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Isometries of Hermitian Symmetric Spaces

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Abstract. We show that every isometry of a canonically embedded hermitian symmetric space extends to an isometry of its ambient transvection Lie algebra.
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1. Introduction and main result

Hermitian symmetric spaces are Riemannian symmetric spaces endowed with a Kähler structure such that the geodesic symmetries are holomorphic. The Kähler structure of a semisimple hermitian symmetric space P gives rise to a canonical embedding of P into its semisimple transvection Lie algebra $\mathfrak{g}$ (see Section 2, [Li58, pp. 165 ff.], [Hi70] and[Na84]). This embedding is extrinsically symmetric (see [Fe74, Fe80, Na84, EH95, KE11]), that is the geodesic symmetry s_p of P at a point $p \in P$ is the restriction of the reflection ρ_p through the affine normal space of $P \subset \mathfrak{g}$ at p. This reflection is an isometry with respect to a suitable inner product on $\mathfrak{g}$. Thus every isometry of P which is generated by geodesic symmetries extends to linear isometries of the ambient space $\mathfrak{g}$. But what about arbitrary isometries of P?

Since semisimple hermitian symmetric spaces are inner symmetric spaces, the geodesic symmetries of P generate the transvection group $\mathfrak{T}(P)$ of P, which is the identity component of the full isometry group $\mathrm{I}(P)$ of P. Transvections of hermitian symmetric spaces are holomorphic isometries. But hermitian symmetric spaces always allow for anti-holomorphic isometries, too (see e.g. [Le79] for the compact case). Thus the isometry group of a semisimple hermitian symmetric space has several connected components. Looking in Loos' list (see [Lo69, p. 156]) one sees that the isometry group of an irreducible hermitian symmetric space of compact type has two connected components, except for the following Grassmannians: the full isometry groups of $G_n(\mathbb{C}^{2n})$, $n \geq 2$, and of $\tilde{G}_2(\mathbb{R}^{2n})$, $n \geq 3$, have four connected components and the full isometry group of $\tilde{G}_2(\mathbb{R}^4)$ has eight

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connected components. The same holds for their non-compact dual symmetric spaces.

In Section 3 we show by an explicit construction that *any* isometry of a canonically embedded semisimple hermitian symmetric space $P \subset \mathfrak{g}$ extends to a linear isometry of the ambient transvection Lie algebra $\mathfrak{g}$:

Theorem 1.1. *Let $P \subset \mathfrak{g}$ be a canonically embedded semisimple hermitian symmetric space and let f be an isometry of P, then there exists a linear isometry F of $\mathfrak{g}$ (w.r.t. a suitable invariant inner product) whose restriction to P coincides with f.*
Moreover F preserves the Lie triple product given by the double Lie bracket $(X, Y, Z) = [X, [Y, Z]]$ on $\mathfrak{g}$.

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2. Canonical embeddings of hermitian symmetric spaces

In this section we explain the canonical embedding of a semisimple hermitian symmetric space. This embedding has been described in [Li58, pp. 165 ff.] and [Hi70].

Let P be a semisimple hermitian symmetric space with complex (Kähler) structure J and semisimple transvection Lie algebra $\mathfrak{g}$. The complex structure J_p at a point $p \in P$ is a skew adjoint derivation of the curvature tensor at p and hence an element of the compact isotropy Lie algebra $\mathfrak{k}_p \subset \mathfrak{g}$ of p. Actually, if P is irreducible, J_p generates the center $\mathfrak{c}(\mathfrak{k}_p)$ of $\mathfrak{k}_p$, that is $\mathfrak{c}(\mathfrak{k}_p) = \mathbb{R}J_p$ (see [He78, pp. 381f.]). Conjugation with the geodesic symmetry s_p of P at p defines an involutive automorphism σ_p of G. Its derivative $(\sigma_p)_*$ at the identity is an involutive automorphism of the Lie algebra $\mathfrak{g}$. Therefore the Lie algebra $\mathfrak{g}$ splits as $\mathfrak{g} = \mathfrak{k}_p \oplus \mathfrak{p}_p$ into the fixed point set of $(\sigma_p)_*$, which coincides with $\mathfrak{k}_p$, and its (-1)-eigenspace $\mathfrak{p}_p$ which is canonically identified with T_pP (see [He78, p. 208]).

The action of $J_p \in \mathfrak{c}(\mathfrak{k}_p)$ on $\mathfrak{p}_p \cong T_pP$ is given by $\operatorname{ad}(J_p)|_{\mathfrak{p}_p}$. Thus $\operatorname{ad}(J_p)$ has eigenvalues $\pm i$ and 0, so that

$$\operatorname{ad}(J_p)^3 = -\operatorname{ad}(J_p). \tag{1}$$

Elements of $\mathfrak{g}$ that satisfy Eq. (1) will be called *extrinsic symmetric.* An extrinsic symmetric element $X \in \mathfrak{g}$ is *compact,* that is the one-parameter subgroup $t \mapsto e^{t\operatorname{ad}(X)}$ is a compact subgroup of $\operatorname{Ad}(G)$ (see [Ne94]).

Let us consider the map

$$\iota : P \to \mathfrak{g}, \quad p \mapsto J_p.$$

Since J is invariant under $G = \mathfrak{I}(P)$, the image of ι is the adjoint orbit $\operatorname{Ad}(G)J_o \subset \mathfrak{g}$, where o is a chosen base point in P. We see that ι is an equivariant covering map. Every semisimple hermitian symmetric space and every adjoint orbit of a compact element in a semisimple Lie algebra is simply connected (see e.g. [He78, Ch. VIII, Thm. 4.6] and [Ne94, Cor. I.16]). Therefore ι is bijective and hence a G-equivariant embedding of P into $\mathfrak{g}$, called the *canonical embedding* of P. The normal space N_pP of $\iota(P)$ at the point $\iota(p) = J_p$ is $\mathfrak{k}_p$ and the tangent space of $\iota(P)$ at J_p is the orthogonal complement $\mathfrak{p}_p$.

Any semisimple hermitian symmetric space P is the de Rham product of irreducible hermitian symmetric spaces

$$P = P_1 \times \ldots \times P_r$$

of either compact or non-compact type (see [He78, Chap. VIII, Prop. 4.4]). Its transvection group G splits accordingly into simple factors as

$$G = G_1 \times \ldots \times G_r,$$

where G_j is the transvection group of the irreducible factor P_j ([Wo84, Thm. 8.3.9]). Therefore the transvection Lie algebra $\mathfrak{g}$ of P is the direct sum

$$\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_r$$

of the simple transvection Lie algebras $\mathfrak{g}_j$ of P_j for $j = 1, \ldots, r$.

Finally, the canonical embedding $\iota : P \to \mathfrak{g}$ decomposes into the canonical embeddings $\iota_j : P_j \to \mathfrak{g}_j$ for $j = 1, \ldots, r$. We endow $\mathfrak{g}$ with an inner product such that the spitting $\mathfrak{g} = \mathfrak{g}_1 \times \ldots \times \mathfrak{g}_r$ is orthogonal and such that the inner product on each $\mathfrak{g}_j$ is proportional to its Cartan-Killing form. The proportionality factor on $\mathfrak{g}_j$ is chosen in such a way that the embedding $\iota_j : P_j \to \mathfrak{g}_j$ is isometric. Recall that if P_j is of compact type, the Cartan-Killing of $\mathfrak{g}_j$ is negative definite. If P_j is of non-compact type, then the Cartan Killing form is positive definite on each tangent space of ι_j and negative definite on each normal space of ι_j.

The canonical embedding ι is extrinsically symmetric (see [Fe74, Fe80, Na84, EH95, KE11]), that is invariant under all reflections through its normal spaces. The reflection ρ_p through the normal space N_pP is given by the inner automorphism

$$\rho_p = e^{\mathrm{ad}(\pi J_p)} = \mathrm{Ad}(\exp(\pi J_p)) \tag{2}$$

of $\mathfrak{g}$ and therefore preserves our chosen inner product on $\mathfrak{g}$.

3. Proof of Theorem 1.1

The full isometry group $\mathrm{I}(P)$ of a semisimple hermitian symmetric space

$$P = P_1 \times \ldots \times P_r$$

(each factor P_j is irreducible and simply connected) is generated by the product $\mathrm{I}(P_1) \times \ldots \times \mathrm{I}(P_r)$ of the full isometry groups of each irreducible factor and by all permutations of isometric irreducible factors of P (see [Wo84, Thm. 8.3.9]). Recall that the embedding ι decomposes into the embeddings of all irreducible factors of P. Therefore permutations of isometric factors of P extend to permutations of the corresponding simple factors of $\mathfrak{g}$ (possibly up to sign on some factors, if the complex structures of the isometric irreducible hermitian symmetric spaces differ by a sign). To prove Theorem 1.1 we may therefore assume that P is irreducible. In this case the transvection Lie algebra $\mathfrak{g}$ of P is simple.

After composition with the geodesic symmetry at a suitable point of P (which extends to the reflection through the normal space, see Eq. (2)) we may assume that our isometry f of P leaves a chosen base point $o \in P$ fixed, that is:

$$f(o) = o.$$

To prove Theorem 1.1, we construct F explicitly. Let $G = \mathfrak{T}(P)$ denote the transvection group of P. Since P is inner, G coincides with the symmetry group of P. Conjugation with f yields an automorphism of G

$$\phi : G \to G, \quad g \mapsto f \circ g \circ f^{-1}$$

Hence, its differential ϕ_* at the identity is an automorphism of $\mathfrak{g}$ and therefore preserves our chosen inner product on $\mathfrak{g}$, which is proportional to the Cartan-Killing form.

Lemma 3.1. $\phi_*(J_o) \in \{\pm J_o\}$.

Proof. Let $K \subset G$ be the stabilizer of J_o (or equivalently of o), that is

$$K := \{g \in G : \ \mathrm{Ad}(g)J_o = J_o\} = \{g \in G : \ go = o\}.$$

Since $P \cong G/K$ is simply connected (see [He78, Ch. VIII, Thm. 4.6]) and G is connected, K must be connected, too.

Let $\mathfrak{k} = \mathfrak{k}_o$ be the Lie algebra of K. As ϕ_* is a Lie algebra automorphism of $\mathfrak{g}$, $\phi_*(J_o)$ is an extrinsic symmetric element. Since the only extrinsic symmetric elements in $\mathfrak{c}(\mathfrak{k})$ are $\pm J_o$, it is sufficient to show that $\phi_*(\mathfrak{c}(\mathfrak{k})) = \mathfrak{c}(\mathfrak{k})$.

Since $\phi(k)o = o$ for all $k \in K$, ϕ restricts to an automorphism of K. Thus ϕ preserves the center of K and therefore $\phi_*(\mathfrak{c}(\mathfrak{k})) = \mathfrak{c}(\mathfrak{k})$. ■

We now look separately at the two cases $\phi_*(J_o) = J_o$ and $\phi_*(J_o) = -J_o$:

Case 1 ($\phi_*(J_o) = J_o$). We want to show that

$$f = \phi_*|_P.$$

Since $\phi_*(P)$ contains J_o and since the Lie algebra automorphism ϕ_* maps adjoint orbits onto adjoint orbits, we have

$$\phi_*(P) = P.$$

As ϕ_* is a linear isometry of $\mathfrak{g}$, its restriction to P is an isometry of P. Recall that an isometry of a connected Riemannian manifold is uniquely determined by its value and its derivative at a single point. Hence, to show that $\phi_*|_P$ equals f, it suffices to verify that ϕ_* coincides on $\mathfrak{p} = \mathfrak{p}_o \cong T_oP$ with the differential of f at the point $o \cong J_o$, that is

$$\phi_*(V) = f.V$$

for all $V \in \mathfrak{p}$. Here $f.V$ denotes the action of f on $\mathfrak{p}$ given by the derivative of f at o.

But, for any connected symmetric space P with base point o, an isometry f of P that fixes o conjugates one-parameter groups of transvections along geodesics that emanate from o as follows:

If $V \in T_oP \cong \mathfrak{p} \subset \mathfrak{g}$, then $\mathbb{R} \to G,\ t \mapsto \exp(tV)$ is the one-parameter group of transvections along the geodesic γ_V of P that emanates from o in direction V and

$$\mathbb{R} \to G,\ t \mapsto \phi(\exp(tV)) = f \circ \exp(tV) \circ f^{-1} = \exp(t(f.V))$$

is the one-parameter group of transvections along the geodesic $f \circ \gamma_V$. Thus $\phi_*(V) = f.V$.

We conclude that $F := \phi_*$ is an extension of f to a linear isometry of $\mathfrak{g}$.

Case 2 ($\phi_*(J_o) = -J_o$). In this case we claim that

$$f = -\phi_*|_P,$$

More precisely, we have to show that $F = -\phi_*$ satisfies $\iota \circ f = F \circ \iota$ where $\iota : P \to \mathfrak{g},\ p \mapsto J_p$ is the embedding of P into $\mathfrak{g}$. For arbitrary $V \in \mathfrak{p}$ let $g_t = \exp tV$. From $\phi_*(\mathrm{Ad}(g_t)J_o) = \mathrm{Ad}\,(\phi(g_t))\,\phi_*J_o$ (which holds for any Lie group homomorphism ϕ) we conclude $F(\mathrm{Ad}(g_t)J_o) = \mathrm{Ad}\,(\phi(g_t))\,J_o$. Using $\iota(go) = \mathrm{Ad}(g)\iota(o) = \mathrm{Ad}(g)J_o$ we obtain

$$F(\iota(g_to)) = F(\mathrm{Ad}(g_t)J_o) = \mathrm{Ad}(fg_tf^{-1})\iota(o) = \iota\left(f(g_to)\right).$$

Thus $F = -\phi_*$ extends f to a linear isometry of $\mathfrak{g}$.

Remark. While in Case 1 f extends to a Lie algebra automorphism of $\mathfrak{g}$, f extends to an anti-automorphism of $\mathfrak{g}$ in Case 2. Nevertheless in both cases f extends to an automorphism of the Lie triple $\mathfrak{g}$.

This finishes the proof of Theorem 1.1.

4. Concluding remarks

In view of the classification of symmetric R-spaces (see e.g. [BCO03, p. 310 f.]) and the list of numbers of connected components of isometry groups of simply connected irreducible symmetric spaces of compact type (see [Lo69, p. 156]), we conjecture that Theorem 1.1 extends to the isometries of *any* extrinsically symmetric submanifold in a Euclidean space.

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