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The Problem of Zero Divisors in Convolution Algebras of Supersolvable Lie Groups

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Abstract. We prove a variant of the Titchmarsh convolution theorem for simply connected supersolvable Lie groups, namely we show that the convolution algebras of compactly supported continuous functions and compactly supported finite measures on such groups do not contain zero divisors. This can be also viewed as a topological version of the zero divisor conjecture of Kaplansky.
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1. Introduction

The Titchmarsh convolution theorem asserts that if $f, g \in L^1(\mathbb{R})$ vanish on $(-\infty, 0)$, and $f * g(x) = 0$ for $x \leq T$, then there exist real numbers α and β, such that $\alpha + \beta = T$, and the functions f and g vanish on $[0, \alpha]$ and $[0, \beta]$ respectively. Existence of numerous proofs of this theorem in the literature ([1, 2, 3, 5, 9, 10, 13]) hints at its significance. One of its corollaries states that the convolution algebra $\mathcal{M}_c(\mathbb{R})$ of compactly supported finite complex measures on $\mathbb{R}$ has no zero divisors. It can be proved directly, by noticing that the holomorphic Fourier transform defines an injective homomorphism of $\mathcal{M}_c(\mathbb{R})$ into $H(\mathbb{C})$, the algebra of entire functions on $\mathbb{C}$ with pointwise multiplication. Unlike the original theorem, which can not be neatly adapted to a more general context of topological groups, the formulation of this corollary makes sense for any such group. In [14] it was proved for locally compact abelian groups without nontrivial compact subgroups.

On the other hand, the Kaplansky zero divisor conjecture states that the group algebra $K[G]$ of a discrete torsion-free group G over a field K has no zero divisors. In the book [12] the proofs of this conjecture in the cases of right-orderable, supersolvable, and polycyclic-by-finite groups are presented.

A topological analogue of a torsion-free group is a compact-free group, i.e. a group without nontrivial compact subgroups. In view of the aforementioned results, it seems reasonable to state the following topological zero divisor problem:

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Problem. Let G be a compact-free topological group. Is it true that the convolution algebra $\mathcal{M}_c(G)$ contains no zero divisors?

Since the group algebra $\mathbb{C}[G]$ of G, viewed as a discrete group, naturally embeds into $\mathcal{M}_c(G)$, an affirmative answer to this problem implies that G satisfies the original Kaplansky zero divisor conjecture for fields of characteristic 0.

A general compact-free Lie group is of the form $R \rtimes \widetilde{SL_2}(\mathbb{R})^n$, where R is simply connected and solvable, and $\widetilde{SL_2}(\mathbb{R})$ is the universal cover of $SL_2(\mathbb{R})$ ([11], Theorem 3.2). We may thus break the original problem into the following three questions:

Question 1. Let G be a simply connected solvable Lie group. Can $\mathcal{M}_c(G)$ contain zero divisors?

Question 2. Does $\mathcal{M}_c(\widetilde{SL_2}(\mathbb{R}))$ contain zero divisors?

Question 3. Let G_1 and G_2 be Lie groups such that $\mathcal{M}_c(G_i)$ contain no zero divisors. Can $\mathcal{M}_c(G_1 \rtimes G_2)$ contain zero divisors?

Negative answers to all three of them would yield a positive solution to the topological zero divisor problem for all Lie groups. Actually, since any linear subspace of a Lie algebra containing its commutant is an ideal, a solvable Lie algebra decomposes as a direct sum of a codimension 1 ideal and a 1-dimensional subalgebra. This gives a decomposition of the corresponding Lie group into a semidirect product, hence negative answers to questions 2 and 3 would be sufficient.

In this short paper we try to attack the first of the questions above. We show the following:

Main Theorem. Let G be a connected, simply connected, supersolvable Lie group. Then $\mathcal{M}_c(G)$ has no zero divisors.

Our proof relies on some properties of holomorphic functions of one variable. For solvable, but not supersolvable Lie groups, we would have to consider holomorphic functions of several variables, in which case the proof would break.

It is worth mentioning that the proof of the Kaplansky conjecture in the polycyclic-by-finite situation required completely different methods than the supersolvable case (and supersolvable Lie groups are somewhat analogous to supersolvable discrete groups). This may lead to the supposition that also in the topological zero divisor problem a different approach is required to deal with non-supersolvable Lie groups.

We also remark that there is a variant of the Kaplansky conjecture in characteristic 0, known as the Linnell conjecture ([6, 7]), which states that for a discrete, torsion-free group G and nonzero functions $f \in \mathbb{C}[G]$ and $g \in \ell^2(G)$ the convolution $f * g$ is nonzero. In [8] it was shown that the topological counterpart of this conjecture, with $f \in C_c(G)$ and $g \in L^2(G)$, fails for every nonabelian connected nilpotent Lie group.

A special case of the Main Theorem, with G being the Heisenberg group H_n, was proved and used in [4] to show that some natural representations of the group of contactomorphisms of an arbitrary contact manifold are irreducible. We are currently working on generalizing these results, so that the full power of the Main Theorem could be utilized.

The paper is organized as follows. In Section 2 we discuss the convolution algebras associated to a locally compact group. In Section 3 we show that if $\mathcal{M}_c(G)$ has no zero divisors, then this property passes to all extensions of G by $\mathbb{R}$. Finally, in Section 4 the solution of the zero divisor problem for connected supersolvable Lie groups is presented.

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2. Convolution algebras of locally compact groups

Let G be a topological group. Denote by $\mathcal{M}_c(G)$ the set of all compactly supported finite complex Borel measures on G. The convolution $\mu * \nu$ of measures $\mu, \nu \in \mathcal{M}_c(G)$ is the measure defined by the condition

$$\int f(x)\, d(\mu * \nu)(x) = \iint f(xy)\, d\mu(x) d\nu(y). \tag{1}$$

It is again compactly supported and finite, hence the operation of convolution turns $\mathcal{M}_c(G)$ into an associative algebra. A continuous group homomorphism $q\colon G \to H$ defines a pushforward map $q_*\colon \mathcal{M}_c(G) \to \mathcal{M}_c(H)$, given by $q_*\mu(A) = \mu(q^{-1}(A))$. It is a homomorphism of algebras.

Suppose that $K \leq G$ is a nontrivial compact subgroup. There exists $1 \neq a \in K$, and we have $(\delta_1 - \delta_a) * \lambda_K = 0$, where δ_x is the Dirac measure supported on x, and λ_K is the Haar measure on K. Since the inclusion of K into G is a continuous homomorphism, $\mathcal{M}_c(G)$ has zero divisors. Hence the condition of being compact-free is necessary for $\mathcal{M}_c(G)$ to contain no zero divisors.

¿From now on, suppose that G is locally compact. We may thus fix a left Haar measure $\lambda = \lambda_G$ on G. The modular function $\Delta_G\colon G \to \mathbb{R}_+$ is then defined by $(R_x)_*\lambda = \Delta_G(x) d\lambda$, where $R_x\colon G \to G$ is the right translation by x. The space $C_c(G)$ of compactly supported continuous complex-valued functions on G with convolution

$$f * g(x) = \int f(y) g(y^{-1}x)\, d\lambda(y) \tag{2}$$

is also an algebra. It is embedded in $\mathcal{M}_c(G)$ through the homomorphism $f \mapsto f d\lambda$. We will identify $C_c(G)$ with a subalgebra of $\mathcal{M}_c(G)$ through this embedding.

Lemma 2.1. *The following conditions are equivalent:*

1. *$\mathcal{M}_c(G)$ has no zero divisors,*

2. *$C_c(G)$ has no zero divisors.*

Proof. Suppose that $\mu, \nu \in \mathcal{M}_c(G)$ are nonzero, and $\mu * \nu = 0$. For any $f, g \in C_c(G)$ we then have

$$(f * \mu) * (\nu * g) = 0. \tag{3}$$

But $f * \mu \in C_c(G)$, namely

$$f * \mu(x) = \int f(xy^{-1})\Delta_G(y)\, d\mu(y). \tag{4}$$

Notice, that the assignment $f(y) \mapsto f(y^{-1})\Delta_G(y)$ is a bijection of $C_c(G)$ with itself. In particular, if we set $x = e$ in equation (4), we may choose f in such a way that $f * \mu(e) \neq 0$.

Similarly, $\nu * g \in C_c(G)$, and we may choose g so that $\nu * g \neq 0$. We therefore obtain a pair of zero divisors in $C_c(G)$. The other implication is obvious. ∎

If N is a closed normal subgroup of G, and $q\colon G \to G/N$ is the quotient map, then the measure pushforward map q_* sends the subalgebra $C_c(G)$ into $C_c(G/N)$. It is defined in this case by the formula

$$q_* f(xN) = \int_N f(xn) d\lambda_N(n). \tag{5}$$

3. Extensions by $\mathbb{R}$

Consider an extension $1 \longrightarrow A \longrightarrow G \xrightarrow{q} Q \longrightarrow 1$ of Lie groups, such that $A \cong \mathbb{R}$. Choose a fixed identification of A with $\mathbb{R}$. Denote by ψ_+ and ψ_- the indicator functions of intervals $[0, \infty)$ and $(-\infty, 0]$ in A, respectively. Now define operations γ_+ and γ_- on $C_c(G)$ by

$$\gamma_\pm(f)(x) = \int_A f(xa^{-1})\psi_\pm(a)\, da, \tag{6}$$

In general they do not preserve compact supports, however we have the following.

Lemma 3.1. *Suppose that $f \in C_c(G)$ satisfies $q_* f = 0$. Then the functions $\gamma_\pm(f)$ are also in $C_c(G)$.*

Proof. Denote by K the support of f. Then $\gamma_+(f)$ vanishes outside KA. Take $x \in G$ such that $\gamma_+(f)(x) \neq 0$ and write $x = ka$, where $k \in K$, and $a \in A$. By definition of $\gamma_+(f)$, there exists $b \in [0, \infty)$ such that $f(kab^{-1}) \neq 0$, i.e. $kab^{-1} \in K$. We thus obtain $a - b \in K^{-1}K \cap A \subseteq [-D, D]$, where $D > 0$ is a positive real number. Since $b > 0$, we get $a \geq -D$, hence $\gamma_+(f)$ is supported in $K \cdot [-D, \infty)$. By a similar argument applied to γ_-, we infer that $\gamma_-(f)$ is supported in $K \cdot (-\infty, D]$. But

$$\gamma_+(f)(x) + \gamma_-(f)(x) = \int_A f(xa^{-1})\, da = q_* f(xA) = 0, \tag{7}$$

which implies that in fact both $\gamma_+(f)$ and $\gamma_-(f)$ have supports in $K \cdot [-D, D]$, which is compact. ∎

For $f \in C_c(G)$ and $x \in G$ define ${}_xf \colon A \to \mathbb{C}$ by ${}_xf(a) = f(xa)$. These functions are compactly supported on A, and thus their Fourier transforms are holomorphic. We have

$$ {}_x\gamma_+(f)(b) = \int_A {}_xf(ba^{-1})\psi_+(a)\,da = \int_{-\infty}^{b} {}_xf(t)\,dt, \tag{8} $$

hence ${}_xf = ({}_x\gamma_+(f))'$. If $q_*f = 0$, then, by Lemma 3.1 the Fourier transform of ${}_x\gamma_+(f)$ is defined and the following identity is satisfied:

$$ ({}_xf)^\wedge(\chi) = i\chi({}_x\gamma_+(f))^\wedge(\chi). \tag{9} $$

Also, note that

$$ ({}_xf)^\wedge(0) = \int_A f(xa)\,da = q_*f(xA). \tag{10} $$

Proposition 3.2. *Consider an extension* $1 \longrightarrow A \longrightarrow G \xrightarrow{q} Q \longrightarrow 1$ *of Lie groups such that* $A \cong \mathbb{R}$. *If* $C_c(Q)$ *has no zero divisors, then neither has* $C_c(G)$.

Proof. Assume to the contrary that there exist nonzero $f, g \in C_c(G)$ such that $f * g = 0$. Suppose first, that $q_*g = 0$. By (10), this means that every Fourier transform $({}_xg)^\wedge$ has a zero of order $n_x > 0$ at $\chi = 0$. At least one ${}_xg$ is nonzero, so at least one n_x is finite. Let $n(g) = \min_x n_x$. By Lemma 3.1 and equation (9), each of the functions $\gamma_+^k(g)$, where $k = 1, \dots, n(g)$, is compactly supported, and $n(\gamma_+^k(g)) = n(g) - k$. Therefore $q_*\gamma_+^{n(g)}(g)$ is nonzero. Furthermore, it is straightforward to see that $\gamma_+(f * g) = f * \gamma_+(g)$, hence $\tilde{g} = \gamma_+^{n(g)}(g)$ is a new zero divisor, such that $f * \tilde{g} = 0$ and $q_*\tilde{g} \neq 0$.

For an arbitrary locally compact group H we may define an involution on $C_c(H)$ by $f^*(x) = \Delta_H(x)\overline{f(x^{-1})}$. It satisfies $f^* * g^* = (g * f)^*$, and commutes with homomorphisms induced by quotient maps. In particular, we have $\tilde{g}^* * f^* = (f * \tilde{g})^* = 0$, and $q_*(\tilde{g}^*) = (q_*\tilde{g})^* \neq 0$. We may proceed as before to replace f^* with $\tilde{f}$ such that $q_*\tilde{f} \neq 0$, and $\tilde{g}^* * \tilde{f} = 0$. This leads to a contradiction, since q_* is a homomorphism, so $q_*\tilde{f}$ and $q_*(\tilde{g}^*)$ are nontrivial zero divisors in $C_c(Q)$. ■

4. Supersolvable Lie groups

A real Lie algebra $\mathfrak{g}$ is said to be supersolvable (also completely solvable or triangular), if it contains a complete flag of ideals, i.e. a chain $\mathfrak{g}_0 < \mathfrak{g}_1 < \cdots < \mathfrak{g}_d = \mathfrak{g}$ of ideals of $\mathfrak{g}$ such that $\dim \mathfrak{g}_i = i$. Such an algebra is solvable and exponential (see e.g. [11], Theorem 6.4). A Lie group G is supersolvable if its Lie algebra is supersolvable.

Supersolvability is a property which interpolates between solvability and nilpotency. Obviously, any nilpotent Lie algebra is supersolvable, as its lower central series can be refined to obtain a complete flag of ideals. The simplest example of a Lie group which is supersolvable, but not nilpotent, is the group "$ax + b$" of affine transformations of the real line. Its Lie algebra $\mathfrak{g}$ is spanned by two vectors X, Y such that $[X, Y] = Y$, and $0 \leq \mathbb{R}Y \leq \mathfrak{g}$ is a complete flag of ideals.

Supersolvability is also strictly stronger than solvability. The simplest example of a solvable Lie group which is not supersolvable is the group $\mathrm{Isom}(\mathbb{R}^2)$ of affine isometries of the Euclidean plane. Its Lie algebra is spanned by three vectors X, Y, Z such that $[X,Y]=0$, $[Z,X]=Y$ and $[Z,Y]=-X$. It does not contain a 1-dimensional ideal, and therefore does not admit a complete flag of ideals.

Lemma 4.1. *Let G be a connected supersolvable Lie group. Then G is compact-free if and only if it is simply connected.*

Proof. Let G be simply connected. Its exponential map $\exp\colon \mathfrak{g}\to G$ is then a diffeomorphism. Suppose that $K\leq G$ is a nontrivial compact subgroup, and let $1\neq k\in K$. There exists $0\neq X\in\mathfrak{g}$ such that $k=\exp X$. Since K is compact, the sequence $\exp nX$ has an accumulation point, which is a contradiction, because the sequence nX has no accumulation points in $\mathfrak{g}$.

Now, if G is not simply connected, then it is a quotient of its universal cover $\widetilde{G}$ by a discrete normal subgroup H. If $1\neq h\in H$, then, since $\widetilde{G}$ is exponential, there exists a one-parameter subgroup of $\widetilde{G}$ containing h. It projects onto a compact one-parameter subgroup of G. ■

Lemma 4.2. *The quotient of a connected, simply connected Lie group G by a connected normal subgroup N is simply connected.*

Proof. Since G is simply connected, the quotient homomorphism $q\colon G\to G/N$ has a lift $\tilde{q}$ to the universal cover $\widetilde{G/N}$. It is surjective, because its image contains a neighborhood of 1, and $\widetilde{G/N}$ is connected. Furthermore, its kernel is a subgroup of N, and $\dim N=\dim\ker\tilde{q}$. Since N is connected, we have $\ker\tilde{q}=N$, hence G/N is isomorphic to its universal cover. ■

The proof of the Main Theorem now becomes a mere formality:

Proof. [Proof of the Main Theorem] By Lemma 2.1 we may consider the convolution algebra $C_c(G)$ instead of $\mathcal{M}_c(G)$. We proceed by induction on $d=\dim G$. If $d=1$, then $G\cong\mathbb{R}$ and $C_c(\mathbb{R})$ has no zero divisors. Now, suppose that $d>1$. Let $0=\mathfrak{g}_0<\mathfrak{g}_1<\cdots<\mathfrak{g}_d=\mathfrak{g}$ be a complete flag of ideals in the Lie algebra of G. The ideal $\mathfrak{g}_1$ corresponds to a closed normal subgroup $A\lhd G$, isomorphic to $\mathbb{R}$. The quotient G/A is supersolvable and simply connected by Lemma 4.2, and $\dim G/A<d$. Hence $C_c(G/A)$ contains no zero divisors, and by Proposition 3.2, $C_c(G)$ also has no zero divisors. ■

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