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Interior Regularity for Degenerate Elliptic Equations with Drift on Homogeneous Groups*

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Abstract. Let G be a homogeneous group and let $X_0, X_1, X_2, \ldots, X_{p_0}$ be left invariant real vector fields on G satisfying Hörmander's rank condition. Assume that $X_1, X_2, \ldots, X_{p_0}$ are homogeneous of degree one and X_0 is homogeneous of degree two. In this paper, we study the following equation with drift: $Lu \equiv \sum_{i,j=1}^{p_0} X_i(a_{ij}(x)X_j u) + a_0 X_0 u = \sum_{j=1}^{p_0} X_j F_j(x)$, where $a_{ij}(x)$ are real valued, bounded measurable functions defined in a domain $\Omega \subset G$, $a_{ij}(x) = a_{ji}(x)$, satisfying the uniform ellipticity condition in $\mathbb{R}^{p_0}$ and $a_0 \in \mathbb{R}\backslash\{0\}$. Moreover, the coefficients a_{ij} belong to the class VMO (Vanishing Mean Oscillation) with respect to the subelliptic metric induced by the vector fields $X_0, X_1, X_2, \ldots, X_{p_0}$. We derive local L^p estimates for second order derivatives and Hölder estimates by establishing the representation formulas and higher order integrability of weak solutions to the above equation.
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1. Introduction and main results

Let $X_0, X_1, X_2, \cdots, X_{p_0} (p_0 < N)$ form a system of C^∞ real vector fields defined in $\mathbb{R}^N$, which are left invariant with respect to translations on the homogeneous group G, and homogeneous with respect to a family of dilations. More precisely, $X_1, X_2, \cdots, X_{p_0}$ are homogeneous of degree one and X_0 is homogeneous of degree two. Assume that these vector fields satisfy Hörmander's condition at every point of $\mathbb{R}^N$, i.e.,

$$rank\mathcal{L}(X_0, X_1, \ldots, X_{p_0})(x) = N, \; x \in \mathbb{R}^N, \tag{1}$$

where $\mathcal{L}(X_0, X_1, \ldots, X_{p_0})$ denotes the Lie algebra generated by $X_0, X_1, \ldots, X_{p_0}$.

We derive local L^p and Hölder estimates of the weak solutions to the equation

$$Lu \equiv \sum_{i,j=1}^{p_0} X_i(a_{ij}(x)X_j u) + a_0 X_0 u = \sum_{j=1}^{p_0} X_j F_j(x), \tag{2}$$

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where $a_0 \in \mathbb{R}\backslash\{0\}$ and $a_{ij}(x)$ are real valued, bounded measurable functions defined in a bounded domain $\Omega \subset G$, belonging to the class VMO, Vanishing Mean Oscillation, with respect to the subelliptic metric induced by the vector fields $X_0, X_1, X_2 \cdots, X_{p_0}$; moreover, the matrix $\{a_{ij}(x)\}$ is symmetric and satisfies the uniformly ellipticity condition:

$$\mu|\xi|^2 \le \sum_{i,j=1}^{p_0} a_{ij}(x)\xi_i\xi_j \le \mu^{-1}|\xi|^2, \tag{3}$$

with a.e. $x \in \Omega,\ \xi \in \mathbb{R}^{p_0},\ \mu > 0$ and

$$\mu \le a_0 \le \mu^{-1}.$$

In particular, when $a_{ij}(x) = \delta_{ij}\,(i, j = 1, \ldots, p_0),\ a_0 = 1,\ L$ can be written as

$$L_1 = \sum_{i=1}^{p_0} X_i^2 + X_0. \tag{4}$$

The operator L_1 has been studied extensively by many authors(see [15, 14, 26]). Hörmander in [15] pointed out that (1) implies the hypoellipticity of (4). In [14], Folland proved that homogeneous hypoelliptic operators on nilpotent groups possess homogeneous fundamental solutions. To study L^p regularity of the operator $\sum_{i=1}^{p_0} X_i^2$, Rothschild and Stein [26] proposed a powerful technique of lifting and approximation.

When $X_i = \partial_{x_i},\ i = 1, 2, \ldots, N-1$, and $X_0 = \partial_t,\ L$ has the form

$$L_2 u = \sum_{i,j=1}^{N-1} \partial_i(a_{ij}(x)\partial_j u) + a_0\partial_t u.$$

More recently, many authors considered the operator L_2. Krylov in [16] gave an approach to the study of the L^p solvability of the equations corresponding to L_2 with leading coefficients measurable in the time variable and VMO in the spatial variables. In [6], Byun proved the L^p estimates for divergence parabolic equations with small BMO coefficients (see also [7, 18] and references therein).

In the case of $a_0X_0 = \sum_{i,j=1}^{n} b_{ij}x_i\partial_{x_j} - \partial_t,\ X_i = \partial_{x_i},\ i = 1, 2, \ldots, p_0,\ L$ becomes

$$L_3 u = \sum_{i,j=1}^{p_0} \partial_i(a_{ij}(x)\partial_j u) + \sum_{i,j=1}^{n} b_{ij}x_i\partial_{x_j} u - \partial_t u,$$

where $(x, t) \in \mathbb{R}^{n+1}$, (b_{ij}) is a constant matrix with a suitable upper triangular structure, while (a_{ij}) is a $p_0 \times p_0$ uniformly elliptic matrix, with $p_0 < n$. The operator L_3 belongs to a class of Kolmogorov-Fokker-Planck ultraparabolic operators and appears in many research fields. For instance, the Kolmogorov equation

$$\partial_{x_1}^2 u + x_1\partial_{x_2} u = \partial_t u,\ (x, t) \in \mathbb{R}^3$$

occurs in a financial problem (see[1, 12]), in the kinetic theory (see[11, 20]) as well as in a visual perception problem (see[22]). Owing to its importance in physics and in mathematical finance, it has been extensively investigated (see e.g. [17, 19, 23, 21, 25, 24] and references therein). If a_{ij} are constant coefficients, a systematic study of a class of equations with respect to L_3 has been carried out in [17, 19, 23]: Pascucci and Polidoro in [23] gave a direct proof of the Harnack inequality and found explicitly the optimal constant in the inequality; a parabolic maximum principle was proved by Kupcov in [17]. If the coefficients a_{ij} are measurable, Pascucci and Polidoro [24] showed that the weak solutions are locally bounded functions by using Moser's iterative method. If the coefficients a_{ij} belong to the Sarason class VMO, Manfredini and Polidoro in [21] investigated the L^p and Hölder estimates of the equation corresponding to L_3 by using the singular integral method; Morrey estimates were obtained by Polidoro and Ragusa in [25]. Wang and Zhang [28] discussed Hölder regularity of the solution of $L_3 u = 0$ with purely measurable coefficients a_{ij}.

Recently, Bramanti and Zhu in [4] introduced the concept of locally homogeneous space and derived the L^p estimates for singular and fractional integrals, as well as L^p estimates on the commutator of a singular or fractional integral with a BMO or VMO function. These results were applied to deduce local L^p and Hölder estimates of nonvariational operators on Hörmander vector fields with drift in [5].

In this paper, we are concerned with the local L^p and Hölder estimates for weak solutions of (2) by establishing the representation formulas and the higher order integrability of weak solutions to (2). To describe main results of this paper, we need to introduce some notions.

Definition 1.1. For a measurable function $f \in L^1_{loc}(G)$, we define

$$\eta_f(R) = \sup_{r \le R} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f(y) - f_{B_r}| dy, \ R > 0, \ a.e. \ x \in G,$$

where f_{B_r} is the average of f over $B_r(x)$, which is given in (9). We say $f \in BMO(G)$ (Bounded Mean Oscillation) if $\|f\|_* := \sup_R \eta_f(R) < +\infty$, while $f \in VMO(G)$ (Vanishing Mean Oscillation) if $\lim_{R\to 0} \eta_f(R) = 0$.

For a given domain Ω, the spaces $BMO(\Omega)$ and $VMO(\Omega)$ are similarly defined, if we take $B_r \cap \Omega$ instead of B_r.

Definition 1.2. We say that u is a weak solution of (2), provided $u, \ X_1 u, \dots, X_{p_0} u, \ X_0 u \in L^2_{loc}(\Omega)$ and

$$\sum_{i,j=1}^{p_0} \int_\Omega a_{ij}(x) X_j u(x) X_i \varphi(x) dx - a_0 \int_\Omega X_0 u(x) \varphi(x) dx = \sum_{j=1}^{p_0} \int_\Omega F_j(x) X_j \varphi(x) dx,$$

for all $\varphi \in C_0^\infty(\Omega)$.

The main results are contained in the following theorems:

Theorem 1.3. *Let $\Omega' \subset\subset \Omega$ and u be a weak solution of (2) in Ω. If $u, \ F_j \in L^p(\Omega)$ for $j = 1, \dots, p_0$, $1 < p < \infty$, then $X_j u \in L^p(\Omega')$ and there* *exists a positive constant c depending only on $G, \mu, p, \Omega', \Omega,\ X_0, X_1, X_2, \cdots, X_{p_0}$ such that*

$$\sum_{j=1}^{p_0} \|X_j u\|_{L^p(\Omega')} \leq c \left(\sum_{i=1}^{p_0} \|F_i\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right). \tag{5}$$

Theorem 1.4. *Let $\Omega' \subset\subset \Omega$ and u be a weak solution of (2) in Ω. If $u,\ F_j \in L^p(\Omega)$ for $j = 1, \ldots, p_0$, $p > Q$, then there exists a positive constant c depending only on $G, \mu, p, \Omega', \Omega,\ X_0, X_1, X_2, \cdots, X_{p_0}$ such that*

$$\frac{|u(x) - u(y)|}{\|y^{-1} \circ x\|^{1-\frac{Q}{p}}} \leq c \left(\sum_{i=1}^{p_0} \|F_i\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)} \right),\ x, y \in \Omega',\ x \neq y. \tag{6}$$

The proof of our results is based on a method introduced by Chiarenza, Frasca and Longo [9, 10] in the study of non-divergence uniformly elliptic equation, then extended by many authors to degenerate PDEs. We first prove a representation formula for solutions to (2) following the lines of the article [21] by Manfredini and Polidoro, and then finish the proof relying on the general estimates proved by Bramanti and Brandolini in [2]. The paper is organized as follows: In Section 2, we introduce some known results about homogenous groups and describe some properties of fundamental solutions for the frozen operator of L. The representation formulas for the weak solutions of (2) are deduced in Section 3. They are given in Theorem 3.1 and Theorem 3.2. By using these formulas, we first establish two important higher order integrability results (see Lemmas 4.1-4.2) and then prove Theorem 1.3 in Section 4. Finally, Hölder estimates are obtained in Section 5.

2. Preliminary

Give a pair of smooth mappings:

$$[(x, y) \mapsto x \circ y] : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N;\ [x \mapsto x^{-1}] : \mathbb{R}^N \to \mathbb{R}^N,$$

$\mathbb{R}^N$ together with these mappings forms a group with the origin being the identity. Suppose that there exists an $N-$ tuple of strictly positive exponents $\omega_1 \leq \omega_2 \leq \ldots \leq \omega_N$, so that the dilations

$$D(\varrho) : (x_1, \ldots, x_N) \mapsto (\varrho^{\omega_1} x_1, \ldots, \varrho^{\omega_N} x_N)$$

are group automorphisms, for all $\varrho > 0$. The space $\mathbb{R}^N$ with this structure is called a homogeneous group and denoted by G.

We define a homogeneous norm $\|\cdot\|$ in G as follows. For any $x \in G \backslash \{0\}$, we set

$$\|x\| = \rho \Leftrightarrow |D(1/\rho)x| = 1,$$

where $|\cdot|$ denotes the Euclidean norm; also let $\|0\| = 0$. Then
(i) $\|D(\varrho)x\| = \varrho\|x\|$ for every $x \in G,\ \varrho > 0$;
(ii) the set $\{x \in G : \|x\| = 1\}$ coincides with the Euclidean unit sphere Σ_N;

(iii) the function $x \mapsto \|x\|$ is smooth outside the origin;
(iv) there exists $c, c_1, c_2 \geq 1$ such that for every $x, y \in G$,

$$\|x^{-1}\| \leq c_1 \|x\|; \tag{7}$$

$$\|x \circ y\| \leq c_2(\|x\| + \|y\|) \tag{8}$$

and

$$\frac{1}{c}|y| \leq \|y\| \leq c|y|^{1/w} \ \text{ if } \|y\| \leq 1, \ \omega = \max(\omega_1, \dots, \omega_N).$$

In view of these properties, we can define the quasidistance d by

$$d(x, y) = \|y^{-1} \circ x\|.$$

It follows from (7) and (8) that for any $x, y, z \in G$ and some positive constants $c_1, c_2 \geq 1$,

$$d(x, y) \geq 0, d(x, y) = 0 \Leftrightarrow x = y;$$
$$c_1^{-1} d(y, x) \leq d(x, y) \leq c_1 d(y, x);$$
$$d(x, y) \leq c_2(d(x, z) + d(z, y)).$$

The ball with respect to d is defined by

$$B_r(x) = \{y \in G : d(x, y) < r\}. \tag{9}$$

Note that $B(0, r) = D(r)B(0, 1)$ and

$$|B(x, r)| = r^Q |B(0, 1)|, \tag{10}$$

where $x \in G$, $r > 0$ and $Q = \omega_1 + \cdots + \omega_N$. We will call Q the homogeneous dimension of G. By (10), the Lebesgue measure dx is a doubling measure with respect to d, that is

$$|B(x, 2r)| \leq c|B(x, r)|, \ x \in G, \ r > 0$$

and therefore (G, dx, d) is a space of homogenous type.

A differential operator Y on G is called homogeneous of degree $\beta > 0$ if

$$Y(\varphi(D(\varrho)x)) = \varrho^\beta (Y\varphi)(D(\varrho)x)$$

for every test function φ, $\varrho > 0$, $x \in G$; a function f is called homogeneous of degree α provided

$$f((D(\varrho)x)) = \varrho^\alpha f(x), \ \varrho > 0, \ x \in G.$$

Clearly, if Y is a homogeneous differential operator of degree β and f is a homogeneous function of degree α, then Yf is homogeneous of degree $\alpha - \beta$.

The convolution of two functions on G is defined by

$$(f * g)(x) = \int_G f(x \circ y^{-1}) g(y) dy = \int_G g(y^{-1} \circ x) f(y) dy,$$

if the above integrals make sense. We see from this definition that if P is any left invariant differential operator, then

$$P(f * g) = f * Pg,$$

provided the integrals converge.

Lemma 2.1 ([2, 13, 14]). *Assume that K_h is a kernel in $C^\infty(G\backslash\{0\})$ and homogeneous of degree $h-Q$, for some integer h with $0<h<Q$; Let T_h be the operator*

$$T_h f = f * K_h$$

and let P^h be a homogeneous left invariant differential operator of degree h. Then

(i) the following representation formula is true:

$$P^h T_h f = P.V.(f * P^h K_h) + \alpha f,$$

where α is a constant depending on P^h and K_h;

(ii) the function $P^h K_h$ belongs to $C^\infty(G\backslash\{0\})$ and is homogeneous of degree $-Q$, satisfying the vanishing property:

$$\int_{r<\|z\|<R} P^h K_h(z)dz = \int_{\|z\|=1} P^h K_h(z)d\sigma(z) = 0,\ 0<r<R<\infty;$$

(iii) the singular integral operator

$$f \mapsto P.V.(f * P^h K_h)$$

is continuous on $L^p(G)$ for $1<p<\infty$.

Lemma 2.2 ([27]). *Let $\alpha \in (0,Q)$ and $K \in C(G\backslash\{0\})$ be a homogeneous function of degree $-\alpha$ with respect to the group $(D(\lambda))_{\lambda>0}$. If $g \in L^q(G)$, then*

$$Tg(x) = \int_G K(x,y)g(y)dy$$

is a.e. defined, and there exists $c>0$ such that

$$\|Tg\|_{L^p(G)} \le c \max_{\|x\|=1} |K(x)| \|g\|_{L^q(G)},$$

for any p verifying $\frac{1}{p}+1=\frac{1}{q}+\frac{\alpha}{Q}$.

Let $\mathcal{A}_\mu$ be the set of $p_0\times p_0$ constant matrices $A=\{a_{ij}\}$, satisfying

$$\mu^2|\xi|^2 \le \sum_{i,j=1}^{p_0} a_{ij}\xi_i\xi_j \le \mu^{-2}|\xi|^2,\ \xi\in\mathbb{R}^{p_0},$$

where μ is the same as in (3). In the sequel, we denote the operator

$$\mathcal{L}_A = \sum_{i,j=1}^{p_0} a_{ij}X_iX_j + X_0,$$

where $A=\{a_{ij}\}\in\mathcal{A}_\mu$. By [2], $\mathcal{L}_A$ has a fundamental solution which is homogeneous of degree $2-Q$, denoted by Γ^A.

For any $x_0 \in G$, we can freeze the coefficients $a_{ij}(x)$ in the operator L of (2) at x_0 and denote

$$L_{x_0} = \sum_{i,j=1}^{p_0} a_{ij}(x_0) X_i X_j + a_0 X_0.$$

By applying the results in [14] to L_{x_0}, we know that L_{x_0} has a fundamental solution which is homogeneous of degree $2 - Q$, denoted by $\Gamma(x_0; \cdot)$. For $i, j = 1, \dots, p_0$, we set

$$\Gamma_{ij}(x_0; y) = X_i X_j [\Gamma(x_0, \cdot)](y).$$

The following results concerning the Fourier expansion of homogeneous functions with respect to spherical harmonics have been given in [2, 21]. Let $\{Y_{km}\}(m = 0, 1, 2, \dots,\ k = 1, \dots, g_m)$ be an orthonormal system of spherical harmonics in G and complete in $L^2(\Sigma_N)$, where Σ_N denotes the unit sphere of G, m is the degree of the polynomial and g_m is the dimension of the space of spherical harmonics of degree m in G. We extend every function $\{Y_{km}\}$ to $G \backslash \{0\}$ by setting $Y_{km}(y) = Y_{km}(D(\|y\|^{-1})y)$ for $y \in G \setminus \{0\}$.

Fixed $x \in G$ and given a homogenous function $K \in C^\infty(G \setminus \{0\})$ of degree α, then there exists a sequence $c^{km}(m = 0, 1, 2, \dots,\ k = 1, \dots, g_m)$ such that

$$K(x, y) = \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} c^{km}(x) \|y\|^\alpha Y_{km}(y). \tag{11}$$

The following bounds about spherical harmonics hold:

$$g_m \le C(N) m^{N-2} \text{ for every } m = 1, 2, \dots \tag{12}$$

$$\left| \left(\frac{\partial}{\partial x} \right)^\beta Y_{km}(x) \right| \le C(N) m^{(\frac{N-2}{2} + |\beta|)} \tag{13}$$

for $x \in \Sigma_N,\ k = 1, \dots, g_m,\ m = 1, 2, \dots$; moreover, for every $r \in \mathbb{N}$ there exists a positive constant $C = C(r)$ such that

$$\sup_{x \in G} |c^{km}(x)| \le C(r) m^{-2r} \tag{14}$$

for every $m = 0, 1, 2, \dots,\ k = 1, \dots, g_m$. These results hold for the functions $\Gamma(x_0; \cdot),\ X_j\Gamma(x_0; \cdot),\ \Gamma_{ij}(x_0; \cdot)$ for $i, j = 1, \dots, p_0$ and $X_0\Gamma(x_0; \cdot)$.

Note that, if we apply the above result to function

$$\Gamma(x_0; \cdot) = a_0 \Gamma^A(\cdot), \tag{15}$$

we get the following estimate proved by Bramanti and Brandolini [2]

$$\sup_{\|x\|=1} \left| \left(\frac{\partial}{\partial x} \right)^\beta \Gamma^A(x) \right| \le c(\beta, G, \mu). \tag{16}$$

The following lemma summarizes the properties of $\Gamma(x_0; \cdot)$ and $\Gamma_{ij}(x_0; \cdot)$ that will be used later.

Lemma 2.3 ([2, 14]). *For every $x_0 \in G$, it follows*

(a) $\Gamma(x_0;\cdot) \in C^\infty(G\backslash\{0\})$;

(b) $\Gamma(x_0;\cdot)$ *is homogeneous of degree* $2-Q$;

(c) *For every test function* u *and every* $x \in G$,

$$u(x) = (L_{x_0}u * \Gamma(x_0;\cdot))(x) = \int_G \Gamma(x_0; y^{-1}\circ x)L_{x_0}u(y)dy. \tag{17}$$

Furthermore, for every $i,j = 1,2,\ldots,p_0$, *there exist constants* $\alpha_{ij}(x_0)$ *such that*

$$X_iX_ju(x) = P.V.\int_G \Gamma_{ij}(x_0; y^{-1}\circ x)L_{x_0}u(y)dy + \alpha_{ij}(x_0)L_{x_0}u(x); \tag{18}$$

(d) $\Gamma_{ij}(x_0;\cdot)$ *is homogeneous of degree* $-Q$;

(e) *For any* $R > r > 0$,

$$\int_{r<\|y\|<R} \Gamma_{ij}(x_0;y)dy = \int_{\|y\|=1} \Gamma_{ij}(x_0;y)d\sigma(y) = 0;$$

(f) *For any multiple index* β, *there exists a positive constant* c *such that for any* $i,j = 1,\ldots,p_0$,

$$\sup_{\|y\|=1, x\in G}\left|\left(\frac{\partial}{\partial y}\right)^\beta \Gamma_{ij}(x;y)\right| \le c.$$

Moreover, for the $\alpha'_{ij}s$ *appearing in (18), a uniform bound holds:*

$$\sup_{x\in G}|\alpha_{ij}(x)| \le c,$$

for some constant $c = c(G,\mu)$.

3. Representation formulas

In this section, we will prove some representation formulas for the weak solutions of (2) in terms of the fundamental solution $\Gamma(\cdot;\cdot)$. For fixed $s,r \in \mathbb{R}$, $0 < s < r$ and a function $\varphi \in C_0^\infty(\mathbb{R})$, we write $[0,s] \prec \varphi \prec [0,r]$ to mean that $0 \le \varphi \le 1$, $\varphi = 1$ on $[0,s]$ and $\mathrm{supp}\varphi \subseteq [0,r]$. For every $y_0 \in \Omega$ and $r > 0$ such that $B_r(y_0) \subset \Omega$, we set

$$\eta(x) = \varphi(\|x^{-1}\circ y_0\|). \tag{19}$$

If u is a weak solution of (2), then ηu is the solution of the following equation

$$L(\eta u) = div_X(\mathcal{G}) + g, \tag{20}$$

where

$$div_X(\mathcal{G}) = \sum_{j=1}^{p_0} X_j\mathcal{G}_j,\ \mathcal{G} = (\mathcal{G}_1,\ldots,\mathcal{G}_{p_0});$$
$$\mathcal{G}_j = \eta F_j + ua_{ij}(x)X_i\eta,\ i,\ j = 1,\ldots,p_0;$$
$$g = \langle ADu, D\eta\rangle - \langle F, D\eta\rangle + a_0uX_0^*\eta$$

and $F = (F_1,\ldots,F_{p_0})$. For convenience, we shall write B_r instead of $B_r(y_0)$ and $\tilde{f}(x) = f(x^{-1})$, $x \in G$.

Theorem 3.1. *If u is a weak solution of (2), then*

$$
\begin{aligned}
(\eta u)(x) = &-\sum_{i,j=1}^{p_0}\int_G K_i(x;y^{-1}\circ x)[(a_{ij}(x)-a_{ij}(y))X_j(\eta u)(y)+\mathcal{G}_i(y)]dy \\
&+\int_G \Gamma(x;y^{-1}\circ x)g(y)dy,
\end{aligned} \tag{21}
$$

where $K_i(x;y^{-1}\circ x) = \tilde{X_i}\tilde{\Gamma}(x;y^{-1}\circ x)(i=1,\dots,p_0)$ is homogeneous of degree $1-Q$.

Proof. Let v be a function in $C_0^\infty(\Omega)$. For every $x_0\in\Omega$ and every function $\phi\in C_0^\infty(\Omega)$, it follows from the definition of weak solutions and (20) that

$$
\begin{aligned}
\int_\Omega L_{x_0}v(y)\phi(y)dy = &\int_\Omega g(y)\phi(y)dy - \int_\Omega \langle \mathcal{G}(y), D\phi(y)\rangle dy \\
&-\int_\Omega \langle [A(x_0)-A(y)]D(\eta u)(y), D\phi(y)\rangle dy \\
&-\int_\Omega \langle A(x_0)D(v-\eta u)(y), D\phi(y)\rangle dy \\
&-\int_\Omega a_0(v-\eta u)(y)X_0^*\phi(y)dy.
\end{aligned} \tag{22}
$$

Now we divide the proof of (21) into three steps.

(i) We shall prove that (22) also holds (almost everywhere) if we replace ϕ with the function $\Gamma(x_0;y^{-1}\circ x)$. First of all, we note that the function v has compact support so that (22) also holds for every $\phi\in C^\infty(G)$.

Assume that φ is a non-increasing function satisfying $[0,1/2]\prec\varphi\prec[0,1]$. For every $x,\ x_0\in\Omega$ and for every $\delta>0$, we denote

$$
\phi_\delta(x_0;x,y) = \left[1-\varphi\left(\frac{\|x^{-1}\circ y\|}{\delta}\right)\right]\tilde{\Gamma}(x_0;x^{-1}\circ y). \tag{23}
$$

By $v,\ L_{x_0}v\in C_0^\infty(G)$ and (17), we have

$$
\lim_{\delta\to 0}\int_\Omega (L_{x_0}v)(y)\phi_\delta(x_0;x,y)dy = \int_\Omega (L_{x_0}v)(y)\Gamma(x_0;y^{-1}\circ x)dy = v(x). \tag{24}
$$

We will consider the term $\int_\Omega g(y)\phi_\delta(y)dy$ in the right hand side of (22). According to the spherical harmonics expansion (11), without loss of generality, $\tilde{\Gamma}(x_0;x^{-1}\circ y)$ has the following form:

$$
\tilde{\Gamma}(x_0;x^{-1}\circ y) = \sum_{m=1}^{\infty}\sum_{k=1}^{g_m} c^{km}(x_0)\frac{Y_{km}(x^{-1}\circ y)}{\|x^{-1}\circ y\|^{Q-2}}.
$$

Choosing $r=2N$ in (14) such that

$$
\sup_{x_0\in G}|c^{km}(x_0)| \le C(N)m^{-4N},
$$

it follows from (12) and (13) that for every $x_0 \in G$,

$$|\tilde{\Gamma}(x_0; x^{-1} \circ y)| \leq \frac{C(N)}{\|x^{-1} \circ y\|^{Q-2}}.$$

Hence, for every $x_0 \in G$, it follows from (23) that

$$\begin{aligned} & \left|\int_\Omega [\phi_\delta(x_0; x, y) - \tilde{\Gamma}(x_0; x^{-1} \circ y)] g(y) dy\right| \\ \leq \;& C(N) \int_\Omega \varphi\left(\frac{\|x^{-1} \circ y\|}{\delta}\right) \left|\frac{1}{\|x^{-1} \circ y\|^{Q-2}} g(y)\right| dy \\ \equiv \;& C(N) R_\delta(g)(x). \end{aligned}$$

Noting that, the function $\delta \mapsto R_\delta(g)(x)$ is non-increasing for every fixed $x \in G$, and for every $x_0 \in G$,

$$\|R_\delta(g)\|_p \leq \|g\|_p \left\| \frac{1}{\|z\|^{Q-2}} \varphi(\frac{\|z\|}{\delta}) \right\|_1 \longrightarrow 0 \text{ as } \delta \to 0.$$

Thus for every $x_0 \in G$, there exists a zero measure set E such that for every $x \in G \backslash E$, we have $R_\delta(g)(x) \to 0$ as $\delta \to 0$, and so for every $x \in G \backslash E$ and every $x_0 \in G$,

$$\lim_{\delta \to 0} \int_\Omega \phi_\delta(x_0; x, y) g(y) dy = \int_\Omega \Gamma(x_0; y^{-1} \circ x) g(y) dy. \tag{25}$$

In the same way, we estimate the next three integrals appearing in the right hand side of (22). Setting

$$\varphi_1(t) = \max\{|\varphi'(s)|, \; s \geq t\},$$

and using the facts that $X_i(\|x\|)$ is homogeneous of degree zero for $i = 1, \ldots, p_0$, and $X_0(\|x\|)$ is homogeneous of degree -1, we obtain that

$$\left|X_{y_i} \varphi\left(\frac{\|x^{-1} \circ y\|}{\delta}\right)\right| = \frac{1}{\delta} \left|\varphi'\left(\frac{\|x^{-1} \circ y\|}{\delta}\right)\right| \cdot |X_{y_i} \|x^{-1} \circ y\|| \leq \frac{c}{\delta} \left|\varphi_1\left(\frac{\|x^{-1} \circ y\|}{\delta}\right)\right|$$

for every $i = 1, \ldots, p_0$. According to the spherical harmonics expansion (11),

$$X_j \tilde{\Gamma}(x_0; x^{-1} \circ y) = \sum_{m=1}^{\infty} \sum_{k=1}^{g_m} c^{j,km}(x_0) \frac{Y_{km}(x^{-1} \circ y)}{\|x^{-1} \circ y\|^{Q-1}}$$

for every $x_0 \in G$ and $j = 1, \ldots, p_0$. Choosing $r = 2N$ in (14) such that

$$\sup_{x_0 \in G} |c^{j,km}(x_0)| \leq C(N) m^{-4N}, \; j = 1, \ldots, p_0,$$

it follows from (12) and (13) that for every $x_0 \in G$ and $j = 1, \ldots, p_0$,

$$|X_j \tilde{\Gamma}(x_0; x^{-1} \circ y)| \leq \frac{C(N)}{\|x^{-1} \circ y\|^{Q-1}}. \tag{26}$$

Similarly, by (23), we can get

$$\lim_{\delta\to 0}\int_\Omega \langle \mathcal{G}(y), D\phi_\delta(x_0;x,y)\rangle dy = \int_\Omega \langle \mathcal{G}(y), D\tilde{\Gamma}(x_0;x^{-1}\circ y)\rangle dy;$$

$$\begin{aligned}&\lim_{\delta\to 0}\int_\Omega \langle [A(x_0)-A(y)]D(\eta u)(y), D\phi_\delta(x_0;x,y)\rangle dy\\ &= \int_\Omega \langle [A(x_0)-A(y)]D(\eta u)(y), D\tilde{\Gamma}(x_0;x^{-1}\circ y)\rangle dy;\end{aligned}$$

$$\begin{aligned}&\lim_{\delta\to 0}\int_\Omega \langle A(x_0)D(v-\eta u)(y), D\phi_\delta(x_0;x,y)\rangle dy\\ =\ &\int_\Omega \langle A(x_0)D(v-\eta u)(y), D\tilde{\Gamma}(x_0;x^{-1}\circ y)\rangle dy,\end{aligned}$$

for every $x\in G\backslash E$ and every $x_0\in G$.

We now deal with the last integral in the right hand side of (22). We claim that for every $x\in G\backslash E$ and for every $x_0\in G$,

$$\int_\Omega a_0(v-\eta u)(y)X_0^*\phi_\delta(x_0;x,y)dy \to T(v-\eta u)(x_0,x) \text{ as } \delta\to 0, \tag{27}$$

where

$$T(v-\eta u)(x_0,x) = \lim_{j\to\infty}\int_{\|x^{-1}\circ y\|\ge\delta_j} a_0(v-\eta u)(y)X_0^*\tilde{\Gamma}(x_0;x^{-1}\circ y)dy, \tag{28}$$

for some sequence $(\delta_j)_{j\in\mathbb{N}}$ such that $\delta_j\to 0$ as $j\to\infty$.

In fact, for every $h\in L^p(G)$,

$$\begin{aligned}\int_G h(y)X_0^*\phi_\delta(x_0;y^{-1}\circ x)dy &= \int_{\|x^{-1}\circ y\|\ge\delta} h(y)X_0^*\tilde{\Gamma}(x_0;x^{-1}\circ y)dy\\ -\ &\int_{\delta/2\le\|x^{-1}\circ y\|\le\delta} h(y)X_0^*\left(\varphi\left(\frac{\|x^{-1}\circ y\|}{\delta}\right)\tilde{\Gamma}(x_0;x^{-1}\circ y)\right)dy.\end{aligned}$$

By using (7), (15) and (16), we have

$$|X_0^*\tilde{\Gamma}(x_0;x^{-1}\circ y)| \le a_0\sup_{\|z\|=1}|X_0^*\Gamma^A(z)|\frac{1}{\|y^{-1}\circ x)\|^Q} \le \frac{c(a_0,Q,G,\mu,c_1)}{\|x^{-1}\circ y)\|^Q},$$

where c_1 is given in (7). Hence, we can find that there exists a bounded function $\hat{\varphi}$ such that $\hat{\varphi}(t)=0$ for $t>1$ and

$$|X_0^*\left(\varphi\left(\frac{\|x^{-1}\circ y\|}{\delta}\right)\tilde{\Gamma}(x_0;x^{-1}\circ y)\right)| \le \frac{1}{\delta^Q}\hat{\varphi}\left(\frac{\|x^{-1}\circ y\|}{\delta}\right)$$

for every $x_0,x,y\in G$ with $\delta/2\le\|x^{-1}\circ y\|\le\delta$. Moreover

$$\int_{\delta/2\le\|x^{-1}\circ y\|\le\delta} X_0^*\left(\varphi\left(\frac{\|x^{-1}\circ y\|}{\delta}\right)\tilde{\Gamma}(x_0;x^{-1}\circ y)\right)dy = 0.$$

Setting

$$T_\delta h(x,x_0) = \int_{\delta/2\le\|x^{-1}\circ y\|\le\delta} h(y) X_0^* \left(\varphi\left(\frac{\|x^{-1}\circ y\|}{\delta}\right)\tilde{\Gamma}(x_0;x^{-1}\circ y)\right)dy,$$

we have

$$\begin{aligned}
&|T_\delta h(x,x_0)| \\
=\ & |\int_{\delta/2\le\|x^{-1}\circ y\|\le\delta} X_0^*\left(\varphi\left(\frac{\|x^{-1}\circ y\|}{\delta}\right)\tilde{\Gamma}(x_0;x^{-1}\circ y)\right)[h(y)-h(x)]dy| \\
\le\ & \frac{1}{\delta^Q}\int_{\delta/2\le\|x^{-1}\circ y\|\le\delta}\hat{\varphi}\left(\frac{\|x^{-1}\circ y\|}{\delta}\right)|h(y)-h(x)|dy \\
\equiv\ & T_\delta^A h(x).
\end{aligned}$$

We claim from Minkowskii's inequality that

$$\|T_\delta^A h\|_p \le \int_{1/2\le\|z\|\le 1}\hat{\varphi}(z)\left(\int_G |h(x\circ D(\delta)z)-h(x)|^p\right)^{1/p} dz \to 0,\ \delta\to 0,$$

and (28) follows.

For convenience, we denote

$$\begin{aligned}
w(x_0,x) = &-\int_\Omega \Gamma(x_0;y^{-1}\circ x)g(y)dy + \int_\Omega\langle G(y), D\tilde{\Gamma}(x_0;x^{-1}\circ y)\rangle dy \\
&+\int_\Omega \langle [A(x_0)-A(y)]D(\eta u)(y), D\tilde{\Gamma}(x_0;x^{-1}\circ y)\rangle dy,
\end{aligned} \tag{29}$$

and

$$S(v-\eta u)(x_0,x) = \int_\Omega \langle A(x_0)D(v-\eta u)(y), D\tilde{\Gamma}(x_0;x^{-1}\circ y)\rangle dy.$$

According to (22), (24)-(29), we derive

$$(\eta u)(x) - w(x_0,x) = (\eta u)(x) - v(x) - S(v-\eta u)(x_0,x) - T(v-\eta u)(x_0,x) \tag{30}$$

for every $x\in G\backslash E$ and every $x_0\in G$.

(ii) We first assert that there exist two operator S^A and T^A such that

$$|S(v-\eta u)(x_0,x)| \le S^A(v-\eta u)(x),\ x_0,\ x\in G, \tag{31}$$

$$\|S^A(v-\eta u)\|_{L^p(G)} \le c\sum_{j=1}^{p_0}\|X_j(v-\eta u)\|_{L^p(G)}, \tag{32}$$

and

$$|T(v-\eta u)(x_0,x)| \le T^A(v-\eta u)(x),\ x\in G\backslash E,\ x_0\in G \tag{33}$$

$$\|T^A(v-\eta u)\|_{L^p(G)} \le c\|v-\eta u\|_{L^p(G)}, \tag{34}$$

for some positive constants c.

In fact, by the uniformly ellipticity condition (3), we have

$$S(v-\eta u)(x_0,x)\le \mu^{-1}\sum_{j=1}^{p_0}\int_\Omega |X_j(v-\eta u)(y)X_j\tilde{\Gamma}(x_0;x^{-1}\circ y)|dy.$$

It follows from (7) and (26) that

$$|X_j\tilde{\Gamma}(x_0;x^{-1}\circ y)|\le \frac{C(N,c_1,Q)}{\|y^{-1}\circ x\|^{Q-1}}$$

for every $x_0\in G$ and $j=1,\dots,p_0$, where c_1 is given in (7). Denoting

$$S^A(v-\eta u)(x)=\int_\Omega \frac{C(N,\mu,p_0,c_1)}{\|y^{-1}\circ x\|^{Q-1}}\sum_{j=1}^{p_0}|X_j(v-\eta u)(y)|dy,$$

it is clear that the operator $S^A(v-\eta u)(x)$ satisfies (31) for every $x_0\in G$ and for every $x\in G$. In view of Lemma 2.2, $S^A(v-\eta u)(x)$ also satisfies (32). In the following, let us consider the expansion in spherical harmonics of the function $X_0^*\tilde{\Gamma}(x_0;\cdot)$:

$$X_0^*\tilde{\Gamma}(x_0;x^{-1}\circ y)=\sum_{m=1}^{\infty}\sum_{k=1}^{g_m}c^{0,km}(x_0)\frac{Y_{km}(x^{-1}\circ y)}{\|x^{-1}\circ y\|^Q}.$$

We can choose $r=2N$ in (14) such that

$$\sup_{x_0\in G}|c^{0,km}(x_0)|\le C(N)m^{-4N}.$$

Denoting

$$\begin{aligned}&T^A(v-\eta u)(x)\\ =\ &\lim_{j\to\infty}\sum_{m=1}^{\infty}\sum_{k=1}^{g_m}a_0C(N)m^{-4N}\left|\int_{\|x^{-1}\circ y\|\ge\delta_j}\frac{Y_{km}(x^{-1}\circ y)}{\|x^{-1}\circ y\|^Q}(v-\eta u)(y)dy\right|,\end{aligned}$$

for every $x,y\in G$ with $x\ne y$, it easily sees that the operator $T^A(v-\eta u)$ satisfies (33) every $x\in G\backslash E$ and for every $x_0\in G$. By [2, Theorem 17] and (12), $T^A(v-\eta u)$ also satisfies (34).

Next, if we set

$$\hat{T}(v-\eta u)=|\eta u(x)-v(x)|+S^A(v-\eta u)(x)+T^A(v-\eta u)(x), \tag{35}$$

then it follows from (30), (31), (33) and (35) that

$$|w(x_0;x)-(\eta u)(x)|\le \hat{T}(v-\eta u)(x),\ x\in G\backslash E,\ x_0\in G,$$

and by (32) and (34), there exists a positive constant c such that

$$\|\hat{T}(v-\eta u)\|_{L^p(G)}\le c\left(\|v-\eta u\|_{L^p(G)}+\sum_{j=1}^{p_0}\|X_j(v-\eta u)\|_{L^p(G)}\right). \tag{36}$$

(iii) Let $\{v_k\}_{k\in\mathbb{N}}$ be a sequence of $C_0^\infty(G)$ functions satisfying

$$\|\eta u - v_k\|_{L^p(G)} \to 0,\ k\to\infty;$$

$$\|X_j(\eta u - v_k)\|_{L^p(G)} \to 0,\ k\to\infty,\ j=1,\ldots,p_0.$$

Denoting

$$\hat{w}(x) = \inf\{\hat{T}(v_k - \eta u)(x) : k\in\mathbb{N}\}$$

and using the results in step (ii), we find a zero measure set $E\subset G$ such that

$$|w(x_0,x) - (\eta u)(x)| \le \hat{w}(x),\ \|\hat{w}\|_{L^p(G)} \le \inf\{\|\hat{T}(v_k-\eta u)\|_{L^p(G)} : k\in\mathbb{N}\},$$

for every $x\in G\backslash E$ and for every $x_0\in G$.

Thus, by using (36) for every function v_k, we conclude that

$$(\eta u)(x) = w(x_0,x) \tag{37}$$

for every $x\in G\backslash E$ and for every $x_0\in G$. Choosing $x_0 = x$ in (37), it follows (21). It completes the proof. ■

Theorem 3.2. *If u is a weak solution of (2), then*

$$\begin{aligned} &X_k(\eta u)(x)\\ &= -\sum_{i,j=1}^{p_0}\lim_{\varepsilon\to 0}\int_{\|y^{-1}\circ x\|\ge\varepsilon} K_{ki}(x;y^{-1}\circ x)[(a_{ij}(x)-a_{ij}(y))X_j(\eta u)(y) + \mathcal{G}_i(y)]dy\\ &+\int_G \Gamma_k(x;y^{-1}\circ x)g(y)dy - \sum_{i=1}^{p_0}\alpha_{ki}(x)\mathcal{G}_i(x), \end{aligned} \tag{38}$$

where $\alpha_{ki}(x) = \int_{\Sigma_N} K_k(x,y)\nu_i(y)d\sigma_y$, ν_k is the outer normal at the unit sphere Σ_N in G, $k=1,\ldots,p_0$.

Proof. It follows from Theorem 3.1 and (37) that

$$\begin{aligned} X_k w(x_0,x) &= \int_G \Gamma_k(x_0;y^{-1}\circ x)g(y)dy\\ &-\sum_{i,j=1}^{p_0}\lim_{\varepsilon\to 0}\int_{\|y^{-1}\circ x\|\ge\varepsilon} K_{ki}(x_0;y^{-1}\circ x)[(a_{ij}(x_0)-a_{ij}(y))X_j(\eta u)(y)+\mathcal{G}_i(y)]dy\\ &-\sum_{i=1}^{p_0}\alpha_{ki}(x_0)\mathcal{G}_i(x) - \sum_{i,j=1}^{p_0}\alpha_{ki}(x_0)(a_{ij}(x_0)-a_{ij}(x))X_j(\eta u)(x) \end{aligned} \tag{39}$$

for every $x_0\in G$ and almost $x\in G$, where $\alpha_{ki}(x_0) = \int_{\Sigma_N} K_k(x_0,y)\nu_i(y)d\sigma_y$, ($\nu_k$ is the outer normal at the set Σ_N, $k,i = 1,\ldots,p_0$) are uniformly bounded for

$x_0 \in G$. For every fixed x_0, let us consider the expansion in spherical harmonics of the terms in (39). In order to simplify the notations we write

$$X_k(\eta u)(x) = X_k w(x_0, x) = \sum_{m=1}^{\infty} b_m(x_0) T_m(g_m)(x), \tag{40}$$

where g_m denotes one of the following functions: g, $\mathcal{G}_i$, $X_j(\eta u)$ or $a_{ij}X_j(\eta u)$, $i, j = 1, \ldots, p_0$, and T_m indicates the convolution of g_m and a suitable homogeneous function. The convergence of the series in (40) is uniform with respect to x_0, then

$$X_k(\eta u)(x) = X_k w(x, x) = \sum_{m=1}^{\infty} b_m(x) T_m(g_m)(x),$$

for almost every $x \in G$. It shows that (38) is valid. ■

Let us define the following singular integral operators

$$K_{ij} f(x) = P.V. \int \Gamma_{ij}(x; y^{-1} \circ x) f(y) dy,$$

where $i, j = 1, \ldots, p_0$ and the commutator

$$C[K, a](f) = K(af) - aK(f),$$

for an operator K and a function $a \in L^\infty(G)$.

Lemma 3.3 ([2]). *For every $p \in (1, \infty)$, there exists a positive constant c such that for every $a \in BMO$, $f \in L^p(G)$, $i, j = 1, \ldots, p_0$,*

$$\|K_{ij} f\|_{L^p(G)} \leq c \|f\|_{L^p(G)};$$

$$\|C[K_{ij}, a] f\|_{L^p(G)} \leq c \|a\|_* \|f\|_{L^p(G)}.$$

We state a local version of Lemma 3.3.

Lemma 3.4. *Let $p \in (1, \infty)$, there exists a positive constant c such that for every $a \in BMO$, $f \in L^p(\Omega)$, $i, j = 1, \ldots, p_0$,*

$$\|K_{ij} f\|_{L^p(\Omega)} \leq c \|f\|_{L^p(\Omega)};$$

$$\|C[K_{ij}, a] f\|_{L^p(\Omega)} \leq c \|a\|_* \|f\|_{L^p(\Omega)}.$$

Lemma 3.5 ([2]). *If the function a belongs to VMO, then for every $\varepsilon > 0$ there exists $r > 0$, depending on ε and the VMO modulus of a, such that for every $f \in L^p(G)$ $(p \in (1, \infty))$ with $\operatorname{supp} f \subset B_r$,*

$$\|C[K_{ij}, a] f\|_{L^p(B_r)} \leq c\varepsilon \|f\|_{L^p(B_r)}$$

for $i, j = 1, 2, \ldots, p_0$.

4. L^p estimates

In this Section, we denote

$$D_0 u = (X_1 u, \dots, X_{p_0} u)$$

and write $\|F\|_{L^p(\Omega)}$, $\|D_0 u\|_{L^p(\Omega)}$, $\|\mathcal{G}\|_{L^p(\Omega)}$ instead of $\sum_{j=1}^{p_0} \|F_j\|_{L^p(\Omega)}$, $\sum_{j=1}^{p_0} \|X_j u\|_{L^p(\Omega)}$, $\sum_{j=1}^{p_0} \|\mathcal{G}_j\|_{L^p(\Omega)}$, respectively. For every $y_0 \in \Omega$ and $r, s > 0$ such that $B_r(y_0) \subset \Omega$ and $s < r$, we will consider the function η defined in (19). For the sake of brevity, we use B_r instead of $B_r(x_0)$ and denote

$$v(x) = \eta(x) u(x).$$

By using (20), we have

$$Lv = div_X(\mathcal{G}) + g.$$

In order to prove Theorem 1.3, we need some preliminary results.

Lemma 4.1. *If u is a weak solution of (2), $u, F \in L^p(B_r)$, $\frac{Q}{Q-1} < p < \infty$ and $D_0 u \in L^q(B_r)(1 < q < Q)$, with $\frac{1}{q} = \frac{1}{p} + \frac{1}{Q}$. Then there exists a positive constant r_0, depending only on the operator L, such that, if $r \le r_0$, then $D_0 u \in L^p_{loc}(B_r)$ and*

$$\|D_0 u\|_{L^p(B_s)} \le c(\|F\|_{L^p(B_r)} + \|u\|_{L^p(B_r)} + \|D_0 u\|_{L^q(B_r)}), \tag{41}$$

for every $s \in (0, r)$, where $c = c(r, s) > 0$.

Proof. We first prove that (41) holds under the hypothesis $D_0 u \in L^p_{loc}(B_r)$. According to Theorem 3.2, it holds

$$X_k v(x) = \sum_{i,j=1}^{p_0} C_{k,i}[a_{i,j}, X_j v](x) + \sum_{i=1}^{p_0} T_{k,i}(\mathcal{G}_i)(x) - Tg(x) - \sum_{i=1}^{p_0} \alpha_{k,i}(x) \mathcal{G}_i(x)$$

for $k = 1, \dots, p_0$. Hence it follows from Lemmas 2.1, 2.2 and 3.4 that

$$\|X_k v\|_{L^p(B_r)} \le c \left(\sum_{i,j=1}^{p_0} \|a_{i,j}\|_* \|X_j v\|_{L^p(B_r)} + \|\mathcal{G}\|_{L^p(B_r)} + \|g\|_{L^q(B_r)} \right),$$

where the norm $\|\cdot\|_*$ is taken over B_r. By $a_{ij} \in VMO$ and Lemma 3.5, there exists $r_0 > 0$ such that

$$\|D_0 v\|_{L^p(B_r)} \le c(\|\mathcal{G}\|_{L^p(B_r)} + \|g\|_{L^q(B_r)}) \tag{42}$$

for every $r \in (0, r_0)$. Since $F, u \in L^p(B_r)$, $D_0 u \in L^q(B_r)$, $p > q$ and η in (19) is a function with support contained in the ball $B_r(x_0)$, it is easy to verify that $\mathcal{G} \in L^p(B_r)$, $g \in L^q(B_r)$ and there exists a positive constant c such that

$$\|\mathcal{G}\|_{L^p(B_r)} \le c(\|u\|_{L^p(B_r)} + \|F\|_{L^p(B_r)}), \tag{43}$$

$$\|g\|_{L^q(B_r)} \le c(\|u\|_{L^p(B_r)} + \|F\|_{L^p(B_r)} + \|D_0 u\|_{L^q(B_r)})). \tag{44}$$

Now (41) immediately follows from (42),(43) and (44).

We next show that the assumption $D_0u \in L^p(B_r)$ can be removed. In fact, let us define the map

$$\tilde{T} : (L^q(B_r))^{p_0} \to (L^q(B_r))^{p_0},$$

by

$$(\tilde{T}U)_k(x) = \sum_{i,j=1}^{p_0} C_{k,i}[a_{i,j}, X_jU](x) + \sum_{i=1}^{p_0} T_{k,i}(\mathcal{G}_i)(x) - Tg(x) - \sum_{i=1}^{p_0} \alpha_{k,i}(x)\mathcal{G}_i(x)$$

for every $U \in (L^q(B_r))^{m_0}$ and $k = 1, \ldots, p_0$.

If r_0 is small enough, then there exists $0 < \varepsilon < 1$ appearing in the L^q estimates for terms $C_{k,i}[a_{i,j}, X_jU]$ such that $\tilde{T}$ is a contraction and so D_0v is its unique fixed point. On the other hand, $\tilde{T}$ is also a contraction map from $(L^p(B_r))^{p_0}$ to $(L^p(B_r))^{p_0}$, thus $\tilde{T}$ has a unique fixed point $U_p \in L^p(B_r)$. Since $L^p(B_r) \subset L^q(B_r)$, the function D_0v must coincide with $U_p \in L^p(B_r)$. It ends the proof. ■

Lemma 4.2. *If u is a weak solution of (2), $u, F \in L^p(B_r)$, $\frac{Q}{Q-1} < p < \infty$. Then there exists a positive constant r_0, depending only on the operator L, such that, if $r \le r_0$, then $D_0u \in L^p_{loc}(B_\sigma)$ for some $s < \sigma < r$ and*

$$\|D_0u\|_{L^p(B_s)} \le c(\|F\|_{L^p(B_r)} + \|u\|_{L^p(B_r)} + \|D_0u\|_{L^{q_0}(B_r)}), \tag{45}$$

for every $s \in (0, r)$ and some $1 < q_0 \le \min(2, p)$, where $c = c(r, s) > 0$.

Proof. If $\frac{Q}{Q-1} < p \le \frac{2Q}{Q-2}$, the result immediately follows from Lemma 4.1. Otherwise we apply the iterative method. Let

$$m = \min\{k \in \mathbb{N} : \frac{k}{Q} \ge \frac{1}{2} - \frac{1}{p}\},\ \delta = \left(\frac{r}{s}\right)^{1/m}$$

and for $h = 1, 2, \ldots, m+1$, let

$$s_h = \delta^{h-1}s,\ r_h = \delta^h s,\ q_h = \frac{pQ}{(h-1)p + Q}.$$

By Lemma 4.1 we derive

$$\|D_0u\|_{L^{q_h}(B_{s_h})} \le c(\|F\|_{L^{q_h}(B_{r_h})} + \|u\|_{L^{q_h}(B_{r_h})} + \|D_0u\|_{L^{q_{h+1}}(B_{r_h})})$$

for every $h = 1, 2, \ldots, m$. Since $s_1 = s$, $r_m = r$ and $q_{m+1} \le 2 < q_m < \ldots < q_1 = p$, we can get (45) from these inequalities and let $q_0 = q_{m+1}$. By applying Lemma 4.1 and let $\sigma = r_1$, we obtain that $D_0u \in L^p_{loc}(B_\sigma)$. The proof is completed. ■

Lemma 4.3 ([8]). *Let ψ be a bounded nonnegative function defined on the interval $[T_0, T_1] (0 \le T_0 < T_1)$. Assume that for any $T_0 \le s \le t \le T_1$, ψ satisfies*

$$\psi(s) \le \theta\psi(t) + \frac{A}{(t-s)^\beta} + B,$$

where $\theta,\ A,\ B,\ \beta$ *are nonnegative constants, and* $\theta < \frac{1}{3}$. *Then*

$$\psi(\rho) \leq c_\beta[\frac{A}{(R-\rho)^\beta} + B],$$

for every ρ *satisfying* $T_0 \leq \rho \leq R \leq T_1$.

Proof of Theorem 1.3. Firstly, by Lemma 4.2, we find that if u is a weak solution of (2) and $u, F \in L^p(\Omega)$ for $\frac{Q}{Q-1} < p < \infty$, then we have $D_0 u \in L^p(\Omega')$ for $\Omega' \subset\subset \Omega$. Otherwise, for $1 < p \leq \frac{Q}{Q-1}$, it is easy to get $D_0 u \in L^p(\Omega')$ by the definition of weak solution.

Next, for any $0 < s < r < r_0$ and $p > q > 1$, we have

$$\begin{aligned}
\|D_0 u\|_{L^q(B_r)} &= \left(\int_{B_r} |D_0 u(x)|^q dx\right)^{1/q} \\
&\leq \left[\left(\int_{B_r} (|D_0 u(x)|^q)^{\frac{p}{q}}\right)^{\frac{q}{p}} \cdot |B_r|^{1-\frac{q}{p}} dx\right]^{1/q} \\
&\leq (r^Q)^{\frac{1}{q}-\frac{1}{p}} \|D_0 u\|_{L^p(B_r)}.
\end{aligned}$$

Then for any $0 < \varepsilon < \frac{1}{3}$, there exists $t > 0$, such that

$$\|D_0 u\|_{L^q(B_r)} \leq \varepsilon \|D_0 u\|_{L^p(B_r)}.$$

Combine with Lemma 4.1, we have

$$\|D_0 u\|_{L^p(B_s)} \leq c(\|F\|_{L^p(B_r)} + \|u\|_{L^p(B_r)} + \varepsilon \|D_0 u\|_{L^p(B_r)}).$$

By Lemma 4.3, we derive

$$\|D_0 u\|_{L^p(B_s)} \leq c(\|F\|_{L^p(B_r)} + \|u\|_{L^p(B_r)}),$$

for any $\frac{Q}{Q-1} < p < \infty$. The case $1 < p < Q$ can be recovered by an elementary duality argument. Finally, (5) follows by using a covering method. It ends the proof. ■

5. Hölder continuity

In this Section, we will derive Theorem 1.4.

Lemma 5.1 ([2]). *Let* $K \in C^1(G \backslash \{0\})$ *be homogenous of degree* $\alpha < 1$ *with respect to the group* $(D(\lambda))_{\lambda>0}$. *Then there exist* $c = c(G, K) > 0$, $M = M(G) > 1$ *such that*

$$|K(x \circ y) - K(x)| + |K(y \circ x) - K(x)| \leq c\|y\| \|x\|^{\alpha-1}$$

for every $x, y \in G$ *with* $\|x\| \geq M\|y\|$. *Here*

$$c = c(G) \cdot \sup_{z \in \Sigma_N} |DK(z)|.$$

By Lemma 5.1, it is easy to conclude the following lemma.

Lemma 5.2. *There exist two constants $c > 0$ and $M > 1$ such that*

$$|\Gamma(x_0; z^{-1} \circ y) - \Gamma(x_0; z^{-1} \circ x)| \le c \frac{\|y^{-1} \circ x\|}{\|x^{-1} \circ z\|^{Q-1}};$$

$$|\Gamma_j(x_0; z^{-1} \circ y) - \Gamma_j(x_0; z^{-1} \circ x)| \le c \frac{\|y^{-1} \circ x\|}{\|x^{-1} \circ z\|^{Q}};$$

$$|\Gamma_{ij}(x_0; z^{-1} \circ y) - \Gamma_{ij}(x_0; z^{-1} \circ x)| \le c \frac{\|y^{-1} \circ x\|}{\|x^{-1} \circ z\|^{Q+1}},$$

for every $x, y, z, x_0 \in G$ such that $\|z^{-1} \circ x\| \ge M \|z^{-1} \circ y\|$ and $i, j = 1, \ldots, p_0$.

Proof of Theorem 1.4. As in Section 4, we prove the statement for the compact set $\overline{B}_s(x_0) \subset \Omega$. It is convenient to write (2) in the following form

$$L_0 u = div_X(JDu) + X_0 u = div_X(\hat{F}),$$

where $\hat{F}_j = F_j + X_j u - \sum_{i=1}^{p_0} a_{ji} X_i u$, $j = 1, \ldots, p_0$, $\hat{F}_j = 0$, $j = p_0 + 1, \ldots, N$ and J is defined by

$$J = \begin{pmatrix} I_{p_0} & 0 \\ 0 & 0 \end{pmatrix}_{N \times N}.$$

Let $r > 0$ be such that $B_r(x_0) \subset \Omega$ and let η be the function defined in (19). If we set

$$v(x) = \eta(x) u(x),$$

then

$$L_0 v = div(\eta \hat{F}) - \langle \hat{F}, D\eta \rangle + u L\eta + 2\langle JDu, D\eta \rangle.$$

By Theorem 1.3, we have

$$\|\hat{F}\|_{L^p(\bar{B}_r)} \le c(\|F\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}). \tag{46}$$

Since u and v coincide in the set $\overline{B}_s$, it is sufficient to prove the statement for the function v. If we denote by $\Gamma^0(x, y) = \Gamma^0(y^{-1} \circ x)$ the fundamental solution of the operator L_0, then the following representation formula for v holds

$$\begin{aligned} v(x) = &\int_G \Gamma^0(x, y)(u(y) L\eta(y) + 2\langle JDu(y), D\eta(y) \rangle - \langle \hat{F}(y), D\eta(y) \rangle) dy \\ &- \sum_{j=1}^{p_0} \int_G K_j^0(x, y) \eta(y) \hat{F}_j(y) dy \equiv v_0(x) - \sum_{j=1}^{p_0} v_j(x), \end{aligned} \tag{47}$$

where $K_j^0(x, y) = \tilde{X_j} \tilde{\Gamma}^0(y^{-1} \circ x)(j = 1, \ldots, p_0)$ is homogeneous of degree $1 - Q$.

Let us consider v_j in (47), for $x, z \in \overline{B}_s$, we derive by Lemma 5.2 that

$$\begin{aligned}
|v_j(x) - v_j(z)| &= |\int_G (K_j^0(x,y) - K_j^0(z,y))\eta(y)\hat{F}_j(y)dy| \\
&\leq \int_{\|x^{-1}\circ y\| \leq M\|x^{-1}\circ z\|} (|K_j^0(x,y)| + |K_j^0(z,y)|)|\eta(y)\hat{F}_j(y)|dy \\
&+ \int_{\|x^{-1}\circ y\| \geq M\|x^{-1}\circ z\|} (|K_j^0(x,y) - K_j^0(z,y)|)|\eta(y)\hat{F}_j(y)|dy \\
&\leq c\int_{\|x^{-1}\circ y\| \leq M\|x^{-1}\circ z\|} \|x^{-1}\circ y\|^{1-Q}|\eta(y)\hat{F}_j(y)|dy \\
&+ c\int_{\|z^{-1}\circ y\| \leq c_0(M+c_1)\|x^{-1}\circ z\|} \|z^{-1}\circ y\|^{1-Q}|\eta(y)\hat{F}_j(y)|dy \\
&+ c\int_{\|x^{-1}\circ y\| \geq M\|x^{-1}\circ z\|} \frac{\|z^{-1}\circ x\|}{\|x^{-1}\circ y\|^Q}|\eta(y)\hat{F}_j(y)|dy \\
&\equiv I_1 + I_2 + I_3.
\end{aligned}$$

Applying the Hölder inequality,

$$\begin{aligned}
|I_1| &\leq \sum_{k=1}^{\infty} \int_{2^{-k}M\|x^{-1}\circ z\| \leq \|x^{-1}\circ y\| < 2^{1-k}M\|x^{-1}\circ z\|} \frac{|\eta(y)\hat{F}_j(y)|}{\|x^{-1}\circ y\|^{Q-1}}dy \\
&\leq \sum_{k=1}^{\infty} \left(\frac{2}{2^{1-k}M\|x^{-1}\circ z\|}\right)^{Q-1} \int_{B_{2^{1-k}M\|x^{-1}\circ z\|}(x)} |\eta(y)\hat{F}_j(y)|dy \\
&\leq c\||\eta\hat{F}_j|\|_{L^p(G)}\|x^{-1}\circ z\|^{\frac{p-Q}{p}} \sum_{k=1}^{\infty}(2^{\frac{p-Q}{p}})^{-k}.
\end{aligned}$$

Since $p > Q$, the above series is convergent. Hence,

$$|I_1| \leq c\||\eta\hat{F}_j|\|_{L^p(G)}\|x^{-1}\circ z\|^{\frac{p-Q}{p}}. \tag{48}$$

Similarly, it follows

$$|I_2| \leq c\||\eta\hat{F}_j|\|_{L^p(G)}\|x^{-1}\circ z\|^{\frac{p-Q}{p}}. \tag{49}$$

Now we have

$$\begin{aligned}
|I_3| &\leq \sum_{k=1}^{\infty} \int_{2^{k-1}M\|x^{-1}\circ z\| \leq \|x^{-1}\circ y\| < 2^{k}M\|x^{-1}\circ z\|} \frac{\|z^{-1}\circ x\|}{\|x^{-1}\circ y\|^Q}|\eta(y)\hat{F}_j(y)|dy \\
&\leq \sum_{k=1}^{\infty} \left(\frac{2}{2^{k}M\|x^{-1}\circ z\|}\right)^{Q} \|x^{-1}\circ z\| \int_{B_{2^{k}M\|x^{-1}\circ z\|}(x)} |\eta(y)\hat{F}_j(y)|dy \\
&\leq c\|\eta\hat{F}_j\|_{L^p(G)}\|x^{-1}\circ z\|^{\frac{p-Q}{p}} \sum_{k=1}^{\infty}(2^{k})^{-\frac{Q}{p}}.
\end{aligned}$$

Since the above series is convergent, it implies

$$|I_3| \leq c\||\eta\hat{F}_j|\|_{L^p(G)}\|x^{-1}\circ z\|^{\frac{p-Q}{p}}. \tag{50}$$

For $x, z \in \overline{B}_s$, it follows from (48), (49) and (50) that

$$|v_j(x) - v_j(z)| \le c\|\hat{F}\|_{L^p(\bar{B}_r)}\|z^{-1} \circ x\|^{1-\frac{Q}{p}},$$

where $j = 1, \ldots, p_0$, c is a positive constant. It follows from (46) that

$$|v_j(x) - v_j(z)| \le c(\|F\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)})\|z^{-1} \circ x\|^{1-\frac{Q}{p}}. \tag{51}$$

With a similar argument, for $x, z \in \overline{B}_s$, we obtain

$$|v_0(x) - v_0(z)| \le c(\|F\|_{L^q(\Omega)} + \|u\|_{L^q(\Omega)})\|z^{-1} \circ x\|^{1-\frac{Q}{p}}, \tag{52}$$

where $\frac{1}{q} = \frac{1}{p} + \frac{1}{Q}$.

Since $q < p$, it follows from (46), (47), (51) and (52) that

$$\frac{|u(x) - u(z)|}{\|z^{-1} \circ x\|^{1-\frac{Q}{p}}} \le c\left(\sum_{i=1}^{p_0} \|F_i\|_{L^p(\Omega)} + \|u\|_{L^p(\Omega)}\right), \ x, z \in \bar{B}_s, \ x \ne z.$$

By using the covering method in [3], we derive (6). The proof is completed. ■

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