Irreducible Representations of a Product of Real Reductive Groups

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Abstract. Let $G_1, G_2$ be real reductive groups and $(\pi, V)$ be a smooth admissible representation of $G_1 \times G_2$. We prove that $(\pi, V)$ is irreducible if and only if it is the completed tensor product of $(\pi_i, V_i)$, $i = 1, 2$, where $(\pi_i, V_i)$ is a smooth, irreducible, admissible representation of moderate growth of $G_i$, $i = 1, 2$. We deduce this from the analogous theorem for Harish-Chandra modules, for which one direction was proven in A. Aizenbud and D. Gourevitch, Multiplicity one theorem for $(GL_{n+1}(\mathbb{R}), GL_n(\mathbb{R}))$, Selecta Mathematica N. S. 15 (2009), 271–294, and the other direction we prove here. As a corollary, we deduce that strong Gelfand property for a pair $H \subset G$ of real reductive groups is equivalent to the usual Gelfand property of the pair $\Delta H \subset G \times H$.

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1. Introduction

Let $G_1, G_2$ be reductive Lie groups, $\mathfrak{g}_i$ be the Lie algebra of $G_i$. Fix $K_i$ - a maximal compact subgroup of $G_i$ ($i = 1, 2$). Let $\mathcal{M}(\mathfrak{g}_i, K_i)$ be the category of Harish-Chandra $(\mathfrak{g}_i, K_i)$-modules and $\mathcal{M}(G_i)$ be the category of smooth admissible Fréchet representations of moderate growth (see [4, 10]). We also denote by $Irr(G_i)$ and $Irr(\mathfrak{g}_i, K_i)$ the isomorphism classes or irreducible objects in the above categories.

In this note we prove

Theorem 1.1. Let $M \in Irr(\mathfrak{g}_1 \times \mathfrak{g}_2, K_1 \times K_2)$. Then there exist $M_i \in Irr(\mathfrak{g}_i, K_i)$ such that $M = M_1 \otimes M_2$.

The converse statement, saying that for irreducible $M_i \in \mathcal{M}(\mathfrak{g}_i, K_i), M_1 \otimes M_2$ is irreducible is [1, Proposition A.0.6]. By the Casselman-Wallach equivalence of categories $\mathcal{M}(\mathfrak{g}, K) \simeq \mathcal{M}(G)$, these two statements imply

Theorem 1.2. A representation $(\pi, V) \in \mathcal{M}(G_1 \times G_2)$ is irreducible if and only if there exist irreducible $(\pi_i, V_i) \in \mathcal{M}(G_i)$ such that $(\pi, V) \simeq (\pi_1, V_1) \otimes (\pi_2, V_2)$. 

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Finally, we deduce a consequence of this theorem concerning Gelfand pairs. A pair \((G, H)\) of reductive groups is called a Gelfand pair if \(H \subset G\) is a closed subgroup and the space \((\pi^*)^H\) of \(H\)-invariant continuous functionals on any \(\pi \in \text{Irr}(G)\) has dimension zero or one. It is called a strong Gelfand pair or a multiplicity-free pair if \(\dim \text{Hom}_H(\pi|_H, \tau) \leq 1\) for any \(\pi \in \text{Irr}(G), \tau \in \text{Irr}(H)\).

**Corollary 1.3.** Let \(H \subset G\) be reductive groups and let \(\Delta H \subset G \times H\) denote the diagonal. Then \((G, H)\) is a multiplicity-free pair if and only if \((G \times H, \Delta H)\) is a Gelfand pair.

An analog of Corollary 1.3 was proven in [7] for generalized Gelfand property of arbitrary Lie groups, with smooth representations replaced by smooth vectors in unitary representations.

An analog of Theorem 1.2 for p-adic groups was proven in [3, §§2.16] and in [5]. For a more detailed exposition see [6, §§10.5].

### 2. Preliminaries

#### 2.1. Harish-Chandra modules and smooth representations.

In this subsection we fix a real reductive group \(G\) and a maximal compact subgroup \(K \subset G\). Let \(\mathfrak{g}, \mathfrak{k}\) denote the complexified Lie algebras of \(G, K\).

**Definition 2.1.** A \((\mathfrak{g}, K)\)-module is a \(\mathfrak{g}\)-module \(\pi\) with a locally finite action of \(K\) such the two induced actions of \(\mathfrak{k}\) coincide and \(\pi(\text{ad}(k)(X)) = \pi(k)\pi(X)\pi(k^{-1})\) for any \(k \in K\) and \(X \in \mathfrak{g}\).

A finitely-generated \((\mathfrak{g}, K)\)-module is called admissible if any representation of \(K\) appears in it with finite (or zero) multiplicity. In this case we also call it a Harish-Chandra module.

**Lemma 2.2** ([9], §§4.2). Any Harish-Chandra module \(\pi\) has finite length.

**Theorem 2.3** (Casselman-Wallach, see [10], §§§11.6.8). The functor of taking \(K\)-finite vectors \(HC : \mathcal{M}(G) \to \mathcal{M}(\mathfrak{g}, K)\) is an equivalence of categories.

In fact, Casselman and Wallach construct an inverse functor \(\Gamma : \mathcal{M}(\mathfrak{g}, K) \to \mathcal{M}(G)\), that is called Casselman-Wallach globalization functor (see [10, Chapter 11] or [4] or, for a different approach, [2]).

**Corollary 2.4.**

(i) The category \(\mathcal{M}(G)\) is abelian.

(ii) Any morphism in \(\mathcal{M}(G)\) has closed image.

**Proof.** (i) \(\mathcal{M}(\mathfrak{g}, K)\) is clearly abelian and by the theorem is equivalent to \(\mathcal{M}(G)\).
(ii) Let $\phi : \pi \to \tau$ be a morphism in $\mathcal{M}(G)$. Let $\tau' = \text{Im} \phi$, $\pi' = \pi / \ker \phi$ and $\phi' : \pi' \to \tau'$ be the natural morphism. Clearly $\phi'$ is monomorphic and epimorphic in the category $\mathcal{M}(G)$. Thus by (i) it is an isomorphism. On the other hand, $\text{Im} \phi' = \text{Im} \phi \subset \text{Im} \phi = \tau'$. Thus $\text{Im} \phi = \text{Im} \phi'$.

We will also use the embedding theorem of Casselman.

**Theorem 2.5.** Any irreducible $(\mathfrak{g}, K)$-module can be imbedded into a $(\mathfrak{g}, K)$-module of principal series.

Lemma 2.2, Theorems 2.3 and 2.5 and Corollary 2.4 have the following corollary.

**Corollary 2.6.** The underlying topological vector space of any admissible smooth Fréchet representation of moderate growth is a nuclear Fréchet space.

**Definition 2.7.** Let $G_1$ and $G_2$ be real reductive groups. Let $(\pi_i, V_i) \in \mathcal{M}(G_i)$ be admissible smooth Fréchet representations of moderate growth of $G_i$. We define $\pi_1 \otimes \pi_2$ to be the natural representation of $G_1 \times G_2$ on the space $V_1 \hat{\otimes} V_2$.

**Proposition 2.8** ([1], Proposition A.0.6). Let $G_1$ and $G_2$ be real reductive groups. Let $\pi_i \in \text{Irr}(\mathfrak{g}_i, K_i)$ be irreducible Harish-Chandra modules of $G_i$. Then $\pi_1 \otimes \pi_2 \in \text{Irr}(\mathfrak{g}_1 \times \mathfrak{g}_2, K_1 \times K_2)$.

We will use the classical statement on irreducible representations of compact groups.

**Lemma 2.9.** Let $K_1, K_2$ be compact groups. A representation $\tau$ of $K_1 \times K_2$ is irreducible if and only if there exist irreducible representations $\tau_i$ of $K_i$ such that $\tau \simeq \tau_1 \otimes \tau_2$. Note that $\tau_i$ are finite-dimensional, and $\otimes$ is the usual tensor product.

**Corollary 2.10.** Let $G_1$ and $G_2$ be real reductive groups and $(\pi_i, V_i) \in \mathcal{M}(G_i)$. Then we have a natural isomorphism $(\pi_1 \otimes \pi_2)^{\text{HC}} \simeq \pi_1^{\text{HC}} \otimes \pi_2^{\text{HC}}$.

### 3. Proof of Theorem 1.1

Throughout the section $\rho_i$ always denote irreducible representations of $K_1$, $\sigma_j$ always denote irreducible representations of $K_2$. For a representation $V$ of $K_1$ (or of $K_2$) we will denote by $V^\rho$ (resp. by $V^\sigma$) the corresponding isotypic component. Let $K := K_1 \times K_2$ and $\mathfrak{g} := \mathfrak{g}_1 \times \mathfrak{g}_2$.

Let $(\pi, V)$ be an irreducible admissible $(\mathfrak{g}, K)$-module. We show that there exist non-zero irreducible and admissible $(\mathfrak{g}_1, K_1)$-module $V_1$ and $(\mathfrak{g}_2, K_2)$-module $V_2$ and a non-zero morphism $V_1 \hat{\otimes} V_2 \to V$. From the irreducibility of $V$ and $V_1 \hat{\otimes} V_2$, we obtain that $V \simeq V_1 \hat{\otimes} V_2$.

Let’s first find the module $V_1$. Choose $\tau \in \text{Irr}(K)$ such that the isotypic component $V^\tau$ is non-zero. By Lemma 2.9 $\tau \simeq \rho \otimes \sigma$ for some $\rho \in \text{Irr}(K_1), \sigma \in \text{Irr}(K_2)$. From $\rho \in \text{Irr}(K_1)$ we infer that $\rho \simeq \rho_i$ for some $i$ and $\sigma \in \text{Irr}(K_2)$, $\sigma \simeq \sigma_j$. Therefore $V \simeq V^\rho \otimes V^\sigma \simeq V_1 \hat{\otimes} V_2$.
Let $W$ be the $(g_1, K_1)$-module generated by $V$. Note that since the actions of $(g_1, K_1)$ and $(g_2, K_2)$ commute, $W$ is also a $K_2$-module and $W = W'$. We claim that $W$ is an admissible $(g_1, K_1)$-module. Indeed, let $\rho_1$ be an irreducible representation of $K_1$. Then $W^{\rho_1} \subseteq V^{\rho_1 \otimes \sigma}$ and as a corollary
\[
\dim(W^{\rho_1}) \leq \dim \left( V^{\rho_1 \otimes \sigma} \right) < \infty,
\]
since $V$ is an admissible $(g, K)$-module.

Now by Lemma 2.2 $W$ has finite length and thus there is an irreducible admissible $(g_1, K_1)$-submodule $V_1 \subseteq W$. Thus, we finished the first stage of the proof. Let
\[
W'_2 := \text{Hom}_{(g_1, K_1)}(V_1, V).
\]
Clearly, $W'_2 \neq 0$. Since actions of $(g_1, K_1)$ and $(g_2, K_2)$ on $V$ commute, $W'_2$ has a natural structure of $(g_2, K_2)$-module. Take any non-zero morphism $L \in W'_2$ and let $W_2 \subset W'_2$ be the $(g_2, K_2)$-module generated by $L$.

Let us show that $W_2$ is admissible. Choose $\sigma_2 \in \text{Irr}(K_2)$. Let $\rho_2 \in \text{Irr}(K_1)$ such that $V'_2^{\rho_2} \neq 0$. Then $V'_2^{\rho_2}$ generates $V_1$ and thus for any $L' , L'' \in W_2^{\sigma_2}$ if $L'$ agrees with $L''$ on $V_1^{\rho_2}$ then $L' = L''$. This gives a linear embedding from $W_2^{\sigma_2}$ into the finite-dimensional space $\text{Hom}_C(V_1^{\rho_2}, V^{\rho_2 \otimes \sigma_2})$. Thus $W_2$ is an admissible $(g_2, K_2)$-module.

Thus $W_2$ has finite length and therefore there is an irreducible admissible submodule $V_2 \subseteq W_2$. Define a linear map $\phi : V_1 \otimes V_2 \to V$ by the formula
\[
\phi(v \otimes l) := l(v)
\]
on the pure tensors. Clearly, this is a non-zero $(g, K)$-map.

The result $V_1 \otimes V_2 \simeq V$ follows now from the irreducibility of $V$ and of $V_1 \otimes V_2$ (Proposition 2.8).

Remark 3.1. An alternative way to prove this theorem is to remark that the category $\mathcal{M}(g, K)$ is equivalent to the category of admissible modules over the idempotented algebra $\mathcal{H}(g, K)$ of $K$-finite distributions on $G$ supported in $K$ (see [5]), then show that this algebra is the tensor product of $\mathcal{H}(g_i, K_i)$ and thus the proofs from [3, 5] extend to this case. We estimate that such proof would be of similar length, but slightly less elementary.

4. Proof of Theorem 1.2 and Corollary 1.3

Proof. [Proof of Theorem 1.2] First take $\pi_i \in \text{Irr}(G_i)$, for $i = 1, 2$. Then $\pi_i^{HC} \in \text{Irr}(g_i, K_i)$ and by Proposition 2.8 $\pi_1^{HC} \otimes \pi_2^{HC} \in \text{Irr}(g_1 \times g_2, K_1 \times K_2)$. By Corollary 2.10 $(\pi_1 \otimes \pi_2)^{HC} \simeq \pi_1^{HC} \otimes \pi_2^{HC} \in \text{Irr}(g_1 \times g_2, K_1 \times K_2)$. This implies $\pi_1 \otimes \pi_2 \in \text{Irr}(G_1 \times G_2)$.

Now take $\pi \in \text{Irr}(G_1 \times G_2)$. Then $\pi^{HC} \in \text{Irr}(g_1 \times g_2, K_1 \times K_2)$ and by Theorem 1.1 there exist $(M_i) \in \text{Irr}(g_i, K_i)$ such that $\pi^{HC} \simeq M_1 \otimes M_2$. By Theorem 2.3 there exist $\pi_i \in \text{Irr}(G_i)$ such that $\pi_i^{HC} \simeq M_i$. Then $\pi^{HC} \simeq \pi_1^{HC} \otimes \pi_2^{HC} \simeq (\pi_1 \otimes \pi_2)^{HC}$ and by Theorem 2.3 this implies $\pi \simeq \pi_1 \otimes \pi_2$. ■
Corollary 1.3 follows from Theorem 1.2 and the following lemma.

**Lemma 4.1.** Let $H \subset G$ be real reductive groups. Let $(\pi, E)$ and $(\tau, W)$ be admissible smooth Fréchet representations of moderate growth of $G$ and $H$ respectively. Then $\text{Hom}_H(\pi, \tau)$ is canonically isomorphic to $\text{Hom}_{\Delta H}(\pi \otimes \tau, \mathbb{C})$, where $\tau$ denotes the contragredient representation.

**Proof.** For a nuclear Fréchet space $V$ we denote by $V'$ its dual space equipped with the strong topology. Let $\tilde{W} \subset W'$ denote the underlying space of $\tilde{\tau}$. By the theory of nuclear Fréchet spaces ([8, Chapter 50], we know $\text{Hom}_C(E, W) \cong E' \hat{\otimes} W$ and $\text{Hom}_C(E \hat{\otimes} \tilde{W}, C) \cong E' \hat{\otimes} \tilde{W}'$. Thus we have canonical embeddings

$$\text{Hom}_H(\pi, \tau) \hookrightarrow \text{Hom}_{\Delta H}(\pi \otimes \tau, \mathbb{C}) \hookrightarrow \text{Hom}_H(\pi, \tilde{\tau}')$$

Since the image of any $H$-equivariant map from $\pi$ to $\tilde{\tau}'$ lies in the space of smooth vectors $\tilde{\tau}$, which is canonically isomorphic to $\tau$, the lemma follows. \qed

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