Integrable Representations for Extended Affine Lie Algebras Coordinated by Quantum Tori

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Abstract. In this paper we realize all irreducible integrable modules for the core of extended affine Lie algebras of type $A$ coordinated by quantum tori with center elements act non-trivially. We also study the sufficient and necessary conditions for such modules to be unitary.

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1. Introduction

In the study of representation theory of Kac-Moody algebras, one of the main ingredients is the classification of irreducible integrable representations. The irreducible integrable modules with finite dimensional weight spaces for affine Kac-Moody algebras were classified by Chari [5] and Chari-Pressly [6], [7]. It was proved that the irreducible integrable modules are the highest weight modules ([19]), their dual, or the loop modules ([6]). As higher dimensional generalization of affine Kac-Moody algebras, extended affine Lie algebras (EALAs) were first introduced in [17] and studied in [1] and [2].

Toroidal Lie algebras are basic examples of EALAs. The classification problem of irreducible integrable modules with finite dimensional weight spaces for the toroidal Lie algebras was studied by Rao [8],[10]. Besides the affine Kac-Moody algebras and toroidal Lie algebras, there are many EALAs whose coordinate algebras are non-commutative or non-associative, examples involving quantum tori, Jordan tori and octonion tori. In [11], Rao studied a class of irreducible integrable modules for certain EALAs coordinated by quantum tori. More precisely, let $\mathcal{L}$ be the core of EALA of type $A_{\nu-1}$ coordinated by quantum tori $\mathbb{C}_q[x^\pm 1, y^\pm 1]$ of two variables. Add two derivations $d_x, d_y$ to $\mathcal{L}$, one has $\tilde{\mathcal{L}} = \mathcal{L} \oplus \mathbb{C}d_x \oplus \mathbb{C}d_y$. Let $c_x$ and $c_y$ denote the two central elements corresponding to the variables $x$ and $y$, respectively. The irreducible integrable $\tilde{\mathcal{L}}$-modules were studied in [11] for...
which $c_x$ or $c_y$ acts non-trivially. It was shown in [11] that any such $\tilde{L}$-module is either a highest weight module or a lowest weight module up to an automorphism. But, the concrete realization problem for those irreducible integrable $\tilde{L}$-modules is open. As pointed out in [11], “it is interesting if one can give a “model” for all known irreducible integrable $\tilde{L}$-modules”. The main goal of this article is to give explicit construction of those integrable $\tilde{L}$-modules.

So far, the known models for integrable $\tilde{L}$-modules were those so-called “basic modules” constructed in [16],[4],[12],[13] and [3] obtained by vertex operators. In addition, a class of “fundamental modules” were given by using Clifford “basic modules” constructed in [16],[4],[12],[13] and [3] obtained by vertex operators. In particular, the $\tilde{L}$-submodule $W_i$ of $V_i$ generated by $w_i = \otimes_{s=1}^{l_i}v_i$ is irreducible.

Let $i = (\tilde{i}_1, \cdots, \tilde{i}_k)$ with $\tilde{i}_s \in \mathbb{Z}^{n_s}, 1 \leq s \leq k$. Then, we have the irreducible $\tilde{L}$-modules $W_{i_1}, \cdots, W_{i_k}$. Thanks to the method developed by Chari and Pressly in [6], we obtain an $\tilde{L}$-module structure on the tensor space $W_{i\alpha} = W_{i_1} \otimes \cdots \otimes W_{i_k} \otimes \mathbb{C}[t,t^{-1}]$ with $k$-tuple $\alpha = (a_1, \cdots, a_k) \in (\mathbb{C}^*)^k$. We need the condition $a_i \neq q^na_j$ for all $1 \leq i \neq j \leq k, n \in \mathbb{Z}$, while it was required that $a_i \neq a_j, \forall i \neq j$ in [6] for the affine Kac-Moody algebras. We prove that the $\tilde{L}$-modules $W_{i\alpha}$ are completely reducible and their irreducible components exhaust all irreducible integrable highest weight $\tilde{L}$-modules up to the actions of $d_x, d_y$. In other words, by changing the actions of $d_x, d_y$ and an automorphism twisting on the $\tilde{L}$-modules $W_{i\alpha}$, we realize all irreducible integrable modules given in [11].

In addition, we obtain necessary and sufficient conditions for those $\tilde{L}$-modules to be unitary when $\vert q \vert = 1$. Several unitary modules related to the algebra $\tilde{L}$ studied in the paper were studied in [18],[9],[15] and [20].

The paper is organized as follows. In Sect.2, we recall some properties for the algebra $\tilde{L}$ from [2] and also recall the classification results from [11]. In Sect.3, we recall the fermionic constructions for $\tilde{L}$-modules given in [14] and prove that the tensor product of such $\tilde{L}$-modules are completely reducible. Then we construct a family of irreducible integrable modules for $\tilde{L}$ in Sect.4, which gives the explicit realization of all irreducible integrable highest weight $\tilde{L}$-modules. In Sect.5, we study the unitarity of those modules constructed in Sect.4 under the condition $\vert q \vert = 1$.

Throughout this paper, we denote the field of complex numbers, the group of non-zero complex numbers, the set of non-negative integers and the ring of integers by $\mathbb{C}, \mathbb{C}^*, \mathbb{N}$ and $\mathbb{Z}$ respectively.

2. Extended affine Lie algebras coordinated by quantum tori

Let $q$ be a non-zero complex number and $\nu \geq 2$ be a positive integer. We begin by recalling the construction of the core of EALA of type $A_{\nu-1}$ coordinated by quantum tori given in [2]. A quantum tori in two variables associated to $q$ is the unital associative $\mathbb{C}$-algebra $\mathbb{C}_q[x^{\pm1}, y^{\pm1}]$ (or, simply $\mathbb{C}_q$) with generators $x^{\pm1}, y^{\pm1}$
and relations
\[ xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1, \quad yx = qxy. \]

Let \( \Lambda(q) = \{ n \in \mathbb{Z} | q^n = 1 \} \), and \( q \) is said to be generic if \( \Lambda(q) = \{0\} \). The center \( Z(C_q) \) of \( C_q \) has a basis consisting of monomials \( x^m y^n \) for \( m, n \in \Lambda(q) \) and the subalgebra \([C_q, C_q]\) has a basis consisting of monomials \( x^m y^n \) for \( m \notin \Lambda(q) \) or \( n \notin \Lambda(q) \). This implies that \( C_q = [C_q, C_q] \oplus Z(C_q) \).

Let \( I \) be the subspace of \( C_q \otimes C_q \) spanned by elements of the form
\[ a \otimes b + b \otimes a, \quad ab \otimes c - a \otimes bc - b \otimes ca \quad \text{for all} \quad a, b, c \in C_q. \]
So we have the quotient space \(<C_q, C_q> = C_q \otimes C_q / I \) and we let \( <a, b> \) denote the element \( a \otimes b + I \). Let \( I_\nu \) be the \( \nu \times \nu \) identity matrix. Define a Lie algebra \( \mathcal{L} = (sl_\nu(C) \otimes C_q) \oplus <C_q, C_q> \) with bracket
\[ [A \otimes a, B \otimes b] = [A, B] \otimes \frac{a \otimes b}{2} + A \circ B \otimes \frac{[a, b]}{2} + \frac{\text{tr}(AB)}{\nu} \quad <a, b>, \]
\[ [<a, b>, <c, d>] = <[a, b], [c, d]>, \]
\[ [<a, b>, A \otimes c] = A \otimes [[a, b], c], \]
where \( A, B \in sl_\nu(C), a, b, c, d \in C_q, [A, B] = AB - BA, \)
\[ A \circ B = AB + BA - \frac{2}{\nu} \text{tr}(AB)I_\nu, \]
\[ [a, b] = ab - ba, a \circ b = ab + ba \quad \text{and} \quad \text{tr} \text{ is the trace form}. \]

There is a natural \( \mathbb{Z}^2 \)-graded structure on \(<C_q, C_q> \) with graded subspaces given as follows:
\[ <C_q, C_q>_{(m, n)} = \text{span}_C \{ <x^{m_1}y^{n_1}, x^{m_2}y^{n_2}> : m_1, m_2, n_1, n_2 \in \mathbb{Z}, \]
\[ m_1 + m_2 = m, n_1 + n_2 = n \}, (m, n) \in \mathbb{Z}^2. \]

Let \( HC_1(C_q) = \{ \sum \langle a_j, b_j \rangle | \sum_{j \in j}[a_j, b_j] = 0 \} \) where \( j \) is any finite index set. It was shown in [2] that \( HC_1(C_q) \) is the center of \( \mathcal{L} \) and \( \mathcal{L} \) is centrally closed. Moreover, by Corollary 3.22 [2], one has
\[ \dim <C_q, C_q>_{(m, n)} = \begin{cases} 1, & \text{if} \quad (m, n) \neq (0, 0); \\ 2, & \text{if} \quad (m, n) = (0, 0), \end{cases} \]
and
\[ HC_1(C_q) = \bigoplus_{m, n \in \Lambda(q)} <C_q, C_q>_{(m, n)}. \]

Thus, we have \(<C_q, C_q> = [C_q, C_q] \oplus HC_1(C_q)\) as vector spaces.

By adding the degree derivations \( d_x, d_y \) to \( \mathcal{L} \), we get a Lie algebra \( \tilde{\mathcal{L}} = \mathcal{L} \oplus Cd_x \oplus Cd_y \) with additional multiplications
\[ [d_x, A \otimes a] = m_1 A \otimes a, \quad [d_y, A \otimes a] = n_1 A \otimes a, \quad [d_x, d_y] = 0, \]
\[ [d_x, <a, b>] = (m_1 + m_2) <a, b>, \quad [d_y, <a, b>] = (n_1 + n_2) <a, b>, \quad (1) \]
where $A \in \mathfrak{sl}_\nu(\mathbb{C})$, $a = x^{m_1}y^{n_1}$, $b = x^{m_2}y^{n_2}$, $m_1, m_2, n_1, n_2 \in \mathbb{Z}$.

Let $\mathcal{K}$ be the vector space spanned by symbols $c_x(m, n), c_y(m, n), m, n \in \Lambda(q)$ with relation $mc_x(m, n) + nc_y(m, n) = 0$. Let $gl_\nu(\mathbb{C}) \otimes \mathbb{C}_q$ be the matrix Lie algebra over the quantum tori $\mathbb{C}_q$. Define a central extension $\mathcal{B}$ of the Lie algebra $gl_\nu(\mathbb{C}) \otimes \mathbb{C}_q$ by central subalgebra $\mathcal{K}$ as follows:

$$[A \otimes x^{m_1}y^{n_1}, B \otimes x^{m_2}y^{n_2}]$$

$$= q^{m_1n_1}AB \otimes x^{m_1+m_2}y^{n_1+n_2} - q^{m_2n_2}BA \otimes x^{m_1+m_2}y^{n_1+n_2}$$

$$+ \text{tr}(AB)\delta_{m_1+m_2, \Lambda(q)}\delta_{n_1+n_2, \Lambda(q)}q^{n_1m_2}$$

$$\cdot (m_1c_x(m_1 + m_2, n_1 + n_2) + n_1c_y(m_1 + m_2, n_1 + n_2)),$$

where $A, B \in gl_\nu(\mathbb{C}), m_1, m_2, n_1, n_2 \in \mathbb{Z}$, and

$$\delta_{m, \Lambda(q)} = \begin{cases} 1, & \text{if } m \in \Lambda(q); \\ 0, & \text{if } m \notin \Lambda(q). \end{cases}$$

It is straightforward to prove the following result.

**Lemma 2.1.** The linear map $\mathcal{L} \to \mathcal{B}$ given by

$$A \otimes x^my^n \mapsto A \otimes x^my^n, \quad A \in \mathfrak{sl}_\nu(\mathbb{C}), \quad m, n \in \mathbb{Z},$$

$$<x^{m_1}y^{n_1}, x^{m_2}y^{n_2}> \mapsto I_\nu \otimes [x^{m_1}y^{n_1}, x^{m_2}y^{n_2}]$$

$$<x, x^{m}y^{n}x^{-1} > \mapsto \nu c_x(m, n), \quad m, n \in \Lambda(q),$$

$$<y, x^{m_1}y^{n_1} > \mapsto \nu c_y(m, n), \quad m, n \in \Lambda(q),$$

is an injective Lie algebra homomorphism.

The result of Lemma 2.1 allows us to identify $\mathcal{L}$ with the subalgebra $\mathcal{B}' := (\mathfrak{sl}_\nu(\mathbb{C}) \otimes \mathbb{C}_q) \oplus (I_\nu \otimes [\mathbb{C}_q, \mathbb{C}_q]) \oplus \mathcal{K}$ of $\mathcal{B}$. We can also define a Lie algebra $\tilde{\mathcal{B}} = \mathcal{B} \oplus \mathbb{C}d_x \oplus \mathbb{C}d_y$ with multiplication given similarly as (1). Then, one can identify $\tilde{\mathcal{L}}$ with the subalgebra $\mathcal{B}' \oplus \mathbb{C}d_x \oplus \mathbb{C}d_y$ of $\tilde{\mathcal{B}}$. Hence, under this identification, we can write $\tilde{\mathcal{B}} = \tilde{\mathcal{L}} \oplus \sum_{m,n \in \Lambda(q)} \mathbb{C}I_\nu \otimes x^m y^n$.

Now we turn to consider the root-space decomposition of $\tilde{\mathcal{L}}$. For $1 \leq i, j \leq \nu$, let $E_{ij}$ be the unit $\nu \times \nu$ matrix which has 1 in the $(i, j)$-entry and 0 elsewhere. Let $h_i = E_{ii} - E_{i+1, i+1}$ for $1 \leq i \leq \nu - 1$ and $H = \text{span}_\mathbb{C}\{h_i \otimes 1 : 1 \leq i \leq \nu - 1\}$. Let $\tilde{\mathcal{H}}$ be a Cartan subalgebra of $\tilde{\mathcal{L}}$ spanned by $\mathcal{H}, c_x := c_x(0, 0), c_y := c_y(0, 0), d_x$ and $d_y$. Then, one has the following root-space decomposition of $\tilde{\mathcal{L}}$ with respect to the Cartan subalgebra $\tilde{\mathcal{H}}$:

$$\tilde{\mathcal{L}} = \bigoplus_{\gamma \in \tilde{\mathcal{H}}^*} \tilde{\mathcal{L}}_{\gamma}, \quad \text{where } \tilde{\mathcal{L}}_{\gamma} = \{A \in \tilde{\mathcal{L}} | [h, A] = \lambda(h)A, \forall h \in \tilde{\mathcal{H}}\}.$$

Let $\Delta = \{\epsilon_i - \epsilon_j | 1 \leq i \neq j \leq \nu\}$ be the root system of $\mathfrak{sl}_\nu(\mathbb{C})$ as usual. View $\Delta$ as a subset of $\tilde{\mathcal{H}}^*$ by setting $(\epsilon_i - \epsilon_j)(z) = 0$, where $z = c_x, d_x, c_y$ or $d_y$. Introduce elements $\delta_x, \delta_y \in \tilde{\mathcal{H}}^*$ by letting $\delta_x(d_x) = 1, \delta_y(d_y) = 1$ and the actions on other
basis elements are zero. Then, the set \(\Delta = \{\alpha + m\delta_x + n\delta_y : \alpha \in \hat{\Delta} \cup \{0\}, m, n \in \mathbb{Z}\}\) is the root system of \(\tilde{L}\) with respect to \(\hat{\mathcal{H}}\). A root of the form \(\alpha + m\delta_x + n\delta_y \in \Delta\) is called real if \(\alpha \in \hat{\Delta}\).

**Definition 2.2.** An \(\tilde{\mathcal{L}}\)-module \(V\) is called integrable if

1. \(V = \bigoplus_{\lambda \in \hat{\mathcal{H}}} V_\lambda\) where \(V_\lambda = \{v \in V | h.v = \lambda(h)v, \forall h \in \hat{\mathcal{H}}\}\).
2. For any weight \(\gamma \in \hat{\mathcal{H}}^*\), one has \(\dim V_\lambda < \infty\).
3. For any real root \(\gamma\) and element \(v \in V\), there exists positive integer \(k\) such that \((\tilde{\mathcal{L}})_{\gamma}^k v = 0\).

Let \(\mathcal{A} = sl_n(\mathbb{C}) \otimes \mathbb{C}[x, x^{-1}] \oplus \mathbb{C}c_x \oplus \mathbb{C}d_x\) be the subalgebra of \(\tilde{\mathcal{L}}\), which is isomorphic to the affine Kac-Moody algebra of type \(A_{n-1}^{(1)}\). Consider the natural triangular decomposition \(\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_0 \oplus \mathcal{A}_-\), where

\[\mathcal{A}_+ = sl_n(\mathbb{C}) \otimes \mathbb{C}[x] \oplus \sum_{i<j} \mathbb{C}E_{ij},\]
\[\mathcal{A}_- = sl_n(\mathbb{C}) \otimes \mathbb{C}[x^{-1}] \oplus \sum_{i>j} \mathbb{C}E_{ij},\]
\[\mathcal{A}_0 = \mathcal{H} \oplus \mathbb{C}c_x \oplus \mathbb{C}d_x.\]

So we have the decomposition \(\tilde{\mathcal{L}} = \tilde{\mathcal{L}}_+ \oplus \tilde{\mathcal{L}}_0 \oplus \tilde{\mathcal{L}}_-\), where

\[\tilde{\mathcal{L}}_+ = \mathcal{A}_+ \oplus \mathcal{C}[y, y^{-1}] \oplus <\mathbb{C}_q, \mathbb{C}_q >_+,\]
\[\tilde{\mathcal{L}}_0 = \mathcal{H} \oplus \mathcal{C}[y, y^{-1}] \oplus <\mathbb{C}_q, \mathbb{C}_q >_0 \oplus \mathbb{C}d_x \oplus \mathbb{C}d_y.\]

We identify \(\mathbb{C}[x^{\pm 1}, y^{\pm 1}]\) with \(\mathbb{C}[x, x^{-1}] \otimes \mathbb{C}[y, y^{-1}]\), and

\[<\mathbb{C}_q, \mathbb{C}_q >_\pm = \oplus_{m,n \in \mathbb{Z}, \pm m > 0} <\mathbb{C}_q, \mathbb{C}_q >_{(m,n)},\]
\[<\mathbb{C}_q, \mathbb{C}_q >_0 = \oplus_{n \in \mathbb{Z}} <\mathbb{C}_q, \mathbb{C}_q >_{(0,n)} .\]

We also introduce a \(\mathbb{Z}\)-grading on \(\tilde{\mathcal{L}}\) as follows:

\[\tilde{\mathcal{L}}(n) = \{A \in \tilde{\mathcal{L}} | [d_y, A] = nA\}, \quad n \in \mathbb{Z}.\]

For any subalgebra \(\mathcal{G}\) of \(\tilde{\mathcal{L}}\), we will use the following notation

\[\mathcal{G}(n) := \tilde{\mathcal{L}}(n) \cap \mathcal{G}, \quad n \in \mathbb{Z}.\]

Let \(\psi\) be a linear function on \(\tilde{\mathcal{L}}_0\) such that \(\psi(c_y) = 0\). Let

\[\tilde{\mathcal{L}}_0 = \mathcal{H} \otimes \mathcal{C}[y, y^{-1}] \oplus <\mathbb{C}_q, \mathbb{C}_q >_0 \oplus \mathbb{C}d_x\]

and define a linear map \(\overline{\psi} : \tilde{\mathcal{L}}_0 \to \mathbb{C}[t, t^{-1}]\) as follows:

\[\overline{\psi}(h) = \psi(h)t^n, \quad \forall h \in \tilde{\mathcal{L}}_0(n).\]

Let \(A_{\overline{\psi}} \subset \mathbb{C}[t, t^{-1}]\) be the image of \(\overline{\psi}\). Then, \(\overline{\psi}\) induces an \(\tilde{\mathcal{L}}_0\)-module structure on \(A_{\overline{\psi}}\) with the actions given by

\[h.t^n = (\overline{\psi}(h))t^n, \quad h \in \tilde{\mathcal{L}}_0, \quad d_y.t^n = (\psi(d_y) + m)t^n, \quad m \in \mathbb{Z}.\]
Remark 2.3. In the paper [11], the linear function $\psi$ was defined on $\hat{L}_0$ and the action of $d_y$ on $A_{\psi}$ was given by $d_y t^m = mt^m$. But, we observe that one can add an extra scalar action of $d_y$ on $A_{\psi}$, that is, we can define the action of $d_y$ by $d_y t^m = (c + m)t^m$ for any fixed scalar $c$.

We recall the definition of highest weight $\hat{L}$-modules, which was introduced by Rao (See [11],[9])

Definition 2.4. An $\hat{L}$-module $V$ is called a highest weight module if there exists a weight vector $v$ in $V$ such that $\hat{L}_-, v = 0, U(\hat{L}) v = V$ and the $\hat{L}_0$-module generated by $v$ is isomorphic to an irreducible $\hat{L}_0$-module $A_{\psi}$ for some linear function $\psi$.

For a given linear function $\psi$, suppose that $A_{\psi}$ is irreducible as $\hat{L}_0$-module. Viewing $A_{\psi}$ as $(\hat{L}_+ \oplus \hat{L}_0)$-module by letting $\hat{L}_+$ acts trivially. So we have an induced $\hat{L}$-module $M(\psi) = U(\hat{L}) \otimes_{U(\hat{L}_+ \oplus \hat{L}_0)} A_{\psi}$. It is easy to see that $M(\psi)$ has a unique irreducible quotient, which we denote by $V(\psi)$.

For $1 \leq i \leq \nu$ and $n \in \mathbb{Z}$, define elements $h_{i,n} \in \hat{L}_0$ as follows

$$
\begin{align*}
    h_{i,n} &= h_i \otimes y^n, \ 1 \leq i \leq \nu - 1, \\
    h_{\nu,n} &= -q^n E_{11} \otimes y^n + E_{\nu\nu} \otimes y^n + \delta_{\nu,\lambda(0)} c_y (0, n).
\end{align*}
$$

(2)

Then $c_y, d_x, d_y, h_{i,n}, 1 \leq i \leq \nu, n \in \mathbb{Z}$ form a basis of $\hat{L}_0$. Let

$$
P_+ = \{ \lambda\in \hat{H}^* : \lambda(h_{i,0}) \in \mathbb{N}, 1 \leq i \leq \nu, \lambda(c_y) = 0 \}
$$

and $l$ be a positive integer. Then, for each pair

$$(\lambda, b) \in (P_+)^l \times (\mathbb{C}^*)^l, \ \lambda = (\lambda_1, \cdots, \lambda_l), \ b = (b_1, \cdots, b_l),$$

such that $b_1, \cdots, b_l$ are distinct, we can define a linear function $\psi_{\lambda,b}$ on $\hat{L}_0$ by requiring that

$$
\begin{align*}
    \psi_{\lambda,b}(h_{i,n}) &= \sum_{j=1}^l \lambda_j(h_{i,0}) b_j^n, \ 1 \leq i \leq d, n \in \mathbb{Z}, \\
    \psi_{\lambda,b}(z) &= \sum_{j=1}^l \lambda_j(z), \ z = d_x, d_y, \text{ or } c_y.
\end{align*}
$$

(3)

Then the resulting $\hat{L}_0$-module $A_{\psi_{\lambda,b}}$ is irreducible and we hence obtain an irreducible highest weight $\hat{L}$-module $V(\psi_{\lambda,b})$. Let $M = \left(\begin{smallmatrix}a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, then $M$ can be extended to an automorphism of $\hat{L}$, which is again denoted by $M$ (See [11], Sect.5 for details). The following result was proved in [11]

Theorem 2.5. (1) Let $V$ be an irreducible $\hat{L}$-module such that $c_x$ acts as positive integer and $c_y$ acts as zero. Then $V$ is integrable if and only if $V \cong V(\psi_{\lambda,b})$ for some pair $(\lambda, b)$. 

In this section we first construct fermionic lowest weight module up to a twist of an automorphism $M$ and then show that the tensor non-trivial. Then $V$ is isomorphic to either a highest weight module $V(\psi_{\lambda,b})$ or a lowest weight module up to a twist of an automorphism $M$.

3. Tensor modules for $\tilde{L}$

In this section we first construct fermionic $\tilde{L} := \mathcal{L} \oplus \mathbb{C}d_\mathcal{L}$-modules $V_k, k \in \mathbb{Z}$ and then show that the tensor $\tilde{L}$-module $\bigotimes_{s=1}^i V_{i_s, i, \cdots, i} \in \mathbb{Z}$ is completely reducible.

Let $R_\nu$ be a unital associative algebra with infinitely many generators $\psi_i(m), \psi_i^*(m)$, for $m \in \mathbb{Z}, 1 \leq i \leq \nu$, subject to the following relations

$$\psi_i(m)\psi_j(n) + \psi_j(n)\psi_i(m) = \psi_i^*(m)\psi_j^*(n) + \psi_j^*(n)\psi_i^*(m) = 0,$$

$$\psi_i(m)\psi_j^*(n) + \psi_j^*(n)\psi_i(m) = \delta_{ij}\delta_{m+n,0}.$$

We define normal ordering as follows

$$: \psi_i(m)\psi_j^*(n) := \begin{cases} \psi_i(m)\psi_j^*(n), & \text{if } m \leq n; \\ -\psi_j^*(n)\psi_i(m), & \text{if } m > n, \end{cases}$$

for $m, n \in \mathbb{Z}, 1 \leq i, j \leq \nu$.

Let $R_\nu^+$ be the subalgebra of $R_\nu$ generated by $\psi_i(m), \psi_i^*(n)$, for $m > 0, n \geq 0$ and $1 \leq i \leq \nu$, and $R_\nu^-$ be the subalgebra generated by $\psi_i(m), \psi_i^*(n)$, for $m \leq 0, n < 0$ and $1 \leq i \leq \nu$. Let $V(\nu)$ be a simple $R_\nu$-module containing an element $v_0$, called “vacuum vector”, and satisfying $R_\nu^-v_0 = 0$. Therefore,

$$V(\nu) = R_\nu^-v_0 \oplus \mathbb{C}v_0.$$

For $m, n \in \mathbb{Z}, 1 \leq i, j \leq \nu$, we set

$$f_{ij}(m, n) = \sum_{p \in \mathbb{Z}} q^{-np} : \psi_i(m-p)\psi_j^*(p) :,$$

$$D = \sum_{i=1}^\nu \sum_{p \in \mathbb{Z}} p : \psi_i(p)\psi_i^*(-p) :,$$

and

$$F_{ij}(m, n) = \begin{cases} f_{ij}(m, n), & \text{for } n \in \Lambda(q), \\ f_{ij}(m, n) - \delta_{ij}\delta_{m,0}\frac{q^n}{q^m-1}, & \text{for } n \notin \Lambda(q). \end{cases}$$

For any vector $v = \psi_{i_1}(m_1)\cdots \psi_{i_s}(m_s)\psi_{j_1}^*(n_1)\cdots \psi_{j_l}^*(n_l)v_0 \in V(\nu)$, we define an linear operator $J$ on $V(\nu)$ by

$$J(v) = (s - t)v.$$

For any $k \in \mathbb{Z}$, let $V_k$ be the $k$-eigenspace of $V(\nu)$ with respect to the operator $J$. Now, as a by-product of Theorem 3.8 [14], we have
Proposition 3.1. \( V(\nu) \) is a module for the Lie algebra \( \hat{B} = B \oplus C_d \) under the actions given by

\[
E_{ij} \otimes x^m y^n \mapsto F_{ij}(m, n), \quad \text{for } 1 \leq i, j \leq \nu, m, n \in \mathbb{Z};
\]

\[
c_x(0, n') \mapsto 1, \quad d_x \mapsto D, \quad \text{for } n' \in \Lambda(q);
\]

\[
c_y(m', n') \mapsto 0, \quad \text{for } m', n' \in \Lambda(q).
\]

Moreover, \( V(\nu) \) is completely reducible and each component \( V_k, k \in \mathbb{Z} \) is irreducible.

For \( k \in \mathbb{Z} \), we define a vector \( v_k \in V_k \) as follows

\[
v_k = \begin{cases} 
\psi_r(-s) \cdots \psi_1(-s) \psi_\nu(1-s) \cdots \psi_\nu(0) \cdots \psi_1(0)v_0, & \text{for } k > 0, \\
\psi_{r+1}^*(s) \cdots \psi_\nu^*(s) \psi_\nu^*(s+1) \cdots \psi_1^*(s+1) \cdots \psi_1^*(-1) \cdots \psi_\nu^*(-1)v_0, & \text{for } k < 0,
\end{cases}
\]

where \((s, r)\) is the unique pair such that \( k = s\nu + r, s \in \mathbb{Z}, 1 \leq r \leq \nu \).

By restriction, one can view \( V_k, k \in \mathbb{Z} \) as an \( \hat{\mathcal{L}} \)-module. Let \( \omega_k \) be an element in \( \hat{\mathcal{L}}^*_0 \) defined by

\[
\omega_k(h_{i,n}) = \delta_{r,q}^{-s}, \omega_k(d_x) = D_k \quad 1 \leq i \leq \nu, n \in \mathbb{Z},
\]

where \( k = s\nu + r, s \in \mathbb{Z}, 1 \leq r \leq \nu \) and \( h_{i,n} \) is defined in (2) and \( D_k \) is the scalar determined by \( Dv_k = D_kv_k \). The following result shows that \( v_k \) is a highest weight vector in the \( \hat{\mathcal{L}} \)-module \( V_k \) with highest weight \( \omega_k \). The verification of this assertion is straightforward, and is omitted.

Lemma 3.2. In the \( \hat{\mathcal{L}} \)-module \( V_k, k \in \mathbb{Z} \), we have \( \hat{\mathcal{L}}^+.v_k = 0 \) and \( h.v_k = \omega_k(h).v_k \) for all \( h \in \hat{\mathcal{L}}_0 \). \( \blacksquare \)

Let \( \hat{H} = \mathcal{H} \oplus C_{dx} \oplus C_{dy} \) and we identify \( \hat{H}^* \) with the set \( \{ \alpha \in \hat{H}^* : \alpha(dy) = 0 \} \). Let \( \alpha_i = \epsilon_i - \epsilon_{i+1}, 1 \leq i \leq \nu - 1, \alpha_\nu = \epsilon_\nu - \epsilon_1 + \delta_x \) and \( Q_+ = N_\alpha \oplus \cdots \oplus N_\alpha \). Clearly, \( V_k \) is a weight module for \( \hat{\mathcal{L}} \) with respect to \( \hat{H} \). Moreover, the previous lemma shows that all weights of \( V_k \) have the form \( \Lambda_k - \eta, \eta \in Q_+ \), where \( \Lambda_k \in \hat{H}^* \) is the restriction of \( \omega_k \) on \( \hat{H} \).

Let \( l \) be a positive integer and \( \bar{i} = \{i_1, \cdots, i_l\} \in \mathbb{Z}^l \). We denote the tensor space \( \bigotimes_{i=1}^{l} V_{i_l} \) by \( V_{\bar{i}} \). In the rest of this section we are going to show that the \( \hat{\mathcal{L}} \)-module \( V_{\bar{i}} \) is completely reducible.

We first consider the case with \( |q| = 1 \). In this case, one can define a conjugate-linear anti-involution \( \theta_q \) of \( \hat{B} \) by letting

\[
\theta_q(E_{ij} \otimes x^m y^n) = q^{mn} E_{ij} \otimes x^{-m} y^{-n}, \quad \theta_q(d_x) = d_x,
\]

\[
\theta_q(c_x(m', n')) = c_x(-m', -n'), \quad \theta_q(c_y(m', n')) = c_y(-m', -n'),
\]

where \( 1 \leq i, j \leq \nu, m, n \in \mathbb{Z} \) and \( m', n' \in \Lambda(q) \).

If \( \mathcal{G} \) is a Lie algebra and \( \theta \) a conjugate-linear anti-involution on \( \mathcal{G} \). Recall that a \( \mathcal{G} \)-module \( W \) is said to be unitary with respect to \( \theta \) if there exists a positive definite Hermitian form \( \langle , \rangle \) on \( W \) such that

\[
\langle A.v, w \rangle = \langle v, \theta(A).w \rangle, \quad A \in \mathcal{G}, v, w \in W.
\]
We now define a positive definite Hermitian structure $< , >$ on $V(\nu)$ by the characteristic conditions

$$
< v_0, v_0 > = 1, \quad < v, w > = \overline{< w, v >},
$$
$$
< \psi_i(m).v, w > = < v, \psi_i^*(m).w >,
$$
where $1 \leq i \leq \nu$, $m \in \mathbb{Z}$ and $v, w \in V(\nu)$. Then, we have

**Lemma 3.3.** If $|q| = 1$, then the $\hat{B}$-module $V(\nu)$ is unitary with respect to $\theta_q$.

**Proof.** For $1 \leq i, j \leq \nu, m, n \in \mathbb{Z}$ and $v, w \in V(\nu)$, one has

$$
< f_{ij}(m, n)v, w > = < \sum p \in \mathbb{Z} q^{-np} : \psi_i(m - p)\psi_j^*(p) : v, w >
$$
$$
= < v, \sum p \in \mathbb{Z} q^{np} : \psi_j(-p)\psi_i^*(p - m) : w > \quad \text{(since $\bar{q} = q^{-1}$)}
$$
$$
= < v, q^{mn} \sum s \in \mathbb{Z} q^{ns} : \psi_j(-m - s)\psi_i^*(s) : w > \quad \text{(let $s = p - m$)}
$$
$$
= < v, q^{mn} f_{ji}(-m, -n)w >.
$$

Since $\frac{q^n}{q^{n-1}} = \overline{\frac{q^n}{q^{n-1}}}$ for $n \notin \Lambda(q)$, we obtain that

$$
< E_{ij} \otimes x^m y^n .v, w > = < v, \theta_q(E_{ij} \otimes x^m y^n) .w >.
$$

For the other cases, we have

$$
< d_x .v, w > = < v, \sum_{i=1}^{\nu} \sum p : \psi_i(p)\psi_i^*(-p) : w > = < v, d_x .w >,
$$
$$
< c_x(0, n').v, w > = < v, w > = < v, c_x(0, -n') .w >,
$$
$$
< c_y(m', n').v, w > = 0 = < v, c_y(-m', -n') .w >,
$$

where $m', n' \in \Lambda(q)$, as required. \[\square\]

Since $\theta_q(\hat{L}) = \hat{L}$, the $\hat{L}$-module $V_i$ is also unitary with respect to $\theta_q$. This implies that the $\hat{L}$-module $V_i$ is completely reducible if $|q| = 1$.

Now we turn to consider the case with $|q| \neq 1$. Let $gl_\infty = \sum_{i,j \in \mathbb{Z}} E_{ij}$ be the usual infinite matrix algebra. It is well-known that each $V_k, k \in \mathbb{Z}$ is a $gl_\infty$-module with the action given by

$$
E_{mv+i,j-nv} \mapsto \psi_i(m)\psi_j^*(n), \quad 1 \leq i, j \leq \nu, m, n \in \mathbb{Z}.
$$

Moreover, $V_k$ is an integrable highest weight $gl_\infty$-module with highest weight vector $v_k$ and the tensor $gl_\infty$-module $V_i$ is completely reducible.

**Lemma 3.4.** Let $q$ be generic. Then $\hat{B}$-submodules and $gl_\infty$-submodules in $V_i$ coincide.
Let \( U \) and each weights of \( \nu, m, n \) \( \in \mathbb{Z} \), \( w \in W \), one has \( \psi_i(m)\psi_j^*(n)w \in W \) and hence : \( \psi_i(m)\psi_j^*(m) : w \in W \). This implies \( f_{ij}(m,n)w = \sum_{p \in \mathbb{Z}} q^{-sp} : \psi_i(m-p)\psi_j^*(p) : w \in W \). Similarly, one has \( F_{ij}(m,n)w \in W \) and \( Dw \in W \), which gives \( W \) is an \( \hat{B} \)-submodule.

Conversely, let \( W \) be an \( \hat{B} \)-submodule of \( V_i' \). For any fixed \( 1 \leq i, j \leq \nu, m, n \in \mathbb{Z}, w \in W \), we need to show that : \( \psi_i(m)\psi_j^*(n) : w \in W \). Note that there exist \( N_1 \leq n \leq N_2 \in \mathbb{Z} \) such that : \( \psi_i(m+n-p)\psi_j^*(p) : w = 0 \) for all \( p < N_1 \) or \( P > N_2 \). Consider now the equations

\[
f_{ij}(m+n,s)w = \sum_{p \in \mathbb{Z}} q^{-sp} : \psi_i(m+n-p)\psi_j^*(p) : w
= \sum_{p=N_1}^{N_2} q^{-sp} : \psi_i(m+n-p)\psi_j^*(p) : w \in W, N_1 \leq s \leq N_2.
\]

Since \( q \) is generic, by solving the above equations one gets : \( \psi_i(m+n-p)\psi_j^*(p) : w \in W, N_1 \leq p \leq N_2 \). In particular, we have : \( \psi_i(m)\psi_j^*(n) : w \in W \), as required.

Observe that \( \hat{B} = \hat{\mathcal{L}} \oplus I_\nu \) if \( q \) is generic and \( I_\nu \) acts as a scalar on \( V_i' \). This implies that the \( \hat{\mathcal{L}} \)-submodules in \( V_i' \) are coincide with the \( \hat{B} \)-submodules and hence with the \( gl_\infty \)-submodules. Therefore, we obtain that the \( \hat{\mathcal{L}} \)-module \( V_i' \) is completely reducible if \( q \) is generic. In summary, we have

**Theorem 3.5.** For \( i_1, \ldots, i_l \in \mathbb{Z} \), the \( \hat{\mathcal{L}} \)-module \( V_i = \bigotimes_{s=1}^{l} V_{i_s} \) is completely reducible.

**4. Realization of integrable highest weight \( \tilde{\mathcal{L}} \)-modules**

Let \( U_1, \ldots, U_k \) be a collection of \( \hat{\mathcal{L}} \)-modules and \( a = (a_1, \ldots, a_k) \in (\mathbb{C}^*)^k \). Due to the work by Chari and Pressly [6], it allows us to define an \( \hat{\mathcal{L}} \)-module structure on the tensor space \( U = \bigotimes_{s=1}^{k} U_s \otimes \mathbb{C}[t, t^{-1}] \) as follows:

\[
A.u_1 \otimes \cdots \otimes u_k \otimes t^l = \sum_{s=1}^{k} a_s^n u_1 \otimes \cdots \otimes u_k \otimes t^{l+n},
\]

\[
d_y.u_1 \otimes \cdots \otimes u_k \otimes t^l = l u_1 \otimes \cdots \otimes u_k \otimes t^l,
\]

where \( u_s \in U_s, 1 \leq s \leq k, A \in \hat{\mathcal{L}}(n) \) and \( n, l \in \mathbb{Z} \).

Let \( W_{\bar{i}} \) be the \( \hat{\mathcal{L}} \)-submodule of \( V_{\bar{i}} = \bigotimes_{s=1}^{l} V_{i_s} \) generated by \( w_{\bar{i}} = \bigotimes_{s=1}^{l} v_{i_s} \).

Combine Theorem 3.5 with Lemma 3.2, we find that the \( \hat{\mathcal{L}} \)-module \( W_{\bar{i}} \) is irreducible, \( \hat{\mathcal{L}}_+ w_{\bar{i}} = 0 \) and \( W_{\bar{i}} = U(\hat{\mathcal{L}_-}) w_{\bar{i}} \). Recall the linear operators \( \omega_k \in \hat{\mathcal{L}}_0^* \) and \( \Lambda_k \in \hat{\mathcal{H}}^* \) defined in Sect. 3. Set \( \omega_{\bar{i}} = \sum_{s=1}^{l} \omega_{i_s} \) and \( \Lambda_{\bar{i}} = \sum_{s=1}^{l} \Lambda_{i_s} \). Then, one has

\[
h.w_{\bar{i}} = \omega_{\bar{i}}(h)w_{\bar{i}}, \ h \in \hat{\mathcal{L}_0},
\]

and each weights of \( W_{\bar{i}} \) has the form \( \Delta_{\bar{i}} - \eta \).
In the following, we shall always take \( U_s = W_{i_s}, 1 \leq s \leq k \) for some \( \vec{i}_s = (i_{1,s}, \cdots, i_{n_s,a}) \in \mathbb{Z}^{n_s} \). Now, for each pair

\[
\mathbf{i} = (\vec{i}_1, \cdots, \vec{i}_k), \quad \mathbf{a} = (a_1, \cdots, a_k),
\]

with the condition that

\[
q^n a_i \neq a_j, \text{ for all } 1 \leq i \neq j \leq k; n \in \mathbb{Z},
\]

we obtain an \( \widehat{L} \)-module structure on the tensor space

\[
W_{\mathbf{i}, \mathbf{a}} = W_{\vec{i}_1} \otimes \cdots \otimes W_{\vec{i}_k} \otimes \mathbb{C}[t, t^{-1}]
\]

with the action given by (5). Notice that, if \( q \) is an \( N \)-th primitive root of unity, then the condition (6) is equivalent to that \( a_i^N \neq a_j^N, \forall i \neq j \). The main purpose of this section is to prove that such \( \widehat{L} \)-modules \( W_{\mathbf{i}, \mathbf{a}} \) are completely reducible and their irreducible components exhaust all of the irreducible integrable highest weight \( \widehat{L} \)-modules classified in Theorem 2.5 by Rao [11].

We define a “character”

\[
\chi_{\mathbf{i}, \mathbf{a}} : U(\widehat{L}_0) \to \mathbb{C}[t, t^{-1}]
\]

of the universal enveloping algebra of \( \widehat{L}_0 \) by extending

\[
\chi_{\mathbf{i}, \mathbf{a}}(h) = \left( \sum_{s=1}^{k} \omega_{\vec{i}_s}(h) a_s^n \right) t^n,
\]

where \( h \in \widehat{L}_0(n) \). Since \( U(\widehat{L}_0) \) inherits a natural \( \mathbb{Z} \)-grading from \( \widehat{L}_0 \), then \( \chi_{\mathbf{i}, \mathbf{a}} \) is a graded algebra homomorphism and the image of \( \chi_{\mathbf{i}, \mathbf{a}} \) is always a Laurent subring \( L_r := \mathbb{C}[t^r, t^{-r}] \) of \( \mathbb{C}[t, t^{-1}] \) for some \( r \geq 1 \). For all \( i \in \mathbb{Z} \), let \( \Omega_i \) denote the element \( \otimes_{s=1}^{k} w_{\vec{i}_s} \otimes t^i \). For \( h \in U(\widehat{L}_0) \) with \( \chi_{\mathbf{i}, \mathbf{a}}(h) = t^m \) for some \( m \in \mathbb{Z} \), one can easily check that \( h.\Omega_i = \Omega_{i+m} \) for \( i \in \mathbb{Z} \).

**Proposition 4.1.** If the image of \( \chi_{\mathbf{i}, \mathbf{a}} \) is \( L_r \) for some \( r \geq 1 \), then the \( \widehat{L} \)-module \( W_{\mathbf{i}, \mathbf{a}} \) is generated by \( \Omega_0, \cdots, \Omega_{r-1} \).

**Proof.** We denote by \( M \) the submodule of \( W_{\mathbf{i}, \mathbf{a}} \) generated by \( \Omega_0, \cdots, \Omega_{r-1} \). Since the image of \( \chi_{\mathbf{i}, \mathbf{a}} \) is \( L_r \), one can get that \( \Omega_n \in U(\widehat{L}_0)\Omega_l \) for all \( n \equiv l \mod r \), where \( 0 \leq l \leq r - 1 \). This forces that \( \Omega_n \in M \) for all \( n \in \mathbb{Z} \).

Let \( A = \sum_{t=1}^{\nu} \lambda_t E_t(t) \otimes x^t \) be an element in \( A_- \), where \( 1 \leq i(t), j(t) \leq \nu, m_t \in \mathbb{Z}, \nu, \lambda_t \in \mathbb{C}^* \). Then, one has

\[
(A \otimes y^n).w_{\vec{i}_s} = \sum_{t=1}^{a} \sum_{p \in \mathbb{Z}} q^{n p} : \psi_{i(t)}(m_t + p) \psi_{j(t)}^* (-p) : w_{\vec{i}_s},
\]

for \( 1 \leq s \leq k \) and \( n \in \mathbb{Z} \). Observe that there are only finite many \( p \), say, \( p_{i_s,t}, 1 \leq l \leq t_s \), such that \( : \psi_{i(t)}(m_t + p) \psi_{j(t)}^* (-p) : w_{\vec{i}_s} \neq 0 \) for some \( t \). For
1 \leq l \leq t_s$, we denote $v_{i,s,t} = \sum_{t=1}^{a} \lambda_k : \psi(t(m_t + p_{i,s,t}))(-p_{i,s,t}) : w_i$, then one has
\begin{equation}
(A \otimes y^n). w_{i,s,t} = \sum_{l=1}^{t_s} q^{np_{i,s,t}}v_{i,s,t},
\end{equation}
for all $1 \leq s \leq k$ and $n \in \mathbb{Z}$.

Let $q^{p_{i,s,t}}, \ldots, q^{p_{i,m_s}}$ be distinct numbers among $q^{p_{i,s,t}}, 1 \leq l \leq t_s$. For $1 \leq s \leq k$ and $1 \leq t \leq m_s$, let $I(s,t) = \{l|1 \leq l \leq t_s, q^{p_{i,s,t}} = q^{p_{i,t}}\}$ and
\begin{equation}
v_{s,t} = \sum_{l \in I(s,t)} v_{i,s,t}.
\end{equation}
Let $p \in \mathbb{Z}$, consider the equation
\begin{equation}
(A \otimes y^n). \Omega_{p-n} = \sum_{s=1}^{k} a_s^n(w_{i,s,t} \otimes \cdots \otimes (A \otimes y^n). w_{i,s,t} \otimes \cdots \otimes w_i) \otimes t^p
\end{equation}
for all $n \in \mathbb{Z}$. Since $a_s q^{p_{i,t}}$ are distinct for all $1 \leq s \leq k, 1 \leq t \leq m_s$, we can solve for $w_{i,s,t} \otimes \cdots \otimes w_{s,t} \otimes \cdots \otimes w_i \otimes t^p$ in term of $(A \otimes y^n). \Omega_{p-n}$. Then, one has
\begin{equation}
w_{i,s,t} \otimes \cdots \otimes w_{s,t} \otimes \cdots \otimes w_i \otimes t^p \in M,
\end{equation}
for all $1 \leq s \leq k, 1 \leq t \leq m_s$. In particular, we have
\begin{equation}
w_{i,s,t} \otimes \cdots \otimes (A \otimes y^n). w_{i,s,t} \otimes \cdots \otimes w_i \otimes t^p \in M,
\end{equation}
for all $n, p \in \mathbb{Z}, 1 \leq s \leq k$ and $A \in \mathcal{A}_-$.

For $I_{\nu} \otimes x^m y^n \in \tilde{\mathcal{C}}_-$, we know $m < 0$, and $m$ or $n \notin \Lambda(q)$. Therefore, we need to divide the argument into two cases.

First, if $m \notin \Lambda(q)$, we have that $I_{\nu} \otimes x^m y^n \in \tilde{\mathcal{C}}_-$ for all $n \in \mathbb{Z}$. Then, a repeated proof of (9) shows that
\begin{equation}
w_{i,s,t} \otimes \cdots \otimes (I_{\nu} \otimes x^m y^n). w_{i,s,t} \otimes \cdots \otimes w_i \otimes t^p \in M,
\end{equation}
for all $n, p \in \mathbb{Z}, 1 \leq s \leq k$ and $m < 0, m \notin \Lambda(q)$.

Next, if $m \in \Lambda(q), n \notin \Lambda(q)$, then $q$ must be a root of unity. Suppose that $q$ is an $N$-th primitive root of unity, then $\Lambda(q) = NZ$. Similar to the proof of (7), we obtain
\begin{equation}(I_{\nu} \otimes x^m y^n). w_{i,s,t} = \sum_{t=1}^{m_s} q^{np_{i,s,t}}v_{s,t},
\end{equation}
for some $v_{s,t} \in W_i, p_{s,t} \in \mathbb{Z}, 1 \leq t \leq m_s$ and $n \notin \Lambda(q)$. This gives
\begin{equation}(I_{\nu} \otimes x^m y^{n+N}). w_{i,s,t} = \sum_{t=1}^{m_s} (q^{p_{i,s,t}})^j v_{s,t},
\end{equation}
for \(0 < j < N\) and \(n \in \mathbb{Z}\). Similar to (8), we have

\[
(I_\nu \otimes x^m y^{nN+j}).\Omega_{p-nN-j} = \sum_{s=1}^{k} \sum_{t=1}^{m_s} (a_s q^{p_a t})^{nN+j} w_{i_1} \otimes \cdots \otimes v_{s,t} \otimes \cdots \otimes w_{i_k} \otimes t^p
\]

\[
= \sum_{s=1}^{k} (a_s)^n \sum_{t=1}^{m_s} (a_s q^{p_a t})^{j} w_{i_1} \otimes \cdots \otimes v_{s,t} \otimes \cdots \otimes w_{i_k} \otimes t^p,
\]

for all \(n, p \in \mathbb{Z}\). Since \(a_1^N, \ldots, a_k^N\) are distinct, this and (11) imply that

\[
a_j^i w_{i_1} \otimes \cdots \otimes (I_\nu \otimes x^m y^{nN+j}).w_{i_s} \otimes \cdots \otimes w_{i_k} \otimes t^p = \sum_{t=1}^{m_s} (a_s q^{p_a t})^{j} w_{i_1} \otimes \cdots \otimes v_{s,t} \otimes \cdots \otimes w_{i_k} \otimes t^p \in M,
\]

(12)

for all \(p, n \in \mathbb{Z}, 0 < j < N, 1 \leq s \leq k\) and \(m \in \mathbb{N}Z, m < 0\).

Note that the elements \(A \otimes y^n, I_\nu \otimes x^m y^n, c_y(m_2, n_2)\) for \(A \in \mathcal{A}_-, n \in \mathbb{Z}, m_1 < 0\) with \(m_1 \notin \Lambda(q)\) or \(n_1 \notin \Lambda(q)\), \(m_2 < 0, n_2 \in \Lambda(q)\) span the subalgebra \(\tilde{\mathcal{L}}_-\). Combining (9), (10), (12) with the fact that \(c_y(m_2, n_2)\) acts on \(W_{i_s}\) as zero, we have

\[
w_{i_1} \otimes \cdots \otimes A.w_{i_s} \otimes \cdots \otimes w_{i_k} \otimes t^p \in M,
\]

for all \(A \in \tilde{\mathcal{L}}_-\) and \(p \in \mathbb{Z}\). This forces

\[
w_{i_1} \otimes \cdots \otimes W_{i_s} \otimes \cdots \otimes w_{i_k} \otimes \mathbb{C}[t, t^{-1}] \subset M,
\]

as \(W_{i_s} = U(\tilde{\mathcal{L}}_-)w_{i_s}\) for \(1 \leq s \leq k\). This completes the proof of the Proposition.

**Proposition 4.2.** For \(l \in \mathbb{Z}\), the \(\tilde{\mathcal{L}}\)-submodule \(W_{i,a}^l\) of \(W_{i,a}\) generated by \(\Omega_l\) is irreducible.

**Proof.** Any weight of \(W_{i,a}^l\) has the form \(\Lambda - \eta + m\delta_\nu\) for some \(\eta \in \mathbb{Q}_+, m \in \mathbb{Z}\), where \(\Lambda = \sum_{i=1}^{k} A_i\). It is sufficient to show that for every non-zero weight vector \(v \in W_{i,a}^l(\Lambda - \eta + m\delta_\nu)\) there exists \(A_v \in \tilde{\mathcal{L}}\) such that \(A_v \cdot v = \Omega_l\). For \(\eta = \sum_{i=1}^{k} k_i \alpha_i \in \mathbb{Q}_+\), we set \(ht\eta = \sum_{i=1}^{k} k_i\). We shall prove this assertion by using induction on \(ht\eta\).

First, we consider the case \(ht\eta = 0\). Assume that the image of \(\chi_{i,a}\) is \(L_r\) for some \(r \geq 1\). So we have \(U(\tilde{\mathcal{L}}_0)\Omega_l = \sum_{m \in \mathbb{Z}+l} \mathbb{C}\Omega_m\), and \(\Lambda + m\delta_\nu\) is a weight of \(W_{i,a}^l\) if and only if \(m \equiv l \pmod{r}\). But, for any \(n \in \mathbb{Z}\), there exists \(Q_n\) in \(U(\tilde{\mathcal{L}}_0)\) such that \(\chi_{i,a}(Q_n) = t^{-nr}\). This implies that \(Q_n\Omega_m = \Omega_l\) if \(m = nr + l\), as required.

Next let \(ht\eta > 0\). Assume for simplicity that \(k = 2\). Write

\[
v = \sum_{\lambda,\mu} c_{\lambda\mu} v_\lambda \otimes w_\mu \otimes t^m \in W_{i,a}^l(\Lambda - \eta + m\delta_\nu),
\]
where \( \{v_\lambda\} \) and \( \{w_\mu\} \) vary over a basis of weight vectors for \( W_{i_1} \) and \( W_{i_2} \) respectively, and \( c_{\lambda\mu} \in \mathbb{C} \). Using induction, it is enough to show that there exists some \( X \in \bar{L}_+ \) such that \( X.v \neq 0 \). So suppose now that

\[
X.v = 0, \ \forall X \in \bar{L}_+.
\]

Similar to (7), for \( A \in A_+ \) or \( A = I_\nu \otimes x^m, m > 0, m \not\in \Lambda(q) \), we have

\[
(A \otimes y^n).v_\lambda = \sum_{r=1}^{n_\lambda} q^{np_{\lambda,r}}v_{\lambda,r}, \quad (A \otimes y^n).w_\mu = \sum_{s=1}^{m_\mu} q^{nd_{\mu,s}}w_{\mu,s},
\]

for some \( v_{\lambda,r} \in W_{i_1}, w_{\mu,s} \in W_{i_2}, p_{\lambda,r}, d_{\mu,s} \in \mathbb{Z}, 1 \leq r \leq n_\lambda, 1 \leq s \leq m_\mu \) and \( n \in \mathbb{Z} \). This gives

\[
\sum_{\lambda,\mu} c_{\lambda\mu} \left( \sum_{r=1}^{n_\lambda} (a_1 q^{p_{\lambda,r}})^n v_{\lambda,r} \right) \otimes w_\mu + \sum_{\lambda,\mu} c_{\lambda\mu} v_\lambda \otimes \left( \sum_{s=1}^{m_\mu} (a_2 q^{d_{\mu,s}})^n w_{\mu,s} \right) = 0, \ \forall n \in \mathbb{Z}.
\]

Let \( q^{p_1}, \ldots, q^{p_{n'}} \) be distinct numbers among \( q^{p_{\lambda,r}}, \ \forall \lambda, 1 \leq r \leq n_\lambda \) and \( q^{d_1}, \ldots, q^{d_{m'}} \) be distinct numbers among \( q^{d_{\mu,s}}, \ \forall \mu, 1 \leq s \leq m_\mu \). Set \( I(r') = \{ (\lambda, r) | q^{p_{\lambda,r}} = q^{p_{r'}} \} \) for \( 1 \leq r' \leq n' \) and \( J(s') = \{ (\mu, s) | q^{d_{\mu,s}} = q^{d_{s'}} \} \) for \( 1 \leq s' \leq m' \). Introduce elements of the form

\[
v_{r'} = \sum_{\mu} c_{\lambda\mu} \left( \sum_{(\lambda,r) \in I(r')} v_{\lambda,r} \right) \otimes w_\mu, \ \ 1 \leq r' \leq n',
\]

\[
w_{s'} = \sum_{\lambda} c_{\lambda\mu} v_\lambda \otimes \left( \sum_{(\mu,s) \in J(s')} w_{\mu,s} \right), \ \ 1 \leq s' \leq m',
\]

so that we have

\[
\sum_{r'=1}^{n'} (a_1 q^{p_{r'}})^n v_{r'} + \sum_{s'=1}^{m'} (a_2 q^{d_{s'}})^n w_{s'} = 0, \ \forall n \in \mathbb{Z}.
\]

This forces

\[
v_{r'} = w_{s'} = 0, \ \ 1 \leq r' \leq n', 1 \leq s' \leq m',
\]

as \( a_1 q^{p_1}, \ldots, a_1 q^{p_{n'}}, a_2 q^{d_1}, \ldots, a_2 q^{d_{m'}} \) are distinct. Since \( \{v_\lambda\} \) and \( \{w_\mu\} \) are sets with linearly independent elements, we get that

\[
\sum_{(\lambda,r) \in I(r')} c_{\lambda\mu} v_{\lambda,r} = 0, \ \forall \mu, 1 \leq r' \leq n',
\]

\[
\sum_{(\mu,s) \in J(s')} c_{\lambda\mu} w_{\mu,s} = 0, \ \forall \lambda, 1 \leq s' \leq m'.
\]

In particular, one has

\[
(A \otimes y^n).\left( \sum_{\lambda} c_{\lambda\mu} v_\lambda \right) = 0 = (A \otimes y^n).\left( \sum_{\mu} c_{\lambda\mu} w_\mu \right), \ \forall n \in \mathbb{Z}.
\] (13)
For the case that \( q \) is an \( N \)-th primitive root of unity, the elements \( B_n^j := I_\nu \otimes x^m y^{nN+j}, m \in \Lambda(q), m > 0, 0 < j < N, n \in \mathbb{Z} \) are also in \( \tilde{L}_+ \). From (11), we write
\[
B_n^j \cdot v_\lambda = \sum_{r=1}^{n_\lambda} (a_1 q^{p_\lambda r})^j v_{\lambda r}, \quad B_n^j \cdot w_\mu = \sum_{s=1}^{m_\mu} (a_2 q^{d_\mu s})^j w_{\mu s},
\]
for some \( v_{\lambda r} \in W_{i_1} \) and \( w_{\mu s} \in W_{i_2} \). These imply
\[
(a_1^N)^n \left( \sum_{\lambda, \mu} c_{\lambda \mu} \left( \sum_{r=1}^{n_\lambda} (a_1 q^{p_\lambda r})^j v_{\lambda r} \right) \otimes w_\mu \right)
\]
\[
+ (a_2^N)^n \left( \sum_{\lambda, \mu} c_{\lambda \mu} v_\lambda \otimes \left( \sum_{s=1}^{m_\mu} (a_2 q^{d_\mu s})^j w_{\mu s} \right) \right) = 0, \quad \forall n \in \mathbb{Z}.
\]
Since \( a_1^N \neq a_2^N \) and \( \{ v_\lambda \}, \{ w_\mu \} \) are linearly independent basis elements, we obtain
\[
\sum_{\lambda} c_{\lambda \mu} \left( \sum_{r=1}^{n_\lambda} (a_1 q^{p_\lambda r})^j v_{\lambda r} \right) = 0, \quad \forall \mu,
\]
\[
\sum_{\mu} c_{\lambda \mu} \left( \sum_{s=1}^{m_\mu} (a_2 q^{d_\mu s})^j w_{\mu s} \right) = 0, \quad \forall \lambda.
\]
These imply that
\[
B_n^j \left( \sum_{\lambda} c_{\lambda \mu} v_\lambda \right) = 0 = B_n^j \left( \sum_{\mu} c_{\lambda \mu} w_\mu \right), \quad \forall n \in \mathbb{Z}, 0 < j < N. \quad (14)
\]
Finally, since the elements \( A \otimes y^n, I_\nu \otimes x^{m_1} y^{n_1}, c_q(m_2, n_2) \) for \( A \in A_+, n \in \mathbb{Z}, m_1 > 0, m_1 \notin \Lambda(q) \) or \( n_1 \notin \Lambda(q) \), \( m_2 > 0, m_2, n_2 \in \Lambda(q) \) span the subalgebra \( \tilde{L}_+ \). Thus, we have from (13) and (14) that
\[
X \left( \sum_{\lambda} c_{\lambda \mu} v_\lambda \right) = 0 = X \left( \sum_{\mu} c_{\lambda \mu} w_\mu \right)
\]
for all \( X \in \tilde{L}_+ \). Choose \( \lambda_0, \mu_0 \) such that \( c_{\lambda_0 \mu_0} \neq 0 \), and set
\[
\tilde{v}_{\mu_0} = \sum_{\lambda} c_{\lambda_0 \mu_0} v_\lambda, \quad \tilde{w}_{\lambda_0} = \sum_{\mu} c_{\lambda_0 \mu} w_\mu.
\]
As \( W_{i_1}, s = 1, 2 \) are irreducible, we obtain \( \tilde{v}_{\mu_0} \in \mathbb{C} W_{i_1} \) and \( \tilde{w}_{\lambda_0} \in \mathbb{C} W_{i_2} \). This implies that \( v \) has weight \( \Lambda + m \delta_y \), which is a contradiction. \( \blacksquare \)

Now we apply Proposition 4.1 and Proposition 4.2 to prove the following result

**Theorem 4.3.** The \( \tilde{L} \)-module \( W_{i,a} \) is completely reducible. Moreover, suppose that the image of \( \chi_{i,a} \) is \( L_r \) for some \( r \geq 1 \), then one has the decomposition
\[
W_{i,a} = \bigoplus_{l=0}^{r-1} W_{i,a}^l,
\]
where $W_{l,a}^{i}$ is the submodule of $W_{i,a}$ generated by the vector $\Omega_{i}$, and each $\tilde{L}$-submodule $W_{l,a}^{i}$ is irreducible.

**Proof.** From Proposition 4.1, we have

$$ W = \sum_{l=0}^{r-1} W_{l,a}^{i}. $$

To see the summation given in (15) is direct, then one needs to check that $W_{i,a}^{l} \cap \sum_{j \neq l} W_{i,a}^{j} = \{0\}$ for $0 \leq l \leq r-1$. Otherwise, due to Proposition 4.2, one has $W_{i,a}^{l} \subseteq \sum_{j \neq l} W_{i,a}^{j}$. But, we know that $\Omega_{l} \notin \sum_{j \neq l} W_{i,a}^{j}$ as $U(\tilde{L})\Omega_{l} = \sum_{n \in \mathbb{Z}} \mathbb{C}\Omega_{nr+j}$ for $0 \leq j \neq l \leq r-1$. This is a contradiction. \[\Box\]

Now, we are going to show that each irreducible $\tilde{L}$-module $W_{i,a}^{l}$ is an integrable highest weight module and that, up to the actions of $d_{x}$ and $d_{y}$, any irreducible integrable highest weight $\tilde{L}$-module classified in Theorem 2.5 must be isomorphic to $W_{i,a}^{0}$ for a suitable choice of pair $(i,a)$.

**Proposition 4.4.** The $\tilde{L}$-module $W_{i,a}$ is integrable.

**Proof.** It is easy to see that for any $1 \leq i \neq j \leq \nu, m, n \in \mathbb{Z}$ and $v_{s} \in W_{i,a}^{1}, 1 \leq s \leq k$, there exists a positive integer $r_{s}$, such that $(E_{ij} \otimes x^{m}y^{n})^{r_{s}} v_{s} = 0$. Set $N = \sum_{s=1}^{k} r_{s}$, then

$$(E_{ij} \otimes x^{m}y^{n})^{N} v_{1} \otimes \cdots \otimes v_{k} \otimes t^{p} = 0,$$

for all $p \in \mathbb{Z}$. This completes the proof. \[\Box\]

Recall the linear function $\psi_{\lambda,b} \in \tilde{L}_{0}$ given by (3). Notice that the value of $\psi_{\lambda,b}$ on $d_{x}, d_{y}$ can be chosen to any complex number. This suggests that we should exploit the extra degree of freedom available in defining the actions of $d_{x}$ and $d_{y}$ on $W_{i,a}$. Namely, for any $\mu_{x}, \mu_{y} \in \mathbb{C}$, we define a new $\tilde{L}$-module structure on the vector space $W_{i,a}$ via changing the actions of $d_{x}, d_{y}$ as follows

$$
\begin{align*}
d_{x}w &= \sum_{s=1}^{k} w_{1} \otimes \cdots \otimes Dw_{s} \otimes \cdots \otimes w_{k} \otimes t^{p} + \mu_{x}w, \\
d_{y}w &= (l + \mu_{y})w,
\end{align*}
$$

where $w = \otimes_{s=1}^{k} w_{s} \otimes t^{l}, w_{s} \in W_{i,a}, 1 \leq s \leq k, l \in \mathbb{Z}$, and $D$ was the operator defined in Sect.3. Denote the resulting $\tilde{L}$-module by $W_{i,a}(\mu_{x}, \mu_{y})$. Furthermore, one can define the “character” $\chi_{i,a}(\mu_{x}, \mu_{y})$ and the irreducible $\tilde{L}$-submodules $W_{i,a}^{l}(\mu_{x}, \mu_{y}), l \in \mathbb{Z}$ in an obvious way. Note that the image of $\chi_{i,a}(\mu_{x}, \mu_{y})$ is the same as that of $\chi_{i,a}$.

Fix a quadruple $(i,a,\mu_{x}, \mu_{y})$, where $i = (i_{1}, \cdots, i_{k}), a = (a_{1}, \cdots, a_{k})$ with $a_{i} \neq ajq^{n}, \forall i \neq j, n \in \mathbb{Z}$ and $\mu_{x}, \mu_{y} \in \mathbb{C}$. We have from Theorem 4.3 and Proposition 4.4 that the $\tilde{L}$-module $W_{i,a}^{0}(\mu_{x}, \mu_{y})$ is irreducible and integrable. This
together with Theorem 2.5 implies that $W_{l,a}^0(\mu_x, \mu_y)$ is an irreducible, integrable highest weight $\widetilde{L}$-module. Conversely, we will show in the following that any irreducible, integrable highest weight $\widetilde{L}$-module is isomorphic to $W_{l,a}^0(\mu_x, \mu_y)$ for a suitable choice of $(i, a, \mu_x, \mu_y)$.

Fix a linear function $\psi_{\lambda, b}$, where $\lambda = (\lambda_1, \ldots, \lambda_l) \in (P_+)^l$, $b = (b_1, \ldots, b_l) \in (\mathbb{C}^*)^l$ and $b_1, \ldots, b_l$ are distinct. Set $m_{s,j} = \lambda_i(h_i, a)$, $1 \leq j \leq \nu, 1 \leq t \leq l$. Let $\{a_1, \ldots, a_k\}$ be a maximal subset of $\{b_1, \ldots, b_l\}$ with the property that $q^{a_i}a_j \neq a_j, \forall i \neq j, n \in \mathbb{Z}$. Let $I(a_s) = \{t|1 \leq t \leq l, b_t = q^{-m_{s,j}}a_s, \text{ for some } t, s \in \mathbb{Z}\}, 1 \leq s \leq k$. Now, for each triple $(s, t, j)$ with $1 \leq s \leq k, t \in I(a_s)$, and $1 \leq j \leq \nu$, we define

$$i_{s,t,j} = (i_{s,t}, j, \ldots, i_{s,t}, j) \in \mathbb{Z}^{m_{s,j}}.$$ 

Suppose that $I(a_s) = \{t_1, \ldots, t_{s,p_s}\}$ and let $n_{s,j} = m_{t_1,j} + \cdots + m_{t_{s,p_s},j}$. For $1 \leq s \leq k$ and $1 \leq j \leq \nu$, we further define

$$i_{s,j} = (i_{s,t_1}, j, \ldots, i_{s,t_{p_s}}, j) \in \mathbb{Z}^{m_{s,j}}.$$ 

For any $1 \leq s \leq k$ with $n_s = \sum_{j=1}^\nu n_{s,j}$, we introduce

$$\tilde{i}_s = (i_{s,1}, \ldots, i_{s,\nu}) \in \mathbb{Z}^n.$$ 

Therefore, we have obtained a pair $(i, a)$, where

$$i = (\tilde{i}_1, \ldots, \tilde{i}_k), \ a = (a_1, \ldots, a_k)$$

with the condition that $a_i q^{\lambda_i} \neq a_j, \forall i \neq j, n \in \mathbb{Z}$. This allows us to construct an $\widetilde{L}$-module $W_{l,a} := W_{i_1} \otimes \cdots \otimes W_{i_k} \otimes \mathbb{C}[t, t^{-1}]$ with the action given by (5). Let $\mu_x = \psi_{\lambda, b}(d_x) - D_0$ and $\mu_y = \psi_{\lambda, b}(d_y)$, where $D_0$ is the scalar determined by $D\Omega_0 = D_0\Omega_0$. Then, we have constructed an irreducible integrable $\widetilde{L}$-module $W_{l,a}^0(\mu_x, \mu_y)$ arising from the linear function $\psi_{\lambda, b}$.

Now, in the $\widetilde{L}$-module $W_{l,a}^0(\mu_x, \mu_y)$, we have $\widetilde{L}_+\Omega_0 = 0$. And, for $1 \leq i \leq \nu, n \in \mathbb{Z}$, one has

$$h_{i,n,\Omega_0} = \left( \sum_{s=1}^k \sum_{t \in I(a_s)} \sum_{j=1}^\nu m_{t,j} \omega_{i_s,1} j (h_i a) a_s^\nu \right) \Omega_n$$

$$= \left( \sum_{s=1}^k \sum_{t \in I(a_s)} \sum_{j=1}^\nu m_{t,j} \delta_{i,j} (a_s q^{-m_{s,j}})^\nu \right) \Omega_n = \left( \sum_{s=1}^k \sum_{t \in I(a_s)} \lambda_t (h_i, a) b_t^\nu \right) \Omega_n$$

$$= \left( \sum_{t=1}^l \lambda_t (h_i, a) b_t^\nu \right) \Omega_n = \psi_{\lambda, b}(h_i, \Omega_0)$$

where the second identity follows from (4). Furthermore, we have $d_x \Omega_0 = \psi_{\lambda, b}(d_x) \Omega_0$ and $d_y \Omega_0 = \psi_{\lambda, b}(d_y) \Omega_0$. We see that the $\widetilde{L}_0$-submodule $W_{l,a}^0(\mu_x, \mu_y)$ generated by $\Omega_0$ is isomorphic to $A_{\psi_{\lambda, b}}$. This gives that $W_{l,a}^0(\mu_x, \mu_y)$ is a highest weight $\widetilde{L}$-module and is isomorphic to $V(\tilde{\psi}_{\lambda, b})$. We observe that $W_{l,a}^0(\mu_x, \mu_y - l)$ is isomorphic to $V(\tilde{\psi}_{\lambda, b})$ as well, so we have

$$W_{l,a}^0(\mu_x, \mu_y) \cong W_{l,a}^0(\mu_x, \mu_y + l), \quad (16)$$
as $\tilde{\mathcal{L}}$-module for all $l \in \mathbb{Z}$.

We summarize the above discussion in the following theorem.

**Theorem 4.5.** Any irreducible integrable highest weight $\tilde{\mathcal{L}}$-module is isomorphic to $W_{i,a}(\mu_x, \mu_y)$ for some suitable $(i, a, \mu_x, \mu_y)$.

---

**5. Unitarity of integrable $\tilde{\mathcal{L}}$-modules**

In this section we shall consider the unitarity of the $\tilde{\mathcal{L}}$-modules $W_{i,a}$ when $|q| = 1$. This in turn determines the unitarity of the irreducible integrable $\tilde{\mathcal{L}}$-modules classified by Rao. The result here is similar to that of affine case which was shown in [6].

Recall the conjugate-linear anti-involution $\theta_q$ of $\hat{\mathcal{L}}$ defined in Sect.3 when $|q| = 1$. Extend $\theta_q$ to a conjugate-linear anti-involution of $\tilde{\mathcal{L}}$, again denoted by $\theta_q$, by letting

$$\theta_q(d_y) = d_y.$$  

We have shown that the $\hat{\mathcal{L}}$-module $V(\nu)$ is unitary with respect to the Hermitian form $< , >$. Thus, $< , >$ can be extended to $W_{i,s}, 1 \leq s \leq k$ in an obvious way, so that

$$< A.v_s, w_s > = < v_s, \theta_q(A).w_s >,$$

where $A \in \hat{\mathcal{L}}, v_s, w_s \in W_{i,s}$.

**Theorem 5.1.** Assume that $|q| = 1$, then the $\tilde{\mathcal{L}}$-module $W_{i,a}$ is unitary with respect to $\theta_q$ if and only if $|a_1| = \cdots = |a_k|$.

**Proof.** Suppose that $|a_s| = c$ for all $1 \leq s \leq k$. We define a positive definite Hermitian form $( , )$ on $W_{i,a}$ by letting

$$(v_1 \otimes \cdots \otimes v_k \otimes t^m, w_1 \otimes \cdots \otimes w_k \otimes t^n) = c^{-2m} \delta_{m, n} < v_1, w_1 > \cdots < v_k, w_k >,$$

where $v_s, w_s \in W_{i,s}, 1 \leq s \leq k, m, n \in \mathbb{Z}$. Then, we have

$$(A.v_1 \otimes \cdots \otimes v_k \otimes t^m, w_1 \otimes \cdots \otimes w_k \otimes t^n)$$

$$= \sum_{s=1}^{k} \left( v_1 \otimes \cdots \otimes a_s^l A.v_s \otimes \cdots \otimes v_k \otimes t^{m+l}, w_1 \otimes \cdots \otimes w_k \otimes t^n \right)$$

$$= \sum_{s=1}^{k} \delta_{m+l,n} c^{-2m-2l} < v_1, w_1 > \cdots < a_s^l A.v_s, w_s > \cdots < v_k, w_k >$$

$$= \sum_{s=1}^{k} c^{-2m} \delta_{m,n-i} < v_1, w_1 > \cdots < v, a_s^{-l} \theta_q(A).w_s > \cdots < v_k, w_k >$$

$$= (v_1 \otimes \cdots \otimes v_k \otimes t^m, \theta_q(A).w_1 \otimes \cdots \otimes w_k \otimes t^n),$$

where $A \in \hat{\mathcal{L}}(l)$. The case for $d_y$ is clearly and hence the $\tilde{\mathcal{L}}$-module $W_{i,a}$ is unitary.
Conversely, suppose that $W_{t,a}$ is unitary with respect to a positive definite Hermitian form $< , >$. Assuming that the image of $\chi_{i,a}$ is $L_r, r \geq 1$, then there exists $Q \in U(\mathcal{L}_0)$ such that $Q \Omega_l = \Omega_{l,r}, l \in \mathbb{Z}$. Note that $\theta_q$ can be (uniquely) extended to a conjugate-linear anti-involution of $U(\mathcal{L})$. One checks easily that

$$c^{2n} || \Omega_l ||^2 = || \Omega_{l+nr} ||^2$$

for some (non-zero) $c \in \mathbb{C}$ and $n \in \mathbb{Z}$, where $|| v ||^2 = < v, v >$ for $v \in W_{t,a}$.

Let

$$i_s = (i_{1,s}, \ldots, i_{n,s})$$

with $i_{j,s} = t_{j,s} \nu + r_{j,s}, t_{j,s} \in \mathbb{Z}, 1 \leq r_{j,s} \leq \nu$,

$$v_s = w_{i_1} \otimes \cdots \otimes A_{i_s} w_{i_\nu} \otimes \cdots \otimes w_{i_k}, 1 \leq s \leq k,$

where $A_{i_s} = E_{r_{1,s}+1,i_s} \otimes 1$ for $1 \leq r_{i_s} \leq \nu - 1$ and $A_{1,s} = E_{1,i_s} \otimes x^{-1}$ for $r_{1,s} = \nu$.

It is easy to see that $v_s \neq 0$ and that

$$[\theta_q(A_{1,s} \otimes y^{-m}), A_{1,s} \otimes y^{-n}] = h_{r_{1,s},m-n} + \delta_{m-n,\Lambda_q} c_q(0, m - n)$$

for all $1 \leq s \leq k, m, n \in \mathbb{Z}$.

For a fixed $\gamma = 1, \cdots, k$, we may write

$$(A_{1,\gamma} \otimes y^{t_i}).w_{t_i} = q^{p_{s,t_i}w_{s,1} + \cdots + q^{p_{s,m_t}}w_{s,m_s}, n \in \mathbb{Z}, 1 \leq s \leq k},$$

for some $p_{s,t_i} \in \mathbb{Z}, w_{s,1} \in W_{t,s}, 1 \leq t \leq m_s$ and $q^{p_{s,1}}, \cdots, q^{p_{s,m_s}}$ are distinct. For any fixed $n \in \mathbb{Z}$, we consider the equation

$$(A_{1,\gamma} \otimes y^{-i}).\Omega_{i+nr} = \sum_{s=1}^{k} a_s^{-i}w_{i_1} \otimes \cdots \otimes A_{1,\gamma} \otimes y^{-i}.w_{i_s} \otimes \cdots \otimes w_{i_k} \otimes t^{nr}$$

$$= \sum_{s=1}^{k} \sum_{t=1}^{m_s} (a_s q^{p_{s,t}})^{-i}w_{i_1} \otimes \cdots \otimes w_{s,t} \otimes \cdots \otimes w_{i_k} \otimes t^{nr},$$

for all $i \in \mathbb{Z}$. Since $a_s q^{p_{s,t}}$ are distinct for all $1 \leq s \leq k, 1 \leq t \leq m_s$. Solving the system of equations to give,

$$u_{\gamma,j} \otimes t^{nr} = \sum_{i=1}^{m} b_{ji}(A_{1,\gamma} \otimes y^{-i}).\Omega_{i+nr},$$

where $u_{\gamma,j} = w_{i_1} \otimes \cdots \otimes w_{\gamma,j} \otimes \cdots \otimes w_{i_k}, 1 \leq j \leq m_\gamma, \bar{m} = m_1 + \cdots + m_k$, and $b_{ji}$ are some scalars which are independent of the choice of $n$. Then, we have for $1 \leq j, j' \leq m_\gamma$

$$< u_{\gamma,j} \otimes t^{nr}, u_{\gamma,j'} \otimes t^{nr} >$$

$$= \sum_{i,i'=1}^{\bar{m}} b_{ji}b_{ji'} < (A_{1,\gamma} \otimes y^{-i}).\Omega_{i+nr}, (A_{1,\gamma} \otimes y^{-i'}).\Omega_{i'+nr} >$$

$$= \sum_{i,i'=1}^{\bar{m}} b_{ji}b_{ji'} < h_{r_{1,\gamma},i'-1}, \Omega_{i+nr}, \Omega_{i'+nr} >$$

$$= \sum_{i,i'=1}^{\bar{m}} b_{ji}b_{ji'} (\sum_{s=1}^{k} \omega_{i,s}(h_{r_{1,\gamma'},i'-1})a_{s}^{r_{i'-1}})c^{2n} || \Omega_{i'} ||^2,$$
where we have used (18) in the second identity, and (17) in the third identity.

Taking $n = 0$ in the previous equations, we have

$$< u_{\gamma,j}, u_{\gamma,j'} > = \sum_{i,j'=1}^{\hat{m}} b_{ij} b_{ij'} (\sum_{s=1}^{k} \omega_{i,s} (h_{r_{1,\gamma,j'-1}})) a_{s}^{j'-i} \| \Omega_{j'} \|^2,$$

which implies that

$$< u_{\gamma,j} \otimes t^{nr}, u_{\gamma,j'} \otimes t^{nr} > = c^{2n} < u_{\gamma,j}, u_{\gamma,j'} >, \quad (19)$$

for all $1 \leq j, j' \leq m_{\gamma}, n \in \mathbb{Z}$.

It is clear that $\theta_q (h_{i,n}) = h_{i,-n}$. Thus, for $n \in \mathbb{Z}$, we have

$$< h_{r_{1,\gamma,nr}, \Omega_0, \Omega_{nr} >} = \langle \Omega_0, h_{r_{1,\gamma,nr}, \Omega_{nr} >}, \quad (20)$$

$$< h_{r_{1,\gamma,nr}, v_{\gamma}, v_{\gamma} \otimes t^{nr} >} = \langle v_{\gamma}, h_{r_{1,\gamma,nr}, v_{\gamma} \otimes t^{nr} >}. \quad (21)$$

For simplicity of notation, we set

$$B_{\gamma,nr} := \sum_{s=1}^{k} \sum_{l=1}^{n_s} \omega_{s,l} (h_{r_{1,\gamma,nr}}) a_{s}^{nr},$$

$$P_{\gamma,nr} := w_{1} \otimes \cdots \otimes A_{1,\gamma} \otimes y_{\gamma} \otimes \cdots \otimes w_{k} = q^{k_{y_{1},nr}} u_{\gamma,1} + \cdots + q^{k_{y_{m_{\gamma}},nr}} u_{\gamma,m_{\gamma}}, \forall n \in \mathbb{Z}.$$

Then, we have

$$h_{r_{1,\gamma,nr}, \Omega_m} = B_{\gamma,nr} \Omega_{m+nr}, \quad m, n \in \mathbb{Z}.$$

Now, from this and (17), (20), we find

$$B_{\gamma,nr} c^{2n} = \overline{B_{\gamma,-nr}}, \quad n \in \mathbb{Z}. \quad (22)$$

By a direct computation, one has

$$[h_{r_{1,s},nr, A_{1,s}}, A_{1,s}] = -2A_{1,s} \otimes y_{nr}, \quad s = 1, \cdots, k, n \in \mathbb{Z}.$$

This gives

$$h_{r_{1,\gamma,nr}, v_{\gamma}} = \sum_{s \neq \gamma} a_{s}^{nr} w_{1} \otimes \cdots \otimes h_{r_{1,\gamma,nr}, w_{s} \otimes \cdots \otimes w_{k} \otimes t^{nr}$$

$$+ a_{\gamma}^{nr} w_{1} \otimes \cdots \otimes h_{r_{1,\gamma,nr}, A_{1,\gamma} w_{1} \otimes \cdots \otimes w_{k} \otimes t^{nr}}$$

$$= B_{\gamma,nr} v_{\gamma} - 2a_{\gamma}^{nr} w_{1} \otimes \cdots \otimes A_{1,\gamma} \otimes y_{\gamma} \otimes \cdots \otimes w_{k} \otimes t^{nr}$$

$$= B_{\gamma,nr} v_{\gamma} - 2a_{\gamma}^{nr} P_{\gamma,nr} \otimes t^{nr}.$$

Therefore, (21) can be rewritten as

$$B_{\gamma,nr} c^{2n} - < 2P_{\gamma,nr} \otimes a_{\gamma}^{nr} t^{nr}, P_{\gamma,0} \otimes t^{nr} >$$

$$= \overline{B_{\gamma,-nr}} - < P_{\gamma,0}, 2P_{\gamma,-nr} a_{\gamma}^{nr} >. \quad (23)$$
Comparing (22) with (23), one has
\[ a_{\gamma}^{nr} < P_{\gamma, nr} \otimes t^{nr}, P_{\gamma, 0} \otimes t^{nr} > = \overline{a_{\gamma}^{-nr}} < P_{\gamma, 0}, P_{\gamma, -nr} > . \]

(24)

We claim that \( < P_{\gamma, nr}, P_{\gamma, 0} > \neq 0 \) for some \( n \neq 0 \). Otherwise, suppose that \( < P_{\gamma, nr}, P_{\gamma, 0} > = 0 \), for all \( n \neq 0 \).

Let \( q^{p_1, \cdots, q^{p_{m'}}} \) are distinct numbers among \( q^{p_1, t}, 1 \leq t \leq m_{\gamma} \). Set \( I(p_i) = \{ t \mid 1 \leq t \leq m_{\gamma}, q^{p_1} = q^{p_i} \} \), \( 1 \leq i \leq m' \) and \( u_i = \sum_{t \in I(p_i)} u_{\gamma, t} \). So we have the following equation
\[ \sum_{i=1}^{m'} (q^{p_i})^n < u_i, P_{\gamma, 0} > = 0, \text{ for all } n \neq 0, \]

which implies
\[ < u_i, P_{\gamma, 0} > = 0, \forall i. \]

In particular, we obtain
\[ < P_{\gamma, 0}, P_{\gamma, 0} > = 0. \]

This gives a contradiction as \( P_{\gamma, 0} = \nu_{\gamma} \neq 0 \).

Note that, by applying (19) and the fact that \( |q| = 1 \), one has
\[ < P_{\gamma, nr} \otimes t^{nr}, P_{\gamma, 0} \otimes t^{nr} > = c^{2n} < P_{\nu, 0}, P_{\gamma, -nr} > . \]

Choose some \( n \neq 0 \) so that \( < P_{\nu, 0}, P_{\gamma, -nr} > \neq 0 \). Therefore we obtain from this and (24) that
\[ |a_{\gamma}|^{2nr} = |c|^{2n} \text{ and } |a_{\gamma}| = |c_\gamma|, \]

for \( 1 \leq s \leq k \), as required.

Proposition 5.2. The irreducible \( \widetilde{L} \)-module \( W_{t,a}^0(\mu_x, \mu_y) \) is unitary with respect to \( \theta_q \) if and only if \( |a_1| = \cdots = |a_k| \) and \( \mu_x, \mu_y \) are real numbers.

Proof. If the image of \( \chi_{t,a} \) is \( L_r \), by (16), we see that all \( W_{t,a}^l, 0 \leq l \leq r-1 \) are isomorphic as \( \widehat{L} \)-modules. Thus, one can complete the proof of this proposition by a similar argument as that given in [6] (Theorem (4.8)).

It was shown in Proposition 2.9 [9] that the automorphism \( M \) (see [11] Sect.5) commutes with \( \theta_q \). Therefore, we have the following result

Theorem 5.3. (1) When \( |q| = 1 \), the irreducible integrable highest weight \( \widetilde{L} \)-module \( V(\psi_{\lambda,b}) \) is unitary with respect to \( \theta_q \) if and only if \( |b_1| = \cdots = |b_l| \) and \( \psi_{\lambda,b}(d_x), \psi_{\lambda,b}(d_y) \) are real.

(2) Let \( V \) be an irreducible integrable \( \widetilde{L} \)-module with \( c_x \) or \( c_y \) acts non-trivially, then \( V \) is unitary with respect to \( \theta_q \) if and only if \( V \) is a highest weight module, and isomorphic to some \( \widetilde{L} \)-module \( V(\psi_{\lambda,b}) \) obtained in (1) or a lowest weight module up to an automorphism \( M \).
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References


