable.

The following lemma is a decisive step towards our main result.

Lemma 3.4. Let $\mathfrak{g}$ be a Lie algebra over an algebraically closed field of characteristic 0 with a product satisfying (PR1) and (PR3). Then $\mathfrak{g}$ is not semi-simple.

Proof. By contradiction, assume that $\mathfrak{g}$ is semi-simple and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\Delta(\mathfrak{g},\mathfrak{h})$ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Choose a system of positive roots, to obtain the root subspace decomposition of $\mathfrak{g}$:

$$
\mathfrak{g} = \mathfrak{h} + \sum_{\alpha > 0} \mathfrak{g}_\alpha + \sum_{\alpha < 0} \mathfrak{g}_\alpha.
$$

Let $\mathfrak{g}^+ = \sum_{\alpha > 0} \mathfrak{g}_\alpha$ and $\mathfrak{g}^- = \sum_{\alpha < 0} \mathfrak{g}_\alpha$. Since $\mathfrak{g} \supseteq \mathfrak{g} \supseteq [\mathfrak{g},\mathfrak{g}]$, it follows that $\mathfrak{g} = \mathfrak{g} \supseteq [\mathfrak{g},\mathfrak{g}]$. Let $\{X_\alpha \in \mathfrak{g}_\alpha \mid \alpha \in \Delta(\mathfrak{g},\mathfrak{h})\}$ be a Chevalley basis for $\mathfrak{g}$ with respect to $\mathfrak{h}$. We will prove the lemma after seven steps.

Step 1: $\mathfrak{hh} \subset \mathfrak{h}$. For arbitrary $h_1, h_2 \in \mathfrak{h}$, suppose that $h_1 h_2 = h_0 + X^+ + X^-$, where $h_0 \in \mathfrak{h}$, $X^+ \in \mathfrak{g}^+$ and $X^- \in \mathfrak{g}^-$. If $X^+ \neq 0$, then there exists $Y^- \in \mathfrak{g}^-$ such that the projection of $[X^+,Y^-]$ to $\mathfrak{h}$ is nonzero. By (PR3),

$$
[h_1 h_2, Y^-] = -[h_1, Y^- h_2].
$$

But the projection to $\mathfrak{h}$ of the left hand side is nonzero while that of the right hand side is zero since $[\mathfrak{h},\mathfrak{g}^-] \subset \mathfrak{g}^-$. Thus $X^+ = 0$. Similarly, $X^- = 0$. Thus $\mathfrak{hh} \subset \mathfrak{h}$.

Step 2: $\mathfrak{a}_\alpha \mathfrak{h} \subset \mathfrak{g}_\alpha$ and $\mathfrak{h} \mathfrak{a}_\alpha \subset \mathfrak{g}_\alpha$. For all $h_1, h_2 \in \mathfrak{h}$,

$$
[X_\alpha h_1, h_2] + [X_\alpha, h_2 h_1] = 0.
$$

Now $\mathfrak{hh} \subset \mathfrak{h}$, so $[X_\alpha, h_2 h_1] \in \mathfrak{g}_\alpha$, which implies that $X_\alpha h_1 \in \mathfrak{g}_\alpha + \mathfrak{h}$. Suppose that $X_\alpha h_1 = cX_\alpha + h_0$, where $c \in \mathbb{C}$ and $h_0 \in \mathfrak{h}$. If $h_0 \neq 0$, then there exists a root $\beta$ such that $\beta(h_0) \neq 0$. We may assume that $\beta \neq \alpha$ since if $\alpha(h_0) \neq 0$, we may choose $\beta = -\alpha$. Then

$$
[X_\alpha h_1, X_\beta] = [cX_\alpha + h_0, X_\beta] = cN_{\alpha,\beta}X_{\alpha + \beta} + \beta(h_0)X_\beta,
$$

where $N_{\alpha,\beta}$ are the Chevalley coefficients. Similarly,

$$
[X_\alpha, X_\beta h_1] = [X_\alpha, cX_\beta + h_0'] = c'N_{\alpha,\beta}X_{\alpha + \beta} - \alpha(h_0')X_\alpha.
$$
Then $[X_\alpha h_1, X_\beta] + [X_\alpha, X_\beta h_1] \neq 0$, which contradicts the identity (PR3). Thus $h_0 = 0$, that is, $g_0 h \subset g_0$. By (PR1), we deduce that $h_g g_0 \subset g_0$.

**Step 3:** $X_\alpha h = f(h)X_\alpha$ and $X_\alpha h = -f(h)X_\alpha$ for some $f \in \mathfrak{h}^*$. From the above discussion, we may assume that

$$X_\alpha h = f(h)X_\alpha \quad \text{and} \quad X_\alpha h = g(h)X_\alpha,$$

for some $f, g \in \mathfrak{h}^*$, since $\dim g_\alpha = 1$. By (PR3),

$$[X_\alpha h, X_\alpha] + [X_\alpha, X_\alpha h] = 0.$$

It follows that

$$[f(h)X_\alpha, X_\alpha] + [X_\alpha, g(h)X_\alpha] = 0.$$

Then $(f(h) + g(h))[X_\alpha, X_\alpha] = 0$. Therefore, $f(h) + g(h) = 0$.

**Step 4:** $g_\alpha = \{0\}$. For every root $\alpha$, there exists $h_1 \in \mathfrak{h}$ such that $\alpha(h_1) \neq 0$.

For all $h_2 \in \mathfrak{h}$,

$$[h_1 h_2, X_\alpha] + [h_1, X_\alpha h_2] = 0 \quad \text{and} \quad [h_1 h_2, X_\alpha] + [h_1, X_\alpha h_2] = 0.$$

Since $\mathfrak{hh} \subset \mathfrak{h}$, $X_\alpha h = f(h)X_\alpha$ and $X_\alpha h = -f(h)X_\alpha$, we see that

$$\alpha(h_1 h_2)X_\alpha + f(h_2)\alpha(h_1)X_\alpha = 0;$$

$$(-\alpha)(h_1 h_2)X_\alpha + (-f)(h_2)(-\alpha)(h_1)X_\alpha = 0.$$

Thus

$$\alpha(h_1 h_2) + f(h_2)\alpha(h_1) = 0 \quad \text{and} \quad (-\alpha)(h_1 h_2) + f(h_2)\alpha(h_1) = 0.$$

Therefore, $f(h_2) = \alpha(h_1 h_2) = 0$. So $X_\alpha h_2 = f(h_2)X_\alpha = 0$. Since $h_2$ is arbitrary, $X_\alpha h = 0$, that is, $g_\alpha h = 0$.

**Step 5:** $\mathfrak{hh} = \{0\}$. Suppose that $h_1 h_2 \neq 0$ for some $h_1, h_2 \in \mathfrak{h}$. Then there is a root $\alpha$ such that $\alpha(h_1 h_2) \neq 0$. Thus

$$[h_1 h_2, X_\alpha] = \alpha(h_1 h_2)X_\alpha \neq 0.$$

By (PR3) and the fact that $g_\alpha \mathfrak{h} = \{0\}$,

$$[h_1 h_2, X_\alpha] = -[h_1, X_\alpha h_2] = 0.$$

This is a contradiction, so $h_1 h_2 = 0$, that is, $\mathfrak{hh} = \{0\}$.

**Step 6:** $g_\alpha g_\beta \subset g_{\alpha + \beta}$, where $g_0 = \mathfrak{h}$. For all $h \in \mathfrak{h}$,

$$[h, X_\alpha X_\beta] + [h X_\beta, X_\alpha] = 0.$$

But $[h X_\beta, X_\alpha] \in g_{\alpha + \beta}$ since $h X_\beta \in g_\beta$. Thus $[h, X_\alpha X_\beta] \in g_{\alpha + \beta}$ for all $h \in \mathfrak{h}$. Therefore

$$X_\alpha X_\beta \in g_{\alpha + \beta} + \mathfrak{h}.$$

Assume that $X_\alpha X_\beta = c_{\alpha, \beta} X_{\alpha + \beta} + h_1$, where $c_{\alpha, \beta} \in \mathbb{C}$ and $h_1 \in \mathfrak{h}$. If $\alpha + \beta = 0$, we are done since $g_{\alpha + \beta} = g_0 = \mathfrak{h}$. So in the following, we assume that $\alpha + \beta \neq 0$. 


If \( h_1 \neq 0 \), then there exists a root \( \gamma \) such that \( \gamma(h_1) \neq 0 \). We may assume that \( \gamma \neq \alpha \), since if \( \alpha(h_1) \neq 0 \), we may choose \( \gamma = -\alpha \). Then

\[
[X_\gamma, X_\alpha X_\beta] = [X_\gamma, c_{\alpha, \beta} X_{\alpha+\beta} + h_1] = c_{\alpha, \beta} [X_\gamma, X_{\alpha+\beta}] - \gamma(h_1) X_\gamma.
\]

Suppose that \( X_\gamma X_\beta = c_{\gamma, \beta} X_{\gamma+\beta} + h_2 \). Then

\[
[X_\gamma X_\beta, X_\alpha] = [c_{\gamma, \beta} X_{\gamma+\beta} + h_2, X_\alpha] = c_{\gamma, \beta} [X_{\gamma+\beta}, X_\alpha] + \alpha(h_2) X_\alpha.
\]

It follows that \( [X_\gamma, X_\alpha X_\beta] + [X_\gamma X_\beta, X_\alpha] \neq 0 \) since \( \alpha + \beta \neq 0 \), \( \alpha \neq \gamma \) and \( \gamma(h_1) \neq 0 \). This is a contradiction, so \( h_1 = 0 \). Hence \( X_\alpha X_\beta \in \mathfrak{g}_{\alpha+\beta} \).

**Step 7:** \( \mathfrak{h} \mathfrak{g}_\alpha = \{0\} \). Suppose that \( h X_\alpha \neq 0 \), for some \( h \in \mathfrak{h} \). Then \( h X_\alpha = f(h) X_\alpha \) and \( f(h) \neq 0 \). By (PR3),

\[
[h, X_{-\alpha} X_\alpha] + [h X_\alpha, X_{-\alpha}] = 0.
\]

Since \( X_{-\alpha} X_\alpha \in \mathfrak{g}_0 = \mathfrak{h} \),

\[
[h, X_{-\alpha} X_\alpha] + [h X_\alpha, X_{-\alpha}] = [h X_\alpha, X_{-\alpha}] = f(h)[X_\alpha, X_{-\alpha}] \neq 0.
\]

This is a contradiction, so \( \mathfrak{h} \mathfrak{g}_\alpha = \{0\} \).

Finally, we have reached a contradiction, since \( [\mathfrak{h}, \mathfrak{g}_\alpha] = \{0\} \) as \( \mathfrak{h} \mathfrak{g}_\alpha = \mathfrak{g}_\alpha \mathfrak{h} = \{0\} \). Thus \( \mathfrak{g} \) is not a semi-simple Lie algebra. \( \square \)

Now we come to the proof of our main result.

**Proof of Theorem 3.1.** First, extend the base field of \( \mathfrak{g} \) to its algebraic closure if necessary. Let \( \mathfrak{g} = \mathfrak{s} + \mathfrak{r} \) be a Levi decomposition of \( \mathfrak{g} \). Then

\[
[s_1 s_2, s_3] + [s_1, s_3 s_2] = 0
\]

for all \( s_1, s_2, s_3 \in \mathfrak{s} \). Let \( s_i s_j = s_{i,j} + r_{i,j} \), where \( s_{i,j} \in \mathfrak{s} \) and \( r_{i,j} \in \mathfrak{r} \). Then

\[
[s_1, s_2 + r_{1,2}, s_3] + [s_1, s_3, s_2 + r_{3,2}] = 0,
\]

that is,

\[
([s_1, s_3] + [s_1, s_3, s_2]) + ([r_{1,2}, s_3] + [s_1, r_{3,2}]) = 0.
\]

Thus

\[
[s_1, s_3] + [s_1, s_3, s_2] = [r_{1,2}, s_3] + [s_1, r_{3,2}] = 0,
\]

since \( \mathfrak{s} \) is a subalgebra and \( \mathfrak{r} \) is an ideal of \( \mathfrak{g} \). Define a product \( \circ : \mathfrak{s} \times \mathfrak{s} \to \mathfrak{s} \) by

\[
s_1 \circ s_2 = P_s(s_1 s_2),
\]

where \( P_s \) denotes the projection from \( \mathfrak{g} \) to \( \mathfrak{s} \) with respect to the Levi decomposition. Then the product \( \circ \) is bilinear.

Further, for all \( s_1, s_2 \in \mathfrak{s} \),

\[
[s_1, s_2] = s_1 s_2 - s_2 s_1 = s_{1,2} + r_{1,2} - s_{2,1} - r_{2,1} = (s_{1,2} - s_{2,1}) + (r_{1,2} - r_{2,1}) \in \mathfrak{s}.
\]

Hence \( r_{1,2} - r_{2,1} = 0 \) and

\[
s_1 \circ s_2 - s_2 \circ s_1 = P_s(s_1 s_2) - P_s(s_2 s_1) = s_{1,2} - s_{2,1} = [s_1, s_2].
\]

Moreover, for all \( s_1, s_2, s_3 \in \mathfrak{s} \),

\[
[s_1 \circ s_2, s_3] + [s_1, s_3 \circ s_2] = [P_s(s_1 s_2), s_3] + [s_1, P_s(s_3 s_2)] = [s_{1,2}, s_3] + [s_1, s_{3,2}] = 0.
\]

Thus, \( (\mathfrak{s}, \circ) \) satisfies the conditions of Lemma 3.4, which implies that \( \mathfrak{s} \) is not semi-simple. Then \( \mathfrak{s} \) must be 0, and \( \mathfrak{g} \) is solvable. \( \square \)
4. A new proof of Boucetta’s results

In this section, we will use our results to classify Riemann–Lie algebras and low dimensional linear pseudo-Riemannian Poisson manifolds. Boucetta [2, 3] proved or claimed similar results. However, our proof is much simpler. For example, Lemma 3.5 in [3] is a trivial consequence of our main theorem.

First, we collect some basic properties of pseudo-Riemannian Lie algebras that will be used frequently. In this section, an ideal of a pseudo-Riemannian Lie algebra $g$ means a subspace of $g$ that is invariant under left and right multiplications in $g$; hence an ideal is automatically a Lie ideal.

To state the next lemmas, let $C(g)$ and $C(g)\perp$ be the center of $g$ and its orthogonal complement:

$$C(g) = \{ a \in g \mid ax = xa \ \forall x \in g \}$$

$$C(g)\perp = \{ u \in g \mid (u, C(g)) = \{0\} \}.$$

**Lemma 4.1.** The subspace $C(g)$ is an ideal of $g$, and $xy = 0$ for all $x, y \in C(g)$.

**Proof.** By (PR3), for all $x \in C(g)$ and $y, z \in g$,

$$[xy, z] + [x, yz] = [xy, z] = 0.$$

It follows that $xy = yx \in C(g)$ for all $x \in g$.

For all $x, y \in C(g)$ and $z \in g$,

$$(xy, z) = -(y, xz) = -(y, zx) = (zy, x) = (yz, x) = -(z, yx) = -(xy, z).$$

It follows that $(xy, z) = 0$ for all $z \in g$, and thus $xy = 0$. $lacksquare$

**Remark 4.2.** This lemma is nontrivial, since the center is not necessarily an ideal for general algebras (for instance, associative algebras).

**Lemma 4.3.** The subspace $C(g)\perp$ is an ideal of $g$. If the restriction of the bilinear form to $C(g)$ is nondegenerate (say, if $g$ is a Riemann–Lie algebra), then $[g, g] \subset C(g)\perp$ and $g = C(g) \oplus C(g)\perp$.

**Proof.** For all $x \in C(g)$, $y \in C(g)\perp$, and $z \in g$,

$$(x, yz) = -(y, xz) = -(xy, z) = (y, xz) = 0,$$

$$(x, zy) = -(xz, y) = 0.$$

So $yz, zy \in C(g)\perp$ since $C(g)$ is an ideal. $lacksquare$

Consequently, we have the following result.

**Corollary 4.4.** If $g$ is a nilpotent Riemann–Lie algebra, then $g$ is abelian.

**Proof.** By Lemma 4.3, $g = C(g) \oplus C(g)\perp$. Then $C(g)\perp$ is also a nilpotent Riemann–Lie algebra. The center of $C(g)\perp$ is contained in the center of $g$, so $C(g)\perp$ must be trivial. $lacksquare$
Henceforth, \( \text{span}\{S\} \) denotes the subspace spanned by \( S \). Further, let

\[ \mathfrak{gg} = \text{span}\{xy \mid x, y \in \mathfrak{g}\} \]

and

\[ Z_r(\mathfrak{g}) = \{ u \in \mathfrak{g} \mid r_u = 0 \}. \]

**Lemma 4.5.** The subspace \( \mathfrak{gg} \) is an ideal of \( \mathfrak{g} \) and \( (\mathfrak{gg})^\perp = Z_r(\mathfrak{g}) \).

**Proof.** The first assertion is trivial. For the second, observe that the following statements are equivalent: first, \( x \in (\mathfrak{gg})^\perp \); second, \( (x, yz) = 0 \) for all \( y, z \in \mathfrak{g} \); third, \( (yx, z) = 0 \) for all \( y, z \in \mathfrak{g} \); fourth, \( yx = 0 \) for all \( y \in \mathfrak{g} \); and finally, \( x \in Z_r(\mathfrak{g}) \).

It is easy to see that \( [\mathfrak{g}, \mathfrak{g}] \subset \mathfrak{gg} \), but, in general, \( [\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{gg} \) and \( [\mathfrak{g}, \mathfrak{g}] \) is not necessarily an ideal of \( \mathfrak{g} \) although it is a Lie ideal. Define the adjoint \( \phi^t \) of \( \phi \in \text{End}(\mathfrak{g}) \) by

\[ (\phi(v), w) = (v, \phi^t(w)) \quad \forall v, w \in \mathfrak{g}, \]

and set

\[ [\mathfrak{g}, \mathfrak{g}]^\perp = \{ x \in \mathfrak{g} \mid (x, [\mathfrak{g}, \mathfrak{g}]) = \{0\} \}. \]

Then the following lemma is easy.

**Lemma 4.6.** The following equality holds:

\[ [\mathfrak{g}, \mathfrak{g}]^\perp = \{ u \in \mathfrak{g} \mid r_u = r^t_u \}. \]

Furthermore, \( uu = 0 \) for all \( u \in [\mathfrak{g}, \mathfrak{g}]^\perp \).

**Proof.** The following are equivalent: first, \( u \in [\mathfrak{g}, \mathfrak{g}]^\perp \); second, \( (u, [v, w]) = 0 \) for all \( v, w \in \mathfrak{g} \); third, \( (u, vw) = (u, wv) \) for all \( v, w \in \mathfrak{g} \); fourth, \( (vu, w) = (v, wu) \) for all \( v, w \in \mathfrak{g} \); and finally, \( r_u = r^t_u \).

Next, \( r_u \) is self-adjoint for all \( u \in [\mathfrak{g}, \mathfrak{g}]^\perp \), so, for all \( w \in \mathfrak{g} \),

\[ (w, uu) = (w, r_u(u)) = (r_u(w), u) = (wu, u) = 0. \]

The last equality follows from (PR2). Thus \( uu = 0 \).

Now we give our classification of Riemann–Lie algebras, which agrees with Theorem 3.1 in [3].

**Theorem 4.7.** Let \( (\mathfrak{g}, (\cdot, \cdot)) \) be a Riemann–Lie algebra. Then \( \mathfrak{g} = Z_r(\mathfrak{g}) \perp [\mathfrak{g}, \mathfrak{g}] \), where \( Z_r(\mathfrak{g}) \) is an abelian subalgebra and \( [\mathfrak{g}, \mathfrak{g}] \) is an abelian ideal.

Conversely, let \( V \) be a real finite-dimensional vector space with an inner product \( (\cdot, \cdot) \) and, as usual, let

\[ \mathfrak{so}(V) = \{ \mathcal{A} \in \text{End} V \mid (\mathcal{A}u, v) + (u, \mathcal{A}v) = 0 \}. \]

Choose an arbitrary torus \( S \subset \mathfrak{so}(V) \) and set \( \mathfrak{g} = S \perp V \). Extend the inner product on \( V \) to an inner product on \( \mathfrak{g} \) such that \( S \perp V \). Then \( \mathfrak{g} \) is a Riemann–Lie algebra and every Riemann–Lie algebra may be obtained in this way.
Proof. The last assertion is clear, so we prove only the first. By Lemmas 4.5 and 4.6, we need to prove that 
\[ [\mathfrak{g}, \mathfrak{g}]^\perp = Z_r(\mathfrak{g}). \]

Now the bilinear form \((\cdot, \cdot)\) is positive definite and \(r_u\) is diagonalizable. Let \(\lambda \in \mathbb{R}\) be an eigenvalue of \(r_u\) and \(v \in \mathfrak{g}\) be an associated eigenvector. Then
\[
\lambda^2(v, v) = \lambda(vu, v) = \lambda([v, u], v) = -([v, uu], v) = 0.
\]
Therefore \(\lambda = 0\). Hence \(r_u = 0\) since the only eigenvalue of \(r_u\) is zero.

By the main theorem, \(\mathfrak{g}\) is solvable, hence \([\mathfrak{g}, \mathfrak{g}]\) is nilpotent. Then \([\mathfrak{g}, \mathfrak{g}]\) is abelian by Corollary 4.4.

Example 4.8. Let \(\mathfrak{g}\) be a 3-dimensional nonabelian Riemann–Lie algebra. Then \(\dim[\mathfrak{g}, \mathfrak{g}] = 2\). There exists an orthonormal basis \(\{s, x, y\}\) of \(\mathfrak{g}\) and \(a \in \mathbb{R}\) such that \(s \in Z_r(\mathfrak{g})\) and \(x, y \in [\mathfrak{g}, \mathfrak{g}]\), and \([s, x] = sx = ay\) and \([s, y] = sy = -ax\).

Define \(\langle u, v \rangle = a^2(u, v)\), and \(s' = a^{-1}s\), \(x' = a^{-1}x\), \(y' = a^{-1}y\). Then \(\{s', x', y'\}\) is an orthonormal basis of \(\mathfrak{g}\); furthermore, \([s', x'] = s'x' = y'\) and \([s', y'] = s'y' = -x'\). In other words, there is a unique inner product on the Lie algebra \(\mathfrak{g}\) (up to a positive constant) such that \(\mathfrak{g}\) is a Riemann–Lie algebra.

In the rest of this paper, we will classify linear pseudo-Riemannian Poisson manifolds of dimension at most 3. Actually, it is enough to give the classification of pseudo-Riemannian Lie algebras of dimension 3 or less. Boucetta claimed the same classification in [2, Theorem 1.6] without proof. Furthermore, Theorem 1.6 [2] is not correct and Boucetta did not describe the product and bilinear form for \(\mathfrak{g}\) to be a pseudo-Riemannian Lie algebra. Using our definition and methods, we will give the classification explicitly in the following.

Theorem 4.9. The unique 2-dimensional pseudo-Riemannian Lie algebra is the 2-dimensional abelian Lie algebra.

There are three 3-dimensional nonabelian pseudo-Riemannian Lie algebras:

(a) The Heisenberg Lie algebra, given by \([x, y] = z\) and \([x, z] = [y, z] = 0\). The bilinear form and the product may be given as follows:
\bullet \ ((x, z) = 1 \text{ and } (y, y) \neq 0); \text{ other undetermined expressions are zero;}
\bullet \ xx = -(y, y)^{-1}y \text{ and } xy = z; \text{ other undetermined products are zero.}

Furthermore, \(\mathfrak{g}\) cannot be a Riemann–Lie algebra.

(b) The Lie algebras \(\mathfrak{g}_\pm\) given by \([x, y] = z\), \([x, z] = \pm y\) and \([y, z] = 0\). The bilinear form and the product may be given as follows:
\bullet \ (x, x) = t, \ (y, y) = 1 \text{ and } (z, z) = \mp 1, \text{ where } t \neq 0; \text{ other undetermined expressions are zero;}
\bullet \ xy = [x, y] \text{ and } xz = [x, z]; \text{ other undetermined products are zero.}
Furthermore, $g_-$ is a Riemann–Lie algebra when $t$ is chosen.

**Remark 4.10.** As we may see from the above theorem, there are essentially three nonabelian pseudo-Riemannian Lie algebras of dimension 3, and $g_{-1}$ is the only nonabelian Riemann–Lie algebra. This may be contrasted with the incorrect statement in Theorem 1.6 of [2].

**Proof.** Assume that $g$ is the nonabelian Lie algebra of dimension 2. We need only show that $g$ cannot be a pseudo-Riemannian Lie algebra.

Choose a basis $\{x, y\}$ for $g$ such that $[x, y] = y$. If $(y, y) = 0$, then $(x, y) \neq 0$. Replacing $x$ by $x - [2(x, y)]^{-1}(x, y)$, we may assume that $(x, x) = 0$. Now $(xy, y) = 0$, so $xy \in \text{span}\{y\}$. Furthermore, $(xx, x) = (yx, x) = 0$, which implies that $xx, yx \in \text{span}\{x\}$. Thus $[x, y] = y$ implies that $xy = y, yx = 0$, and $[xx, y] + [x, yx] = 0$ implies that $xx = 0$. So $(xy, x) + (y, xx) = 0$ implies that $(y, x) = 0$, a contradiction. If $(y, y) \neq 0$, a similar argument also leads to a contradiction.

Now we assume that $g$ is a nonabelian 3-dimensional Lie algebra. Then $\dim C(g) \leq 1$. There are two cases to consider.

**Case 1:** $\dim C(g) = 1$. In this case, $g$ is a Heisenberg Lie algebra or a direct sum of $C(g)$ and the two dimensional nonabelian Lie algebra.

Assume that $\{z\}$ is a basis of $C(g)$. One may easily see that $(z, z) = 0$. Otherwise, $g = C(g) + C(g) \perp$ and $C(g) \perp$ is a 2-dimensional pseudo-Riemannian Lie algebra, hence $g$ is abelian, which is a contradiction. Therefore, $C(g) \subset C(g) \perp$.

Assume that $\{y, z\}$ is a basis of $C(g) \perp$. Then $(y, y) \neq 0$. Choose $x \in g$ such that $(x, x) = (x, y) = 0$ and $(x, z) = 1$. Then $xz = xz = 0$ since $(xz, x) = 0$. Furthermore, $zy = yz = 0$ since $(zy, x) = -(y, zx) = 0$. So

$$gg \subset C(g) \perp = \text{span}\{y, z\}.$$ 

Since $(xx, x) = (xx, z) = 0$, we have $xx \in \text{span}\{y\}$. Thus $[x, yx] = -[xx, y] = 0$. Now $(yx, x) = 0$, so $yx = 0$, for otherwise it would follow that $x \in C(g)$. Therefore $[x, y] = xy \in \text{span}\{z\}$ since $(xy, y) = 0$. It means that $g$ is the Heisenberg Lie algebra. We may assume that $[x, y] = z$. Similarly, we may show that $yy = 0$ and $xx = -(y, y)^{-1}y$.

**Case 2:** $\dim C(g) = 0$. Since $g$ is solvable, there exists a basis $\{x, y, z\}$ of $g$ such that

$$[x, y] = ay + bz, \quad [x, z] = cy + dz, \quad [y, z] = 0,$$

where $ad - bc \neq 0$.

First we prove that the bilinear form restricted to $[g, g] = \text{span}\{y, z\}$ is nondegenerate. If not, we may assume that $z \in \text{span}\{y, z\} \perp$. So $(y, y) \neq 0$. Choose $x$ such that $(x, y) = 0$, $(x, x) = 0$ and $(x, z) = 1$. Then we claim that $gg = [g, g]$. To see this, one may easily deduce from (PR2) that $yz, yz, yz, zz \in \text{span}\{y, z\} \perp = \text{span}\{z\}$. Furthermore, $xz \in \text{span}\{y, z\}$ and $xz \in \text{span}\{y, z\}$ since $(xz, z) = 0$, so

$$0 = [xz, y] = -[x, yz] = -[x, zy] = [xy, z],$$

which implies that $yz = 0$ and $xy \in \text{span}\{y, z\}$. Thus $(xz, y) = -(x, yz) = 0$ and $(xz, x) = 0$ imply that $xz = 0$ since $xz \in \text{span}\{y, z\}$. Finally, $[xx, z] = [x, zz] = \ldots$
0, so \( xx \in \text{span}\{y, z\} \). Hence we see that \( gg \subset [g, g] \), therefore \( gg = [g, g] \). So \( Z_r(g) = (gg)^\perp = [g, g]^\perp = \text{span}\{z\} \). So \( [x, z] = xx = 0 \), a contradiction.

Now that the restriction to \( \text{span}\{y, z\} \) of the bilinear form is nondegenerate, we may choose \( x, y, z \) orthogonal. Since \( x \in [g, g]^\perp \), we have \( r^g_x = r_x \) and \( xx = 0 \). For all \( u \in g \), \( (ux, x) = 0 \), thus \( ux \in \text{span}\{y, z\} \). By \( [ux, x] = -[u, xx] = 0 \), we have \( ux \in \ker \text{ad}_x \cap \text{span}\{y, z\} = \{0\} \), that is, \( x \in Z_r(g) \). Then \( gg = [g, g] \), which is abelian as a 2-dimensional pseudo-Riemannian Lie algebra. Furthermore, since \( (xy, y) = 0 \), we have \( [x, y] = xy \in \text{span}\{z\} \), thus \( a = 0 \). Similarly, \( d = 0 \). Replacing \( x \) by \( b^{-1}x \), we see the Lie algebra structure of \( g \) is given by \( [x, y] = z \), \( [x, z] = cy \) and \( [y, z] = 0 \). Since \( (xz, y) = -(z, xy) \), one has \( c(y, y) = -(z, z) \). Replacing the bilinear form by a suitable multiple, we may assume that \( (y, y) = 1 \), then \( (z, z) = -c \). Replacing \( x \) by \( |c|^{-1/2}x \) and \( y \) by \( |c|^{1/2}y \), we may take \( c = \pm 1 \).

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References


