Journal of Lie Theory
Volume **23** (2013) 953–977

$\mathbf{L^p}$-Boundedness of Flag Kernels on Homogeneous Groups via Symbolic Calculus

Paweł Głowacki

Communicated by J. Faraut

Abstract. We prove that the flag kernel singular integral operators of Nagel-Ricci-Stein on a homogeneous group are bounded on L^p, $1 < p < \infty$. The gradation associated with the kernels is the natural gradation of the underlying Lie algebra. Our main tools are the Littlewood-Paley theory and a symbolic calculus combined in the spirit of Duoandikoetxea and Rubio de Francia.
Mathematics Subject Classification 2010: 42B20, 42B25.
Key Words and Phrases: Homogeneous groups, singular integrals, multipliers, flag kernels, Fourier transform, maximal functions, L^p-spaces, Littlewood-Paley theory.

1. Introduction

Flag kernels on homogeneous groups have been introduced by Nagel-Ricci-Stein [11] in their study of quadratic CR-manifolds. They can be regarded as a generalisation of Calderón-Zygmund singular kernels with singularities extending over the whole of the hyperplane $x_1 = 0$, where x_1 is the top level variable. The definition is complex, as it involves cancellation conditions for each variable separately. However, the description of flag kernels in terms of their Fourier transforms is much simpler and bears a striking resemblance to that of the symbols of convolution operators considered independently by the author in, e.g. [8]. As a matter of fact, the kernels of [8] turn out to be flag kernels smoothed out at infinity.

In Nagel-Ricci-Stein [11] we find an L^p-boundedness theorem for the very special flag kernels where the associated gradation consists of commuting subalgebras of the underlying Lie algebra of the homogeneous group. The natural question of what happens if the gradation is that of a homogeneous Lie algebra has been left open. The aim of this paper is to answer the question in the affirmative. We prove that such flag kernels give rise to bounded operators.

The smooth symbolic calculus mentioned above has been adapted to an extended class of flag kernels of small (positive and negative) orders and combined with a variant of the Littlewood-Paley theory built on a decomposition of the Dirac delta due to Dziubański [4]. The decomposition is obtained by a functional calculus

ISSN 0949–5932 / \$2.50

of a generalised heat kernel very similar to the Poisson kernel on the Euclidean space (see [7]). The strong maximal function of Christ [1] is also instrumental. The approach has been inspired by the well-known paper by Duoandikoetxea and Rubio de Francia [3]. The influence of this paper and, of course, Nagel-Ricci-Stein [11] is evident throughout.

A preliminary step is the L^2-boundedness of operators with flag kernels (see [9]) which we reproduce here for the convenience of the reader. This is proved solely by means of symbolic calculus.

After this paper had been completed, a preprint of Nagel-Ricci-Stein-Wainger [12] has been made available, where the L^p-boundedness theorem for flag kernels is proved. This comprehensive treatment of flag kernels on homogeneous groups has been announced for some time. Professor Stein has lectured a couple of times on the subject, see, e.g. [13]. The authors also use a version of Littlewood-Paley theory but otherwise the approach differs from the one presented here in many respects, the most important being our use of symbolic calculus and partition of the Dirac delta related to a generalised heat kernel. That is why we believe that what is presented here has an independent value and may count as a contribution to the theory.

Contents

2. Preliminaries

Let $\mathfrak{g}$ be a nilpotent Lie algebra of dimension D and $\mathfrak{g}^\star$ its dual. Let $\{\delta_t\}_{t>0}$ be a family of dilations on $\mathfrak{g}$ and let

$$\mathfrak{g}_j = \{x \in \mathfrak{g} : \delta_t x = t^{p_j} x\}, \qquad 1 \le j \le d,$$

where $1 \leq p_1 < p_2 < \cdots < p_d$. Denote by n_j the dimension of $\mathfrak{g}_j$ and by $Q_j = p_j n_j$ the *homogenous* dimension of $\mathfrak{g}_j$. The homogeneous dimension of $\mathfrak{g}$ is

$$Q = \sum_{j=1}^{d} Q_j.$$

We have

$$\mathfrak{g} = \bigoplus_{j=1}^{d} \mathfrak{g}_j, \qquad \mathfrak{g}^\star = \bigoplus_{j=1}^{d} \mathfrak{g}_j^\star \tag{1}$$

and

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \begin{cases} \mathfrak{g}_k, & \text{if } p_i + p_j = p_k, \\ \{0\}, & \text{if } p_i + p_j \notin \mathcal{P}, \end{cases}$$

where $\mathcal{P} = \{p_j : 1 \leq j \leq d\}$.

We shall also regard $\mathfrak{g}$ as a Lie group with the Campbell-Hausdorff multiplication

$$xy = x + y + r(x, y),$$

where $r(x, y)$ is the (finite) sum of terms of order at least 2 in the Campbell-Hausdorff series for $\mathfrak{g}$ (cf., e.g. Corwin-Greenleaf [2]). In particular, 0 becomes the group identity, and $x^{-1} = -x$, for $x \in \mathfrak{g}$. Under this identification the homogeneous ideals

$$\mathfrak{g}^{(k)} = \bigoplus_{j=k}^{d} \mathfrak{g}_j$$

are normal subgroups.

We also pick an auxilliary Euclidean norm $\|\cdot\|$ such that the decomposition (1) is orthogonal and fix an orthonormal basis $\{e_{kj}\}_{j=1}^{n_k}$ in $\mathfrak{g}_k$. Thus the variable $x \in \mathfrak{g}$ splits into $x = (x_1, x_2, \ldots, x_d)$, where

$$x_k = (x_{k1}, x_{k2}, \ldots, x_{kn_k}) \in \mathfrak{g}_k.$$

A similar notation will be applied to the variable $\xi \in \mathfrak{g}^\star$ and to the multiindices α. In particular,

$$|\alpha| = \sum_{k=1}^{d} |\alpha_k|, \qquad |\alpha_k| = \sum_{j=1}^{n_k} |\alpha_{kj}|$$

and

$$d(\alpha) = \sum_{k=1}^{d} p_k |\alpha_k|,$$

for

$$\alpha = (\alpha_k)_{k=1}^{d} \qquad \alpha_k = (\alpha_{kj})_{j=1}^{n_k}.$$

Let also

$$D^\alpha = \prod_{k=1}^{d} \prod_{j=1}^{n_k} D_{kj}^{\alpha_{kj}}, \qquad D_{kj} F(x) = F'(x) e_{kj}.$$

These are usual vector space derivatives and should not be confused with left-invariant group derivatives which play no explicit role here.

A nonnegative function $x \mapsto |x|$ is called a *homogeneous norm*, if

a) $|x| = 0$ if and only if $x = 0$,

b) $|-x| = |x|$, for $x \in \mathfrak{g}$,

c) $|\delta_t x| = t|x|$, for $x \in \mathfrak{g}, t > 0$.

Such a function is by no means unique. However, if $|\cdot|$ and $|\cdot|'$ are homogeneous norms, then there exists a constant $C > 0$ such that

$$C^{-1}|x| \leq |x|' \leq C|x|, \qquad x \in \mathfrak{g}.$$

For most purposes the following homogeneous norm

$$x \to |x| = \sum_{j=1}^{d} \|x_j\|^{1/p_j}$$

is sufficient. We fix it once and for all. Define also partial homogeneous norms

$$|x|_k = \sum_{j=1}^{k} \|x_j\|^{1/p_j} = \sum_{j=1}^{k} |x_j|, \qquad 1 \leq k \leq d.$$

In particular, $|x|_1 = |x_1|$, and $|x|_d = |x|$. Another notation will be applied to $\mathfrak{g}^\star$ where we have the dual dilations and the dual homogeneous norm, both denoted in the same way as those on $\mathfrak{g}$. For $\xi \in \mathfrak{g}^\star$,

$$|\xi|_k = \sum_{j=k}^{d} \|\xi_j\|^{1/p_j} = \sum_{j=k}^{d} |\xi_j|, \qquad 1 \leq k \leq d.$$

In particular, $|\xi|_1 = |\xi|$, and $|\xi|_d = |\xi_d|$.

The Schwartz space of smooth functions which vanish rapidly at infinity along with their derivatives will be denoted by $\mathcal{S}(\mathfrak{g})$. This is a Fréchet space with the usual countable set of seminorms. Let $\mathcal{S}'(\mathfrak{g})$ be the space of tempered distributions, that is the dual to $\mathcal{S}(\mathfrak{g})$. If $T \in \mathcal{S}'(\mathfrak{g})$ is a tempered distribution on $\mathfrak{g}$, we let

$$\langle \widetilde{T}, f \rangle = \langle T, \widetilde{f} \rangle,$$

where $\widetilde{f}(x) = f(x^{-1})$. For $t > 0$, we let

$$f_t(x) = t^{-Q} f(\delta_{t^{-1}} x), \qquad x \in \mathfrak{g},$$

so that

$$D^\alpha f_t(x) = t^{-d(\alpha)} (D^\alpha f)_t(x) = t^{-d(\alpha)-Q} (D^\alpha f)(\delta_{t^{-1}} x).$$

This extends to distributions by

$$\langle M_t, f \rangle = \langle M, f \circ \delta_t \rangle, \qquad t > 0.$$

Convolution on $\mathfrak{g}$ is defined by

$$f \star g(x) = \int_{\mathfrak{g}} f(xy^{-1})g(y)\,dy, \qquad f, g \in \mathcal{S}(\mathfrak{g}),$$

where dy is a Lebesgue measure which is also invariant under left and right group translations. The Lebesgue spaces of functions integrable with p-th power will be denoted by $L^p(\mathfrak{g})$. Convolution is easily extended to distributions in the following way. If $T \in \mathcal{S}'(\mathcal{F})$ and $g \in \mathcal{S}(\mathfrak{g})$, then $g \star T$ is a distribution acting by

$$\langle g \star T, f\rangle = \langle T, \widetilde{g} \star f\rangle, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

If, furthermore, $S \in \mathcal{S}'(\mathfrak{g})$ has the property that $\operatorname{Op}(S)f = f \star \widetilde{S}$ is a continuous endomorphism of $\mathcal{S}(\mathfrak{g})$, then the distribution $T \star S$ is defined by

$$\langle T \star S, f\rangle = \langle T, f \star \widetilde{S}\rangle, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

Convolution $T \star S$ also makes sense when T, S are *convolvers*, that is when the operators $\operatorname{Op}(T)$ and $\operatorname{Op}(S)$ extend from $\mathcal{S}(\mathfrak{g})$ to bounded operators on $L^2(\mathfrak{g})$. Then, there exists a tempered distribution U such that

$$\operatorname{Op}(U)f = \operatorname{Op}(T)\operatorname{Op}(S)f = \operatorname{Op}(T)(f \star \widetilde{S}), \qquad f \in \mathcal{S}(\mathfrak{g}).$$

The distribution $U = T \star S$ is called the convolution of T and S. In both cases

$$(T \star S)_t = T_t \star S_t, \qquad t > 0.$$

The Fourier transforms are

$$\widehat{f}(\xi) = \int_{\mathfrak{g}} e^{-i\langle x,\xi\rangle} f(x)\,dx, \qquad f^{\vee}(\xi) = \int_{\mathfrak{g}^\star} e^{i\langle x,\xi\rangle} f(\xi)\,d\xi,$$

where the Lebesgue measures dx and $d\xi$ are normalised so that the Plancherel formula

$$\|\widehat{f}\|_2^2 = \int_{\mathfrak{g}^\star} |\widehat{f}(\xi)|^2\,d\xi = \int_{\mathfrak{g}} |f(x)|^2\,dx = \|f\|_2^2$$

holds. If M is a distribution such that $\widehat{M}$ is a locally integrable function, then

$$\widehat{M_t}(\xi) = \widehat{M}(\delta_t\xi), \qquad \xi \in \mathfrak{g}^\star,\ t > 0.$$

Whenever we use the symbol $\star$ or refer to *convolution*, we mean the group convolution.

Recall also the integration in polar coordinates formula (Folland-Stein [6], Proposition 1.15). There exists a unique Borel measure σ on the unit sphere $\Sigma = \{x \in \mathfrak{g} : |x| = 1\}$ such that, for every $f \in L^1(\mathfrak{g})$,

$$\int_{\mathfrak{g}} f(x)\,dx = \int_0^\infty r^{Q-1} \int_\Sigma f(\delta_r z)\,\sigma(dz).$$

In particular, if $f(x) = \varphi(|x|)$, then

$$\int_{\mathfrak{g}} \varphi(|x|)\,dx = C \int_0^\infty r^{Q-1}\varphi(r)\,dr,$$

where $C = \sigma(\Sigma)$.

Remark 2.1. We shall write

$$A(s) \approx B(s), \qquad s \in S,$$

whenever $A(s)$, $B(s)$ are quantities dependent on s and there exists a constant $C > 0$ such that

$$C^{-1}A(s) \le B(s) \le CA(s), \qquad s \in S.$$

3. The class $\mathcal{S}_0(\mathfrak{g})$ of test functions

Let $\mathcal{S}_0(\mathfrak{g})$ be the subspace of $\mathcal{S}(\mathfrak{g})$ of those f whose Fourier transform vanishes for ξ_d in a neighbourhood of $0 \in \mathfrak{g}_d^\star$ and also for ξ_d outside a compact subset (which may depend on f) of $\mathfrak{g}_d^\star$. Note that if $f \in \mathcal{S}_0(\mathfrak{g})$, then it has zero integral, hence $\mathcal{S}_0(\mathfrak{g})$ cannot be dense in $L^1(\mathfrak{g})$.

Lemma 3.1. *The class $\mathcal{S}_0(\mathfrak{g})$ is a dense subspace of $L^p(\mathfrak{g})$, for $1 < p < \infty$.*

Proof. Let $1/q + 1/p = 1$, and let $g \in L^q(\mathfrak{g})$ be such that

$$\int_{\mathfrak{g}} f(x)g(x)\,dx = 0, \qquad f \in \mathcal{S}_0(\mathfrak{g}).$$

We are going to show that $g = 0$, which implies the required density. For $\varphi \in \mathcal{S}(\mathfrak{g}')$, where $\mathfrak{g}' = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{d-1}$, let

$$g_1(x_d) = \int_{\mathfrak{g}'} \varphi(x')g(x', x_d)\,dx'.$$

Then, $g_1 \in L^q(\mathfrak{g}_d)$. Observe that, for every $\psi \in C_c^\infty(\mathfrak{g}_d)$ such that $\widehat{\psi}$ vanishes in a neighbourhood of the origin,

$$\int_{\mathfrak{g}_d^\star} \widehat{g_1}(\xi_d)\widehat{\psi}(\xi_d)\,d\xi_d = \int_{\mathfrak{g}_d} g_1(x_d)\psi(x_d)\,dx_d = \int_{\mathfrak{g}} g(x)(\varphi \otimes \psi)(x)\,dx = 0$$

so $\widehat{g_1} \in \mathcal{S}'(\mathfrak{g}_g)$ is supported at the origin. Therefore, g_1 is also a polynomial, hence it must be zero. Since $\varphi \in \mathcal{S}(\mathfrak{g}')$ was arbitrary, g itself must be zero. ■

Lemma 3.2. *Let K be a convolver on $\mathfrak{g}$. If $f \in \mathcal{S}_0(\mathfrak{g})$, then $K \star f, f \star \widetilde{K} \in \mathcal{S}_0(\mathfrak{g})$. In particular, $\mathcal{S}_0(\mathfrak{g})$ is invariant under left and right group translations.*

Proof. If $\varphi \in C_c^\infty(\mathfrak{g}_d)$ and $\Phi(\xi) = \varphi(\xi_d)$, then

$$\Phi^\vee = \delta \otimes \varphi^\vee, \tag{2}$$

where δ is the Dirac delta at the origin in $\mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_{d-1}$. Thus, $\Phi^\vee$ is a measure supported in $\{0\} \times \mathfrak{g}_d$, hence it is central. Therefore, $f \in \mathcal{S}_0(\mathfrak{g})$ if and only if there exists a neighbourhood $U \subset \mathfrak{g}_d$ of the origin and a compact set $C \subset \mathfrak{g}_d^\star$ such that $f \star \Phi^\vee = 0$, for all $\varphi \in C_c^\infty(U)$ and all $\varphi \in C_c^\infty(\mathfrak{g}_d^\star \setminus C)$. Thus, our claim follows from the following identities

$$(K \star f) \star \Phi^\vee = K \star (f \star \Phi^\vee), \qquad (f \star \widetilde{K}) \star \Phi^\vee = (f \star \Phi^\vee) \star \widetilde{K},$$

valid for every convolver K on $\mathfrak{g}$. ■

4. Multipliers in $\mathcal{M}(\mu)$

Let $\mu = (\mu_1, \mu_2, \ldots, \mu_d) \in \boldsymbol{R}^d$. We say that a distribution $A \in \mathcal{S}'(\mathfrak{g})$ belongs to the class $S(\mu)$, if its Fourier transform $\widehat{A}$ is a smooth function which satisfies the estimates

$$|D^\alpha \widehat{A}(\xi)| \leq C_\alpha \prod_{k=1}^{d} (1 + |\xi|_k)^{\mu_k - p_k|\alpha_k|}, \qquad \text{all } \alpha.$$

The space $S(\mu)$ is a locally convex space if endowed with the family of seminorms

$$\|A\|_{S(\mu),l} = \sup_{|\alpha|\leq l} \sup_{\xi\in\mathfrak{g}^\star} \prod_{k=1}^{d} (1 + |\xi|_k)^{-\mu_k + p_k|\alpha_k|} |D^\alpha \widehat{A}(\xi)|,$$

for $l \in \boldsymbol{N}$. Apart from the locally convex topology, one also considers *the topology of bounded convergence* in $S(\mu)$, that is the topology of uniform convergence on compact subsets of $\mathfrak{g}^\star$ of Fourier transforms and all their derivatives of sequences of elements of $S(\mu)$ bounded in the locally convex topology. Note that $\mathcal{S}(\mathfrak{g})$ is dense in $S(\mu)$ with respect to the topology of bounded convergence.

Proposition 4.1. *The mapping*

$$\mathcal{S}(\mathfrak{g}) \times \mathcal{S}(\mathfrak{g}) \ni (f, g) \mapsto f \star g \in S(\mu + \nu)$$

is continuous if the space $\mathcal{S}(\mathfrak{g}) \times \mathcal{S}(\mathfrak{g})$ is considered as a subspace of $S(\mu) \times S(\nu)$. It is also continuous when the spaces $S(\mu)$, $S(\nu)$, and $S(\mu + \nu)$ are endowed with the topology of bounded convergence. By continuity, it extends to a mapping $S(\nu) \times S(\mu) \to S(\mu + \nu)$ which is continuous in the twofold sense.

Proof. This follows from Corollary 5.2 of [8]. ∎

Corollary 4.2. *Let $A \in S(\mu)$. Then $f \mapsto f \star \widetilde{A}$ is a continuous endomorphism of the Schwartz space $\mathcal{S}(\mathfrak{g})$.*

Remark 4.3. Let $A \in S(\mu)$, $B \in S(\nu)$. Then, by Corollary 4.2, we can define

$$\langle A \star B, f\rangle = \langle A, f \star \widetilde{B}\rangle.$$

so that $A \star B \in S(\mu + \nu)$. The mapping $(A, B) \mapsto A \star B$ is the extension mapping $S(\nu) \times S(\mu) \to S(\nu + \mu)$ of Proposition 4.1.

Let

$$\mathcal{N} = \{\mu = (\mu_1, \mu_2, \ldots, \mu_d) : |\mu_k| < 1,\ 1 \leq k \leq d\}.$$

Let $\mu \in \mathcal{N}$. We say that a distribution $M \in \mathcal{S}'(\mathfrak{g})$ belongs to the class $\mathcal{M}(\mu)$, if its Fourier transform is a locally integrable function which is smooth where $\xi_d \neq 0$ and satisfies the estimates

$$|D^\alpha \widehat{M}(\xi)| \leq C_\alpha \prod_{k=1}^{d} |\xi|_k^{\mu_k - p_k|\alpha_k|}, \qquad \xi_d \neq 0,\ \text{all } \alpha.$$

The space $\mathcal{M}(\mu)$ is a locally convex space if endowed with the family of seminorms

$$\|M\|_{\mathcal{M}(\mu),l} = \sup_{|\alpha|\le l} \sup_{\xi_d\neq 0} \prod_{k=1}^{d} |\xi|_k^{-\mu_k+p_k|\alpha_k|} |D^\alpha \widehat{M}(\xi)|,$$

for $l \in \boldsymbol{N}$. We have

$$\|M_t\|_{\mathcal{M}(\mu),l} = t^{s(\mu)} \|M\|_{\mathcal{M}(\mu),l}, \qquad t>0,\ l\in \boldsymbol{N}, \tag{3}$$

where $s(\mu) = \sum_{k=1}^d \mu_k$.

Let $u : \mathfrak{g}^\star \to [0,1]$ be a smooth even function depending only on ξ_d, supported in the set $1/2 \le |\xi_d| \le 4$, and such that

$$\sum_{k\in\boldsymbol{Z}} u_k(\xi)^2 = \sum_{k\in\boldsymbol{Z}} u(2^{-k}\xi)^2 = 1, \qquad \xi_d \neq 0. \tag{4}$$

Let $U_k = u_k^\vee$. Then U_k is a bounded central measure (see the beginning of the proof of Lemma 3.2). Note that $U_k = (U_0)_{2^{-k}}$. Let $V_k = U_k \star U_k$. For any $T \in \mathcal{S}'(\mathfrak{g})$ such that $\widehat{T}$ is locally integrable on $\mathfrak{g}^\star$,

$$T = \sum_{k\in\boldsymbol{Z}} V_k \star T = \lim_{n\to\infty} \sum_{|k|\le n} V_k \star T, \tag{5}$$

where the series of the Fourier transforms is convergent almost everwhere on $\mathfrak{g}^\star$. By Lebesgue's dominated convergence theorem, it is convergent also in $\mathcal{S}'(\mathfrak{g}^\star)$. Therefore, the series itself is convergent in $\mathcal{S}'(\mathfrak{g})$.

Lemma 4.4. *If $M \in \mathcal{M}(\mu)$, then, for every k, $U_k\star M, V_k\star M \in \mathcal{M}(\mu)\cap S(\mu)$, and*

$$\|U_0\star M\|_{\mathcal{M}(\mu),l} \approx \|U_0\star M\|_{S(\mu),l}, \qquad \|V_0\star M\|_{\mathcal{M}(\mu),l} \approx \|V_0\star M\|_{S(\mu),l}, \tag{6}$$

for every $l\in\boldsymbol{N}$. Furthermore,

$$\|M\|_{\mathcal{M}(\mu),l} \approx \sup_{k\in\boldsymbol{Z}} \|V_k\star M\|_{\mathcal{M}(\mu),l}, \qquad l\in\boldsymbol{N}. \tag{7}$$

Proof. Note that U_0 is central and $\widehat{U_0} = u$ so

$$(U_0\star M)^\wedge(\xi) = u(\xi)\widehat{M}(\xi), \qquad \xi\in\mathfrak{g}^\star.$$

Thus (6) follows from the fact that $u(\xi)$ vanishes for ξ_d near the origin.

We have

$$\|V_0\star M\|_{\mathcal{M}(\mu),l} \le C\|M\|_{\mathcal{M}(\mu,l}$$

and

$$V_k\star M = \Big(V_0\star M_{2^k}\Big)_{2^{-k}},$$

which combined with (3) implies

$$\|V_k\star M\|_{\mathcal{M}(\mu),l} \le C\|M\|_{\mathcal{M}(\mu),l}.$$

To complete the proof observe that the partition of unity (4) is uniformly locally finite. ∎

Proposition 4.5. *Let* $\mu, \nu, \mu+\nu \in \mathcal{N}$. *For every* $M \in \mathcal{M}(\mu)$ *and every* $N \in \mathcal{M}(\nu)$, *the sequence*

$$T_n = \sum_{|k|\leq n} (U_k \star M) \star (U_k \star N)$$

is convergent in $\mathcal{S}'(\mathfrak{g})$ *to an element* $T \in \mathcal{M}(\mu+\nu)$, *and, for every* $l \in \boldsymbol{N}$, *there exist* $l_1, l_2 \in \boldsymbol{N}$ *and a constant* $C > 0$ *such that*

$$\|T\|_{\mathcal{M}(\mu+\nu),l} \leq C\|M\|_{\mathcal{M}(\mu),l_1}\|N\|_{\mathcal{M}(\nu),l_2}. \tag{8}$$

Proof. By the argument used at the end of the previous proof,

$$\|T_n\|_{\mathcal{M}(\mu+\nu),l} \leq C \sup_{|k|\leq n} \|(U_k \star M) \star (U_k \star N)\|_{\mathcal{M}(\mu+\nu),l}.$$

Furthermore, by (3), (6), and Proposition 4.1,

$$\begin{aligned}
\|(U_k \star M) \star (U_k \star N)\|_{\mathcal{M}(\mu+\nu),l} &= \|\Big((U_0 \star M_{2^k}) \star (U_0 \star N_{2^k})\Big)_{2^{-k}}\|_{\mathcal{M}(\mu+\nu),l} \\
&= 2^{-ks(\mu+\nu)}\|(U_0 \star M_{2^k}) \star (U_0 \star N_{2^k})\|_{\mathcal{M}(\mu+\nu),l} \\
&\leq C_1 2^{-ks(\mu+\nu)}\|(U_0 \star M_{2^k}) \star (U_0 \star N_{2^k})\|_{S(\mu+\nu),l} \\
&\leq C_2 2^{-ks(\mu+\nu)}\|U_0 \star M_{2^k}\|_{S(\mu),l_1}\|U_0 \star N_{2^k}\|_{S(\nu),l_2} \\
&\leq C_3 2^{-ks(\mu)}\|M_{2^k}\|_{\mathcal{M}(\mu),l_1} 2^{-ks(\nu)}\|N_{2^k}\|_{\mathcal{M}(\nu),l_2} \\
&= C_3\|M\|_{\mathcal{M}(\mu),l_1}\|N\|_{\mathcal{M}(\nu),l_2}.
\end{aligned}$$

Now, for every $\xi \in \mathfrak{g}^\star$ with $\xi_d \neq 0$, there exists n_0 such that

$$\widehat{T}(\xi) = \widehat{T_n}(\xi) = \widehat{T_{n_0}}(\xi), \qquad n \geq n_0, \tag{9}$$

which shows that $T \in \mathcal{M}(\mu)$ and gives the bound (8). By definition of $\|\cdot\|_{\mathcal{M}(\mu),l}$ and (8),

$$|\widehat{T_n}(\xi)| \leq C \prod_{j=1}^{d} |\xi|_j^{\mu_j+\nu_j},$$

where the function on the right is locally integrable, so, by (9) and Lebesgue's dominated convergence theorem, $T_n \to T$ in $\mathcal{S}'(\mathfrak{g})$. ∎

5. More on the multipliers

Most properties of distributions in the classes $\mathcal{M}(\mu)$ are expressed in terms of their Fourier transforms. However, we need also an estimate on the growth of $M \in \mathcal{M}(\mu)$ directly on the group. This is the objective of this section.

Proposition 5.1. *For* $\nu \in \mathcal{N}$, *let* $M \in \mathcal{M}(\nu) \cap L^2(\mathfrak{g})$. *Then, there exists* $l \in \boldsymbol{N}$ *and a constant* $A > 0$ *such that*

$$|M(x)| \leq A\|M\|_{\mathcal{M}(\nu),l} \prod_{k=1}^{d} |x|_k^{-Q_k-\nu_k}, \qquad x_1 \neq 0. \tag{10}$$

Proof. There is no harm in assuming that $M \in \mathcal{S}(\mathfrak{g})$ as long as the final estimate depends only on a seminorm $\|M\|_{\mathcal{M}(\nu),l}$. We start with the case $d = 1$. Let $\nu = a$, where $|a| < 1$. The dilations are $\delta_t x = t^p x$, where $p \geq 1$. Let φ be a compactly supported smooth function on $\mathfrak{g}^\star$ equal to 1 in the neighbourhood of zero $|\xi| < 1$. Then,

$$M(x) = \int_{\mathfrak{g}^\star} e^{i\langle x,\xi\rangle} \widehat{M}(\xi)\, d\xi = \int_{\mathfrak{g}^\star} e^{i\langle x,\xi\rangle} \varphi(\delta_{|x|}\xi)\widehat{M}(\xi)\, d\xi$$
$$+ \int_{\mathfrak{g}^\star} e^{i\langle x,\xi\rangle}(1-\varphi(\delta_{|x|}\xi))\widehat{M}(\xi)\, d\xi = I_1(x) + I_2(x).$$

The first integral is estimated by a simple change of variable:

$$|I_1(x)| \leq \|M\|_{\mathcal{M}(a),0}|x|^{-Q-a} \int_{\mathfrak{g}^\star} |\varphi(\xi)||\xi|^a\, d\xi \leq C\|M\|_{\mathcal{M}(a),0}|x|^{-Q-a}.$$

Let u be a unit vector in $\mathfrak{g}$ such that $\langle x, u\rangle = \|x\| = |x|^p$. Denote by ∂_u the derivative in the direction of u. Let m be an integer such that $mp > Q + a$. By m integrations by parts,

$$(-i)^m\langle x,u\rangle^m I_2(x) = -\sum_{r=1}^{m} \binom{m}{r} \int e^{i\langle x,\xi\rangle}|x|^{rp}\, \partial_u^r\varphi(\delta_{|x|}\xi)\partial_u^{m-r}\widehat{M}(\xi)\, d\xi$$
$$+ \int e^{i\langle x,\xi\rangle}(1-\varphi(\delta_{|x|}\xi)\partial_u^m\widehat{M}(\xi)\, d\xi = J_1(x) + J_2(x).$$

The integrands in the above integrals vanish for $|x|\,|\xi| < 1$ so, by the same change of variable as in I_1,

$$|J_1(x)| \leq \|M\|_{\mathcal{M}(a),m}|x|^{mp-Q-a} \sum_{r=1}^{m} \int_{|\xi|\geq 1} |\partial_u^r\varphi(\xi)||\xi|^{-(m-r)p+a}\, d\xi$$
$$\leq C\|M\|_{\mathcal{M}(a),m}|x|^{mp-Q-a},$$

since $|\xi|^{-rp} \leq |\xi|^a \leq C$ on the support of the derivatives of φ. Similarly,

$$|J_2(x)| \leq \|M\|_{\mathcal{M}(a),m}|x|^{mp-Q-a} \int_{|\xi|\geq 1} |\xi|_k^{a-mp}\, d\xi$$
$$\leq C\|M\|_{\mathcal{M}(a),m}|x|^{mp-Q-a},$$

where the integral on the right is convergent since $mp > Q+a$. Collecting together the estimates for $J_1(x)$ and $J_2(x)$, we get

$$|x|^{mp}|I_2(x)| \leq |\langle x,u\rangle|^m|I_2(x)| \leq |J_1(x)| + |J_2(x)|$$
$$\leq C\|M\|_{\mathcal{M}(a),m}|x|^{mp-Q-a},$$

and finally

$$|M(x)| \leq |I_1(x)| + |I_2(x)| \leq C\|M\|_{\mathcal{M}(a),m}|x|^{-Q-a}.$$

This proves our proposition for $d = 1$.

We continue by induction on d. Suppose that $d > 1$ and the claim holds for every $1 \leq d' < d$. For a given $x \in \mathfrak{g}$, let

$$k_0 = \max\{k : |x_j| \leq |x_1| : 1 \leq j \leq k_0\}.$$

Let

$$x' = (x_1, x_2, \ldots, x_{k_0}) \in \mathfrak{g}', \qquad x'' = (x_{k_0+1}, \ldots, x_d) \in \mathfrak{g}''.$$

Similarly, we split $\xi = (\xi', \xi'') \in (\mathfrak{g}^\star)' \oplus (\mathfrak{g}^\star)''$, $\nu = (\nu', \nu'')$, and $\alpha = (\alpha', \alpha'')$. For $x'' \in \mathfrak{g}''$, let

$$M_{x''}(x') = M(x', x''), \qquad x' \in \mathfrak{g}'.$$

Assume for the time being that $\nu_k \leq 0$. By induction hypothesis,

$$|M(x)| = |M_{x''}(x')| \leq A_1 \|M_{x''}\|_{\mathcal{M}(\nu'), l_1} \prod_{k=1}^{k_0} |x|_k^{-Q_k - \nu_k}, \tag{11}$$

where

$$\|M_{x''}\|_{\mathcal{M}(\nu'), l_1} = \max_{|\alpha'| \leq l_1} \sup_{\xi_{k_0} \neq 0} |D_{\xi'}^{\alpha'} M(\overset{\wedge}{\xi'}, x'')| \prod_{k=1}^{k_0} |\xi|_k^{-\nu_k + p_k |\alpha_k|}. \tag{12}$$

We have used the fact that

$$(|\xi_1| + |\xi_2| + \cdots + |\xi_k|)^{-\nu_k} \leq |\xi|_k^{-\nu_k}, \qquad 1 \leq k \leq k_0,$$

because $\nu_k \leq 0$. By induction hypothesis again,

$$|D_{\xi'}^{\alpha'} M(\overset{\wedge}{\xi'}, x'')| \leq A_2 C(M) \prod_{k=k_0+1}^{d} |x|_k^{-Q_k - \nu_k}, \tag{13}$$

where

$$\begin{aligned} C(M) &= \|D_{\xi'}^{\alpha'} M(\overset{\wedge}{\xi'}, \cdot)\|_{\mathcal{M}(\nu''), l_2} \\ &= \max_{|\alpha''| \leq l_2} \sup_{\xi_d \neq 0} |D_{\xi''}^{\alpha''} D_{\xi'}^{\alpha'} \widehat{M}(\xi', \xi'')| \prod_{k=k_0+1}^{d} |\xi|_k^{-\nu_k + p_k |\alpha_k|}. \end{aligned}$$

Here again we have taken advantage of $\nu_k \leq 0$. Even more importantly, by the choice of k_0,

$$|x|_k \leq k_0 (|x_{k_0+1}| + \cdots + |x_k|), \qquad k_0 < k \leq d,$$

so we could put $|x|_k^{-Q_k - \nu_k}$ on the right hand side of (13). Thus, by (12), and (13),

$$\|M_{x''}\|_{\mathcal{M}(\nu'), l_1} \leq A_2 \|M\|_{\mathcal{M}(\nu), l_1 + l_2} \prod_{k=k_0+1} |x|_k^{-Q_k - \nu_k},$$

which combined with (11) gives

$$|M(x)| \leq A_1 A_2 \, \|M\|_{\mathcal{M}(\nu), l_1 + l_2} \prod_{k=1}^{d} |x|_k^{-Q_k - \nu_k}.$$

We are almost done. To complete the proof, we only need to remove the additional assumption $\nu_k \le 0$. Fix $x \in \mathfrak{g}$ with $x_1 \neq 0$. For each k, let $k_1 \le k$ be such that

$$|x|_k \le d|x_{k_1}|.$$

Let u_{k_1} be a unit vector in $\mathfrak{g}_{k_1}^\star$ such that

$$\langle x_{k_1}, u_{k_1}\rangle = \|x_{k_1}\| = |x_{k_1}|^{p_{k_1}}.$$

Then

$$M_1(y) = \prod_{k=1}^{d} \langle y, u_{k_1}\rangle M(y)$$

is in $\mathcal{M}(\mu)$, where $\mu_k = \nu_k - p_{k_1} \le 0$ so, by our proof so far,

$$d^{-d}\prod_{k=1}^{d} |x|_k^{p_{k_1}} |M(x)| \le |M_1(x)| \le A_l \|M_1\|_{\mathcal{M}(\mu),l} \prod_{k=1}^{d} |x|_k^{-Q_k-\nu_k+p_{k_1}}.$$

Note that

$$\|M_1\|_{\mathcal{M}(\mu),l} \le \|M\|_{\mathcal{M}(\nu),l+d},$$

so that, finally,

$$|M(x)| \le d^d A_l \|M\|_{\mathcal{M}(\nu),l+d} \prod_{k=1}^{d} |x|_k^{-Q_k-\nu_k}.$$

■

Remark 5.2. In the estimate (10) the exponents $-Q_k - \nu_k$ are all negative and $|x_k| \le |x|_k$ so, under the hypothesis of Proposition 5.1,

$$|M(x)| \le A\|M\|_{\mathcal{M}(\nu),l} \prod_{k=1}^{d} |x_k|^{-Q_k-\nu_k}. \qquad x_k \neq 0. \tag{14}$$

It is this estimate that we really need (see Lemma 10.1 below).

6. L^2-boundedness of flag kernels

In the context of this paper it is natural to work with a description of flag kernels given in terms of flag multipliers (see Nagel-Ricci-Stein [11], Theorem 2.3.9). We say that a tempered distribution K on $\mathfrak{g}$ is a *flag kernel* if its Fourier transform $\widehat{K}$ is a smooth function where $\xi_d \neq 0$ and satisfies the following estimates

$$|D^\alpha \widehat{K}(\xi)| \le C_\alpha \prod_{k=1}^{d} |\xi|_k^{-p_k|\alpha_k|}, \qquad \xi_d \neq 0, \text{ all } \alpha.$$

Thus, K is a flag kernel if and only if $K \in \mathcal{M}(\mathbf{0})$, where $\mathbf{0} = (0, 0, \ldots, 0)$.

The following is Theorem 2.5 of [9]. For the convenience of the reader we reproduce its proof here.

Proposition 6.1. *Let K be a flag kernel on $\mathfrak{g}$. The convolution operator $f \mapsto f \star \widetilde{K}$ defined initially on $\mathcal{S}_0(\mathfrak{g})$ extends uniquely to a bounded operator $\boldsymbol{K}$ on $L^2(\mathfrak{g})$ and there exists a constant $C > 0$ and an integer l such that*

$$\|\boldsymbol{K}\| \leq C\|K\|_{\mathcal{M}(\mathbf{0}),l}.$$

Proof. It is sufficient to know that

$$\|f \star \widetilde{K}\|_2 \leq C\|K\|_{\mathcal{M}(\mathbf{0}),l}\|f\|_2,$$

for f in the dense class $\mathcal{S}_0(\mathfrak{g})$. By Theorem 5.5 of [8], there exists a constant $C > 0$ and an integer $l \in \boldsymbol{N}$ such that

$$\|f \star T\|_2 \leq C\|T\|_{S(\mathbf{0}),l}\|f\|_2, \qquad f \in \mathcal{S}(\mathfrak{g}), \tag{15}$$

for $T \in S(\mathbf{0})$. By the definition of the distributions V_k,

$$f = \sum_{k\in\boldsymbol{Z}} f \star V_k = \sum_{k\in\boldsymbol{Z}} V_k \star f, \qquad f \in \mathcal{S}_0(\mathfrak{g}),$$

where the supports of the Fourier transforms $(V_k \star f)^\wedge$ form a uniformly locally finite family. Therefore, by Plancherel's theorem,

$$\|f\|_2^2 \approx \sum_{k\in\boldsymbol{Z}} \|V_k \star f\|_2^2, \qquad f \in \mathcal{S}_0(\mathfrak{g}). \tag{16}$$

Recall also that $V_k = U_k \star U_k$ and $U_k = (U_0)_{2^{-k}}$. Therefore, by Lemma 4.4, (15), and (3),

$$\begin{aligned}
\|V_k \star f \star \widetilde{K}\|_2^2 &= \|U_k \star U_k \star f \star \widetilde{K}\|_2^2 \\
&= 2^{kQ}\|(U_0 \star f_{2^k}) \star (U_0 \star \widetilde{K}_{2^k})\|_2^2 \leq 2^{kQ}\|U_0 \star \widetilde{K}_{2^k}\|_{S(\mathbf{0}),l}^2\|U_0 \star f_{2^k}\|_2^2 \\
&\approx 2^{kQ}\|U_0 \star \widetilde{K}_{2^k}\|_{\mathcal{M}(\mathbf{0}),l}^2\|U_0 \star f_{2^k}\|_2^2 \leq C\|K\|_{\mathcal{M}(\mathbf{0}),l}^2\|U_k \star f\|_2^2.
\end{aligned}$$

By (16),

$$\sum_{k\in\boldsymbol{Z}} \|U_k \star f\|_2^2 = \sum_{k\in\boldsymbol{Z}} \langle V_k \star f, f\rangle \leq C\|f\|_2^2,$$

which, in virtue of (16) and the above calculus, gives the required estimate. ■

If $M, N \in \mathcal{M}(\mathbf{0})$, then they are convolvers. Therefore, if $f \in \mathcal{S}_0(\mathfrak{g})$, then, by Lemma 3.2, $f \star \widetilde{N} \in \mathcal{S}_0(\mathfrak{g})$ and

$$f \star \widetilde{M \star N} = (f \star \widetilde{N}) \star \widetilde{M},$$

that is

$$\langle M \star N, {}_x f\rangle = \langle M, {}_x f \star \widetilde{N}\rangle, \qquad x \in \mathfrak{g}, \tag{17}$$

where ${}_x f(y) = f(xy)$.

Corollary 6.2. *Let $M \in \mathcal{M}(\mu) \cap \mathcal{M}(\mathbf{0})$, $N \in \mathcal{M}(\nu) \cap \mathcal{M}(\mathbf{0})$, where $\mu, \nu, \mu+\nu \in \mathcal{N}$. Then, M, N are convolvers and*

$$M \star N = \sum_{k \in \boldsymbol{Z}} (U_k \star M) \star (U_k \star N) \in \mathcal{M}(\mu + \nu),$$

where the series is convergent in $\mathcal{S}'(\mathfrak{g})$. Moreover, for every l, there exist l_1, l_2 and a constant $C > 0$ such that

$$\|M \star N\|_{\mathcal{M}(\mu+\nu),l} \leq C \|M\|_{\mathcal{M}(\mu),l_1} \|N\|_{\mathcal{M}(\nu),l_2}.$$

Proof. Let

$$T = \sum_{k \in \boldsymbol{Z}} (U_k \star M) \star (U_k \star N).$$

By Proposition 4.5, $T \in \mathcal{M}(\mu+\nu) \cap \mathcal{M}(\mathbf{0})$ and the series is convergent in the sense of tempered distributions. Since $M \star N$ and T are convolvers, $\mathcal{S}_0(\mathfrak{g})$ is dense in $L^2(\mathfrak{g})$ and invariant under translations, and T satisfies (8), it is enough to show that

$$\langle M \star N, f \rangle = \langle T, f \rangle, \qquad f \in \mathcal{S}_0(\mathfrak{g}),$$

which follows by (17). In fact,

$$\begin{aligned} \langle M \star N, f \rangle &= \langle M, f \star \widetilde{N} \rangle \\ &= \sum_{k \in \boldsymbol{Z}} \langle V_k \star M, f \star \widetilde{N} \rangle = \sum_{k \in \boldsymbol{Z}} \langle U_k \star M, f \star (U_k \star N)^{\sim} \rangle \\ &= \sum_{k \in \boldsymbol{Z}} \langle (U_k \star M) \star (U_k \star N), f \rangle = \langle T, f \rangle. \end{aligned}$$

■

Corollary 6.3 (Theorem 2.2 of [9]). *If K_1, K_2 are flag kernels on $\mathfrak{g}$, then their convolution $K_1 \star K_2$ is also a flag kernel.*

7. A representation of the Dirac delta

There always exists a homogeneous norm ρ on $\mathfrak{g}$ which is smooth away from the origin. In fact, we may take advantage of the implicit function theorem and let

$$\|\delta_{\rho(x)} x\| = 1, \qquad x \in \mathfrak{g} \setminus \{0\}, \ \rho(x) > 0,$$

and $\rho(0) = 0$. By Folland-Stein [6], page 8, ρ is a homogeneous norm.

Let us define a tempered distribution

$$\langle P, f \rangle = \lim_{\varepsilon \to} \int_{|x| \geq \varepsilon} \Big(f(0) - f(x) \Big) \frac{dx}{\rho(x)^{Q+1}}, \qquad f \in \mathcal{S}(\mathfrak{g}),$$

which, as is easily seen, coincides with the smooth function $x \mapsto \rho(x)^{-Q-1}$ on $\mathfrak{g} \setminus \{0\}$ and is homogeneous of degree $-Q-1$, that is

$$\langle P, f_t \rangle = t^{-Q-1} \langle P, f \rangle, \qquad f \in \mathcal{S}(\mathfrak{g}),$$

see, e.g. [7]. The distribution $-P$ is a *generalised laplacian* and, therefore, the operator $f \mapsto f \star P$ is positive and essentially selfadjoint with $\mathcal{S}(\mathfrak{g})$ for its core domain (Hunt [10]). Let us denote by $\mathbf{P}$ its selfadjoint closure and by $E(d\lambda)$ the corresponding spectral measure. If $m \in C_c^\infty(0,\infty)$, then, by Proposition 4.3 of Dziubański [4], there exists $\varphi \in \mathcal{S}(\mathfrak{g})$, denoted by $m^\vee$, such that

$$m(\mathbf{P})f = \int_0^\infty m(\lambda)E(d\lambda)f = f \star m^\vee = f \star \varphi, \qquad f \in L^2(\mathfrak{g}).$$

In a conversation, Jacek Dziubański explained how to prove the following lemma.

Lemma 7.1. *If $m \in C_c^\infty(1,2)$, then $\varphi = m^\vee$ has all moments vanishing.*

Proof. Let N be a large integer. By hypothesis, we can write $m(\lambda) = \lambda^N m_1(\lambda)$, where $m_1 \in C_c^\infty(1,2)$. Therefore,

$$\varphi = \psi \star P^N,$$

where $\psi = m_1^\vee \in \mathcal{S}(\mathfrak{g})$, and P^N is the convolution power of P which is a homogeneous distribution of degree $-Q-N$ smooth away from the origin. Note that P^N can be represented as

$$P^N = K_N + k_N,$$

where K_N is a compactly supported distribution, and $k_N \in C^\infty(\mathfrak{g})$ satisfies

$$|k_N(x)| \le C(1+|x|)^{-Q-N}, \qquad x \in \mathfrak{g}.$$

If F is a smooth function of polynomial growth, say, $|F(x)| \le C(1+|x|)^a$, then, for $N > a$, the convolution

$$F \star P^N = F \star K_N + F \star k_N$$

makes sense, and defines a function of polynomial growth. Moreover, for every $f \in \mathcal{S}(\mathfrak{g})$,

$$\langle F \star P^N, f\rangle = \langle F, f \star P^N\rangle.$$

Now, we are ready to conclude our argument. If F is a homogeneous polynomial of homogeneous degree a, then

$$\langle F, \varphi\rangle = \langle F, \psi \star P^N\rangle = \langle F \star P^N, \psi\rangle = 0,$$

since $F \star P^N$ is a smooth function which is homogeneous of degree $a - Q - N < -Q < 0$, hence must be zero. ∎

The following corollary is the objective of this section.

Corollary 7.2. *There exists a real even function $\varphi \in \mathcal{S}(\mathfrak{g})$ such that*

$$\int_{\mathfrak{g}} x^\alpha \varphi(x)\,dx = 0,$$

for every α, *and*

$$f=\int_0^\infty f\star\varphi_t\star\varphi_t\,\frac{dt}{t}, \tag{18}$$

where the integral is convergent in $L^2(\mathfrak{g})$*-norm, for every* $f\in\mathcal{S}(\mathfrak{g})$. *Moreover,*

$$\varphi_t\star\varphi_s=\varphi_s\star\varphi_t,\qquad t,s>0.$$

Proof. Let $m\in C_c^\infty(1,2)$ be real and such that

$$\int_1^2 m(t)^2\,\frac{dt}{t}=1.$$

Let $\varphi=m^\vee$. By Dziubański's theorem, $\varphi\in\mathcal{S}(\mathfrak{g})$ is real and $\varphi^\star=\varphi$. The vanishing moments condition is a consequence of Lemma 7.1. Since P is homogeneous of degree $-Q-1$, we have

$$m(t\mathbf{P})f=f\star\varphi_t,\qquad t>0, \tag{19}$$

cf. Dziubański [5], Lemma 3.4. Therefore, if $f\in\mathcal{S}(\mathfrak{g})$, then

$$\begin{aligned} f=\int_0^\infty E(d\lambda)f&=\int_0^\infty\int_0^\infty m(t\lambda)^2\frac{dt}{t}E(d\lambda)f\\ &=\int_0^\infty m(t\mathbf{P})^2f\frac{dt}{t}=\int_0^\infty f\star\varphi_t\star\varphi_t\frac{dt}{t}. \end{aligned}$$

That dilations of φ commute follows by (19) and spectral theorem. ■

8. Littlewood-Paley theory

From now on we fix the function φ of Corollary 7.2.

Proposition 8.1. *Let*

$$g_\varphi(f)(x)=\left(\int_0^\infty|f\star\varphi_t(x)|^2\,\frac{dt}{t}\right)^{1/2},$$

be the Littlewood-Paley square function operator. Then,

$$\|g_\varphi(f)\|_2=\|f\|_2,\qquad f\in\mathcal{S}(\mathfrak{g}).$$

Proof. Let $f\in\mathcal{S}(\mathfrak{g})$. By (18),

$$\begin{aligned} \|f\|_2^2=\langle f,\bar f\,\rangle&=\int_0^\infty\langle f\star\varphi_t\star\varphi_t,\bar f\,\rangle\frac{dt}{t}=\int_0^\infty\int_{\mathfrak{g}}|f\star\varphi_t(x)|^2dx\,\frac{dt}{t}\\ &=\int_{\mathfrak{g}}\int_0^\infty|f\star\varphi_t(x)|^2\,\frac{dt}{t}\,dx=\|g_\varphi(f)\|_2^2. \end{aligned}$$

■

Recall that $\mathfrak{g}^{(k)}$ is a subgroup of $\mathfrak{g}$. Let φ_k be the counterpart of φ for $\mathfrak{g}$ replaced by $\mathfrak{g}^{(k)}$, $1\le k\le d$. Let

$$\Phi_k=\delta_k\otimes\varphi_k,$$

where δ_k stands for the Dirac delta at $0\in\oplus_{j=1}^{k-1}\mathfrak{g}_j$.

Corollary 8.2. *If*

$$g_{\Phi_k}(f)(x) = \left(\int_0^\infty |f \star (\Phi_k)_t(x)|^2 \, \frac{dt}{t} \right)^{1/2},$$

then

$$\|g_{\Phi_k} f\|_2 = \|f\|_2, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

Proof. This is a direct consequence of Proposition 8.1. ■

Let

$$\Phi = \Phi_1 \star \Phi_2 \star \cdots \star \Phi_d,$$

and

$$\Phi_T = (\Phi_1)_{t_1} \star \cdots \star (\Phi_d)_{t_d}, \qquad T = (t_1, \dots, t_d) \in \boldsymbol{R}_+^d.$$

Corollary 8.3. *For every T,*

$$\Phi_T \in \mathcal{S}(\mathfrak{g}),$$

and for every $\nu \in \mathcal{N}$,

$$\|\Phi_T\|_{\mathcal{M}(\nu),l} \leq C_l \prod_{k=1}^d t_k^{\nu_k}.$$

Proof. That $\Phi_T \in \mathcal{S}(\mathfrak{g})$ is a simple exercise. It is clear that,

$$\Phi_k \in \bigcap_{|a|<1} \mathcal{M}(0, \dots 0, a, 0, \dots, 0),$$

where the only nonzero term stands on the k-th position, and

$$\|(\Phi_k)_{t_k}\|_{\mathcal{M}(0,\dots 0,\nu_k,0,\dots,0),l} \leq c_l t_k^{\nu_k}, \qquad |\nu_k| < 1.$$

Therefore the assertion follows by Corollary 6.2. ■

Corollary 8.4. *We have*

$$\langle f, g \rangle = \int_{\boldsymbol{R}_+^d} \langle f \star \Phi_T, g \star \Phi_T \rangle \, \frac{dT}{T}, \qquad f, g \in L^2(\mathfrak{g}),$$

where

$$\frac{dT}{T} = \frac{dt_1 dt_2 \dots dt_d}{t_1 t_2 \dots t_d}.$$

Proof. This follows from (18) by iteration. ■

Proposition 8.5. *The Littlewood-Paley square function operator*

$$G_\Phi(f)(x) = \left(\int_{\boldsymbol{R}_+^d} |f \star \Phi_T(x)|^2 \, \frac{dT}{T} \right)^{1/2},$$

is of type (p,p), for every $1 < p < \infty$. That is, there exists a constant C (dependent on p) such that

$$\|G_\Phi(f)\|_p \le C\|f\|_p, \qquad f \in \mathcal{S}(\mathfrak{g}).$$

Proof. The proof is based on standard techniques of Littlewood-Paley theory for vector-valued functions as contained in Folland-Stein [6] (see Theorem 6.20.b and Theorem 7.7).

We start with defining some Hilbert spaces and operators. Let $X_0 = \boldsymbol{C}$ and

$$X_k = L^2(\boldsymbol{R}_+^k, \frac{dT}{T}), \qquad 1 \le k \le d.$$

Let $W_k : L^2(\mathfrak{g}, X_{k-1}) \to L^2(\mathfrak{g}, X_k)$ be the operator

$$W_k f(x)(T, t_k) = f \star (\Phi_k)_{t_k}(x)(T) = \int_{\mathfrak{g}^{(k)}} (\varphi_k)_{t_k}(y) f(xy)(T)\, dy,$$

where $T = (t_1, \dots, t_{k-1})$. Note that W_k acts only on the $(x_k, \dots, x_d)$-variable. We can also write

$$W_k f(x) = \int_{\mathfrak{g}^{(k)}} w_k(y) f(xy)\, dy,$$

where, for every $y \in \mathfrak{g}^{(k)}$ and every $m \in X_{k-1}$,

$$w_k(y) : X_{k-1} \to X_k, \qquad w_k(y)(m)(T, t_k) = (\varphi_k)_{t_k}(y) m(T).$$

We claim that W_k is a bounded operator, even an isometry. In fact, by the definition of Φ_k and Corollary 8.2,

$$\begin{aligned}
\|W_k f\|^2_{L^2(\mathfrak{g}, X_k)} &= \int_{\mathfrak{g}} \|W_k f(x)\|^2_{X_k}\, dx \\
&= \int_{\mathfrak{g}} dx \int_0^\infty \frac{dt}{t} \int_{\boldsymbol{R}_+^{k-1}} \frac{dT}{T} \left| \int_{\mathfrak{g}^{(k)}} (\varphi_k)_t(y) f(xy)(T)\, dy \right|^2 \\
&= \int_{\boldsymbol{R}_+^{k-1}} \frac{dT}{T} \int_0^\infty \frac{dt}{t} \int_{\mathfrak{g}} \left| \int_{\mathfrak{g}^{(k)}} (\varphi_k)_t(y) f(xy)(T)|\, dy \right|^2 dx, \\
&= \int_{\boldsymbol{R}_+^{k-1}} \frac{dT}{T} \int_0^\infty \frac{dt}{t} \langle f_T \star (\Phi_k)_t, f_T \star (\Phi_k)_t \rangle \\
&= \int_{\boldsymbol{R}_+^{k-1}} \|g_{\Phi_k}(f_T)\|_2^2 \frac{dT}{T} = \int_{\boldsymbol{R}_+^{k-1}} \|f_T\|_2^2 \frac{dT}{T} = \|f\|^2_{L^2(\mathfrak{g}, X_{k-1})},
\end{aligned}$$

where $f_T(x) = f(x)(T)$.

Another property of the kernel w_k of W_k that is needed is the following. For every α,

$$\|D^\alpha w_k(x)\|_{X_{k-1} \to X_k} \le C_\alpha |x|_k^{-Q^{(k)} - d(\alpha)}, \tag{20}$$

where $Q^{(k)} = Q_k + Q_{k+1} + \cdots + Q_d$. This follows readily from the fact that $\varphi_k \in \mathcal{S}(\mathfrak{g}^{(k)})$ and therefore satisfies

$$|D^\alpha \varphi_k(x)| \le C_\alpha (1 + |x|_k)^{-Q^{(k)} - d(\alpha)}.$$

As a bounded operator from $L^2(\mathfrak{g}, X_{k-1})$ to $L^2(\mathfrak{g}, X_k)$ satisfying (20) W_k is a vector-valued kernel of type 0, and, by Theorem 6.20.b of Folland-Stein [6], it maps $L^p(\mathfrak{g}, X_{k-1})$ into $L^p(\mathfrak{g}, X_k)$ boundedly, for every $1 < p < \infty$.

This implies our assertion. In fact,

$$G_\Phi(f)(x) = \|W_d W_{d-1} \dots W_1 f(x)\|_{X_d},$$

and therefore

$$\|G_\Phi(f)\|_{L^p(\mathfrak{g})} = \|W_d W_{d-1} \dots W_1 f\|_{L^p(\mathfrak{g}, X_d)} \leq C\|f\|_{L^p(\mathfrak{g}, X_0)} = C\|f\|_p. \qquad \blacksquare$$

A word of comment on the symbol Φ_T would be appropriate here. The notation may suggest that the functions Φ_T are dilates of a single function. They are not, but they have estimates of this form, which is our justification. In the next section we are going to use the same notation for the "real" dilates of a function. We hope the reader will not get confused.

9. The strong maximal function

Let

$$\Delta_T x = (\delta_{t_1} x_1, \delta_{t_2} x_2, \dots, \delta_{t_d} x_d), \qquad x \in \mathfrak{g}, T \in \boldsymbol{R}_+^d.$$

For a function F on $\mathfrak{g}$ and $T \in \boldsymbol{R}_+^d$, let

$$F_T(x) = T^{-Q} F(\Delta_{T^{-1}} x) = t_1^{-Q_1} t_2^{-Q_2} \dots t_d^{-Q_d} F(\delta_{t_1^{-1}} x_1, \dots, \delta_{t_d^{-1}} x_d),$$

where

$$T^{-Q} = t_1^{-Q_1} \dots t_d^{-Q_d}, \qquad T^{-1} = (t_1^{-1}, \dots, t_d^{-1}).$$

Let $B_j = \{x \in \mathfrak{g}_j : |x| \leq 1\}$, and let $D = B_1 \times \cdots \times B_d$. Let $|D|$ be the Lebesgue measure of D. *The strong maximal function* on $\mathfrak{g}$ is defined by

$$\mathbf{M} f(x) = \sup_{T \in \boldsymbol{R}_+^d} \int_D |f(x \Delta_T y)| \, dy = \sup_T |f| \star (\chi_D)_T(x). \tag{21}$$

A theorem of Michael Christ asserts that, for every $1 < p < \infty$, there exists a constant $C > 0$ such that

$$\|\mathbf{M} f\|_p \leq C\|f\|_p, \qquad f \in L^p(\mathfrak{g}),$$

that is, $\mathbf{M}$ is of (p,p) type (see Christ [1]). Actually, Christ considers a slightly different but obviously equivalent maximal function, where χ_D is replaced with χ_B, where $B = \{x \in \mathfrak{g} : |x| \leq 1\}$.

We shall need the following corollary of Christ's theorem.

Corollary 9.1. *Let $0 < a < 1$, and let*

$$F(x) = \prod_{j=1}^d \gamma(|x_j|)^a |x_j|^{-Q_j}, \qquad x_j \neq 0,$$

where $\gamma(s)=\min\{s,s^{-1}\}$, *for* $s>0$. *Then the maximal function*

$$M_F f(x)=\sup_{T\in\boldsymbol{R}_+^d}|f|\star F_T(x)$$

is of (p,p) *type for* $1<p<\infty$.

Proof. The function F is radially decreasing in each variable so

$$F(x)=\sup_{h_{\mathcal{R}}\le F}h_{\mathcal{R}}(x),$$

and

$$\int F(x)\,dx=\sup_{h_{\mathcal{R}}\le F}\int h_{\mathcal{R}}(x)\,dx,$$

where

$$h_{\mathcal{R}}=\sum_{R\in\mathcal{R}}c_R\chi_D(\Delta_{R^{-1}}x),\quad C_R>0,\qquad R=(r_1,r_2,\dots,r_d)\in\boldsymbol{R}_+^d,$$

and $\mathcal{R}\subset\boldsymbol{R}_+^d$ is a finite set. For $f\ge 0$, we have

$$\begin{aligned}f\star(h_{\mathcal{R}})_T(x)&=\sum_R c_R r_1^{Q_1}r_2^{Q_2}\dots r_d^{Q_d}f\star(\chi_D)_{RT}(x)\\&\le\Big(\frac{1}{|D|}\sum_R c_R r_1^{Q_1}r_2^{Q_2}\dots r_d^{Q_d}|D|\Big)\mathbf{M}f(x)\\&=\frac{\|h_{\mathcal{R}}\|_1}{|D|}\mathbf{M}f(x)\end{aligned}$$

and therefore

$$M_F f(x)\le\frac{\|F\|_1}{|D|}\mathbf{M}f(x),$$

which completes the proof. ■

10. L^p-boundedness of flag kernels

We keep the notation established in previous sections. For $S\in\boldsymbol{R}_+^d$, let

$$\gamma(S)=\gamma(s_1)\gamma(s_2)\cdots\gamma(s_d),\qquad\gamma(s)=\min\{s,s^{-1}\}.$$

Lemma 10.1. *Let* K *be a flag kernel. Let*

$$K_{T,S}=\widetilde{\Phi}_{TS}\star K\star\Phi_T,\qquad T,S\in\boldsymbol{R}_+^d.$$

There exists a constant $C>0$ *and an integer* l *such that, for every* $T,S\in\boldsymbol{R}_+^d$,

$$|K_{T,S}(x)|\le C\|K\|_{\mathcal{M}(\mathbf{0}),l}\,\gamma(S)^{1/4}F_T(x),\tag{22}$$

where

$$F(x)=\prod_{k=1}^d\gamma(|x_k|)^{1/4}|x_k|^{-Q_k},\tag{23}$$

In particular, there exists a constant $C_1>0$ *independent of* T *such that*

$$\int_{\mathfrak g}|K_{T,S}(x)|\,dx\le C_1\|K\|_{\mathcal M(\mathbf 0),l}\,\gamma(S)^{1/4}.$$

Proof. By Corollary 8.3, for $\mu,\nu,\mu+\nu\in[-1/2,1/2]^d$,

$$\|\widetilde{\Phi_{TS}}\|_{\mathcal M(\mu),l_1}\le C_{l_1}\prod_{k=1}^d(t_ks_k)^{\mu_k},\qquad \|\Phi_T\|_{\mathcal M(\nu),l_2}\le C_{l_2}\prod_{k=1}^d(t_k)^{\nu_k},$$

hence, by Corollary 6.2 and the fact that $K\in\mathcal M(\mathbf 0)$, for every l, there exists l' such that

$$\|K_{T,S}\|_{\mathcal M(\mu+\nu),l}\le C\|K\|_{\mathcal M(\mathbf 0),l'}\prod_{k=1}^d(s_kt_k)^{\mu_k}t_k^{\nu_k}.$$

Note also that, by Propositions 8.3 and 6.1, $K_{T,S}\in L^2(\mathfrak g)$. Hence, by Proposition 5.1,

$$|K_{T,S}(x)|\le C\|K\|_{\mathcal M(\mathbf 0),l}\prod_{k=1}^d s_k^{\mu_k}t_k^{\mu_k+\nu_k}|x|_k^{-Q_k-\mu_k-\nu_k},$$

for some C,l. Since $Q_k+\mu_k+\nu_k\ge 0$,

$$|K_{T,S}(x)|\le C\|K\|_{\mathcal M(\mathbf 0),l}\prod_{k=1}^d s_k^{\mu_k}t_k^{\mu_k+\nu_k}|x_k|^{-Q_k-\mu_k-\nu_k}.$$

By choosing appropriately $\mu_k=\pm1/4$ and $\nu_k=0,\pm1/2$ depending on whether s_k and $t_k^{-1}|x_k|$ are smaller or greater than 1, we get

$$\begin{aligned}|K_{T,S}(x)|&\le C\|K\|_{\mathcal M(\mathbf 0),l}\prod_{k=1}^d\gamma(s_k)^{1/4}\gamma(t_k^{-1}|x|_k)^{1/4}|x|^{-Q_k}\\&=C\|K\|_{\mathcal M(\mathbf 0),l}\,\gamma(S)^{1/4}F_T(x).\end{aligned}$$

Finally,

$$\int_{\mathfrak g}|K_{T,S}(x)|\,dx\le C\|K\|_{\mathcal M(\mathbf 0),l}\,\gamma(S)^{1/4}\int_{\mathfrak g}F(x)\,dx=C_1\|K\|_{\mathcal M(\mathbf 0),l}\,\gamma(S)^{1/4}.\qquad\blacksquare$$

Corollary 10.2. *Let* $1<p<\infty$. *For a given* $S\in\boldsymbol R_+^d$, *the maximal operator*

$$K_S^\star f(x)=\sup_T|f|\star|K_{T,S}^{\sim}|(x)$$

is of type (p,p) *with*

$$\|K_S^\star f\|_p\le C\|K\|_{\mathcal M(\mathbf 0),l}\,\gamma(S)^{1/4}\|f\|_p,\qquad f\in L^p(\mathfrak g),$$

for some $C>0$ *and* $l\in\boldsymbol N$.

Proof. The assertion follows by Lemma 10.1 and Corollary 9.1. $\blacksquare$

Lemma 10.3. *Let* $K \in L^1(\mathfrak{g})$, $g \in L^2(\mathfrak{g})$. *Then,*

$$|g \star K(x)|^2 \leq \|K\|_1 \, |g|^2 \star |K|(x), \qquad x \in \mathfrak{g}.$$

Proof. In fact,

$$\begin{aligned} |g \star K(x)|^2 &\leq \left(\int_{\mathfrak{g}} |g(xy^{-1})||K(y)|^{1/2}|K(y)|^{1/2}\, dy \right)^2 \\ &\leq \int_{\mathfrak{g}} |g|^2(xy^{-1})|K|(y)\, dy \int_{\mathfrak{g}} |K(y)|\, dy \\ &\leq \|K\|_1 \, |g|^2 \star |K|(x), \end{aligned}$$

for every $x \in \mathfrak{g}$. ∎

We turn to the main result of this paper. The reader may wish to compare the proof we give with that of Theorem B and the preceding lemma of Duoandikoetxea-Rubio de Francia [3].

Theorem 10.4. *Let* K *be a flag kernel on* $\mathfrak{g}$. *For every* $1 < p < \infty$, *the singular integral operator*

$$f \to f \star \widetilde{K}, \qquad f \in \mathcal{S}_0(\mathfrak{g}),$$

extends uniquely to a bounded operator $\boldsymbol{K}$ *on* $L^p(\mathfrak{g})$, *and there exists a constant* $C > 0$ *and an integer* l *such that*

$$\|\boldsymbol{K}\|_{L^p \to L^p} \leq C\|K\|_{\mathcal{M}(\mathbf{0}),l}.$$

Proof. By Lemma 3.1, the space $\mathcal{S}_0(\mathfrak{g})$ is dense in $L^p(\mathfrak{g})$, for every $1 < p < \infty$. Let $f, g \in \mathcal{S}_0(\mathfrak{g})$. By Corollary 8.4, Proposition 6.3, and Lemma 3.2,

$$\begin{aligned} \langle f \star \widetilde{K}, g \rangle = \langle f, g \star K \rangle &= \int_{\boldsymbol{R}^d_+} \langle f \star \Phi_T, g \star K \star \Phi_T \rangle \frac{dT}{T} \\ &= \int_{\boldsymbol{R}^d_+} \int_{\boldsymbol{R}^d_+} \langle f \star \Phi_T, g \star \Phi_{TS} \star \widetilde{\Phi}_{TS} \star K \star \Phi_T \rangle \frac{dS}{S} \frac{dT}{T} \\ &= \int_{\boldsymbol{R}^d_+} \int_{\boldsymbol{R}^d_+} \langle f_T, g_{TS} \star K_{T,S} \rangle \frac{dT}{T} \frac{dS}{S}, \end{aligned}$$

where

$$f_T = f \star \Phi_T, \qquad g_{TS} = g \star \Phi_{TS}, \qquad K_{T,S} = \widetilde{\Phi}_{TS} \star K \star \Phi_T.$$

We are going to estimate

$$L_S(f, g) = \int_{\boldsymbol{R}^d_+} \langle f_T, g_{TS} \star K_{T,S} \rangle \frac{dT}{T},$$

for a given S. Recall that, by Proposition 8.5, the square function G_Φ of Section 8 is of type (p, p), for every $1 < p < \infty$. Let $1 < p < 2$ and $f, g \in \mathcal{S}_0(\mathfrak{g})$. By the

Schwartz and Hölder inequalities,

$$\begin{aligned} |\langle L_S f, g\rangle| &\le \int_{\mathfrak{g}} \left(\int_{\boldsymbol{R}_+^d} |f_T(x)|^2 \frac{dT}{T} \right)^{1/2} \left(\int_{\boldsymbol{R}_+^d} |g_{TS} \star K_{T,S}(x)|^2 \frac{dT}{T} \right)^{1/2} dx \\ &\le \|G_\Phi(f)\|_p \left(\int_{\mathfrak{g}} \left(\int_{\boldsymbol{R}_+^d} |g_{TS} \star K_{T,S}(x)|^2 \frac{dT}{T} \right)^{q/2} dx \right)^{1/q} \\ &\le C_1 \|f\|_p \left\| \int_{\boldsymbol{R}_+^d} |g_{TS} \star K_{T,S}(\cdot)|^2 \frac{dT}{T} \right\|_{q/2}^{1/2}, \end{aligned}$$

where $1/p + 1/q = 1$. Note that $q > 2$. Thus, there exists a nonnegative function u with $\|u\|_r = 1$, where $2/q + 1/r = 1$, such that

$$A = \left\| \int_{\boldsymbol{R}_+^d} |g_{TS} \star K_{T,S}(\cdot)|^2 \frac{dT}{T} \right\|_{q/2} = \int_{\mathfrak{g}} \int_{\boldsymbol{R}_+^d} |g_{TS} \star K_{T,S}(x)|^2 \frac{dT}{T} u(x)\, dx.$$

Therefore, by Lemmas 10.3, 10.1 and Corollary 10.2, there exists $l \in \boldsymbol{N}$ such that

$$\begin{aligned} A &\le C \|K\|_{\mathcal{M}(\mathbf{0}),l}\, \gamma(S)^{1/4} \int_{\boldsymbol{R}_+^d} \int_{\mathfrak{g}} |g_{TS}|^2 \star |K_{T,S}|(x)\, u(x)\, dx \frac{dT}{T} \\ &\le C \|K\|_{\mathcal{M}(\mathbf{0}),l}\, \gamma(S)^{1/4} \int_{\mathfrak{g}} \int_{\boldsymbol{R}_+^d} |g_{TS}(x)|^2 \frac{dT}{T} K_S^\star u(x)\, dx \\ &\le C_1 \|K\|_{\mathcal{M}(\mathbf{0}),l}\, \gamma(S)^{1/4} \|G_\Phi(g)\|_q^2 \|K_S^\star u\|_r \le C_2 \|K\|_{\mathcal{M}(\mathbf{0}),l}^2 \gamma(S)^{1/2} \|g\|_q^2, \end{aligned}$$

whence, by Proposition 8.5,

$$|L_S(f,g)| \le C \|K\|_{\mathcal{M}(\mathbf{0}),l} \gamma(S)^{1/4} \|f\|_p \|g\|_q. \tag{24}$$

Finally,

$$\begin{aligned} |\langle \boldsymbol{K} f, g\rangle| &\le C \|K\|_{\mathcal{M}(\mathbf{0}),l} \left(\int_{\boldsymbol{R}_+^d} \gamma(S)^{1/4} \frac{dS}{S} \right) \|f\|_p \|g\|_q \\ &= C_1 \|K\|_{\mathcal{M}(\mathbf{0}),l} \|f\|_p \|g\|_q, \end{aligned}$$

which proves our case for $1 < p < 2$. The result for $2 < p < \infty$ follows by duality. The case $p = 2$ has already been established in Proposition 6.1. ∎

Corollary 10.5. *Let $1 < p < \infty$. Let K be a flag kernel on $\mathfrak{g}$. For each $n \in \boldsymbol{N}$, let*

$$K_n = \sum_{|k| \le n} V_k \star K, \qquad \boldsymbol{K}_n f = f \star K_n.$$

Then, for every $f \in L^p(\mathfrak{g})$,

$$\|\boldsymbol{K}_n f - \boldsymbol{K} f\|_p \to 0, \qquad n \to \infty.$$

Proof. By Lemma 4.4,

$$\|K_n\|_{\mathcal{M}(\mathbf{0}),l} \leq C \max_{|k|\leq n} \|V_k \star K\|_{\mathcal{M}(\mathbf{0}),l} \leq C_1 \|K\|_{\mathcal{M}(\mathbf{0}),l}$$

so the family $\{K_n\}$ is bounded in $\mathcal{M}(\mathbf{0})$. By Theorem 10.4, the norms of the operators $\boldsymbol{K}_n$ acting on $L^p(\mathfrak{g})$ are uniformly bounded. Recall from Section 4 that, for every $f \in \mathcal{S}_0(\mathfrak{g})$,

$$\boldsymbol{K}f = \boldsymbol{K}_n f,$$

if n is large enough so that, by Lemma 3.1, the convergence holds on a dense subset of $L^p(\mathfrak{g})$. ■

Acknowledgements. I wish to extend thanks to Alexander Nagel, Fulvio Ricci, and Elias M. Stein for their interest in my work and an inspiring conversation. I am also indebted to the referee for his helpful suggestions. Finally, I thank Françoise Piquard for pointing out an editorial omission in the initial version of the manuscript.

References

[1] Christ, M., *The strong maximal function on a nilpotent group*, Trans. Amer. Math. Soc. **331** (1992), 1–13.

[2] Corwin, L., and F. P. Greenleaf, "Representations of nilpotent Lie groups and their applications. Part 1: Basic theorey and examples," Cambridge University Press, Cambridge 1990.

[3] Duoandikoetxea J., and Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Inv. math. **84** (1986), 541–561.

[4] Dziubański, J., *Schwartz spaces associated with some non-differential convolution operators on homogeneous groups*, Colloq. Math. **63** (1992), 153–157.

[5] —, *A note on Schrödinger operators with polynomial potentials*, Colloq. Math. **78** (1998), 149–161.

[6] Folland, G. B., and E. M. Stein, "Hardy spaces on homogeneous groups," Princeton University Press, Princeton 1982.

[7] Głowacki, P., *Stable semigroups of measures as commutative approximate identities on non-graded homogeneous groups*, Inv. Math. **83** (1986), 557–582.

[8] —, *The Melin calculus for general homogeneous groups*, Arkiv för matematik **45** (2007), 31–48,

[9] —, *Composition and L^2-boundedness of flag kernels*, Colloq. Math. **118** (2010), 581–585. *Correction*, Colloq. Math., **120** (2010), 331.

[10] Hunt, G. A., *Semigroups of measures on Lie groups*, Trans. Amer. Math. Soc. **81** (1956), 264–293.

[11] Nagel, A., F. Ricci, and E. M. Stein, *Singular integrals with flag kernels and analysis on quadratic CR–manifolds*, J. Func. Analysis **181** (2001), 29–18.

[12] Nagel, A., F. Ricci, E. M. Stein, and S. Wainger, *Singular integrals with flag kernels on homogeneous groups I*, arXiv: 1108.0177v1.

[13] Stein, E. M., *Algebras of operators*, http://www.math.ucla.edu/dls/2009/-stein.shtml,

[14] Yosida, K., "Functional analysis," Springer-Verlag, Berlin-Heidelberg-New York, 1980.

Pawel Głowacki
Institute of Mathematics
University of Wrocław
pl. Grunwaldzki 2/4
51-386 Wrocław, Poland
glowacki@math.uni.wroc.pl

Received September 10, 2012
and in final form February 7, 2013