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Integrable Representations for Extended Affine Lie Algebras Coordinated by Quantum Tori

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Abstract. In this paper we realize all irreducible integrable modules for the core of extended affine Lie algebras of type A coordinated by quantum tori with center elements act non-trivially. We also study the sufficient and necessary conditions for such modules to be unitary.
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1. Introduction

In the study of representation theory of Kac-Moody algebras, one of the main ingredients is the classification of irreducible integrable representations. The irreducible integrable modules with finite dimensional weight spaces for affine Kac-Moody algebras were classified by Chari [5] and Chari-Pressly [6], [7]. It was proved that the irreducible integrable modules are the highest weight modules ([19]), their dual, or the loop modules ([6]). As higher dimensional generalization of affine Kac-Moody algebras, extended affine Lie algebras (EALAs) were first introduced in [17] and studied in [1] and [2].

Toroidal Lie algebras are basic examples of EALAs. The classification problem of irreducible integrable modules with finite dimensional weight spaces for the toroidal Lie algebras was studied by Rao [8],[10]. Besides the affine Kac-Moody algebras and toroidal Lie algebras, there are many EALAs whose coordinate algebras are non-commutative or non-associative, examples involving quantum tori, Jordan tori and octonion tori. In [11], Rao studied a class of irreducible integrable modules for certain EALAs coordinated by quantum tori. More precisely, let $\mathcal{L}$ be the core of EALA of type $A_{\nu-1}$ coordinated by quantum tori $\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]$ of two variables. Add two derivations d_x, d_y to $\mathcal{L}$, one has $\widetilde{\mathcal{L}} = \mathcal{L} \oplus \mathbb{C}d_x \oplus \mathbb{C}d_y$. Let c_x and c_y denote the two central elements corresponding to the variables x and y, respectively. The irreducible integrable $\widetilde{\mathcal{L}}$-modules were studied in [11] for

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which c_x or c_y acts non-trivially. It was shown in [11] that any such $\widetilde{\mathcal{L}}$-module is either a highest weight module or a lowest weight module up to an automorphism. But, the concrete realization problem for those irreducible integrable $\widetilde{\mathcal{L}}$-modules is open. As pointed out in [11], "it is interesting if one can give a "model" for all known irreducible integrable $\widetilde{\mathcal{L}}$-modules". The main goal of this article is to give explicit construction of those integrable $\widetilde{\mathcal{L}}$-modules.

So far, the known models for integrable $\widetilde{\mathcal{L}}$-modules were those so-called "basic modules" constructed in [16],[4],[12],[13] and [3] obtained by vertex operators. In addition, a class of "fundamental modules" were given by using Clifford algebras with infinitely many generators in [14]. Let $\widehat{\mathcal{L}}$ denote the subalgebra $\mathcal{L} \oplus \mathbb{C}d_x$ of $\widetilde{\mathcal{L}}$. Motivated by [14], we obtain a series of highest weight $\widehat{\mathcal{L}}$-modules $V_k, k \in \mathbb{Z}$ with v_k as highest weight vectors. It is proved that the tensor $\widehat{\mathcal{L}}$-module $V_{\bar{i}} = \bigotimes_{s=1}^{l} V_{i_s}$ is completely reducible, where $\bar{i} = (i_1, \cdots, i_l) \in \mathbb{Z}^l$. In particular, the $\widehat{\mathcal{L}}$-submodule $W_{\bar{i}}$ of $V_{\bar{i}}$ generated by $w_{\bar{i}} = \otimes_{s=1}^{l} v_{i_s}$ is irreducible.

Let $\mathbf{i} = (\bar{i}_1, \cdots, \bar{i}_k)$ with $\bar{i}_s \in \mathbb{Z}^{n_s}, 1 \leq s \leq k$. Then, we have the irreducible $\widehat{\mathcal{L}}$-modules $W_{\bar{i}_1}, \cdots, W_{\bar{i}_k}$. Thanks to the method developed by Chari and Pressly in [6], we obtain an $\widetilde{\mathcal{L}}$-module structure on the tensor space $W_{\mathbf{i},\mathbf{a}} = W_{\bar{i}_1} \otimes \cdots \otimes W_{\bar{i}_k} \otimes \mathbb{C}[t, t^{-1}]$ with k-tuple $\mathbf{a} = (a_1, \cdots, a_k) \in (\mathbb{C}^*)^k$. We need the condition $a_i \neq q^n a_j$ for all $1 \leq i \neq j \leq k, n \in \mathbb{Z}$, while it was required that $a_i \neq a_j, \forall i \neq j$ in [6] for the affine Kac-Moody algebras. We prove that the $\widetilde{\mathcal{L}}$-modules $W_{\mathbf{i},\mathbf{a}}$ are completely reducible and their irreducible components exhaust all irreducible integrable highest weight $\widetilde{\mathcal{L}}$-modules up to the actions of d_x, d_y. In other words, by changing the actions of d_x, d_y and an automorphism twisting on the $\widetilde{\mathcal{L}}$-modules $W_{\mathbf{i},\mathbf{a}}$, we realize all irreducible integrable modules given in [11].

In addition, we obtain necessary and sufficient conditions for those $\widetilde{\mathcal{L}}$-modules to be unitary when $|q| = 1$. Several unitary modules related to the algebra $\widetilde{\mathcal{L}}$ studied in the paper were studied in [18],[9],[15] and [20].

The paper is organized as follows. In Sect.2, we recall some properties for the algebra $\widetilde{\mathcal{L}}$ from [2] and also recall the classification results from [11]. In Sect.3, we recall the fermionic constructions for $\widehat{\mathcal{L}}$-modules given in [14] and prove that the tensor product of such $\widehat{\mathcal{L}}$-modules are completely reducible. Then we construct a family of irreducible integrable modules for $\widetilde{\mathcal{L}}$ in Sect.4, which gives the explicit realization of all irreducible integrable highest weight $\widetilde{\mathcal{L}}$-modules. In Sect.5, we study the unitarity of those modules constructed in Sect.4 under the condition $|q| = 1$.

Throughout this paper, we denote the field of complex numbers, the group of non-zero complex numbers, the set of non-negative integers and the ring of integers by $\mathbb{C}, \mathbb{C}^*, \mathbb{N}$ and $\mathbb{Z}$ respectively.

2. Extended affine Lie algebras coordinated by quantum tori

Let q be a non-zero complex number and $\nu \geq 2$ be a positive integer. We begin by recalling the construction of the core of EALA of type $A_{\nu-1}$ coordinated by quantum tori given in [2]. A quantum tori in two variables associated to q is the unital associative $\mathbb{C}$-algebra $\mathbb{C}_q[x^{\pm 1}, y^{\pm 1}]$ (or, simply $\mathbb{C}_q$) with generators $x^{\pm 1}, y^{\pm 1}$

and relations

$$xx^{-1} = x^{-1}x = yy^{-1} = y^{-1}y = 1, \; yx = qxy.$$

Let $\Lambda(q) = \{n \in \mathbb{Z} | q^n = 1\}$, and q is said to be generic if $\Lambda(q) = \{0\}$. The center $Z(\mathbb{C}_q)$ of $\mathbb{C}_q$ has a basis consisting of monomials $x^m y^n$ for $m, n \in \Lambda(q)$ and the subalgebra $[\mathbb{C}_q, \mathbb{C}_q]$ has a basis consisting of monomials $x^m y^n$ for $m \notin \Lambda(q)$ or $n \notin \Lambda(q)$. This implies that $\mathbb{C}_q = [\mathbb{C}_q, \mathbb{C}_q] \oplus Z(\mathbb{C}_q)$.

Let I be the subspace of $\mathbb{C}_q \otimes \mathbb{C}_q$ spanned by elements of the form

$$a \otimes b + b \otimes a, ab \otimes c - a \otimes bc - b \otimes ca \quad \text{for all} \quad a, b, c \in \mathbb{C}_q.$$

So we have the quotient space $< \mathbb{C}_q, \mathbb{C}_q > := \mathbb{C}_q \otimes \mathbb{C}_q / I$ and we let $< a, b >$ denote the element $a \otimes b + I$. Let I_ν be the $\nu \times \nu$ identity matrix. Define a Lie algebra $\mathcal{L} = (sl_\nu(\mathbb{C}) \otimes \mathbb{C}_q) \oplus < \mathbb{C}_q, \mathbb{C}_q >$ with bracket

$$\begin{aligned}
&[A \otimes a, B \otimes b] = [A, B] \otimes \frac{a \circ b}{2} + A \circ B \otimes \frac{[a, b]}{2} + \frac{\operatorname{tr}(AB)}{\nu} < a, b >, \\
&[< a, b >, < c, d >] = < [a, b], [c, d] >, \\
&[< a, b >, A \otimes c] = A \otimes [[a, b], c],
\end{aligned}$$

where $A, B \in sl_\nu(\mathbb{C}), a, b, c, d \in \mathbb{C}_q, [A, B] = AB - BA$,

$$A \circ B = AB + BA - \frac{2}{\nu} \operatorname{tr}(AB) I_\nu,$$

$[a, b] = ab - ba, a \circ b = ab + ba$ and tr is the trace form.

There is a natural $\mathbb{Z}^2$-graded structure on $< \mathbb{C}_q, \mathbb{C}_q >$ with graded subspaces given as follows:

$$\begin{aligned}
< \mathbb{C}_q, \mathbb{C}_q >_{(m,n)} = \operatorname{span}_{\mathbb{C}} \{ < x^{m_1} y^{n_1}, x^{m_2} y^{n_2} >: m_1, m_2, n_1, n_2 \in \mathbb{Z}, \\
m_1 + m_2 = m, n_1 + n_2 = n\}, \; (m, n) \in \mathbb{Z}^2.
\end{aligned}$$

Let $HC_1(\mathbb{C}_q) = \{\sum < a_j, b_j > | \sum_{j \in \mathbb{J}} [a_j, b_j] = 0\}$ where $\mathbb{J}$ is any finite index set. It was shown in [2] that $HC_1(\mathbb{C}_q)$ is the center of $\mathcal{L}$ and $\mathcal{L}$ is centrally closed. Moreover, by Corollary 3.22 [2], one has

$$\dim < \mathbb{C}_q, \mathbb{C}_q >_{(m,n)} = \begin{cases} 1, & \text{if } (m, n) \neq (0, 0); \\ 2, & \text{if } (m, n) = (0, 0), \end{cases}$$

and

$$HC_1(\mathbb{C}_q) = \bigoplus_{m, n \in \Lambda(q)} < \mathbb{C}_q, \mathbb{C}_q >_{(m,n)} .$$

Thus, we have $< \mathbb{C}_q, \mathbb{C}_q > = [\mathbb{C}_q, \mathbb{C}_q] \oplus HC_1(\mathbb{C}_q)$ as vector spaces.

By adding the degree derivations d_x, d_y to $\mathcal{L}$, we get a Lie algebra $\widetilde{\mathcal{L}} = \mathcal{L} \oplus \mathbb{C} d_x \oplus \mathbb{C} d_y$ with additional multiplications

$$\begin{aligned}
&[d_x, A \otimes a] = m_1 A \otimes a, \; [d_y, A \otimes a] = n_1 A \otimes a, \; [d_x, d_y] = 0, \\
&[d_x, < a, b >] = (m_1 + m_2) < a, b >, \; [d_y, < a, b >] = (n_1 + n_2) < a, b >,
\end{aligned} \tag{1}$$

where $A\in sl_\nu(\mathbb{C}), a=x^{m_1}y^{n_1}, b=x^{m_2}y^{n_2}, m_1,m_2,n_1,n_2\in\mathbb{Z}$.

Let $\mathcal{K}$ be the vector space spanned by symbols $c_x(m,n), c_y(m,n), m,n\in\Lambda(q)$ with relation $mc_x(m,n)+nc_y(m,n)=0$. Let $gl_\nu(\mathbb{C})\otimes\mathbb{C}_q$ be the matrix Lie algebra over the quantum tori $\mathbb{C}_q$. Define a central extension $\mathcal{B}$ of the Lie algebra $gl_\nu(\mathbb{C})\otimes\mathbb{C}_q$ by central subalgebra $\mathcal{K}$ as follows:

$$\begin{aligned}&[A\otimes x^{m_1}y^{n_1}, B\otimes x^{m_2}y^{n_2}]\\ =&q^{m_2n_1}AB\otimes x^{m_1+m_2}y^{n_1+n_2}-q^{n_2m_1}BA\otimes x^{m_1+m_2}y^{n_1+n_2}\\ &+\operatorname{tr}(AB)\delta_{m_1+m_2,\Lambda(q)}\delta_{n_1+n_2,\Lambda(q)}q^{n_1m_2}\\ &\cdot(m_1c_x(m_1+m_2,n_1+n_2)+n_1c_y(m_1+m_2,n_1+n_2)),\end{aligned}$$

where $A,B\in gl_\nu(\mathbb{C}), m_1,m_2,n_1,n_2\in\mathbb{Z}$, and

$$\delta_{m,\Lambda(q)}=\begin{cases}1, & \text{if } m\in\Lambda(q);\\ 0, & \text{if } m\notin\Lambda(q).\end{cases}$$

It is straightforward to prove the following result.

Lemma 2.1. *The linear map $\mathcal{L}\to\mathcal{B}$ given by*

$$\begin{aligned}A\otimes x^my^n&\mapsto A\otimes x^my^n,\ A\in sl_\nu(\mathbb{C}), m,n\in\mathbb{Z},\\ <x^{m_1}y^{n_1},x^{m_2}y^{n_2}>&\mapsto I_\nu\otimes[x^{m_1}y^{n_1},x^{m_2}y^{n_2}],\ m_1+m_2\notin\Lambda(q)\ \textit{or}\ n_1+n_2\notin\Lambda_q,\\ <x,x^my^nx^{-1}>&\mapsto\nu c_x(m,n),\ m,n\in\Lambda(q),\\ <y,x^my^ny^{-1}>&\mapsto\nu c_y(m,n),\ m,n\in\Lambda(q),\end{aligned}$$

is an injective Lie algebra homomorphism. ■

The result of Lemma 2.1 allows us to identify $\mathcal{L}$ with the subalgebra $\mathcal{B}':=(sl_\nu(\mathbb{C})\otimes\mathbb{C}_q)\oplus(I_\nu\otimes[\mathbb{C}_q,\mathbb{C}_q])\oplus\mathcal{K}$ of $\mathcal{B}$. We can also define a Lie algebra $\widetilde{\mathcal{B}}=\mathcal{B}\oplus\mathbb{C}d_x\oplus\mathbb{C}d_y$ with multiplication given similarly as (1). Then, one can identify $\widetilde{\mathcal{L}}$ with the subalgebra $\mathcal{B}'\oplus\mathbb{C}d_x\oplus\mathbb{C}d_y$ of $\widetilde{\mathcal{B}}$. Hence, under this identification, we can write $\widetilde{\mathcal{B}}=\widetilde{\mathcal{L}}\oplus\sum_{m,n\in\Lambda(q)}\mathbb{C}I_\nu\otimes x^my^n$.

Now we turn to consider the root-space decomposition of $\widetilde{\mathcal{L}}$. For $1\le i,j\le\nu$, let E_{ij} be the unit $\nu\times\nu$ matrix which has 1 in the (i,j)-entry and 0 elsewhere. Let $h_i=E_{ii}-E_{i+1,i+1}$ for $1\le i\le\nu-1$ and $\mathcal{H}=\operatorname{span}_{\mathbb{C}}\{h_i\otimes 1: 1\le i\le\nu-1\}$. Let $\widetilde{\mathcal{H}}$ be a Cartan subalgebra of $\widetilde{\mathcal{L}}$ spanned by $\mathcal{H}, c_x:=c_x(0,0), c_y:=c_y(0,0), d_x$ and d_y. Then, one has the following root-space decomposition of $\widetilde{\mathcal{L}}$ with respect to the Cartan subalgebra $\widetilde{\mathcal{H}}$:

$$\widetilde{\mathcal{L}}=\bigoplus_{\gamma\in\widetilde{\mathcal{H}}^*}\widetilde{\mathcal{L}}_\gamma,\qquad\text{where }\widetilde{\mathcal{L}}_\gamma=\{A\in\widetilde{\mathcal{L}}|[h,A]=\lambda(h)A,\forall h\in\widetilde{\mathcal{H}}\}.$$

Let $\dot\Delta=\{\epsilon_i-\epsilon_j|1\le i\ne j\le\nu\}$ be the root system of $sl_\nu(\mathbb{C})$ as usual. View $\dot\Delta$ as a subset of $\widetilde{\mathcal{H}}^*$ by setting $(\epsilon_i-\epsilon_j)(z)=0$, where $z=c_x,d_x,c_y$ or d_y. Introduce elements $\delta_x,\delta_y\in\widetilde{\mathcal{H}}^*$ by letting $\delta_x(d_x)=1,\delta_y(d_y)=1$ and the actions on other basis elements are zero. Then, the set $\Delta = \{\alpha + m\delta_x + n\delta_y : \alpha \in \dot{\Delta} \cup \{0\}, m, n \in \mathbb{Z}\}$ is the root system of $\widetilde{\mathcal{L}}$ with respect to $\widetilde{\mathcal{H}}$. A root of the form $\alpha + m\delta_x + n\delta_y \in \Delta$ is called real if $\alpha \in \dot{\Delta}$.

Definition 2.2. An $\widetilde{\mathcal{L}}$-module V is called integrable if

(1) $V = \oplus_{\lambda \in \widetilde{\mathcal{H}}^*} V_\lambda$ where $V_\lambda = \{v \in V | h.v = \lambda(h)v, \forall h \in \widetilde{\mathcal{H}}\}$.

(2) For any weight $\gamma \in \widetilde{\mathcal{H}}^*$, one has $\dim V_\lambda < \infty$.

(3) For any real root γ and element $v \in V$, there exists positive integer k such that $(\widetilde{\mathcal{L}}_\gamma)^k.v = 0$.

Let $\mathcal{A} = sl_\nu(\mathbb{C}) \otimes \mathbb{C}[x, x^{-1}] \oplus \mathbb{C}c_x \oplus \mathbb{C}d_x$ be the subalgebra of $\widetilde{\mathcal{L}}$, which is isomorphic to the affine Kac-Moody algebra of type $A^{(1)}_{\nu-1}$. Consider the natural triangular decomposition $\mathcal{A} = \mathcal{A}_+ \oplus \mathcal{A}_0 \oplus \mathcal{A}_-$, where

$$\mathcal{A}_+ = sl_\nu(\mathbb{C}) \otimes x\mathbb{C}[x] \oplus \sum_{i<j} \mathbb{C}E_{ij},$$

$$\mathcal{A}_- = sl_\nu(\mathbb{C}) \otimes x^{-1}\mathbb{C}[x^{-1}] \oplus \sum_{i>j} \mathbb{C}E_{ij},$$

$$\mathcal{A}_0 = \mathcal{H} \oplus \mathbb{C}c_x \oplus \mathbb{C}d_x.$$

So we have the decomposition $\widetilde{\mathcal{L}} = \widetilde{\mathcal{L}}_+ \oplus \widetilde{\mathcal{L}}_0 \oplus \widetilde{\mathcal{L}}_-$, where

$$\widetilde{\mathcal{L}}_\pm = \mathcal{A}_\pm \otimes \mathbb{C}[y, y^{-1}] \oplus < \mathbb{C}_q, \mathbb{C}_q >_\pm,$$

$$\widetilde{\mathcal{L}}_0 = \mathcal{H} \otimes \mathbb{C}[y, y^{-1}] \oplus < \mathbb{C}_q, \mathbb{C}_q >_0 \oplus \mathbb{C}d_x \oplus \mathbb{C}d_y.$$

We identify $\mathbb{C}[x^{\pm 1}, y^{\pm 1}]$ with $\mathbb{C}[x, x^{-1}] \otimes \mathbb{C}[y, y^{-1}]$, and

$$< \mathbb{C}_q, \mathbb{C}_q >_\pm = \oplus_{m,n \in \mathbb{Z}, \pm m > 0} < \mathbb{C}_q, \mathbb{C}_q >_{(m,n)},$$

$$< \mathbb{C}_q, \mathbb{C}_q >_0 = \oplus_{n \in \mathbb{Z}} < \mathbb{C}_q, \mathbb{C}_q >_{(0,n)} .$$

We also introduce a $\mathbb{Z}$-grading on $\widetilde{\mathcal{L}}$ as follows:

$$\widetilde{\mathcal{L}}(n) = \{A \in \widetilde{\mathcal{L}} | [d_y, A] = nA\}, \ n \in \mathbb{Z}.$$

For any subalgebra $\mathcal{G}$ of $\widetilde{\mathcal{L}}$, we will use the following notation

$$\mathcal{G}(n) := \widetilde{\mathcal{L}}(n) \cap \mathcal{G}, \ n \in \mathbb{Z}.$$

Let ψ be a linear function on $\widetilde{\mathcal{L}}_0$ such that $\psi(c_y) = 0$. Let

$$\widehat{\mathcal{L}}_0 = \mathcal{H} \otimes \mathbb{C}[y, y^{-1}] \oplus < \mathbb{C}_q, \mathbb{C}_q >_0 \oplus \mathbb{C}d_x$$

and define a linear map $\overline{\psi} : \widehat{\mathcal{L}}_0 \to \mathbb{C}[t, t^{-1}]$ as follows:

$$\overline{\psi}(h) = \psi(h)t^n, \ \forall h \in \widehat{\mathcal{L}}_0(n).$$

Let $A_{\overline{\psi}} \subset \mathbb{C}[t, t^{-1}]$ be the image of $\overline{\psi}$. Then, $\overline{\psi}$ induces an $\widetilde{\mathcal{L}}_0$-module structure on $A_{\overline{\psi}}$ with the actions given by

$$h.t^m = (\overline{\psi}(h))t^m, \ h \in \widehat{\mathcal{L}}_0, \qquad d_y.t^m = (\psi(d_y) + m)t^m, \ m \in \mathbb{Z}.$$

Remark 2.3. In the paper [11], the linear function ψ was defined on $\widehat{\mathcal{L}}_0$ and the action of d_y on $A_{\overline{\psi}}$ was given by $d_y.t^m = mt^m$. But, we observe that one can add an extra scalar action of d_y on $A_{\overline{\psi}}$, that is, we can define the action of d_y by $d_y.t^m = (c+m)t^m$ for any fixed scalar c.

We recall the definition of highest weight $\widetilde{\mathcal{L}}$-modules, which was introduced by Rao (See [11],[9])

Definition 2.4. An $\widetilde{\mathcal{L}}$-module V is called a highest weight module if there exists a weight vector v in V such that $\widetilde{\mathcal{L}}_+.v = 0, U(\widetilde{\mathcal{L}})v = V$ and the $\widetilde{\mathcal{L}}_0$-module generated by v is isomorphic to an irreducible $\widetilde{\mathcal{L}}_0$-module $A_{\overline{\psi}}$ for some linear function ψ.

For a given linear function ψ, suppose that $A_{\overline{\psi}}$ is irreducible as $\widetilde{\mathcal{L}}_0$-module. Viewing $A_{\overline{\psi}}$ as $(\widetilde{\mathcal{L}}_+ \oplus \widetilde{\mathcal{L}}_0)$-module by letting $\widetilde{\mathcal{L}}_+$ acts trivially. So we have an induced $\widetilde{\mathcal{L}}$-module $M(\overline{\psi}) = U(\widetilde{\mathcal{L}}) \otimes_{U(\widetilde{\mathcal{L}}_+\oplus\widetilde{\mathcal{L}}_0)} A_{\overline{\psi}}$. It is easy to see that $M(\overline{\psi})$ has a unique irreducible quotient, which we denote by $V(\overline{\psi})$.

For $1 \le i \le \nu$ and $n \in \mathbb{Z}$, define elements $h_{i,n} \in \widetilde{\mathcal{L}}_0$ as follows

$$\begin{aligned} h_{i,n} &= h_i \otimes y^n, \ 1 \le i \le \nu - 1, \\ h_{\nu,n} &= -q^n E_{11} \otimes y^n + E_{\nu\nu} \otimes y^n + \delta_{n,\Lambda(q)} c_x(0,n). \end{aligned} \tag{2}$$

Then $c_y, d_x, d_y, h_{i,n}, 1 \le i \le \nu, n \in \mathbb{Z}$ form a basis of $\widetilde{\mathcal{L}}_0$. Let

$$P_+ = \{\lambda \in \widetilde{\mathcal{H}}^* : \lambda(h_{i,0}) \in \mathbb{N}, 1 \le i \le \nu, \lambda(c_y) = 0\}$$

and l be a positive integer. Then, for each pair

$$(\lambda, \mathbf{b}) \in (P_+)^l \times (\mathbb{C}^*)^l, \ \lambda = (\lambda_1, \cdots, \lambda_l), \ \mathbf{b} = (b_1, \cdots, b_l),$$

such that $b_1, \cdots, b_l$ are distinct, we can define a linear function $\psi_{\lambda,\mathbf{b}}$ on $\widetilde{\mathcal{L}}_0$ by requiring that

$$\begin{aligned} \psi_{\lambda,\mathbf{b}}(h_{i,n}) &= \sum_{j=1}^{l} \lambda_j(h_{i,0}) b_j^n, \ 1 \le i \le d, n \in \mathbb{Z}, \\ \psi_{\lambda,\mathbf{b}}(z) &= \sum_{j=1}^{l} \lambda_j(z), \ z = d_x, d_y, \text{ or } c_y. \end{aligned} \tag{3}$$

Then the resulting $\widetilde{\mathcal{L}}_0$-module $A_{\overline{\psi}_{\lambda,\mathbf{b}}}$ is irreducible and we hence obtain an irreducible highest weight $\widetilde{\mathcal{L}}$-module $V(\overline{\psi}_{\lambda,\mathbf{b}})$. Let $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{Z})$, then M can be extended to an automorphism of $\widetilde{\mathcal{L}}$, which is again denoted by M (See [11], Sect.5 for details). The following result was proved in [11]

Theorem 2.5. (1) *Let V be an irreducible $\widetilde{\mathcal{L}}$-module such that c_x acts as positive integer and c_y acts as zero. Then V is integrable if and only if $V \cong V(\overline{\psi}_{\lambda,\mathbf{b}})$ for some pair $(\lambda, \mathbf{b})$.*

(2) *Let V be an irreducible integrable $\widetilde{\mathcal{L}}$-module such that c_x or c_y acts non-trivial. Then V is isomorphic to either a highest weight module $V(\overline{\psi}_{\lambda,\mathbf{b}})$ or a lowest weight module up to a twist of an automorphism M.*

3. Tensor modules for $\widehat{\mathcal{L}}$

In this section we first construct fermionic $\widehat{\mathcal{L}} := \mathcal{L} \oplus \mathbb{C}d_x$-modules $V_k, k \in \mathbb{Z}$ and then show that the tensor $\widehat{\mathcal{L}}$-module $\bigotimes_{s=1}^{l} V_{i_s}, i_1, \cdots, i_l \in \mathbb{Z}$ is completely reducible.

Let $\mathcal{R}_\nu$ be a unital associative algebra with infinitely many generators $\psi_i(m), \psi_i^*(m)$, for $m \in \mathbb{Z}, 1 \le i \le \nu$, subject to the following relations

$$\psi_i(m)\psi_j(n) + \psi_j(n)\psi_i(m) = \psi_i^*(m)\psi_j^*(n) + \psi_j^*(n)\psi_i^*(m) = 0,$$
$$\psi_i(m)\psi_j^*(n) + \psi_j^*(n)\psi_i(m) = \delta_{ij}\delta_{m+n,0}.$$

We define normal ordering as follows

$$:\psi_i(m)\psi_j^*(n): = \begin{cases} \psi_i(m)\psi_j^*(n), & \text{if } m \le n; \\ -\psi_j^*(n)\psi_i(m), & \text{if } m > n, \end{cases}$$

for $m, n \in \mathbb{Z}, 1 \le i, j \le \nu$.

Let $\mathcal{R}_\nu^+$ be the subalgebra of $\mathcal{R}_\nu$ generated by $\psi_i(m), \psi_i^*(n)$, for $m > 0$, $n \ge 0$ and $1 \le i \le \nu$, and $\mathcal{R}_\nu^-$ be the subalgebra generated by $\psi_i(m), \psi_i^*(n)$, for $m \le 0$, $n < 0$ and $1 \le i \le \nu$. Let $V(\nu)$ be a simple $\mathcal{R}_\nu$-module containing an element v_0, called "vacuum vector", and satisfying $\mathcal{R}_\nu^+ v_0 = 0$. Therefore,

$$V(\nu) = \mathcal{R}_\nu^- v_0 \oplus \mathbb{C}v_0.$$

For $m, n \in \mathbb{Z}, 1 \le i, j \le \nu$, we set

$$f_{ij}(m,n) = \sum_{p \in \mathbb{Z}} q^{-np} :\psi_i(m-p)\psi_j^*(p):,$$
$$D = \sum_{i=1}^{\nu} \sum_{p \in \mathbb{Z}} p :\psi_i(p)\psi_i^*(-p):,$$

and

$$F_{ij}(m,n) = \begin{cases} f_{ij}(m,n), & \text{for } n \in \Lambda(q), \\ f_{ij}(m,n) - \delta_{ij}\delta_{m,0}\frac{q^n}{q^n-1}, & \text{for } n \notin \Lambda(q). \end{cases}$$

For any vector $v = \psi_{i_1}(m_1)\cdots\psi_{i_s}(m_s)\psi_{j_1}^*(n_1)\cdots\psi_{j_t}^*(n_t)v_0 \in V(\nu)$, we define an linear operator $\mathcal{J}$ on $V(\nu)$ by

$$\mathcal{J}(v) = (s-t)v.$$

For any $k \in \mathbb{Z}$, let V_k be the k-eigenspace of $V(\nu)$ with respect to the operator $\mathcal{J}$. Now, as a by-product of Theorem 3.8 [14], we have

Proposition 3.1. *$V(\nu)$ is a module for the Lie algebra $\widehat{\mathcal{B}} = \mathcal{B} \oplus \mathbb{C}d_x$ under the actions given by*

$$\begin{aligned} E_{ij} \otimes x^m y^n &\mapsto F_{ij}(m,n), \text{ for } 1 \le i,j \le \nu, m,n \in \mathbb{Z}; \\ c_x(0,n') &\mapsto 1,\ d_x \mapsto D, \text{ for } n' \in \Lambda(q); \\ c_y(m',n') &\mapsto 0, \text{ for } m',n' \in \Lambda(q). \end{aligned}$$

Moreover, $V(\nu)$ is completely reducible and each component $V_k, k \in \mathbb{Z}$ is irreducible.

For $k \in \mathbb{Z}$, we define a vector $v_k \in V_k$ as follows

$$v_k = \begin{cases} \psi_r(-s)\cdots\psi_1(-s)\psi_\nu(1-s)\cdots\psi_1(1-s)\cdots\psi_\nu(0)\cdots\psi_1(0)v_0, & \text{for } k>0, \\ \psi^*_{r+1}(s)\cdots\psi^*_\nu(s)\psi^*_1(s+1)\cdots\psi^*_\nu(s+1)\cdots\psi^*_1(-1)\cdots\psi^*_\nu(-1)v_0, & \text{for } k<0, \end{cases}$$

where (s,r) is the unique pair such that $k = s\nu + r, s \in \mathbb{Z}, 1 \le r \le \nu$.

By restriction, one can view $V_k, k \in \mathbb{Z}$ as an $\widehat{\mathcal{L}}$-module. Let ω_k be an element in $\widehat{\mathcal{L}}_0^*$ defined by

$$\omega_k(h_{i,n}) = \delta_{r,i}q^{-sn}, \omega_k(d_x) = \mathcal{D}_k \ 1 \le i \le \nu, n \in \mathbb{Z}, \tag{4}$$

where $k = s\nu + r, s \in \mathbb{Z}, 1 \le r \le \nu$, $h_{i,n}$ is defined in (2) and $\mathcal{D}_k$ is the scalar determined by $Dv_k = \mathcal{D}_k v_k$. The following result shows that v_k is a highest weight vector in the $\widehat{\mathcal{L}}$-module V_k with highest wight ω_k. The verification of this assertion is straightforward, and is omitted.

Lemma 3.2. *In the $\widehat{\mathcal{L}}$-module $V_k, k \in \mathbb{Z}$, we have $\widetilde{\mathcal{L}}_+.v_k = 0$ and $h.v_k = \omega_k(h).v_k$ for all $h \in \widehat{\mathcal{L}}_0$.* ■

Let $\widehat{\mathcal{H}} = \mathcal{H} \oplus \mathbb{C}c_x \oplus \mathbb{C}d_x \oplus \mathbb{C}c_y$ and we identify $\widehat{\mathcal{H}}^*$ with the set $\{\alpha \in \widetilde{\mathcal{H}}^* : \alpha(d_y) = 0\}$. Let $\alpha_i = \epsilon_i - \epsilon_{i+1}, 1 \le i \le \nu-1, \alpha_\nu = \epsilon_\nu - \epsilon_1 + \delta_x$ and $Q_+ = \mathbb{N}\alpha_1 \oplus \cdots \oplus \mathbb{N}\alpha_\nu$. Clearly, V_k is a weight module for $\widehat{\mathcal{L}}$ with respect to $\widehat{\mathcal{H}}$. Moreover, the previous lemma shows that all weights of V_k have the form $\Lambda_k - \eta, \eta \in Q_+$, where $\Lambda_k \in \widehat{\mathcal{H}}^*$ is the restriction of ω_k on $\widehat{\mathcal{H}}$.

Let l be a positive integer and $\bar{i} = \{i_1, \cdots, i_l\} \in \mathbb{Z}^l$. We denote the tensor space $\bigotimes_{s=1}^l V_{i_s}$ by $V_{\bar{i}}$. In the rest of this section we are going to show that the $\widehat{\mathcal{L}}$-module $V_{\bar{i}}$ is completely reducible.

We first consider the case with $|q| = 1$. In this case, one can define a conjugate-linear anti-involution θ_q of $\widehat{\mathcal{B}}$ by letting

$$\begin{aligned} &\theta_q(E_{ij} \otimes x^m y^n) = q^{mn} E_{ji} \otimes x^{-m}y^{-n},\ \theta_q(d_x) = d_x, \\ &\theta_q(c_x(m',n')) = c_x(-m',-n'),\ \theta_q(c_y(m',n')) = c_y(-m',-n'), \end{aligned}$$

where $1 \le i,j \le \nu, m,n \in \mathbb{Z}$ and $m',n' \in \Lambda(q)$.

If $\mathcal{G}$ is a Lie algebra and θ a conjugate-linear anti-involution on $\mathcal{G}$. Recall that a $\mathcal{G}$-module W is said to be unitary with respect to θ if there exists a positive definite Hermitian form $< , >$ on W such that

$$< A.v, w > = < v, \theta(A).w >,\ A \in \mathcal{G}, v,w \in W.$$

We now define a positive definite Hermitian structure $< \, , \, >$ on $V(\nu)$ by the characteristic conditions

$$\begin{aligned} &< v_0, v_0 >= 1, \; < v, w >= \overline{< w, v >}, \\ &< \psi_i(m).v, w >=< v, \psi_i^*(-m).w >, \end{aligned}$$

where $1 \leq i \leq \nu, m \in \mathbb{Z}$ and $v, w \in V(\nu)$. Then, we have

Lemma 3.3. *If $|q| = 1$, then the $\widehat{\mathcal{B}}$-module $V(\nu)$ is unitary with respect to θ_q.*

Proof. For $1 \leq i, j \leq \nu, m, n \in \mathbb{Z}$ and $v, w \in V(\nu)$, one has

$$\begin{aligned} &< f_{ij}(m,n)v, w >=< \sum_{p\in\mathbb{Z}} q^{-np} : \psi_i(m-p)\psi_j^*(p) : v, w > \\ &=< v, \sum_{p\in\mathbb{Z}} q^{np} : \psi_j(-p)\psi_i^*(p-m) : w > \;\; (\text{since } \bar{q} = q^{-1}) \\ &=< v, q^{mn} \sum_{s\in\mathbb{Z}} q^{ns} : \psi_j(-m-s)\psi_i^*(s) : w > \;\; (\text{let } s = p-m) \\ &=< v, q^{mn} f_{ji}(-m,-n)w > . \end{aligned}$$

Since $\frac{q^n}{q^n-1} = \overline{\frac{q^{-n}}{q^{-n}-1}}$ for $n \notin \Lambda(q)$, we obtain that

$$< E_{ij} \otimes x^m y^n.v, w >=< v, \theta_q(E_{ij} \otimes x^m y^n).w > .$$

For the other cases, we have

$$\begin{aligned} &< d_x.v, w >=< v, \sum_{i=1}^{\nu} \sum_{p\in\mathbb{Z}} p : \psi_i(p)\psi_i^*(-p) : w >=< v, d_x.w >, \\ &< c_x(0,n').v, w >=< v, w >=< v, c_x(0,-n').w >, \\ &< c_y(m',n').v, w >= 0 =< v, c_y(-m',-n').w >, \end{aligned}$$

where $m', n' \in \Lambda(q)$, as required. ■

Since $\theta_q(\widehat{\mathcal{L}}) = \widehat{\mathcal{L}}$, the $\widehat{\mathcal{L}}$-module $V_{\bar{i}}$ is also unitary with respect to θ_q. This implies that the $\widehat{\mathcal{L}}$-module $V_{\bar{i}}$ is completely reducible if $|q| = 1$.

Now we turn to consider the case with $|q| \neq 1$. Let $gl_\infty = \sum_{i,j\in\mathbb{Z}} E_{i,j}$ be the usual infinite matrix algebra. It is well-known that each $V_k, k \in \mathbb{Z}$ is a gl_∞-module with the action given by

$$E_{m\nu+i, j-n\nu} \mapsto \psi_i(m)\psi_j^*(n), \; 1 \leq i, j \leq \nu, m, n \in \mathbb{Z}.$$

Moreover, V_k is an integrable highest weight gl_∞-module with highest weight vector v_k and the tensor gl_∞-module $V_{\bar{i}}$ is completely reducible.

Lemma 3.4. *Let q be generic. Then $\widehat{\mathcal{B}}$-submodules and gl_∞-submodules in $V_{\bar{i}}$ coincide.*

Proof. Let W be a gl_∞-submodule of $V_{\bar{i}}$. Then, for any $1 \le i,j \le \nu, m,n \in \mathbb{Z}, w \in W$, one has $\psi_i(m)\psi_j^*(n)w \in W$ and hence $:\psi_i(m)\psi_j^*(m): w \in W$. This implies $f_{ij}(m,n)w = \sum_{p\in\mathbb{Z}} q^{-np} :\psi_i(m-p)\psi_j^*(p): w \in W$. Similarly, one has $F_{ij}(m,n)w \in W$ and $Dw \in W$, which gives W is an $\widehat{\mathcal{B}}$-submodule.

Conversely, let W be an $\widehat{\mathcal{B}}$-submodule of $V_{\bar{i}}$. For any fixed $1 \le i,j \le \nu, m,n \in \mathbb{Z}, w \in W$, we need to show that $:\psi_i(m)\psi_j^*(n): w \in W$. Note that there exist $N_1 \le n \le N_2 \in \mathbb{Z}$ such that $:\psi_i(m+n-p)\psi_j^*(p): w = 0$ for all $p < N_1$ or $P > N_2$. Consider now the equations

$$\begin{aligned} f_{ij}(m+n,s)w &= \sum_{p\in\mathbb{Z}} q^{-sp} :\psi_i(m+n-p)\psi_j^*(p): w \\ &= \sum_{p=N_1}^{N_2} q^{-sp} :\psi_i(m+n-p)\psi_j^*(p): w \in W,\ N_1 \le s \le N_2. \end{aligned}$$

Since q is generic, by solving the above equations one gets $:\psi_i(m+n-p)\psi_j^*(p): w \in W, N_1 \le p \le N_2$. In particular, we have $:\psi_i(m)\psi_j^*(n): w \in W$, as required. ∎

Observe that $\widehat{\mathcal{B}} = \widehat{\mathcal{L}} \oplus I_\nu$ if q is generic and I_ν acts as a scalar on $V_{\bar{i}}$. This implies that the $\widehat{\mathcal{L}}$-submodules in $V_{\bar{i}}$ are coincide with the $\widehat{\mathcal{B}}$-submodules and hence with the gl_∞-submodules. Therefore, we obtain that the $\widehat{\mathcal{L}}$-module $V_{\bar{i}}$ is completely reducible if q is generic. In summary, we have

Theorem 3.5. *For $i_1, \cdots, i_l \in \mathbb{Z}$, the $\widehat{\mathcal{L}}$-module $V_{\bar{i}} = \bigotimes_{s=1}^l V_{i_s}$ is completely reducible.* ∎

4. Realization of integrable highest weight $\widetilde{\mathcal{L}}$-modules

Let $U_1, \cdots, U_k$ be a collection of $\widehat{\mathcal{L}}$-modules and $\mathbf{a} = (a_1, \cdots, a_k) \in (\mathbb{C}^*)^k$. Due to the work by Chari and Pressly [6], it allows us to define an $\widetilde{\mathcal{L}}$-module structure on the tensor space $U = \bigotimes_{s=1}^k U_s \bigotimes \mathbb{C}[t,t^{-1}]$ as follows:

$$\begin{aligned} A.u_1 \otimes \cdots \otimes u_k \otimes t^l &= \sum_{s=1}^k a_s^n u_1 \otimes \cdots \otimes A.u_s \otimes \cdots \otimes u_k \otimes t^{l+n}, \\ d_y.u_1 \otimes \cdots \otimes u_k \otimes t^l &= l u_1 \otimes \cdots \otimes u_k \otimes t^l, \end{aligned} \tag{5}$$

where $u_s \in U_s, 1 \le s \le k, A \in \widehat{\mathcal{L}}(n)$ and $n,l \in \mathbb{Z}$.

Let $W_{\bar{i}}$ be the $\widehat{\mathcal{L}}$-submodule of $V_{\bar{i}} = \otimes_{s=1}^l V_{i_s}$ generated by $w_{\bar{i}} = \otimes_{s=1}^l v_{i_s}$. Combine Theorem 3.5 with Lemma 3.2, we find that the $\widehat{\mathcal{L}}$-module $W_{\bar{i}}$ is irreducible, $\widetilde{\mathcal{L}}_+.w_{\bar{i}_s} = 0$ and $W_{\bar{i}_s} = U(\widetilde{\mathcal{L}}_-)w_{\bar{i}_s}$. Recall the linear operators $\omega_k \in \widehat{\mathcal{L}}_0^*$ and $\Lambda_k \in \widehat{\mathcal{H}}^*$ defined in Sect. 3. Set $\omega_{\bar{i}} = \sum_{s=1}^l \omega_{i_s}$ and $\Lambda_{\bar{i}} = \sum_{s=1}^l \Lambda_{i_s}$. Then, one has

$$h.w_{\bar{i}} = \omega_{\bar{i}}(h)w_{\bar{i}},\ h \in \widehat{\mathcal{L}}_0,$$

and each weights of $W_{\bar{i}}$ has the form $\Lambda_{\bar{i}} - \eta$.

In the following, we shall always take $U_s = W_{\bar{i}_s}, 1 \leq s \leq k$ for some $\bar{i}_s = (i_{1,s}, \cdots, i_{n_s,s}) \in \mathbb{Z}^{n_s}$. Now, for each pair

$$\mathbf{i} = (\bar{i}_1, \cdots, \bar{i}_k),\ \mathbf{a} = (a_1, \cdots, a_k),$$

with the condition that

$$q^n a_i \neq a_j, \text{ for all } 1 \leq i \neq j \leq k, n \in \mathbb{Z}, \tag{6}$$

we obtain an $\widetilde{\mathcal{L}}$-module structure on the tensor space

$$W_{\mathbf{i},\mathbf{a}} = W_{\bar{i}_1} \otimes \cdots \otimes W_{\bar{i}_k} \otimes \mathbb{C}[t, t^{-1}]$$

with the action given by (5). Notice that, if q is an N-th primitive root of unity, then the condition (6) is equivalent to that $a_i^N \neq a_j^N, \forall i \neq j$. The main purpose of this section is to prove that such $\widetilde{\mathcal{L}}$-modules $W_{\mathbf{i},\mathbf{a}}$ are completely reducible and their irreducible components exhaust all of the irreducible integrable highest weight $\widetilde{\mathcal{L}}$-modules classified in Theorem 2.5 by Rao [11].

We define a "character"

$$\chi_{\mathbf{i},\mathbf{a}} : U(\widehat{\mathcal{L}}_0) \to \mathbb{C}[t, t^{-1}]$$

of the universal enveloping algebra of $\widehat{\mathcal{L}}_0$ by extending

$$\chi_{\mathbf{i},\mathbf{a}}(h) = (\sum_{s=1}^{k} \omega_{\bar{i}_s}(h)) a_s^n) t^n,$$

where $h \in \widehat{\mathcal{L}}_0(n)$. Since $U(\widehat{\mathcal{L}}_0)$ inherits a natural $\mathbb{Z}$-grading from $\widehat{\mathcal{L}}_0$, then $\chi_{\mathbf{i},\mathbf{a}}$ is a graded algebra homomorphism and the image of $\chi_{\mathbf{i},\mathbf{a}}$ is always a Laurent subring $L_r := \mathbb{C}[t^r, t^{-r}]$ of $\mathbb{C}[t, t^{-1}]$ for some $r \geq 1$. For all $i \in \mathbb{Z}$, let Ω_i denote the element $\otimes_{s=1}^{k} w_{\bar{i}_s} \otimes t^i$. For $h \in U(\widehat{\mathcal{L}}_0)$ with $\chi_{\mathbf{i},\mathbf{a}}(h) = t^m$ for some $m \in \mathbb{Z}$, one can easily check that $h.\Omega_i = \Omega_{i+m}$ for $i \in \mathbb{Z}$.

Proposition 4.1. *If the image of $\chi_{\mathbf{i},\mathbf{a}}$ is L_r for some $r \geq 1$, then the $\widetilde{\mathcal{L}}$-module $W_{\mathbf{i},\mathbf{a}}$ is generated by $\Omega_0, \cdots, \Omega_{r-1}$.*

Proof. We denote by M the submodule of $W_{\mathbf{i},\mathbf{a}}$ generated by $\Omega_0, \cdots, \Omega_{r-1}$. Since the image of $\chi_{\mathbf{i},\mathbf{a}}$ is L_r, one can get that $\Omega_n \in U(\widehat{\mathcal{L}}_0)\Omega_l$ for all $n \equiv l (\mod r)$, where $0 \leq l \leq r-1$. This forces that $\Omega_n \in M$ for all $n \in \mathbb{Z}$.

Let $A = \sum_{t=1}^{a} \lambda_t E_{i(t),j(t)} \otimes x^{m_t}$ be an element in $\mathcal{A}_-$, where $1 \leq i(t), j(t) \leq \nu, m_t \in \mathbb{Z}, \lambda_t \in \mathbb{C}^*$. Then, one has

$$(A \otimes y^n).w_{\bar{i}_s} = \sum_{t=1}^{a} \sum_{p \in \mathbb{Z}} q^{np} : \psi_{i(t)}(m_t + p) \psi^*_{j(t)}(-p) : w_{\bar{i}_s},$$

for $1 \leq s \leq k$ and $n \in \mathbb{Z}$. Observe that there are only finite many p, say, $p_{\bar{i}_s,l}, 1 \leq l \leq t_s$, such that $: \psi_{i(t)}(m_t + p) \psi^*_{j(t)}(-p) : w_{\bar{i}_s} \neq 0$ for some t. For

$1 \le l \le t_s$, we denote $v_{\bar{i}_s,l} = \sum_{t=1}^{a} \lambda_k : \psi_{i(t)}(m_t + p_{\bar{i}_s,l})\psi_{j(t)}(-p_{\bar{i}_s,l}) : w_{\bar{i}_s}$, then one has

$$(A \otimes y^n).w_{\bar{i}_s} = \sum_{l=1}^{t_s} q^{np_{\bar{i}_s,l}} v_{\bar{i}_s,l}, \tag{7}$$

for all $1 \le s \le k$ and $n \in \mathbb{Z}$.

Let $q^{p_{s,1}}, \cdots, q^{p_{s,m_s}}$ be distinct numbers among $q^{p_{\bar{i}_s,l}}, 1 \le l \le t_s$. For $1 \le s \le k$ and $1 \le t \le m_s$, let $I(s,t) = \{l | 1 \le l \le t_s, q^{p_{\bar{i}_s,l}} = q^{p_{s,t}}\}$ and

$$v_{s,t} = \sum_{l \in I(s,t)} v_{\bar{i}_s,l}.$$

Let $p \in \mathbb{Z}$, consider the equation

$$\begin{aligned}(A \otimes y^n).\Omega_{p-n} &= \sum_{s=1}^{k} a_s^n (w_{\bar{i}_1} \otimes \cdots \otimes (A \otimes y^n).w_{\bar{i}_s} \otimes \cdots \otimes w_{\bar{i}_k}) \otimes t^p \\ &= \sum_{s=1}^{k} \sum_{t=1}^{m_s} (a_s q^{p_{s,t}})^n w_{\bar{i}_1} \otimes \cdots \otimes v_{s,t} \otimes \cdots \otimes w_{\bar{i}_k} \otimes t^p,\end{aligned} \tag{8}$$

for all $n \in \mathbb{Z}$. Since $a_s q^{p_{s,t}}$ are distinct for all $1 \le s \le k, 1 \le t \le m_s$, we can solve for $w_{\bar{i}_1} \otimes \cdots \otimes v_{s,t} \otimes \cdots \otimes w_{\bar{i}_k} \otimes t^p$ in term of $(A \otimes y^n).\Omega_{p-n}$. Then, one has

$$w_{\bar{i}_1} \otimes \cdots \otimes v_{s,t} \otimes \cdots \otimes w_{\bar{i}_k} \otimes t^p \in M,$$

for all $1 \le s \le k, 1 \le t \le m_s$. In particular, we have

$$w_{\bar{i}_1} \otimes \cdots \otimes (A \otimes y^n).w_{\bar{i}_s} \otimes \cdots \otimes w_{\bar{i}_k} \otimes t^p \in M, \tag{9}$$

for all $n, p \in \mathbb{Z}, 1 \le s \le k$ and $A \in \mathcal{A}_-$.

For $I_\nu \otimes x^m y^n \in \widetilde{\mathcal{L}}_-$, we know $m < 0$, and m or $n \notin \Lambda(q)$. Therefore, we need to divide the argument into two cases.

First, if $m \notin \Lambda(q)$, we have that $I_\nu \otimes x^m y^n \in \widetilde{\mathcal{L}}_-$ for all $n \in \mathbb{Z}$. Then, a repeated proof of (9) shows that

$$w_{\bar{i}_1} \otimes \cdots \otimes (I_\nu \otimes x^m y^n).w_{\bar{i}_s} \otimes \cdots \otimes w_{\bar{i}_k} \otimes t^p \in M, \tag{10}$$

for all $n, p \in \mathbb{Z}, 1 \le s \le k$ and $m < 0, m \notin \Lambda(q)$.

Next, if $m \in \Lambda(q), n \notin \Lambda(q)$, then q must be a root of unity. Suppose that q is an N-th primitive root of unity, then $\Lambda(q) = N\mathbb{Z}$. Similar to the proof of (7), we obtain

$$(I_\nu \otimes x^m y^n).w_{\bar{i}_s} = \sum_{t=1}^{m_s} q^{np_{s,t}} v_{s,t},$$

for some $v_{s,t} \in W_{\bar{i}_s}, p_{s,t} \in \mathbb{Z}, 1 \le t \le m_s$ and $n \notin N\mathbb{Z}$. This gives

$$(I_\nu \otimes x^m y^{nN+j}).w_{\bar{i}_s} = \sum_{t=1}^{m_s} (q^{p_{s,t}})^j v_{s,t}, \tag{11}$$

for $0 < j < N$ and $n \in \mathbb{Z}$. Similar to (8), we have

$$
\begin{aligned}
(I_\nu \otimes x^m y^{nN+j}).\Omega_{p-nN-j} &= \sum_{s=1}^{k}\sum_{t=1}^{m_s}(a_s q^{p_{s,t}})^{nN+j} w_{\bar{i}_1} \otimes \cdots \otimes v_{s,t} \otimes \cdots \otimes w_{\bar{i}_k} \otimes t^p \\
&= \sum_{s=1}^{k}(a_s^N)^n \sum_{t=1}^{m_s}(a_s q^{p_{s,t}})^{j} w_{\bar{i}_1} \otimes \cdots \otimes v_{s,t} \otimes \cdots \otimes w_{\bar{i}_k} \otimes t^p,
\end{aligned}
$$

for all $n, p \in \mathbb{Z}$. Since $a_1^N, \cdots, a_k^N$ are distinct, this and (11) imply that

$$
\begin{aligned}
& a_s^j w_{\bar{i}_1} \otimes \cdots \otimes (I_\nu \otimes x^m y^{nN+j}).w_{\bar{i}_s} \otimes \cdots \otimes w_{\bar{i}_k} \otimes t^p \\
=& \sum_{t=1}^{m_s}(a_s q^{p_{s,t}})^j w_{\bar{i}_1} \otimes \cdots \otimes v_{s,t} \otimes \cdots \otimes w_{\bar{i}_k} \otimes t^p \in M,
\end{aligned} \tag{12}
$$

for all $p, n \in \mathbb{Z}, 0 < j < N, 1 \le s \le k$ and $m \in N\mathbb{Z}, m < 0$.

Note that the elements $A \otimes y^n, I_\nu \otimes x^{m_1} y^{n_1}, c_y(m_2, n_2)$ for $A \in \mathcal{A}_-, n \in \mathbb{Z}, m_1 < 0$ with $m_1 \notin \Lambda(q)$ or $n_1 \notin \Lambda(q)$, $m_2 < 0, m_2, n_2 \in \Lambda(q)$ span the subalgebra $\widetilde{\mathcal{L}}_-$. Combining (9),(10),(12) with the fact that $c_y(m_2, n_2)$ acts on $W_{\bar{i}_s}$ as zero, we have

$$
w_{\bar{i}_1} \otimes \cdots \otimes A.w_{\bar{i}_s} \otimes \cdots \otimes w_{\bar{i}_k} \otimes t^p \in M,
$$

for all $A \in \widetilde{\mathcal{L}}_-$ and $p \in \mathbb{Z}$. This forces

$$
w_{\bar{i}_1} \otimes \cdots \otimes W_{\bar{i}_s} \otimes \cdots \otimes w_{\bar{i}_k} \otimes \mathbb{C}[t, t^{-1}] \subset M,
$$

as $W_{\bar{i}_s} = U(\widetilde{\mathcal{L}}_-) w_{\bar{i}_s}$ for $1 \le s \le k$. This completes the proof of the Proposition. ■

Proposition 4.2. *For $l \in \mathbb{Z}$, the $\widetilde{\mathcal{L}}$-submodule $W^l_{\mathbf{i},\mathbf{a}}$ of $W_{\mathbf{i},\mathbf{a}}$ generated by Ω_l is irreducible.*

Proof. Any weight of $W^l_{\mathbf{i},\mathbf{a}}$ has the form $\Lambda - \eta + m\delta_y$ for some $\eta \in Q_+, m \in \mathbb{Z}$, where $\Lambda = \sum_{s=1}^k \Lambda_{\bar{i}_s}$. It is sufficient to show that for every non-zero weight vector $v \in W^l_{\mathbf{i},\mathbf{a}}(\Lambda - \eta + m\delta_y)$ there exists $A_v \in \widetilde{\mathcal{L}}$ such that $A_v.v = \Omega_l$. For $\eta = \sum_{i=1}^{\nu} k_i \alpha_i \in Q_+$, we set $\mathrm{ht}\eta = \sum_{i=1}^{\nu} k_i$. We shall prove this assertion by using induction on $\mathrm{ht}\eta$.

First, we consider the case $\mathrm{ht}\eta = 0$. Assume that the image of $\chi_{\mathbf{i},\mathbf{a}}$ is L_r for some $r \ge 1$. So we have $U(\widehat{\mathcal{L}}_0)\Omega_l = \sum_{m \in r\mathbb{Z}+l} \mathbb{C}\Omega_m$, and $\Lambda + m\delta_y$ is a weight of $W^l_{\mathbf{i},\mathbf{a}}$ if and only if $m \equiv l (\mod r)$. But, for any $n \in \mathbb{Z}$, there exists Q_n in $U(\widehat{\mathcal{L}}_0)$ such that $\chi_{\mathbf{i},\mathbf{a}}(Q_n) = t^{-nr}$. This implies that $Q_n \Omega_m = \Omega_l$ if $m = nr + l$, as required.

Next let $\mathrm{ht}\eta > 0$. Assume for simplicity that $k = 2$. Write

$$
v = \sum_{\lambda,\mu} c_{\lambda\mu} v_\lambda \otimes w_\mu \otimes t^m \in W^l_{\mathbf{i},\mathbf{a}}(\Lambda - \eta + m\delta_y),
$$

where $\{v_\lambda\}$ and $\{w_\mu\}$ vary over a basis of weight vectors for $W_{\bar{i}_1}$ and $W_{\bar{i}_2}$ respectively, and $c_{\lambda\mu} \in \mathbb{C}$. Using induction, it is enough to show that there exists some $X \in \widetilde{\mathcal{L}}_+$ such that $X.v \neq 0$. So suppose now that

$$X.v = 0,\ \forall X \in \widetilde{\mathcal{L}}_+.$$

Similar to (7), for $A \in \mathcal{A}_+$ or $A = I_\nu \otimes x^m, m > 0, m \notin \Lambda(q)$, we have

$$(A \otimes y^n).v_\lambda = \sum_{r=1}^{n_\lambda} q^{np_{\lambda,r}} v_{\lambda,r},\ (A \otimes y^n).w_\mu = \sum_{s=1}^{m_\mu} q^{nd_{\mu,s}} w_{\mu,s},$$

for some $v_{\lambda,r} \in W_{\bar{i}_1}, w_{\mu,s} \in W_{\bar{i}_2}, p_{\lambda,r}, d_{\mu,s} \in \mathbb{Z}, 1 \le r \le n_\lambda, 1 \le s \le m_\mu$ and $n \in \mathbb{Z}$. This gives

$$\sum_{\lambda,\mu} c_{\lambda\mu}(\sum_{r=1}^{n_\lambda}(a_1 q^{p_{\lambda,r}})^n v_{\lambda,r}) \otimes w_\mu + \sum_{\lambda,\mu} c_{\lambda\mu} v_\lambda \otimes (\sum_{s=1}^{m_\mu}(a_2 q^{d_{\mu,s}})^n w_{\mu,s}) = 0,\ \forall n \in \mathbb{Z}.$$

Let $q^{p_1}, \cdots, q^{p_{n'}}$ be distinct numbers among $q^{p_{\lambda,r}}$, $\forall \lambda, 1 \le r \le n_\lambda$ and $q^{d_1}, \cdots, q^{d_{m'}}$ be distinct numbers among $q^{d_{\mu,s}}$, $\forall \mu, 1 \le s \le m_\mu$. Set $I(r') = \{(\lambda, r) | q^{p_{\lambda,r}} = q^{p_{r'}}\}$ for $1 \le r' \le n'$ and $J(s') = \{(\mu, s) | q^{d_{\mu,s}} = q^{d_{s'}}\}$ for $1 \le s' \le m'$. Introduce elements of the form

$$v'_{r'} = \sum_\mu c_{\lambda\mu}(\sum_{(\lambda,r)\in I(r')} v_{\lambda,r}) \otimes w_\mu,\ 1 \le r' \le n',$$

$$w'_{s'} = \sum_\lambda c_{\lambda\mu} v_\lambda \otimes (\sum_{(\mu,s)\in J(s')} w_{\mu,s}),\ 1 \le s' \le m',$$

so that we have

$$\sum_{r'=1}^{n'}(a_1 q^{p_{r'}})^n v'_{r'} + \sum_{s'=1}^{m'}(a_2 q^{d_{s'}})^n w'_{s'} = 0,\ \forall n \in \mathbb{Z}.$$

This forces

$$v'_{r'} = w'_{s'} = 0,\ 1 \le r' \le n', 1 \le s' \le m',$$

as $a_1 q^{p_1}, \cdots, a_1 q^{p_{n'}}, a_2 q^{d_1}, \cdots, a_2 q^{d_{m'}}$ are distinct. Since $\{v_\lambda\}$ and $\{w_\mu\}$ are sets with linearly independent elements, we get that

$$\sum_{(\lambda,r)\in I(r')} c_{\lambda\mu} v_{\lambda,r} = 0,\ \forall \mu, 1 \le r' \le n',$$

$$\sum_{(\mu,s)\in J(s')} c_\mu w_{\mu,s} = 0,\ \forall \lambda, 1 \le s' \le m'.$$

In particular, one has

$$(A \otimes y^n).(\sum_\lambda c_{\lambda\mu} v_\lambda) = 0 = (A \otimes y^n).(\sum_\mu c_{\lambda\mu} w_\mu),\ \forall n \in \mathbb{Z}. \tag{13}$$

For the case that q is an N-th primitive root of unity, the elements $B_n^j := I_\nu \otimes x^m y^{nN+j}, m \in \Lambda(q), m > 0, 0 < j < N, n \in \mathbb{Z}$ are also in $\widetilde{\mathcal{L}}_+$. From (11), we write

$$B_n^j.v_\lambda = \sum_{r=1}^{n_\lambda} (q^{p_{\lambda,r}})^j v_{\lambda,r},\ B_n^j.w_\mu = \sum_{s=1}^{m_\mu} (q^{d_{\mu,s}})^j w_{\mu,s},$$

for some $v_{\lambda,r} \in W_{\bar{i}_1}$ and $w_{\mu,s} \in W_{\bar{i}_2}$. These imply

$$\begin{aligned}&(a_1^N)^n(\sum_{\lambda,\mu} c_{\lambda\mu}(\sum_{r=1}^{n_\lambda}(a_1 q^{p_{\lambda,r}})^j v_{\lambda,r}) \otimes w_\mu)\\ +&(a_2^N)^n(\sum_{\lambda,\mu} c_{\lambda\mu} v_\lambda \otimes (\sum_{s=1}^{m_\mu}(a_2 q^{d_{\mu,s}})^j w_{\mu,s})) = 0,\ \forall n \in \mathbb{Z}.\end{aligned}$$

Since $a_1^N \neq a_2^N$ and $\{v_\lambda\}$, $\{w_\mu\}$ are linearly independent basis elements, we obtain

$$\sum_\lambda c_{\lambda\mu}(\sum_{r=1}^{n_\lambda}(a_1 q^{p_{\lambda,r}})^j v_{\lambda,r}) = 0,\ \forall \mu,$$

$$\sum_\mu c_{\lambda\mu}(\sum_{s=1}^{m_\mu}(a_2 q^{p_{\mu,s}})^j w_{\mu,s}) = 0,\ \forall \lambda.$$

These imply that

$$B_n^j.(\sum_\lambda c_{\lambda\mu} v_\lambda) = 0 = B_n^j.(\sum_\mu c_{\lambda\mu} w_\mu),\ \forall n \in \mathbb{Z}, 0 < j < N. \tag{14}$$

Finally, since the elements $A \otimes y^n, I_\nu \otimes x^{m_1} y^{n_1}, c_y(m_2, n_2)$ for $A \in \mathcal{A}_+, n \in \mathbb{Z}$, $m_1 > 0, m_1 \notin \Lambda(q)$ or $n_1 \notin \Lambda(q)$, $m_2 > 0, m_2, n_2 \in \Lambda(q)$ span the subalgebra $\widetilde{\mathcal{L}}_+$. Thus, we have from (13) and (14) that

$$X.(\sum_\lambda c_{\lambda\mu} v_\lambda) = 0 = X.(\sum_\mu c_{\lambda\mu} w_\mu)$$

for all $X \in \widetilde{\mathcal{L}}_+$. Choose λ_0, μ_0 such that $c_{\lambda_0\mu_0} \neq 0$, and set

$$\tilde{v}_{\mu_0} = \sum_\lambda c_{\lambda\mu_0} v_\lambda,\ \tilde{w}_{\lambda_0} = \sum_\mu c_{\lambda_0\mu} w_\mu.$$

As $W_{\bar{i}_s}, s = 1, 2$ are irreducible, we obtain $\tilde{v}_{\mu_0} \in \mathbb{C}w_{\bar{i}_1}$ and $\tilde{w}_{\lambda_0} \in \mathbb{C}w_{\bar{i}_2}$. This implies that v has weight $\Lambda + m\delta_y$, which is a contradiction. ∎

Now we apply Proposition 4.1 and Proposition 4.2 to prove the following result

Theorem 4.3. *The $\widetilde{\mathcal{L}}$-module $W_{\boldsymbol{i},\boldsymbol{a}}$ is completely reducible. Moreover, suppose that the image of $\chi_{\boldsymbol{i},\boldsymbol{a}}$ is L_r for some $r \geq 1$, then one has the decomposition*

$$W_{\boldsymbol{i},\boldsymbol{a}} = \bigoplus_{l=0}^{r-1} W_{\boldsymbol{i},\boldsymbol{a}}^l,$$

where $W^l_{\mathbf{i},\mathbf{a}}$ is the submodule of $W_{\mathbf{i},\mathbf{a}}$ generated by the vector Ω_l, and each $\widetilde{\mathcal{L}}$-submodule $W^l_{\mathbf{i},\mathbf{a}}$ is irreducible.

Proof. From Proposition 4.1, we have

$$W = \sum_{l=0}^{r-1} W^l_{\mathbf{i},\mathbf{a}}. \tag{15}$$

To see the summation given in (15) is direct, then one needs to check that $W^l_{\mathbf{i},\mathbf{a}} \cap \sum_{j\neq l} W^j_{\mathbf{i},\mathbf{a}} = \{0\}$ for $0 \le l \le r-1$. Otherwise, due to Proposition 4.2, one has $W^l_{\mathbf{i},\mathbf{a}} \subseteq \sum_{j\neq l} W^j_{\mathbf{i},\mathbf{a}}$. But, we know that $\Omega_l \notin \sum_{j\neq l} W^j_{\mathbf{i},\mathbf{a}}$ as $U(\widetilde{\mathcal{L}}_0)\Omega_j = \sum_{n\in\mathbb{Z}} \mathbb{C}\Omega_{nr+j}$ for $0 \le j \neq l \le r-1$. This is a contradiction. ■

Now, we are going to show that each irreducible $\widetilde{\mathcal{L}}$-module $W^l_{\mathbf{i},\mathbf{a}}$ is an integrable highest weight module and that, up to the actions of d_x and d_y, any irreducible integrable highest weight $\widetilde{\mathcal{L}}$-module classified in Theorem 2.5 must be isomorphic to $W^0_{\mathbf{i},\mathbf{a}}$ for a suitable choice of pair $(\mathbf{i}, \mathbf{a})$.

Proposition 4.4. *The $\widetilde{\mathcal{L}}$-module $W_{\mathbf{i},\mathbf{a}}$ is integrable.*

Proof. It is easy to see that for any $1 \le i \neq j \le \nu, m, n \in \mathbb{Z}$ and $v_s \in W_{\bar{i}_s}, 1 \le s \le k$, there exists a positive integer r_s, such that $(E_{ij} \otimes x^m y^n)^{r_s}.v_s = 0$.

Set $N = \sum_{s=1}^k r_s$, then

$$(E_{ij} \otimes x^m y^n)^N.v_1 \otimes \cdots \otimes v_k \otimes t^p = 0,$$

for all $p \in \mathbb{Z}$. This completes the proof. ■

Recall the linear function $\psi_{\lambda,\mathbf{b}} \in \widetilde{\mathcal{L}}_0^*$ given by (3). Notice that the value of $\psi_{\lambda,\mathbf{b}}$ on d_x, d_y can be chosen to any complex number. This suggests that we should exploit the extra degree of freedom available in defining the actions of d_x and d_y on $W_{\mathbf{i},\mathbf{a}}$. Namely, for any $\mu_x, \mu_y \in \mathbb{C}$, we define a new $\widetilde{\mathcal{L}}$-module structure on the vector space $W_{\mathbf{i},\mathbf{a}}$ via changing the actions of d_x, d_y as follows

$$\begin{aligned} d_x.w &= \sum_{s=1}^k w_1 \otimes \cdots \otimes Dw_s \otimes \cdots \otimes w_k \otimes t^p + \mu_x w, \\ d_y.w &= (l + \mu_y)w, \end{aligned}$$

where $w = \otimes_{s=1}^k w_s \otimes t^l, w_s \in W_{\bar{i}_s}, 1 \le s \le k, l \in \mathbb{Z}$ and D was the operator defined in Sect.3. Denote the resulting $\widetilde{\mathcal{L}}$-module by $W_{\mathbf{i},\mathbf{a}}(\mu_x, \mu_y)$. Furthermore, one can define the "character" $\chi_{\mathbf{i},\mathbf{a}}(\mu_x, \mu_y)$ and the irreducible $\widetilde{\mathcal{L}}$-submodules $W^l_{\mathbf{i},\mathbf{a}}(\mu_x, \mu_y), l \in \mathbb{Z}$ in an obvious way. Note that the image of $\chi_{\mathbf{i},\mathbf{a}}(\mu_x, \mu_y)$ is the same as that of $\chi_{\mathbf{i},\mathbf{a}}$.

Fix a quadruple $(\mathbf{i}, \mathbf{a}, \mu_x, \mu_y)$, where $\mathbf{i} = (\bar{i}_1, \cdots, \bar{i}_k), \mathbf{a} = (a_1, \cdots, a_k)$ with $a_i \neq a_j q^n, \forall i \neq j, n \in \mathbb{Z}$ and $\mu_x, \mu_y \in \mathbb{C}$. We have from Theorem 4.3 and Proposition 4.4 that the $\widetilde{\mathcal{L}}$-module $W^0_{\mathbf{i},\mathbf{a}}(\mu_x, \mu_y)$ is irreducible and integrable. This together with Theorem 2.5 implies that $W^0_{\mathbf{i},\mathbf{a}}(\mu_x,\mu_y)$ is an irreducible, integrable highest weight $\widetilde{\mathcal{L}}$-module. Conversely, we will show in the following that any irreducible, integrable highest weight $\widetilde{\mathcal{L}}$-module is isomorphic to $W^0_{\mathbf{i},\mathbf{a}}(\mu_x,\mu_y)$ for a suitable choice of $(\mathbf{i},\mathbf{a},\mu_x,\mu_y)$.

Fix a linear function $\psi_{\lambda,\mathbf{b}}$, where $\lambda=(\lambda_1,\cdots,\lambda_l)\in(P_+)^l, \mathbf{b}=(b_1,\cdots,b_l)\in(\mathbb{C}^*)^l$ and $b_1,\cdots,b_l$ are distinct. Set $m_{t,j}=\lambda_t(h_{j,0}), 1\le j\le\nu, 1\le t\le l$. Let $\{a_1,\cdots,a_k\}$ be a maximal subset of $\{b_1,\cdots,b_l\}$ with the property that $q^n a_i\ne a_j,\forall i\ne j, n\in\mathbb{Z}$. Set $I(a_s)=\{t|1\le t\le l, b_t=q^{-\iota_{s,t}}a_s,$ for some $\iota_{s,t}\in\mathbb{Z}\}, 1\le s\le k$. Now, for each triple (s,t,j) with $1\le s\le k, t\in I(a_s)$, and $1\le j\le\nu$, we define

$$i_{s,t,j}=(\iota_{s,t}\nu+j,\cdots,\iota_{s,t}\nu+j)\in\mathbb{Z}^{m_{t,j}}.$$

Suppose that $I(a_s)=\{t_{s,1},\cdots,t_{s,p_s}\}$ and let $n_{s,j}=m_{t_{s,1},j}+\cdots+m_{t_{s,p_s},j}$. For $1\le s\le k$ and $1\le j\le\nu$, we further define

$$i_{s,j}=(i_{s,t_{s,1},j},\cdots,i_{s,t_{s,p_s},j})\in\mathbb{Z}^{n_{s,j}}.$$

For any $1\le s\le k$ with $n_s=\sum_{j=1}^{\nu}n_{s,j}$, we introduce

$$\bar{i}_s=(i_{s,1},\cdots,i_{s,\nu})\in\mathbb{Z}^{n_s}.$$

Therefore, we have obtained a pair $(\mathbf{i},\mathbf{a})$, where

$$\mathbf{i}=(\bar{i}_1,\cdots,\bar{i}_k),\ \ \mathbf{a}=(a_1,\cdots,a_k)$$

with the condition that $a_iq^n\ne a_j,\forall i\ne j, n\in\mathbb{Z}$. This allows us to construct an $\widetilde{\mathcal{L}}$-module $W_{\mathbf{i},\mathbf{a}}:=W_{\bar{i}_1}\otimes\cdots\otimes W_{\bar{i}_k}\otimes\mathbb{C}[t,t^{-1}]$ with the action given by (5). Let $\mu_x=\psi_{\lambda,\mathbf{b}}(d_x)-D_0$ and $\mu_y=\psi_{\lambda,\mathbf{b}}(d_y)$, where D_0 is the scalar determined by $D\Omega_0=D_0\Omega_0$. Then, we have constructed an irreducible integrable $\widetilde{\mathcal{L}}$-module $W^0_{\mathbf{i},\mathbf{a}}(\mu_x,\mu_y)$ arising from the linear function $\psi_{\lambda,\mathbf{b}}$.

Now, in the $\widetilde{\mathcal{L}}$-module $W^0_{\mathbf{i},\mathbf{a}}(\mu_x,\mu_y)$, we have $\widetilde{\mathcal{L}}_+.\Omega_0=0$. And, for $1\le i\le\nu, n\in\mathbb{Z}$, one has

$$\begin{aligned}h_{i,n}.\Omega_0=&\big(\sum_{s=1}^{k}\sum_{t\in I(a_s)}\sum_{j=1}^{\nu}m_{t,j}\omega_{\iota_{s,t}\nu+j}(h_{i,n})a_s^n\big)\Omega_n\\=&\big(\sum_{s=1}^{k}\sum_{t\in I(a_s)}\sum_{j=1}^{\nu}m_{t,j}\delta_{i,j}(a_sq^{-\iota_{s,t}})^n\big)\Omega_n=\big(\sum_{s=1}^{k}\sum_{t\in I(a_s)}\lambda_t(h_{i,0})b_t^n\big)\Omega_n\\=&\big(\sum_{t=1}^{l}\lambda_t(h_{i,0})b_r^n\big)\Omega_n=\psi_{\lambda,\mathbf{b}}(h_{i,n})\Omega_n,\end{aligned}$$

where the second identity follows from (4). Furthermore, we have $d_x.\Omega_0=\psi_{\lambda,b}(d_x)\Omega_0$ and $d_y.\Omega_0=\psi_{\lambda,b}(d_y)\Omega_0$. We see that the $\widetilde{\mathcal{L}}_0$-submodule $W^0_{\mathbf{i},\mathbf{a}}(\mu_x,\mu_y)$ generated by Ω_0 is isomorphic to $A_{\bar{\psi}_{\lambda,\mathbf{b}}}$. This gives that $W^0_{\mathbf{i},\mathbf{a}}(\mu_x,\mu_y)$ is a highest weight $\widetilde{\mathcal{L}}$-module and is isomorphic to $V(\bar{\psi}_{\lambda,\mathbf{b}})$. We obverse that $W^l_{\mathbf{i},\mathbf{a}}(\mu_x,\mu_y-l)$ is isomorphic to $V(\bar{\psi}_{\lambda,\mathbf{b}})$ as well, so we have

$$W^l_{\mathbf{i},\mathbf{a}}(\mu_x,\mu_y)\cong W^0_{\mathbf{i},\mathbf{a}}(\mu_x,\mu_y+l), \tag{16}$$

as $\widetilde{\mathcal{L}}$-module for all $l \in \mathbb{Z}$.

We summarize the above discussion in the following theorem.

Theorem 4.5. *Any irreducible integrable highest weight $\widetilde{\mathcal{L}}$-module is isomorphic to $W^0_{\boldsymbol{i},\boldsymbol{a}}(\mu_x, \mu_y)$ for some suitable $(\boldsymbol{i},\boldsymbol{a}, \mu_x, \mu_y)$.* ■

5. Unitarity of integrable $\widetilde{\mathcal{L}}$-modules

In this section we shall consider the unitarity of the $\widetilde{\mathcal{L}}$-modules $W_{\mathbf{i},\mathbf{a}}$ when $|q| = 1$. This in turn determines the unitarity of the irreducible integrable $\widetilde{\mathcal{L}}$-modules classified by Rao. The result here is similar to that of affine case which was shown in [6].

Recall the conjugate-linear anti-involution θ_q of $\widehat{\mathcal{L}}$ defined in Sect.3 when $|q| = 1$. Extend θ_q to a conjugate-linear anti-involution of $\widetilde{\mathcal{L}}$, again denoted by θ_q, by letting

$$\theta_q(d_y) = d_y.$$

We have shown that the $\widehat{\mathcal{L}}$-module $V(\nu)$ is unitary with respect to the Hermitian form $<\ ,\ >$. Thus, $<\ ,\ >$ can be extended to $W_{\bar{i}_s}, 1 \leq s \leq k$ in an obvious way, so that

$$< A.v_s, w_s >=< v_s, \theta_q(A).w_s >,$$

where $A \in \widehat{\mathcal{L}}, v_s, w_s \in W_{\bar{i}_s}$.

Theorem 5.1. *Assume that $|q| = 1$, then the $\widetilde{\mathcal{L}}$-module $W_{\boldsymbol{i},\boldsymbol{a}}$ is unitary with respect to θ_q if and only if $|a_1| = \cdots = |a_k|$.*

Proof. Suppose that $|a_s| = c$ for all $1 \leq s \leq k$. We define a positive definite Hermitian form $(\ ,\ )$ on $W_{\mathbf{i},\mathbf{a}}$ by letting

$$(v_1 \otimes \cdots \otimes v_k \otimes t^m, w_1 \otimes \cdots \otimes w_k \otimes t^n) = c^{-2m}\delta_{m,n} < v_1, w_1 > \cdots < v_k, w_k >,$$

where $v_s, w_s \in W_{\bar{i}_s}, 1 \leq s \leq k, m, n \in \mathbb{Z}$. Then, we have

$$\begin{aligned}
&(A.v_1 \otimes \cdots \otimes v_k \otimes t^m, w_1 \otimes \cdots \otimes w_k \otimes t^n)\\
=&\sum_{s=1}^{k}(v_1 \otimes \cdots \otimes a_s^l A.v_s \otimes \cdots \otimes v_k \otimes t^{m+l}, w_1 \otimes \cdots \otimes w_k \otimes t^n)\\
=&\sum_{s=1}^{k} \delta_{m+l,n}c^{-2m-2l} < v_1, w_1 > \cdots < a_s^l A.v_s, w_s > \cdots < v_k, w_k >\\
=&\sum_{s=1}^{k} c^{-2m}\delta_{m,n-l} < v_1, w_1 > \cdots < v, a_s^{-l}\theta_q(A).w_s > \cdots < v_k, w_k >\\
=&(v_1 \otimes \cdots \otimes v_k \otimes t^m, \theta_q(A).w_1 \otimes \cdots \otimes w_k \otimes t^n),
\end{aligned}$$

where $A \in \widehat{\mathcal{L}}(l)$. The case for d_y is clearly and hence the $\widetilde{\mathcal{L}}$-module $W_{\mathbf{i},\mathbf{a}}$ is unitary.

Conversely, suppose that $W_{\mathbf{i},\mathbf{a}}$ is unitary with respect to a positive definite Hermitian form $<\ ,\ >$. Assuming that the image of $\chi_{\mathbf{i},\mathbf{a}}$ is $L_r, r \geq 1$, then there exists $Q \in U(\widehat{\mathcal{L}}_0)$ such that $Q.\Omega_l = \Omega_{l+r}, l \in \mathbb{Z}$. Note that θ_q can be (uniquely) extended to a conjugate-linear anti-involution of $U(\widetilde{\mathcal{L}})$. One checks easily that $\theta_q(Q).\Omega_l = c^2\Omega_{l-r}$ for some $c \in \mathbb{C}$. This implies

$$c^{2n} \parallel \Omega_l \parallel^2 = \parallel \Omega_{l+nr} \parallel^2 \tag{17}$$

for some (non-zero) $c \in \mathbb{C}$ and $n \in \mathbb{Z}$, where $\parallel v \parallel^2 = < v, v >$ for $v \in W_{\mathbf{i},\mathbf{a}}$.

Let

$$\bar{i}_s = (i_{1,s}, \cdots, i_{n_s,s}) \text{ with } i_{j,s} = t_{j,s}\nu + r_{j,s}, t_{j,s} \in \mathbb{Z}, 1 \leq r_{j,s} \leq \nu,$$
$$v_s = w_{\bar{i}_1} \otimes \cdots \otimes A_{1,s}.w_{\bar{i}_s} \otimes \cdots \otimes w_{\bar{i}_k}, 1 \leq s \leq k,$$

where $A_{1,s} = E_{r_{1,s}+1,r_{1,s}} \otimes 1$ for $1 \leq r_{1,s} \leq \nu - 1$ and $A_{1,s} = E_{1,\nu} \otimes x^{-1}$ for $r_{1,s} = \nu$. It is easy to see that $v_s \neq 0$ and that

$$[\theta_q(A_{1,s} \otimes y^{-m}), A_{1,s} \otimes y^{-n}] = h_{r_{1,s},m-n} + \delta_{m-n,\Lambda_q} c_y(0, m-n) \tag{18}$$

for all $1 \leq s \leq k, m, n \in \mathbb{Z}$.

For a fixed $\gamma = 1, \cdots, k$, we may write

$$(A_{1,\gamma} \otimes y^n).w_{\bar{i}_s} = q^{p_{s,1}n} w_{s,1} + \cdots + q^{p_{s,m_s}n} w_{s,m_s},\ n \in \mathbb{Z}, 1 \leq s \leq k,$$

for some $p_{s,t} \in \mathbb{Z}, w_{s,t} \in W_{\bar{i}_s}, 1 \leq t \leq m_s$ and $q^{p_{s,1}}, \cdots, q^{p_{s,m_s}}$ are distinct. For any fixed $n \in \mathbb{Z}$, we consider the equation

$$\begin{aligned}(A_{1,\gamma} \otimes y^{-i}).\Omega_{i+nr} &= \sum_{s=1}^{k} a_s^{-i} w_{\bar{i}_1} \otimes \cdots \otimes A_{1,\gamma} \otimes y^{-i}.w_{\bar{i}_s} \otimes \cdots \otimes w_{\bar{i}_k} \otimes t^{nr} \\ &= \sum_{s=1}^{k} \sum_{t=1}^{m_s} (a_s q^{p_{s,t}})^{-i} w_{\bar{i}_1} \otimes \cdots \otimes w_{s,t} \otimes \cdots \otimes w_{\bar{i}_k} \otimes t^{nr},\end{aligned}$$

for all $i \in \mathbb{Z}$. Since $a_s q^{p_{s,t}}$ are distinct for all $1 \leq s \leq k, 1 \leq t \leq m_s$. Solving the system of equations to give,

$$u_{\gamma,j} \otimes t^{nr} = \sum_{i=1}^{\tilde{m}} b_{ji} (A_{1,\gamma} \otimes y^{-i}).\Omega_{i+nr},$$

where $u_{\gamma,j} = w_{\bar{i}_1} \otimes \cdots \otimes w_{\gamma,j} \otimes \cdots \otimes w_{\bar{i}_k}, 1 \leq j \leq m_\gamma, \tilde{m} = m_1 + \cdots + m_k$, and b_{ji} are some scalars which are independent of the choice of n. Then, we have for $1 \leq j, j' \leq m_\gamma$

$$\begin{aligned}&< u_{\gamma,j} \otimes t^{nr}, u_{\gamma,j'} \otimes t^{nr} > \\ &= \sum_{i,i'=1}^{\tilde{m}} b_{ji} \overline{b_{j'i'}} < (A_{1,\gamma} \otimes y^{-i}).\Omega_{i+nr}, (A_{1,\gamma} \otimes y^{-i'}).\Omega_{i'+nr} > \\ &= \sum_{i,i'=1}^{\tilde{m}} b_{ji} \overline{b_{j'i'}} < h_{r_{1,\gamma},i'-i}.\Omega_{i+nr}, \Omega_{i'+nr} > \\ &= \sum_{i,i'=1}^{\tilde{m}} b_{ji} \overline{b_{j'i'}} (\sum_{s=1}^{k} (\sum_{l=1}^{n_s} \omega_{i_{l,s}}(h_{r_{1,\gamma},i'-i})) a_s^{i'-i}) c^{2n} \parallel \Omega_{i'} \parallel^2,\end{aligned}$$

where we have used (18) in the second identity, and (17) in the third identity. Taking $n = 0$ in the previous equations, we have

$$< u_{\gamma,j}, u_{\gamma,j'} >= \sum_{i,i'=1}^{\tilde{m}} b_{ji}\overline{b_{j'i'}}(\sum_{s=1}^{k}(\sum_{j=1}^{n_s} \omega_{i_{j,s}}(h_{r_{1,\gamma,i'-i}}))a_s^{i'-i}) \parallel \Omega_{i'} \parallel^2,$$

which implies that

$$< u_{\gamma,j} \otimes t^{nr}, u_{\gamma,j'} \otimes t^{nr} >= c^{2n} < u_{\gamma,j}, u_{\gamma,j'} >, \tag{19}$$

for all $1 \leq j, j' \leq m_\gamma, n \in \mathbb{Z}$.

It is clear that $\theta_q(h_{i,n}) = h_{i,-n}$. Thus, for $n \in \mathbb{Z}$, we have

$$< h_{r_{1,\gamma,nr}}.\Omega_0, \Omega_{nr} > =< \Omega_0, h_{r_{1,\gamma,-nr}}.\Omega_{nr} >, \tag{20}$$

$$< h_{r_{1,\gamma,nr}}.v_\gamma, v_\gamma \otimes t^{nr} > =< v_\gamma, h_{r_{1,\gamma,-nr}}.v_\gamma \otimes t^{nr} > . \tag{21}$$

For simplicity of notation, we set

$$\begin{aligned}
B_{\gamma,nr} :=& \sum_{s=1}^{k}\sum_{l=1}^{n_s} \omega_{i_{l,s}}(h_{r_{1,\gamma,nr}})a_s^{nr}, \\
P_{\gamma,nr} :=& w_{\bar{i}_1} \otimes \cdots \otimes A_{1,\gamma} \otimes y^{nr}.w_{\bar{i}_\gamma} \otimes \cdots \otimes w_{\bar{i}_k} \\
=& q^{p_{\gamma,1}rn}u_{\gamma,1} + \cdots + q^{p_{\gamma,m_\gamma}rn}u_{\gamma,m_\gamma}, \ \forall n \in \mathbb{Z}.
\end{aligned}$$

Then, we have

$$h_{r_{1,\gamma,nr}}.\Omega_m = B_{\gamma,nr}\Omega_{m+nr}, \ m, n \in \mathbb{Z}.$$

Now, from this and (17),(20), we find

$$B_{\nu,nr}c^{2n} = \overline{B_{\nu,-nr}}, n \in \mathbb{Z}. \tag{22}$$

By a direct computation, one has

$$[h_{r_{1,s,nr}}, A_{1,s}] = -2A_{1,s} \otimes y^{nr}, \ s = 1, \cdots, k, n \in \mathbb{Z}.$$

This gives

$$\begin{aligned}
& h_{r_{1,\gamma,nr}}.v_\gamma \\
=& \sum_{s\neq\gamma} a_s^{nr} w_{\bar{i}_1} \otimes \cdots \otimes h_{r_{1,\gamma,nr}}.w_{\bar{i}_s} \otimes \cdots \otimes w_{\bar{i}_k} \otimes t^{nr} \\
& + a_\gamma^{nr} w_{\bar{i}_1} \otimes \cdots \otimes h_{r_{1,\gamma,nr}}.A_{1,\gamma}.w_{\bar{i}_\gamma} \otimes \cdots \otimes w_{\bar{i}_k} \otimes t^{nr} \\
=& B_{\gamma,nr}v_\gamma - 2a_\gamma^{nr} w_{\bar{i}_1} \otimes \cdots \otimes A_{1,\gamma} \otimes y^n.w_{\bar{i}_\gamma} \otimes \cdots \otimes w_{\bar{i}_k} \otimes t^{nr} \\
=& B_{\gamma,nr}v_\gamma - 2a_\gamma^{nr} P_{\gamma,nr} \otimes t^{nr}.
\end{aligned}$$

Therefore, (21) can be rewritten as

$$\begin{aligned}
& B_{\gamma,nr}c^{2n} - < 2P_{\gamma,nr} \otimes a_\gamma^{nr}t^{nr}, P_{\gamma,0} \otimes t^{nr} > \\
=& \overline{B_{\gamma,-nr}} - < P_{\gamma,0}, 2P_{\gamma,-nr}a_\gamma^{-nr} > .
\end{aligned} \tag{23}$$

Comparing (22) with (23), one has

$$a_\gamma^{nr} < P_{\gamma,nr} \otimes t^{nr}, P_{\gamma,0} \otimes t^{nr} >= \overline{a_\gamma^{-nr}} < P_{\gamma,0}, P_{\gamma,-nr} > . \tag{24}$$

We claim that $< P_{\gamma,nr}, P_{\gamma,0} > \neq 0$ for some $n \neq 0$. Otherwise, suppose that

$$< P_{\gamma,nr}, P_{\gamma,0} >= 0, \text{ for all } n \neq 0.$$

Let $q^{p_1 r}, \cdots, q^{p_{m'} r}$ are distinct numbers among $q^{p_{\gamma,t} r}, 1 \leq t \leq m_\gamma$. Set $I(p_i) = \{t | 1 \leq t \leq m_\gamma, q^{p_i r} = q^{p_{\gamma,t} r}\}, 1 \leq i \leq m'$ and $u_i = \sum_{t \in I(p_i)} u_{\gamma,t}$. So we have the following equation

$$\sum_{i=1}^{m'} (q^{p_i r})^n < u_i, P_{\gamma,0} >= 0, \text{ for all } n \neq 0,$$

which implies

$$< u_i, P_{\gamma,0} >= 0, \ \forall i.$$

In particular, we obtain

$$< P_{\gamma,0}, P_{\gamma,0} >= 0.$$

This gives a contradiction as $P_{\gamma,0} = v_\gamma \neq 0$.

Note that, by applying (19) and the fact that $|q| = 1$, one has

$$< P_{\gamma,nr} \otimes t^{nr}, P_{\gamma,0} \otimes t^{nr} >= c^{2n} < P_{\nu,0}, P_{\gamma,-nr} > .$$

Choose some $n \neq 0$ so that $< P_{\nu,0}, P_{\gamma,-nr} > \neq 0$. Therefore we obtain from this and (24) that

$$|a_\gamma|^{2nr} = |c|^{2n} \text{ and } |a_\gamma| = |c^{\frac{1}{r}}|,$$

for $1 \leq s \leq k$, as required. ■

Proposition 5.2. *The irreducible $\widetilde{\mathcal{L}}$-module $W^0_{\mathbf{i},\mathbf{a}}(\mu_x, \mu_y)$ is unitary with respect to θ_q if and only if $|a_1| = \cdots = |a_k|$ and μ_x, μ_y are real numbers.*

Proof. If the image of $\chi_{\mathbf{i},\mathbf{a}}$ is L_r, by (16), we see that all $W^l_{\mathbf{i},\mathbf{a}}, 0 \leq l \leq r-1$ are isomorphic as $\widehat{\mathcal{L}}$-modules. Thus, one can complete the proof of this proposition by a similar argument as that given in [6] (Theorem (4.8)). ■

It was shown in Proposition 2.9 [9] that the automorphism M (see [11] Sect.5) commutes with θ_q. Therefore, we have the following result

Theorem 5.3. (1) *When $|q| = 1$, the irreducible integrable highest weight $\widetilde{\mathcal{L}}$-module $V(\bar{\psi}_{\lambda,\mathbf{b}})$ is unitary with respect to θ_q if and only if $|b_1| = \cdots = |b_l|$ and $\psi_{\lambda,\mathbf{b}}(d_x), \psi_{\lambda,\mathbf{b}}(d_y)$ are real.*

(2) *Let V be an irreducible integrable $\widetilde{\mathcal{L}}$-module with c_x or c_y acts non-trivially, then V is unitary with respect to θ_q if and only if V is a highest weight module, and isomorphic to some $\widetilde{\mathcal{L}}$-module $V(\bar{\psi}_{\lambda,\mathbf{b}})$ obtained in (1) or a lowest weight module up to an automorphism M.* ■

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References

[1] Allison, B., S. Azam, S. Berman, Y. Gao, and A. Pianzola, *Extended affine Lie algebras and their root systems,* Mem. Amer. Math. Soc. **603**, 1997

[2] Berman, S., Y. Gao, and Y. Krylyuk, *Quantum tori and the structure of elliptic quasi-simple Lie algebras,* J. Funct. Anal. **135** (1996), 339–389.

[3] Berman, S., Y. Gao, and S. Tan, *A unified view of some vertex operator constructions,* Israel J. Math. **134** (2003), 29–60.

[4] Berman S., and J. Szmigielski, *Principal realization for extended affine Lie algebra of type sl_2 with coordinates in a simple quantum torus with two variables,* Contemp. Math. **248** (1999), 39–67.

[5] Chari, V., *Integrable representations of affine Lie algebras,* Invent. Math. **85** (1986), 317–335.

[6] Chari, V., and A. Pressley, *New unitary representations of loop groups,* Math. Ann. **275** (1986), 87–104.

[7] —, *Integrable representations of twisted affine Lie algebras,* J. Alg. **113** (1988), 438–464.

[8] Eswara Rao, S., *Classification of irreducible integrable modules for multi-loop Lie algebras with finite dimensional weight spaces,* J. Alg. **246** (2001), 215–225.

[9] —, *Unitary modules for EALAs co-ordinatized by a quantum torus,* Comm. Alg. **31** (2003), 2245–2256.

[10] —, *Classification of irreducible integrable modules for toroidal Lie algebras with finite dimensional weight spaces,* J. Alg. **277** (2004), 318–348.

[11] —, *A class of integrable modules for the core of EALA coordinatized by quantum tori*, J. Alg. **275** (2004), 59–74.

[12] Gao, Y., *Representations of extended affine Lie algebras coordinatized by certain quantum tori*, Compos. Math. **123** (2000), 1–25.

[13] —, *Vertex operators arising from the homogeneous realization for $\hat{gl}_N$*, Comm. Math. Phys. **211** (2000), 745–777.

[14] —, *Fermionic and bosonic representations of the extended affine Lie algebra $\widetilde{gl_N(\mathbb{C}_q)}$*, Canada Math. Bull. **45** (2002), 623–633.

[15] Gao, Y., and Z. Zeng, *Hermitian representations of the extended affine Lie algebra* $\widetilde{gl_2(\mathbb{C}_q)}$, Adv. Math. **207** (2006), 244–265.

[16] Golenishcheva-Kutuzova, M., and D. Lebedev, *Vertex operator representation of some quantum tori Lie algebras,* Comm. Math. Phys. **148** (1992), 403–416.

[17] Høegh-Krohn, R., and B. Torresani, *Classification and construction of quasisimple Lie algebras,* J. Funct. Anal. **89** (1990), 106–136.

[18] Jakobsen, H., and V. Kac, *A new class of unitarizable highest weight representations of infinite-dimensional Lie algebras. II,* J. Funct. Anal. **82** (1989), 69–90.

[19] Kac, V., *Infinite dimensional Lie algebras, 3rd edition,* Cambridge University Press, 1990.

[20] Zeng, Z., *Unitary representations of the extended affine Lie algebra* $\widetilde{gl_3(\mathbb{C}_q)}$, Pacific J. Math. **233** (2007), 481–509.

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