Journal of Lie Theory
Volume **23** (2013) 921–935

On the Isospectral Sixth Order Sturm-Liouville Equation

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Communicated by P. Olver

Abstract. In this paper we investigate families of sixth-order Sturm-Liouville equations having the same spectrum. We factorize the Sturm-Liouville operator as the product of a third order linear differential operator and its adjoint. By reversing the order of the factors we obtain another sixth-order Sturm-Liouville operator which is isospectral with the initial operator. The factorization is possible provided the coefficients of the factors satisfy a system of nonlinear third-order ordinary differential equations so called *principal system*. The coefficients in the factorization products are solutions of the principal system. We study this system by using Lie group of symmetries and we show that it may admit a one or two parameter Lie group of transformations. One of the cases leads to Chazy's equation which admits a three parameter Lie group of transformations. In some cases, we solve the system and obtain an isospectral operator.
Mathematics Subject Classification 2010: 34B24, 70G65.
Key Words and Phrases: Sixth order Sturm-Liouville equation, isospectral, Lie group symmetries.

1. Introduction

We consider a sixth-order Sturm-Liouville equation of the form

$$Ly := -y^{(6)} + (A(z)y'')'' + (B(z)y')' + C(z)y = \lambda y, \quad a \leq z \leq b, \tag{1}$$

with six end conditions which make a self-adjoint problem. If λ is such that this problem has a nontrivial solution, then λ is called an eigenvalue and nontrivial solution for that λ is called an eigenfunction. In equation (1) the interval (a,b) is finite and functions $A(z)$, $B(z)$ and $C(z)$ are in $L^1(a,b)$. Under this assumptions the eigenvalues are bounded below and can be ordered as

$$\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_k \leq \cdots ,$$

where $\lim_{k\to\infty} \lambda_k = \infty$ [9, 8]. The set of all eigenvalues of the operator L is called the spectrum of L and is denote by $\sigma(L)$. Spectral problems for differential equations arise in many different physical applications. Sixth-order sturm-Liouville

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ISSN 0949–5932 / $2.50

problems arise in astrophysics, i.e., the narrow convecting layers bounded by stable layers which are believed to surround A-type stars may be modeled by sixth-order boundary value problems, also this problem arise in hydrodynamic and magnetohydrodynamic stability theory. Eq.(1) is often referred to as the circular ring structure with constraints which has rectangular cross-sections of constant width and parabolic variable thickness. For more detail see [10, 11]. If two such problems have the same spectrum i.e.,

$$\sigma(L) = \sigma(\widehat{L}),$$

then we say that they are isospectral. In this paper we seek sixth - order operators $\widehat{L}$ such that L and $\widehat{L}$ are isospectral. The analysis is based on the fundamental result that if A and B are two linear operators then AB and BA have the same eigenvalues except for zero. For if AB has eigenpair (λ, u) then (λ, Bu) is eigenpair of BA. To apply this to our situation we factorize the operator L as $L = H^*H$. This method first proposed by Poschel and Trubowitz [7, 13] to a second order Sturm-Liouville operator. Ghanbari [6] applied this method on vibrating beams, where he factorized the beam operator as a product of two second order differential operators, one of the factors is the adjoint of the other. Wafo Soh [15] used the factorization given by [6] to find isospectral beams using Lie symmetry methods. In fact resulting system of nonlinear ordinary differential equation in [15] reduced and solved by Lie symmetry methods. In this paper we use the factorization method for sixth-order Sturm-Liouville operator which leads to the principal system of nonlinear third-order ordinary differential equations and we analyze the principal system by Lie symmetry methods.

2. Factorization of sixth-order Sturm-Liouville equation

According to the equation (1) we define the operator L as follows

$$L = -D^6 + D^2(AD^2) + D(BD) + C. \tag{2}$$

Every linear differential operator of the form

$$H(y) = \sum_{k=0}^{n} p_k(z) D^k(y), \tag{3}$$

has the adjoint operator H^* of the form

$$H^*(y) = \sum_{k=0}^{n} D^k[(-1)^k p_k(z) y]. \tag{4}$$

For more detail see [5]. We want to write L as a product of the form $L = H^*H$, where $H = -D^3 + rD^2 + sD + t$. The idea is to find the isospectral sixth-order Sturm-Liouville equations by factorizing the operator L and reversing the factors. Suppose that L can be factorized as follows

$$L = H^*H = [D^3 + rD^2 + (2r' - s)D + (r'' - s' + t)][-D^3 + rD^2 + sD + t]. \tag{5}$$

Comparing (1) and (5) we have nonlinear system:

$$\begin{cases} r' + r^2 + 2s = A \\ -s'' - (rs)' - 3t' - 2rt + s^2 = B \\ t''' + (rt)'' - (st)' + t^2 = C \end{cases} \tag{6}$$

Reversing the factors in the factorization we obtain the operator

$$\widehat{L} = HH^* = -D^6 + D^2(\widehat{A}D^2) + D(\widehat{B}D) + \widehat{C}, \tag{7}$$

where

$$\begin{cases} \widehat{A} = -5r' + r^2 + 2s \\ \widehat{B} = 5r''' + 2(r')^2 - 4rr'' - 4s'' + 3t' + 3rs' - 2rt - 3sr' + s^2 \\ \widehat{C} = -r^{(5)} + s^{(4)} - t''' + rr^{(4)} - rs''' + rt'' + sr''' - ss'' + st' + tr'' - ts' + t^2. \end{cases} \tag{8}$$

The solutions of nonlinear system (6) will produce isospectral operators $\widehat{L}$. We consider a self-adjoint boundary value problem containing the Eq.(1) and six end conditions. These boundary conditions are linear combination of y and derivatives up to order 5 at end points a and b. Corresponding boundary conditions for $\widehat{L}$ are linear combination of $H^*\widehat{y}$ $(\widehat{y} = Hy)$ and derivatives. For example if (λ, y) is an eigenpair of L with end conditions

$$y^{(j)}(a) = y^{(j)}(b) = 0, \qquad j = 0, 1, 2 \tag{9}$$

then $(\lambda, \widehat{y})$ is an eigenpair of $\widehat{L}$ with end conditions

$$(H^*\widehat{y})^{(j)}(a) = (H^*\widehat{y})^{(j)}(b) = 0, \qquad j = 0, 1, 2. \tag{10}$$

Similarly, if $(\lambda, \widehat{y})$ is an eigenpair of $\widehat{L}$ with end conditions

$$\widehat{y}^{(j)}(a) = \widehat{y}^{(j)}(b) = 0, \qquad j = 0, 1, 2 \tag{11}$$

then $(\lambda, H^*\widehat{y})$ is an eigenpair of L with boundary conditions

$$(Hy)^{(j)}(a) = (Hy)^{(j)}(b) = 0, \qquad j = 0, 1, 2. \tag{12}$$

Solving the Eq.(6)-1 for s and substituting into Eq.(6)-2 and Eq.(6)-3 by some calculations, we find a system of nonlinear third-order ordinary differential equation for r and t as follows

$$\begin{cases} r^{(3)} = -\frac{7}{2}(r')^2 - 3rr'' - r'(4r^2 - A) - \frac{r^4}{2} + 6t' + 4rt \\ \qquad\quad +(rA)' + A'' + Ar^2 + 2B - \frac{A^2}{2} \\ t^{(3)} = -rt'' - \frac{3}{2}tr'' - \frac{5}{2}r't' - trr' - \frac{1}{2}r^2t' - t^2 \\ \qquad\quad +\frac{1}{2}At' + \frac{1}{2}tA' + C. \end{cases} \tag{13}$$

This system is called the *principal system*. In the next sections we try to solve the principal system by Lie symmetry methods .

3. Lie group of transformations

In this section we find admitted symmetries of nonlinear *principal system* (13). Consider a system of two ordinary differential equations (ODEs) of order n with dependent variable r,t and independent variable z,

$$\begin{cases} r^{(n)} = F(z,r,t,r',t',\cdots,r^{n-1}) \\ t^{(n)} = G(z,r,t,r',t',\cdots,r^{n-1}) \end{cases} \tag{14}$$

each solution of system lies on the surface defined by the intersection of surfaces

$$\begin{cases} r_n = F(z,r,t,r_1,t_1,\cdots,r_{n-1}) \\ t_n = G(z,r,t,r_1,t_1,\cdots,r_{n-1}) \end{cases} \tag{15}$$

where $r_j = \frac{d^j r}{dz^j}$, $t_j = \frac{d^j t}{dz^j}$.
A vector field

$$X = \xi(z,r,t)\frac{\partial}{\partial z} + \eta_1(z,r,t)\frac{\partial}{\partial r} + \eta_2(z,r,t)\frac{\partial}{\partial t}, \tag{16}$$

is the infinitesimal generator of system (14) if and only if

$$X^{(n)}(r_n - F) = 0,\ X^{(n)}(t_n - G) = 0 \tag{17}$$

when $r_n = F$ and $t_n = G$, where

$$X^{(k)} = \xi(z,r,t)\tfrac{\partial}{\partial z} + \eta_1(z,r,t)\tfrac{\partial}{\partial r} + \eta_2(z,r,t)\tfrac{\partial}{\partial t} + \cdots + \eta_1^{(k)}\tfrac{\partial}{\partial r_k} + \eta_2^{(k)}\tfrac{\partial}{\partial t_k}, \tag{18}$$

with

$$\eta_1^{(k)} = D(\eta_1^{(k-1)}) - r_k D(\xi),\ \eta_2^{(k)} = D(\eta_2^{(k-1)}) - t_k D(\xi), \tag{19}$$

and

$$D = \frac{\partial}{\partial z} + r_1\frac{\partial}{\partial r} + t_1\frac{\partial}{\partial t} + \cdots\ . \tag{20}$$

It is easy to see that $\eta_1^{(k)}$ and $\eta_2^{(k)}$ are a polynomial in $r,t,r_1,t_1,\cdots,r_k,t_k$ with coefficients that are linear homogeneous in ξ,η_1,η_2 and their partial derivatives. If F and G are polynomials in $r,t,r_1,t_1,\cdots,r_k,t_k$, then Eqs.(17) are polynomials in $r,t,r_1,t_1,\cdots,r_{n-1},t_{n-1}$, by using (15). Therefore, the coefficients of these polynomial equation must vanish. This yields to system of linear homogeneous PDEs for ξ,η_1 and η_2. This linear system defines the set of determining equations for the point symmetries admitted by system (14) see [2, 12]. The goal is to classify the point symmetries of principal system (13) in terms of the functions $A(z), B(z)$ and $C(z)$. Specifically, the determining equations are to be solved for ξ,η_1,η_2, as well as $A(z), B(z)$ and $C(z)$. Applying the conditions (17) for principal system (13) and solving determining equations we have the following theorem:

Theorem 3.1. *The point symmetries of principal system (13) are generated by*

$$X = a\left(z\frac{\partial}{\partial z} - r\frac{\partial}{\partial r} - 3t\frac{\partial}{\partial t}\right) + b\frac{\partial}{\partial z},$$

if A, B *and* C *are as follows*

$$A = \frac{k_1}{(z+\lambda)^2}, \quad B = \frac{k_2}{(z+\lambda)^4}, \quad C = \frac{k_3}{(z+\lambda)^6},$$

where a, b *and* k_i *are constants and* $\lambda = \frac{b}{a}$.

Proof. For principal system (13) the symmetry conditions (17) are

$$\begin{array}{c} X^{(3)}\{r_3 + \frac{7}{2}(r_1)^2 + 3rr_2 + r_1(4r^2 - A) + \frac{r^4}{2} - 6t_1 \\ -4rt - rA' - r_1 A - A' - Ar^2 - 2B + \frac{A^2}{2}\} = 0 \end{array} \tag{21}$$

and

$$\begin{array}{c} X^{(3)}\{t_3 + rt_2 + \frac{3}{2}tr_2 + \frac{5}{2}r_1 t_1 + trr_1 + \frac{1}{2}r^2 t_1 \\ +t^2 - \frac{1}{2}At_1 - \frac{1}{2}tA' - C\} = 0 \end{array} \tag{22}$$

which leads to

$$\begin{array}{l} \xi(-2r_1A' - rA'' - r^2A' - A''' - 2B' + AA') \\ +\eta_1(3r_2 + 8rr_1 + 2r^3 - 4t - A' - 2Ar) - 4r\eta_2 \\ +\eta_1^{(1)}(7r_1 + 4r^2 - 2A) - 6\eta_2^{(1)} + 3r\eta_1^{(2)} + \eta_1^{(3)} = 0 \end{array} \tag{23}$$

and

$$\begin{array}{l} \xi(-\frac{1}{2}A''t - \frac{1}{2}t_1A' - C') + \eta_1(t_2 + rt_1 + tr_1) \\ +\eta_2(\frac{3}{2}r_2 + rr_1 + 2t - \frac{1}{2}A') + \eta_1^{(1)}(\frac{5}{2}t_1 + rt) \\ +\eta_2^{(1)}(\frac{5}{2}r_1 + \frac{1}{2}r^2 - \frac{1}{2}A) + \frac{3}{2}t\eta_1^{(2)} + r\eta_2^{(2)} + \eta_2^{(3)} = 0, \end{array} \tag{24}$$

whenever (13) is satisfied and $\eta_1^{(k)}$, $\eta_2^{(k)}$ are computed from (19). After replacing r_3 and t_3 in Eq.(23) and Eq.(24) that appear in $\eta_1^{(3)}$ and $\eta_2^{(3)}$ by the right- hand side of principal system (13) we obtain polynomial equations in r, t, r_1, t_1, r_2, t_2. Since we can assign arbitrary values to each of r, t, r_1, t_1, r_2, t_2 at any fixed value of z, it follows that the coefficients of each monomial term in (23) and (24) must vanish since this polynomial equations must hold for arbitrary values of $z, r, t, r_1, t_1, r_2, t_2$. This leads to a system of linear homogeneous PDEs for $\xi(z,r,t), \eta_1(z,r,t)$ and $\eta_2(z,r,t)$. This leads to the following equations:

$$\xi = az + b, \quad \eta_1 = -ar, \quad \eta_2 = -3at, \tag{25}$$

$$\begin{array}{c} -2A'\xi + 7\frac{\partial \eta_1}{\partial z} - 2A\frac{\partial \eta_1}{\partial r} + 2A\frac{\partial \xi}{\partial z} - 6\frac{\partial \eta_2}{\partial r} \\ +3\frac{\partial^3 \eta_1}{\partial r \partial z^2} - \frac{\partial^3 \xi}{\partial z^3} + 2A\left(\frac{\partial \eta_1}{\partial r} - 3\frac{\partial \xi}{\partial z}\right) = 0, \end{array} \tag{26}$$

$$A'''\xi - 2B'\xi + AA'\xi - 4A''\frac{\partial \xi}{\partial z} - 8B\frac{\partial \xi}{\partial z} + 2A^2\frac{\partial \xi}{\partial z} = 0, \tag{27}$$

$$\xi C' + (\frac{\partial \eta_2}{\partial t} - 3\frac{\partial \xi}{\partial z})C = 0. \tag{28}$$

Taking ξ, η_1 and η_2 from (25) and replacing in Eqs.(26)-(28) we obtain

$$A = \frac{k_1}{(z+\lambda)^2}, \quad B = \frac{k_2}{(z+\lambda)^4}, \quad C = \frac{k_3}{(z+\lambda)^6}, \tag{29}$$

The corresponding infinitesimal generator X is

$$X = a\left(z\frac{\partial}{\partial z} - r\frac{\partial}{\partial r} - 3t\frac{\partial}{\partial t}\right) + b\frac{\partial}{\partial z}, \tag{30}$$

where $a, b,\ k_i$ are constants and $\lambda = \frac{b}{a}$. ■

In fact (30) is infinitesimal generator of (13) if A, B and C are of the form (29). In case 1 in section 4 we use infinitesimal generator (30) to solve principal system (13).

4. Isospectral operators $\widehat{L}$

In this section we use the Lie group of symmetries to find solutions of the principal system (13) and obtain isospectral operator $\widehat{L}$.

Case 1. Suppose that A, B and C are given by (29). By Theorem 3.1 the corresponding infinitesimal generator is

$$X = (az+b)\frac{\partial}{\partial z} - ar\frac{\partial}{\partial r} - 3at\frac{\partial}{\partial t}, \tag{31}$$

for arbitrary constants a, b. Thus the system (13) admits two-parameter Lie group of transformations

$$\begin{cases} z^* = e^{\epsilon_1} z + \epsilon_2 \\ r^* = e^{-\epsilon_1} r \\ t^* = e^{-3\epsilon_1} t \end{cases} \tag{32}$$

For $a = 0$, the resulting invariant solution is $r = C_1, t = C_2$ for any constants C_1 and C_2. For $a \neq 0$, let $\lambda = \frac{b}{a}$. Invariant solutions satisfy

$$r' = \frac{r}{z+\lambda}, \quad t' = \frac{-3t}{z+\lambda},$$

which leads to

$$r = \frac{c_1}{(z+\lambda)}, \quad t = \frac{c_2}{(z+\lambda)^3}, \tag{33}$$

where c_1, c_2 are arbitrary constants. Substituting (33) in the principal system (13) we have

$$\begin{aligned} &c_1^4 - 8c_1^3 + (19 - 2k_1)c_1^2 + (8k_1 - 12)c_1 + 36c_2 - 8c_1c_2 - 4k_2 + k_1^2 = 0 \\ &c_2(2c_2 - 5c_1^2 + 45c_1 - 120 + 5k_1) - 2k_3 = 0. \end{aligned} \tag{34}$$

For arbitrary constants k_i, solving (34) we obtain c_1 and c_2. Then from (33) we find r, t and by (8) we obtain an isospectral operator $\widehat{L}$.

For $k_3 = 0$, the system (34) can be solved explicitly for c_1 and c_2 by Maple. From Eq.(34)-1 we find

$$c_2 = \frac{1}{4}\frac{{c_1}^4 - 8{c_1}^3 + 19{c_1}^2 - 2{c_1}^2 k_1 + 8c_1 k_1 - 12c_1 - 4k_2 + {k_1}^2}{-9 + 2c_1}, \tag{35}$$

and Eq.(34)-2 can be factorized into a product of 4th degree equations for c_1. Solving these equations by Maple we obtain

$$(c_1, c_2) \in \{(2+\alpha, 0), (2-\alpha, 0), (2+\beta, 0), (2-\beta, 0)\}, \tag{36}$$

$$c_1 \in \{7+\alpha, 7-\alpha, 7+\beta, 7-\beta\}, \tag{37}$$

where

$$\alpha = \frac{1}{2}\sqrt{10 + 2\sqrt{9 - 12k_1 + 16k_2} + 4k_1}, \quad \beta = \frac{1}{2}\sqrt{10 - 2\sqrt{9 - 12k_1 + 16k_2} + 4k_1},$$

and c_2 corresponding to c_1 in (37) can be found from (35). Using (33) and (6)-1 we find

$$s = \frac{c_1 - c_1^2}{2(z+\lambda)^2} \tag{38}$$

and by (8) we obtain

$$\begin{cases} \widehat{A} = \dfrac{6c_1}{(z+\lambda)^2} \\ \widehat{B} = -\dfrac{156c_1 + 72c_2}{4(z+\lambda)^4} \\ \widehat{C} = \dfrac{-36c_1^3 + 3c_1c_2 + c_1^2c_2 + 2c_2^2 + 228c_2 + 324c_1}{2(z+\lambda)^6} \end{cases} \tag{39}$$

In summary, we have the following theorem.

Theorem 4.1. *The operator*

$$L = -D^6 + D^2\left(\frac{k_1}{(z+\lambda)^2}D^2\right) + D\left(\frac{k_2}{(z+\lambda)^4}D\right)$$

is isospectral with the operator $\widehat{L}$, *where coefficients* $\widehat{A}, \widehat{B}$ *and* $\widehat{C}$ *are obtained by Eqs.(39), and* c_1, c_2 *are given by (36) and (37).*

For instance if $k_1 = k_2 = k_3 = 0$, then we find

$$(c_1, c_2) \in \{(5,10), (6,15), (8,40), (9,60), (4,0), (3,0), (1,0), (0,0)\}. \tag{40}$$

For example, for $(c_1, c_2) = (5, 10)$ we have

$$\widehat{A} = \frac{30}{(z+\lambda)^2}, \qquad \widehat{B} = \frac{-375}{(z+\lambda)^4}, \qquad \widehat{C} = 0$$

and operator $L=-D^6$ is isospectral with

$$\widehat{L}=-D^6+D^2\left(\frac{30}{(z+\lambda)^2}D^2\right)-D\left(\frac{375}{(z+\lambda)^4}D\right),$$

and for $(c_1,c_2)=(6,15)$, the operator $L=-D^6$ is isospectral with

$$\widehat{L}=-D^6+D^2\left(\frac{36}{(z+\lambda)^2}D^2\right)-D\left(\frac{504}{(z+\lambda)^4}D\right)-\frac{576}{(z+\lambda)^6}.$$

Case 2. ($A=B=C=0$ and $t=0$). It is obvious that $t=0$ satisfies in Eq.(13)-2. Therefore, we have nonlinear equation

$$r'''=-3rr''-\frac{7}{2}(r')^2-4r^2r'-\frac{r^4}{2}. \tag{41}$$

Changing variables $z=-\frac{2}{3}x$ and $r=y$, the Eq.(41) can be transformed to

$$y'''=2yy''-3(y')^2+\frac{4}{27}(6y'-y^2)^2, \tag{42}$$

which is called Chazy's equation with parameter $\alpha=\frac{4}{27}$. This equation admits infinitesimal generators

$$X_1=\frac{\partial}{\partial x},\quad X_2=x\frac{\partial}{\partial x}-y\frac{\partial}{\partial y},\quad X_3=x^2\frac{\partial}{\partial x}-(2xy+6)\frac{\partial}{\partial y},$$

for more detail see [4]. General form of Chazy's equation is

$$y'''=2yy''-3(y')^2+\alpha(6y'-y^2)^2 \tag{43}$$

and general solution of (43) is determined by the following theorem.

Theorem 4.2. *[3] Assume that ϕ and ψ are two arbitrary linearly independent solutions of the hypergeometric equation*

$$t(1-t)\frac{d^2\chi}{dt^2}+\left(\frac{1}{2}-\frac{7}{6}t\right)\frac{d\chi}{dt}-\sigma\chi=0,\qquad \sigma=\frac{1}{144(1-9\alpha)}, \tag{44}$$

then the general solution of the Eq.(43) is given in parametric form by

$$x=\frac{\phi(t)}{\psi(t)},\qquad y=\frac{6}{\psi}\frac{d\psi}{dx}=\frac{6}{\psi}\frac{d\psi}{dt}\frac{dt}{dx} \tag{45}$$

For Eq.(42), we have $\sigma=-\frac{1}{48}$. Using [1] we find

$$\chi(t)=c_1\,{}_2\tilde{F}_1(\frac{1}{4},-\frac{1}{12},\frac{2}{3},1-t)+c_2(1-t)^{\frac{1}{3}}\,{}_2\tilde{F}_1(\frac{1}{4},\frac{7}{12},\frac{4}{3},1-t) \tag{46}$$

with exact formulas [14]

$${}_2\tilde{F}_1\left(\frac{1}{4},-\frac{1}{12},\frac{2}{3},\frac{\tau(\tau+4)^3}{4(2\tau-1)^3}\right)=(1-2\tau)^{-\frac{1}{4}}, \tag{47}$$

$${}_2\tilde{F}_1\left(\frac{1}{4},\frac{7}{12},\frac{4}{3},\frac{\tau(\tau+4)^3}{4(2\tau-1)^3}\right)=\frac{1}{1+\frac{1}{4}\tau}(1-2\tau)^{\frac{3}{4}}. \tag{48}$$

By parametrization

$$1-t=\frac{\tau(\tau+4)^3}{4(2\tau-1)^3} \tag{49}$$

we obtain

$$\chi(\tau)=(c_1+c_2\tau^{\frac{1}{3}})(1-2\tau)^{\frac{-1}{4}}. \tag{50}$$

Therefore, two arbitrary linearly independent solutions ϕ and ψ are

$$\phi(\tau)=(c_1+c_2\tau^{\frac{1}{3}})(1-2\tau)^{\frac{-1}{4}}, \quad \psi(\tau)=(c_3+c_4\tau^{\frac{1}{3}})(1-2\tau)^{\frac{-1}{4}}. \tag{51}$$

Wronskian ϕ and ψ is

$$W(\phi,\psi)=-\frac{c_2c_3-c_4c_1}{3\tau^{\frac{2}{3}}(1-2\tau)^{\frac{1}{2}}}, \tag{52}$$

and to be linear independent we assume that $c_2c_3-c_4c_1$ equal to a nonzero constant and therefore the solution of Eq.(42) has three arbitrary constants. Here we consider $c_2c_3-c_4c_1=1$. By Theorem 4.2 we obtain

$$x=\frac{c_1+c_2\tau^{\frac{1}{3}}}{c_3+c_4\tau^{\frac{1}{3}}} \tag{53}$$

$$y=\frac{3(c_3+c_4\tau^{\frac{1}{3}})(3c_3\tau^{\frac{2}{3}}+c_4(2-\tau))}{(1-2\tau)}. \tag{54}$$

Solving Eq.(53) for τ and substituting into (54) we find

$$y=\frac{3[3c_3(c_3x-c_1)^2(c_4x-c_2)+2c_4(c_4x-c_2)^3+c_4(c_3x-c_1)^3]}{(c_4x-c_2)[2(c_1-c_3x)^3-(c_4x-c_2)^3]}. \tag{55}$$

$$r=\frac{18c_3(3c_3z+2c_1)^2(3c_4z+2c_2)+12c_4(3c_4z+2c_2)^3+6c_4(3c_3z+2c_1)^3}{(3c_4z+2c_2)[2(2c_1+3c_3z)^3+(3c_4z+2c_2)^3]}, \tag{56}$$

and

$$s=-\frac{r^2+r'}{2}. \tag{57}$$

Thus we have the following result.

Theorem 4.3. *The operator $L=-D^6$ is isospectral with the operator $\widehat{L}$, where*

$$\begin{aligned}\widehat{A}&=-6r',\\ \widehat{B}&=7r'''+\tfrac{31}{4}r'^2-\tfrac{3}{2}rr''-r^2r'+\tfrac{r^4}{4},\\ \widehat{C}&=-\tfrac{3}{2}r^{(5)}-3r''^2-\tfrac{17}{4}r'r'''+\tfrac{7}{2}rr'r''+\tfrac{3}{4}r^2r'''\\ &\quad+\tfrac{1}{2}(r^2r'^2+r^3r''+r'^3+rr^{(4)}),\end{aligned}$$

and r is given by (56).

For example with $c_2 = c_3 = 1, c_1 = c_4 = 0$ we obtain

$$\begin{aligned}
\widehat{A} &= \frac{486z(27z^3-8)}{(27z^3+4)^2},\\
\widehat{B} &= -\frac{26244z^2(729z^6-5616z^3+448)}{(27z^3+4)^4},\\
\widehat{C} &= \frac{78732(531441z^{12}-15746400z^9+6683472z^6-335232z^3+1024)}{(27z^3+4)^6}.
\end{aligned}$$

Case 3. ($A \neq 0, B \neq 0, C = 0, t = 0$). In this case for $t \neq 0$ the principal system (13) has generator (30) if A, B are of the form (29). This arises in case 1 with $k_3 = 0$, but for $t = 0$ the principal system (13) leads to

$$\begin{aligned}
r^{(3)} = &-\tfrac{7}{2}(r')^2 - 3rr'' - r'(4r^2 - A) - \tfrac{r^4}{2}\\
&+(rA)' + A'' + Ar^2 + 2B - \tfrac{A^2}{2}.
\end{aligned} \tag{58}$$

By procedure mentioned in section 3, Eq. (58) admits the infinitesimal generator

$$X = \xi(r,z)\frac{\partial}{\partial z} + \eta(r,z)\frac{\partial}{\partial r}, \tag{59}$$

if and only if

$$\begin{aligned}
X^{(3)}\{&r^{(3)} + \tfrac{7}{2}(r')^2 + 3rr'' + r'(4r^2 - A) + \tfrac{r^4}{2}\\
&-(rA)' - A'' - Ar^2 - 2B + \tfrac{A^2}{2}\} = 0,
\end{aligned} \tag{60}$$

whenever Eq.(58) is satisfied, where

$$X^{(3)} = \xi\frac{\partial}{\partial z} + \eta\frac{\partial}{\partial r} + \eta^{(1)}\frac{\partial}{\partial r'} + \eta^{(2)}\frac{\partial}{\partial r''} + \eta^{(3)}\frac{\partial}{\partial r'''}, \tag{61}$$

and

$$\eta^{(1)} = \eta_z + (\eta_r - \xi_z)r' - \xi_r(r')^2, \tag{62}$$

$$\begin{aligned}
\eta^{(2)} = \eta_{zz} &+ (2\eta_{rz} - \xi_{zz})r' + (\eta_{rr} - 2\xi_{rz})(r')^2\\
&-\xi_{rr}(r')^3 + (\eta_r - 2\xi_z)r'' - 3\xi_r r' r'',
\end{aligned} \tag{63}$$

$$\begin{aligned}
\eta^{(3)} = \eta_{zzz} &+ (3\eta_{rzz} - \xi_{zzz})r' + 3(\eta_{rrz} - \xi_{rzz})(r')^2\\
&+(\eta_{rrr} - 3\xi_{rrz})(r')^3 - \xi_{rrr}(r')^4 + 3(\eta_{rz} - \xi_{zz})r''\\
&+3(\eta_{rr} - 3\xi_{rz})r'r'' - 6\xi_{rr}(r')^2 r''\\
&-3\xi_r(r'')^2 + (\eta_r - 3\xi_z)r''' - 4\xi_r r' r'''.
\end{aligned} \tag{64}$$

Eq.(60) is equivalent to

$$\begin{aligned}
&\xi(-2r'A' - rA'' - A''' - r^2A' - 2B' + AA')\\
&+\eta(3r'' + 8rr' + 2r^3 - A' - 2rA) + \eta^{(1)}(7r' + 4r^2 - 2A)\\
&+\eta^{(2)}(3r) + \eta^{(3)} = 0,
\end{aligned} \tag{65}$$

when Eq.(58) is satisfied. After Eq.(58) is used to eliminate r''' in $\eta^{(3)}$ in (65) we find a polynomial equation in r, r', r''. Therefore, the coefficient of this polynomial equation must vanish. This yields to system of linear homogeneous PDEs for ξ and η. Putting the coefficients of r''^2 and $r'r''$ in Eq.(65) equal to zero, we find

$$\xi_r = 0, \quad \eta_{rr} = 0.$$

Hence by setting the coefficients of r'' in Eq.(65) equal to zero, we obtain

$$\xi = -a(z), \quad \eta = a'(z)r - 2a''(z),$$

and the corresponding infinitesimal generator is

$$X = -a(z)\frac{\partial}{\partial z} + (a'(z) - 2a''(z))\frac{\partial}{\partial r}, \tag{66}$$

where a is an arbitrary smooth function of z. If we put the coefficients of r' in Eq.(65) equal to zero, we have

$$aA' + 2a'A - 5a^{(3)} = 0. \tag{67}$$

Setting the independent terms of r, r', r'' equal to zero we obtain

$$B' + \frac{4a'}{a}B = \frac{a'}{a}A^2 + \frac{AA'}{2} - \frac{a''}{a}A' - \frac{2a'}{a}A'' - \frac{2a^{(3)}}{a}A - \frac{A^{(3)}}{2} + \frac{a^{(5)}}{a}. \tag{68}$$

For $a \neq 0$ Eq.(67) and Eq.(68) are first order linear differential equations for A and B, respectively. Thus we obtain

$$A = \frac{-5(a')^2 + 10aa'' + 2c_1}{2a^2}, \tag{69}$$

$$\begin{aligned} B \;=\; & \frac{81a'^4 + 12a'^2(3c_1 - 17aa'') + 72a^2a'a^{(3)}}{16a^4} \\ & + \frac{c_1^2 + 4c_2 - 6c_1aa'' + 21a^2a''^2 - 6a^3a^{(4)}}{4a^4} \end{aligned} \tag{70}$$

where c_1 and c_2 are arbitrary constants. We use differential invariants of generator (66) to reduce order of the Eq.(58). First extension of the infinitesimal generator X is

$$X^{(1)} = -a\frac{\partial}{\partial z} + (a'r - 2a'')\frac{\partial}{\partial r} + (2a'r' - 2a^{(3)} + a''r)\frac{\partial}{\partial r'}. \tag{71}$$

Invariants of the first extension are solutions of characteristic equation

$$\frac{dz}{-a(z)} = \frac{dr}{a'(z) - 2a''(z)} = \frac{dr'}{(2a'r' - 2a^{(3)} + a''r)}. \tag{72}$$

These solutions are

$$u = ar - 2a', \quad v = a^2r' + aa'r - 2aa''. \tag{73}$$

Thus ODE (58) reduces to the second order ODE

$$2v^2\frac{d^2v}{du^2} = -6uv\frac{dv}{du} - 2v\left(\frac{dv}{du}\right)^2 - u^4 - 8u^2v - 7v^2 + 2c_1u^2 + 4c_1v + 4c_2, \quad (74)$$

for more detail see Appendix. For $c_1 \neq 0$ or $c_2 \neq 0$ Eq.(74) has infinitesimal generator $X = 0$. Therefore in this case Eq.(74) has no Lie symmetry group. We assume that $c_1 = 0, c_2 = 0$. In this case Eq.(74) has infinitesimal generator $X = k_1u\frac{\partial}{\partial u} + 2k_1v\frac{\partial}{\partial v}$ and the corresponding Lie group of transformation is

$$u^* = e^{\epsilon}u, \qquad v^* = e^{2\epsilon}v. \quad (75)$$

Invariant solution satisfy

$$\frac{du}{u} = \frac{dv}{2v},$$

which leads to

$$v = ku^2, \quad (76)$$

where k is a constant. Substituting (76) into (74) and solving the resulting equation for k we obtain

$$k \in \left\{-1, -\frac{1}{3}, -\frac{1}{4}\right\}. \quad (77)$$

Using (73) and (76) we have

$$\frac{d}{dz}\left(r - 2\frac{a'}{a}\right) + \frac{a'}{a}\left(r - 2\frac{a'}{a}\right) = k\left(r - 2\frac{a'}{a}\right)^2. \quad (78)$$

The natural change of variable $w = r - 2\frac{a'}{a}$, Eq.(78) becomes Bernoulli's equation

$$w' + \frac{a'}{a}w = kw^2. \quad (79)$$

This Bernoulli's equation has solution

$$w = \frac{1}{C_1a - ka\int_0^z a^{-1}dt}. \quad (80)$$

Thus

$$r = \frac{1}{C_1a - ka\int_0^z a^{-1}dx} + 2\frac{a'}{a}, \quad k \in \{-1, -\frac{1}{3}, -\frac{1}{4}\}, \quad (81)$$

where C_1 is an arbitrary constant. Summarizing the previous results we come to the following theorem.

Theorem 4.4. *The operator* $L = -D^6 + D^2(AD^2) + D(BD)$ *, where*

$$A = -\frac{5(a')^2 - 10aa''}{2a^2}$$

and B *is given by (70), is isospectral with the operator* $\widehat{L}$*, where* $\widehat{A}, \widehat{B}$ *and* $\widehat{C}$ *are obtained by (8) ,* r *is given by Eq.(81),* $s = -\frac{r^2+r'-A}{2}$ *and* $t = 0$*.*

For example with $k=1, C_1=0$ and $a=e^z$, we obtain

$$A=\frac{5}{2},\ B=\frac{9}{16},\ r=\frac{2e^z-1}{e^z-1},\ s=-\frac{3(e^z+1)}{4(e^z-1)}$$

and the coefficients of the isospectral operator $\widehat{L}$ are

$$\begin{aligned}\widehat{A} &= \frac{5e^{2z}+2e^z+5}{2(e^z-1)^2},\quad \widehat{B}=-\frac{2\cosh(z)-9\cosh^2(z)+151}{64\sinh^4(z/2)}\\ \widehat{C} &= \frac{110\cosh(z)-15\cosh^2(z)+193}{128\sinh^6(z/2)}.\end{aligned}$$

Note that the Eq.(58) by change of variable $z=-x$ and replacing A with $-A$ becomes

$$\begin{array}{r}r^{(3)}=\frac{7}{2}(r')^2+3rr''-r'(2r^2+A)+\frac{r^4}{2}\\ -(rA)'+A''+Ar^2-2B+\frac{A^2}{2}.\end{array} \tag{82}$$

The Eq.(82) is the same principal equation which reduced and solved using Lie symmetry method by Wafo Soh [15].

5. Conclusion

In this paper, we developed factorization method by using Lie group of transformations for finding sixth-order isospectral operators. Factorization of operator L leads to the nonlinear principal system (13). The principal system admits a two-parameter Lie group of transformations. Using this fact in case 1 we obtained a class of isospectral operators. In case 2 with $t=0$ the principal system is transformed to Chazy's equation which admits a three parameters group of transformations. Using the general solution of Chazy's equation given by [4], we find another class of isospectral operators. In case 3 we have a nonlinear third-order ordinary differential equation. This nonlinear equation with differential invariants reduced and solved. Therefore we produced sixth-order isospectral operators in case 3.

6. Appendix

Reduction of ODE (58): Invariants of the first extension of the generator (66), are

$$u=ar-2a',\quad v=a^2r'+aa'r-2aa''.$$

Then

$$r=\frac{u+2a'}{a},\quad r'=\frac{v-aa'r+2aa''}{a^2}, \tag{83}$$

and

$$\frac{du}{dz}=a'r+ar'-2a''=\frac{v}{a}.$$

Hence

$$r'' = \frac{dr'}{dz} = -\frac{3a'}{a^3}v + \frac{v}{a^3}\frac{dv}{du} - \frac{a''}{a^2}u + \frac{2a'^2}{a^3}u - 6\frac{a'a''}{a^2} + 4\frac{a'^3}{a^3} + 2\frac{a'''}{a}, \tag{84}$$

and

$$\begin{aligned} r''' = {} & \tfrac{1}{a^4}v^2\tfrac{d^2v}{du^2} + \tfrac{1}{a^4}v(\tfrac{dv}{du})^2 - 6\tfrac{a'}{a^4}v\tfrac{dv}{du} + (-4\tfrac{a''}{a^3} + 11\tfrac{a'^2}{a^4})v \\ & +(-\tfrac{a'''}{a^2} + 6\tfrac{a'a''}{a^3} - 6\tfrac{a'^3}{a^4})u - 8\tfrac{a'a'''}{a^2} - 6\tfrac{a''^2}{a^2} \\ & +24\tfrac{a'^2a''}{a^3} - 12\tfrac{a'^4}{a^4} + 2\tfrac{a^{(4)}}{a}. \end{aligned} \tag{85}$$

Thus ODE (58) reduces to the second order ODE

$$2v^2\frac{d^2v}{du^2} = -6uv\frac{dv}{du} - 2v\left(\frac{dv}{du}\right)^2 - u^4 - 8u^2v - 7v^2 + 2c_1u^2 + 4c_1v + 4c_2.$$

Acknowledgment. The authors would like to thank the referee for a careful reading of the paper and valuable comments to improve it.

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Received July 24, 2012
and in final form February 5, 2013