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Lie Superalgebras of Differential Operators

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Abstract. We describe explicitly Lie superalgebra isomorphisms between the Lie superalgebras of first-order superdifferential operators on supermanifolds, showing in particular that any such isomorphism induces a diffeomorphism of the supermanifolds. We also prove that the group of automorphisms of such a Lie superalgebra is a semi-direct product of the subgroup of automorphisms induced by the supermanifold diffeomorphisms and another subgroup which consists of automorphisms determined by even superdivergences. We prove the existence of such superdivergences on any supermanifold and we describe their local form.
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1. Introduction

In [GKP09] the authors described all Lie superalgebra isomorphisms $\mathcal{X}(\mathcal{M}_1) \to \mathcal{X}(\mathcal{M}_2)$ of the Lie superalgebras of supervector fields on smooth supermanifolds $\mathcal{M}_i$, $i = 1, 2$. It was shown that every such isomorphism is induced by a diffeomorphism of supermanifolds $\mathcal{M}_1 \to \mathcal{M}_2$, unless the dimension of the manifolds is $1|1$ or $0|2$. In the latter low-dimensional cases there exist additional isomorphisms which have been also listed in [GKP09].

In the present paper the authors address a similar question, this time about isomorphisms of another natural and important Lie superalgebra attached to supermanifolds – the Lie superalgebra $\mathcal{D}^1(\mathcal{M})$ of first-order superdifferential operators. The structure of the Lie superalgebra automorphism group is much more complicated in this case, see Abstract.

Our work fits into a series of papers on algebraic characterization of space. It is well known that isomorphisms of the associative algebras of smooth functions living on second countable manifolds are implemented by diffeomorphisms between these manifolds. The classical result by Pursell and Shanks [PS54], which states

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that the Lie algebra structure of the space of compactly supported vector fields characterizes the differential structure of the underlying manifold, is the starting point of a multitude of papers by, among others, Koriyama, Maeda, Omori (other complete and transitive Lie algebras of vector fields), see e.g. [Omo76], Amemiya, Masuda, Shiga, Duistermaat, Singer, Grabowski, Poncin (differential and pseudodifferential operators) [AMS75, DS76, GP04, GP05], Abe, Atkin, Grabowski, Fukui, Tomita, Hauser, Müller, Rybicki (special geometric situations), Amemiya, Grabowski (real and analytic cases) [Ame75, Gra78], Skryabin (modular Lie algebras) [Skr87], Grabowska, Grabowski, Kotov, Poncin (Lie algebras associated with Lie algebroids) [GG01, GKP11], and Lecomte, Leuther, Mushengezi, Grabowski, Poncin (characterization of vector bundles) [GP08, Lec81, LLM12]. See also [BHMP02, FGKOR95, GKO96, Pon99].

The paper is organized as follows. In section 2 we review some basic facts about (super)differential operators on a smooth supermanifold and prove an important theorem (Theorem 1) about a representation of a supervector field as a sum of supercommutators. We also explain why we focus on the first-order operators.

In section 3 we prove that, if $\dim \mathcal{M}$ differs from $0|1$, the algebra $\mathcal{A}(\mathcal{M}) \subset \mathcal{D}^1(\mathcal{M})$ of smooth (super)functions on $\mathcal{M}$ can be characterized as the unique maximal super Lie ideal in $\mathcal{D}^1(\mathcal{M})$ consisting of ad-nilpotent elements. In particular, any automorphism of the Lie superalgebra $\mathcal{D}^1(\mathcal{M})$ preserves $\mathcal{A}(\mathcal{M})$, thus induces an automorphism of the Lie superalgebra of supervector fields $\mathcal{X}(\mathcal{M}) \simeq \mathcal{D}^1(\mathcal{M})/\mathcal{A}(\mathcal{M})$.

In section 4 we prove that, according to the canonical splitting $\mathcal{D}^1(\mathcal{M}) = \mathcal{X}(\mathcal{M}) \oplus \mathcal{A}(\mathcal{M})$, any automorphism of the Lie superalgebra $\mathcal{D}^1(\mathcal{M})$ is a product of an automorphism induced by a superdiffeomorphism of $\mathcal{M}$ and an automorphism of the form $(X+f) \mapsto (X+f+c(X))$, determined by a 1-cocycle $c : \mathcal{X}(\mathcal{M}) \to \mathcal{A}(\mathcal{M})$ on the Lie superalgebra of vector fields $\mathcal{X}(\mathcal{M})$, with values in the algebra of functions on $\mathcal{M}$. We also explain why only those automorphisms of $\mathcal{X}(\mathcal{M})$ which are induced by supermanifold diffeomorphisms can be extended from $\mathcal{X}(\mathcal{M})$ to $\mathcal{D}^1(\mathcal{M})$.

In section 5 we compute the first cohomology of the Lie superalgebra $\mathcal{X}(\mathcal{M})$ of supervector fields with values in $\mathcal{A}$. We show that every 1-cocycle is a combination of a closed super 1-form and a fixed divergence.

Finally, in section 6, we review necessary facts about measures (Berezinian volumes) and divergences on supermanifolds and prove the existence of a divergence for each supermanifold.

2. Sheaves of superdifferential operators

Let us recall that a *smooth supermanifold* $\mathcal{M}$ of dimension $p|q$ is a (local) ringed superspace $(M, \mathcal{A})$ over a topological space M that is locally isomorphic to $(\mathbb{R}^p, C^{\infty\, p|q})$, where, for any open subset $\mathfrak{U} \subset \mathbb{R}^p$, $C^{\infty\, p|q}(\mathfrak{U}) := C^\infty(\mathfrak{U})[\xi^1, \ldots, \xi^q]$ – the ξ^α being formal anticommuting generators. More precisely, we assume that $\mathcal{A}$ is a sheaf of associative supercommutative $\mathbb{R}$-algebras with unit. The superalgebra $\mathcal{A}(M) = \Gamma(M, \mathcal{A})$ of global sections of $\mathcal{A}$ is the algebra $C^\infty(\mathcal{M})$ of *functions* of the supermanifold $\mathcal{M}$. It is well-known that, due to the local model condition, the locality condition for the stalks is automatically satisfied. Further, the considered data induce a smooth manifold structure of dimension p on M. More precisely, there exists a short exact sequence of sheaves

$$0 \to \mathcal{J} \to \mathcal{A} \xrightarrow{\varepsilon} C^\infty \to 0,$$

where $\mathcal{J} = \ker \varepsilon$ and where C^∞ is the function sheaf of a smooth manifold structure on the base M of $\mathcal{M}$. The projection $\varepsilon : \mathcal{A} \to C^\infty$, of the structure sheaf of $\mathcal{M}$ onto the structure sheaf of M, provides an embedding $i : M \to \mathcal{M}$ of the classical base manifold M into the supermanifold $\mathcal{M}$. We can thus view a supermanifold as a "classical manifold surrounded by a cloud of odd stuff" [Var04]. The base manifold M is often called the *body* of $\mathcal{M}$.

An important result of smooth supergeometry [Gaw77, Bat79], which is usually referred to as Batchelor's theorem, asserts that there exists a vector bundle V of rank q over M such that $\mathcal{M}$ is diffeomorphic as a supermanifold to ΠV, that is, to the total space of V with the reversed parity of fibres. This implies that the algebra of smooth functions on $\mathcal{M}$ is isomorphic (as a commutative superalgebra) to the algebra of functions on ΠV, which is canonically identified with $\Gamma(\Lambda^\bullet V^*)$. This isomorphism is not canonical but it gives us an identification

$$\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1 \simeq \Gamma(\Lambda^\bullet V^*)\,, \tag{2.1}$$

with

$$\mathcal{A}_0 = \bigoplus_{i\geq 0} \mathcal{A}^{2i}\,,\;\; \mathcal{A}_1 = \bigoplus_{i\geq 0} \mathcal{A}^{2i+1}\,,\;\; \text{where } \mathcal{A}^k \simeq \Gamma(\Lambda^k V^*)\,. \tag{2.2}$$

In particular, this gives a (non-canonical) embedding of the algebra $C^\infty(M)$ into $C^\infty(\mathcal{M})$ and a super-version of the partition of unity.

Remark 2.1.
- As usual, we have denoted the $\mathbb{Z}_2$-grading by subscripts and the $\mathbb{N}$-grading by superscripts.
- For simplicity, we assume in this text that M is connected.
- The prefix 'super' will be understood, whenever possible.

For any open subset $U \subset M$, we denote by $(\operatorname{Der}\mathcal{A})(U)$ the $\mathcal{A}(U)$-module $\operatorname{Der}(\mathcal{A}(U))$ of derivations of the algebra $\mathcal{A}(U)$. If $X \in (\operatorname{Der}\mathcal{A})(U)$, there is, in view of the localization principle, for any open subset $V \subset U$, a unique derivation $X|_V \in (\operatorname{Der}\mathcal{A})(V)$ such that $(Xf)|_V = X|_V f|_V$, for all $f \in \mathcal{A}(U)$. The assignment $U \to (\operatorname{Der}\mathcal{A})(U)$ is actually a locally free sheaf of $\mathcal{A}$-modules, called the derivation sheaf of the structure sheaf $\mathcal{A}$ and denoted by $\operatorname{Der}\mathcal{A}$, or, also, the tangent sheaf $T\mathcal{M}$ of the manifold $\mathcal{M}$. The module $(T\mathcal{M})(M)$ of global sections of the vector bundle $T\mathcal{M}$ is the $C^\infty(\mathcal{M})$-module $\mathcal{X}(\mathcal{M})$ of *vector fields* of $\mathcal{M}$ – which carries an obvious Lie algebra structure.

In the following we denote by $\underline{\operatorname{End}}(\mathcal{A}(U))$ the $\mathcal{A}(U)$-module of even and odd $\mathbb{R}$-linear maps from $\mathcal{A}(U)$ to itself. We can identify $\mathcal{A}(U)$ with a subalgebra of $\underline{\mathrm{End}}(\mathcal{A}(U))$ using the left-regular representation $f \mapsto m_f$, $m_f(g) = fg$. The $\mathcal{A}(U)$-module of k-th order differential operators $\mathcal{D}^k(U)$, $k \in \mathbb{N}$, is then defined inductively by

$$\mathcal{D}^k(U) := \{D \in \underline{\mathrm{End}}(\mathcal{A}(U)) : [D, \mathcal{A}(U)] \subset \mathcal{D}^{k-1}(U)\}, \tag{2.3}$$

where $[-,-]$ is the commutator and where $\mathcal{D}^{-1}(U) = \{0\}$.

Of course, $\mathcal{D}^0(U) = \mathcal{A}(U)$, and thus 0-order operators are local. This entails by induction that any superdifferential operator is local. Indeed, if $D \in \mathcal{D}^k(U)$, if the restriction $f|_V$ of $f \in \mathcal{A}(U)$ to an open $V \subset U$ vanishes, and if $v \in V$, let $\gamma \in \mathcal{A}_0(U)$ be a bump function with support $\operatorname{supp}\gamma \subset V$ (in the supercontext the support can be defined as usual as the complement in U of the set of those points $u \in U$ for which the restriction of γ to some neighborhood of u vanishes) and restriction $\gamma|_W = 1$, for some neighborhood $W \subset V$ of v, see localization principle [Lei80, Corollary 3.1.8]. It then follows from the defining property of differential operators applied to $[D,\gamma]f$, the induction assumption, and the fact $\gamma f = 0$, that $(Df)|_W = 0$. We can now show that there exists, just as in the case of vector fields, for any $D \in \mathcal{D}^k(U)$ and any open $V \subset U$, a unique $D|_V \in \mathcal{D}^k(V)$ such that $(Df)|_V = D|_V f|_V$, for all $f \in \mathcal{A}(U)$. Indeed, if $f \in \mathcal{A}(V)$ and $v \in V$, it is possible to choose a function $F \in \mathcal{A}(U)$ (of the same parity as f) such that $\operatorname{supp} F \subset \operatorname{supp} f$ and $F|_W = f|_W$, for some neighborhood $W \subset V$ of v. Locality entails that $(DF)|_W \in \mathcal{A}(W)$ and $(DF')|_{W'} \in \mathcal{A}(W')$, defined for two points $v, v' \in V$, depend only on f and coincide in the intersection $W \cap W'$. Thus these local functions define a unique global function $D|_V f \in \mathcal{A}(V)$ such that

$$(D|_V f)|_W = (DF)|_W.$$

Since, obviously, $D|_V \in \underline{\mathrm{End}}(\mathcal{A}(V))$ (note that $D|_V$ has the same parity as D), it suffices – to prove the above claim – to observe that, for any $f_1, \ldots, f_{k+1} \in \mathcal{A}(V)$, we have

$$[\ldots[[D|_V, f_1], f_2], \ldots, f_{k+1}]|_W = [\ldots[[D, F_1], F_2], \ldots, F_{k+1}]|_W = 0,$$

with self-explaining notations.

In view of the properties of the restrictions of differential operators, the assignment $U \to \mathcal{D}^k(U)$ is a presheaf and obviously also a sheaf – as $\mathcal{A}$ is a sheaf.

Proposition 2.2. *For any $k \in \mathbb{N}$, the presheaf $\mathcal{D}^k$ of k-th order superdifferential operators over the base manifold M of a smooth supermanifold $\mathcal{M} = (M, \mathcal{A})$ of dimension $p|q$ is a locally free sheaf of $\mathcal{A}$-modules, with local basis*

$$\partial_x^\alpha \partial_\xi^\beta := \partial_{x^1}^{\alpha^1} \ldots \partial_{x^p}^{\alpha^p} \partial_{\xi^q}^{\beta^q} \ldots \partial_{\xi^1}^{\beta^1},$$

where (x, ξ) are local coordinates, $\beta^a \in \{0,1\}$, and $|\alpha| + |\beta| \leq k$.

Proof. The method used to prove local freeness of the sheaf of vector fields goes through in the case of differential operators. Let us give some details because of the increased technicality.

If $M = U^{p|q}$ is a superdomain, if $D \in \mathcal{D}^k(U)$ is of the type

$$\sum_{i=0}^{k} D^i = \sum_{i=0}^{k} \sum_{|\alpha|+|\beta|=i} D^i_{\alpha\beta}(x,\xi)\, \partial_x^\alpha \partial_\xi^\beta \in \mathcal{D}^k(U), \tag{2.4}$$

and if $m_{\alpha\beta} = (1/\alpha!)x^\alpha\xi^\beta$, where the odd coordinates are written in increasing order, then necessarily

$$D^i_{\alpha\beta} = D^i m_{\alpha\beta} = Dm_{\alpha\beta} - \sum_{j=0}^{i-1} D^j m_{\alpha\beta}, \tag{2.5}$$

and an induction on i immediately shows that the coefficients $D^i_{\alpha\beta}$, if they exist, are unique.

Take now an arbitrary $D \in \mathcal{D}^k(U)$ and set $\Delta = D - \sum \in \mathcal{D}^k(U)$, where $\sum$ denotes the RHS of (2.4) with the coefficients defined in (2.5). This operator Δ vanishes by construction on the polynomials of degree $\leq k$ in x, ξ.

For any $f_1, \ldots, f_{\ell-1}, h \in \mathcal{A}(U)$, $\ell \geq k+1$, we have

$$\Delta(f_1 \ldots f_{\ell-1}h) = \sum_{b=1}^{\ell-1} \sum \pm f_{i_1} \ldots f_{i_b} \Delta(f_{i_{b+1}} \ldots f_{i_{\ell-1}}h) + F(h), \tag{2.6}$$

as immediately seen when developing $F(h) := [\ldots[[\Delta, f_1], f_2], \ldots, f_{\ell-1}]h$. If $\ell > k+1$, the term $F(h)$ vanishes, whereas in the case $\ell = k+1$ it is given by $F(h) = F(1)h$. Equation (2.6) shows that $\Delta = 0$ on any polynomial of degree $k+1$, then, by induction, that $\Delta = 0$ on an arbitrary polynomial in x, ξ. Further, this equation entails that $\Delta \mathcal{I}_m^{k+c} \subset \mathcal{I}_m^c$, $m \in U$, $c \geq 1$, where $\mathcal{I}_m$ is the unique homogeneous maximal ideal of the stalk $\mathcal{A}_m$. However, in view of Hadamard's lemma [Lei80], we can, for any $f \in \mathcal{A}(U)$ and any $m \in U$, find a polynomial $P_{f,m}$ in x, ξ such that $f - P_{f,m} \in \mathcal{I}_m^{k+q+1}$. It follows that $\Delta f = \Delta(f - P_{f,m}) \in \mathcal{I}_m^{q+1}$, for all $m \in U$, so that $\Delta f = 0$. ■

Remark 2.3. It follows easily from the above proof that the order of a differential operator can be determined locally by looking at commutators with the multiplications by coordinate functions: $D \in \mathcal{D}^k(U)$ if and only if, for a given system $(u^1, u^2, \ldots, u^{p+q})$ of local coordinates in U,

$$[[\ldots[[D|_U, u^{i_1}], u^{i_2}], \ldots], u^{i_{k+1}}] = 0$$

for any sequence $i_1, \ldots, i_{k+1} \in \{1, \ldots, p+q\}$.

Remark 2.4. The idea of defining differential operators on an abstract commutative algebra by the formula (2.3) goes back to Grothendieck [Gro67] and Vinogradov [Vin72].

The $\mathbb{R}$-vector space $\underline{\mathrm{End}}(\mathcal{A}(U))$ carries natural associative and Lie algebra structures $\circ$ and $[-,-]$ (we often omit the symbol $\circ$). An induction on $k+\ell$ allows seeing that $\mathcal{D}^k(U) \circ \mathcal{D}^\ell(U) \subset \mathcal{D}^{k+\ell}(U)$ and $[\mathcal{D}^k(U), \mathcal{D}^\ell(U)] \subset \mathcal{D}^{k+\ell-1}(U)$, so that the vector space $\mathcal{D}(U) := \cup_{k\in\mathbb{N}}\mathcal{D}^k(U)$ of all differential operators inherits associative and Lie algebra structures that have weight 0 and -1, respectively, with respect to the filtration degree. It is easily checked that $\mathcal{D} : U \to \mathcal{D}(U)$ (resp. $\mathcal{D}^1 : U \to \mathcal{D}^1(U)$) is a locally free sheaf of $\mathcal{A}$-modules and of associative and Lie algebras (resp. of $\mathcal{A}$-modules and sub Lie algebras) over M. The algebra $\mathcal{D}(M)$ (resp. $\mathcal{D}^1(M)$) is the Lie algebra of *differential operators* (resp. *first-order differential operators*) of the manifold $\mathcal{M}$. In the sequel we denote this algebra also by $\mathcal{D}(\mathcal{M})$ or even by $\mathcal{D}$ (resp. by $\mathcal{D}^1(\mathcal{M})$ or $\mathcal{D}^1$).

The usual splitting of the space of first-order differential operators holds true in the case of supermanifolds.

Proposition 2.5. *Let $\mathcal{M} = (M, \mathcal{A})$ be a smooth manifold. An endomorphism $D \in \underline{\mathrm{End}}(\mathcal{A})$ is a first-order differential operator if and only if, for any $f, g \in \mathcal{A}$, we have*

$$D(fg) = (Df)g + (-1)^{|D||f|} f(Dg) - (D1)fg, \tag{2.7}$$

so that vector fields are those first-order differential operators D that verify $D1 = 0$. Moreover, the vector space $\mathcal{D}^1$ admits a canonical splitting

$$\mathcal{D}^1 = \mathcal{A} \oplus \mathcal{X} \tag{2.8}$$

given by $D \mapsto D1 + (D - D1)$.

Proof. To prove the direct implication (resp. converse implication), it suffices to compute $[[D, f], g](1)$ (resp. $[D, f](g)$), where $[-,-]$ denotes the commutator of endomorphisms and where 1 is the unit of $\mathcal{A}$. If $X \in \mathcal{X}$, we have $[X, f] = Xf \in \mathcal{A}$, so that $\mathcal{X} \subset \mathcal{D}^1$. The second claim now follows from Equation (2.7). It entails that the sum $\mathcal{A} + \mathcal{X}$ is direct. Finally, any $D \in \mathcal{D}^1$ decomposes in the form $D = D1 + (D - D1)$, where $D1 \in \mathcal{A}$ and $D - D1 \in \mathcal{X}$. ∎

For purely even manifolds it is a well-known fact that the derived ideal $\mathcal{X}(M)' = [\mathcal{X}(M), \mathcal{X}(M)]$ is the whole Lie algebra $\mathcal{X}(M)$ of vector fields of M [Gra78]. More precisely, if $X \in \mathcal{X}(M)$, $\operatorname{supp} X \subset U$, U open in M, then $X = \sum_{i=1}^n [X_i, Y_i]$, where n is independent of the considered X and where X_i, Y_i are vector fields of M with support $\operatorname{supp} X_i$, $\operatorname{supp} Y_i \subset U$ [Pon04]. The next theorem extends these result to the supercontext and will be used as a technical tool in the sequel. But let us begin with the following

Remark 2.6. The algebras of functions and of vector fields of manifolds of the type ΠV carry a $\mathbb{Z}_2$-compatible $\mathbb{N}$-grading. As mentioned before, we denote the parity-grading by subscripts and the $\mathbb{N}$-grading by superscripts. The grading is recognized by the *Euler vector field* ε, in the sense that $[\varepsilon, X] = nX$, for $X \in \mathcal{X}^n(\Pi V)$. Let us recall that the vector fields of degree 0 which, in local coordinates (x, ξ), read

$$X = \sum_i X^i(x)\partial_{x^i} + \sum_{a,b} X^a_b(x)\xi^b\partial_{\xi^a},$$

can be identified with the sections of the Atiyah algebroid of V. This identification is a Lie algebra isomorphism, so that

$$0 \to \ker\pi \to \mathcal{X}^0(\Pi V) \xrightarrow{\pi} \mathcal{X}(M) \to 0$$

is a split short exact sequence of Lie algebras. Hence, $\mathcal{X}^0(\Pi V) = \ker\pi \oplus \mathcal{X}(M)$. Note that in coordinates

$$\pi\left(\sum_i X^i(x)\partial_{x^i} + \sum_{a,b} X^a_b(x)\xi^b\partial_{\xi^a}\right) = \sum_i X^i(x)\partial_{x^i}$$

and $\varepsilon = \sum_a \xi^a\partial_{\xi^a}$.

Theorem 2.7. *Let $\mathcal{M}$ be a smooth manifold. Every $X \in \mathcal{X}(\mathcal{M})$ can be written as a finite sum of commutators,*

$$X = \sum_i [X_i, Y_i]\,, \tag{2.9}$$

where $X_i \in \mathcal{X}(\mathcal{M})$ and $Y_i \in \mathcal{X}_0(\mathcal{M})$. Moreover, if $\operatorname{supp} X \subset U$, U open in M, then X_i, Y_i can be chosen so that $\operatorname{supp} Y_i \subset \operatorname{supp} X$ and $\operatorname{supp}(\pi X_i) \subset U$ for any i. In particular, the derived algebra $\mathcal{X}'(\mathcal{M}) = [\mathcal{X}(\mathcal{M}), \mathcal{X}(\mathcal{M})]$ equals $\mathcal{X}(\mathcal{M})$.

Proof. Using the theorem stating that any smooth supermanifold $\mathcal{M} = (M, \mathcal{A})$ is diffeomorphic to the supermanifold ΠV, for some vector bundle V over M, we can assume that $\mathcal{M} = \Pi V$.

We first prove the theorem for $X \in \mathcal{X}(M)$, $\operatorname{supp} X \subset U$. According to [Gra93, Theorems (3.2) and (4.1)],

$$X \in [\mathcal{X}(M), [\mathcal{X}(M), X]]\,,$$

so that there are $X_i, Y_i' \in \mathcal{X}(M)$ such that

$$X = \sum_i [X_i, [Y_i', X]]\,.$$

Putting $Y_i{=}[Y_i', X]$, we can write X in the form (2.9) with $\operatorname{supp} Y_i{=}\operatorname{supp}[Y_i', X] \subset \operatorname{supp} X$. Let us observe that this is also true for any $X \in \mathcal{X}(\Pi V)$. Indeed, if $X \in \mathcal{X}^k(\Pi V)$, $k \neq 0$, then

$$X = \frac{1}{k}[\epsilon, X]\,,$$

where ϵ is the Euler vector field. If $X \in \mathcal{X}^0(\Pi V)$, $X \in \ker\pi$, then X vanishes on $C^\infty(M)$. It is well known that there are vector fields $Z_i \in \mathcal{X}(M)$ and smooth functions $f_i \in C^\infty(M)$ such that $\sum_i Z_i(f_i) = 1$ [Gra78]. As $X(f_i) = 0$ by assumption, we can write

$$X = \left(\sum_i Z_i(f_i)\right) X = \sum_i \left([Z_i, f_iX] - [f_iZ_i, X]\right)$$

and it is clear that $\operatorname{supp} f_i X \subset \operatorname{supp} X$. We can therefore assume the decomposition (2.9) with $\operatorname{supp} Y_i \subset \operatorname{supp} X$. Consider now a function $\psi \in C^\infty(M)$ with support in U that takes the value 1 on $\operatorname{supp} X$. We have

$$\sum_i [\psi X_i, Y_i] = \sum_i (\psi[X_i, Y_i] \pm (Y_i \psi) X_i) = \psi \sum_i [X_i, Y_i] = \psi X = X\,,$$

since $Y_i(\psi) = 0$, as $\psi = 1$ on the support of Y_i. ■

Our next goal is to explain how the manifold structure of $\mathcal{M}$ is encoded in the Lie algebra structure of $\mathcal{D}^1(\mathcal{M})$ and, further, to describe all the automorphisms of this algebra. More precisely, we will show that every Lie algebra isomorphism $\mathcal{D}^1(\mathcal{M}_1) \to \mathcal{D}^1(\mathcal{M}_2)$, where the dimension of the manifolds is different from $0|1$, respects the filtration $\mathcal{D}^0(\mathcal{M}_i) \subset \mathcal{D}^1(\mathcal{M}_i)$, $i = 1, 2$, and covers a diffeomorphism $\mathcal{M}_1 \to \mathcal{M}_2$. We prove that the group of automorphisms of the Lie algebra $\mathcal{D}^1(\mathcal{M})$ is a semi-direct product of the subgroup induced by supermanifold diffeomorphisms and another subgroup which consists of automorphisms determined by a 1-cocycle on the Lie algebra of vector fields $\mathcal{X}(\mathcal{M})$ with values in the algebra of functions on $\mathcal{M}$.

Remark 2.8. In contrast with even manifolds, the structure of a Lie algebra on all differential operators $\mathcal{D}(\mathcal{M})$ for a general supermanifold $\mathcal{M}$ does not "remember" its origin as the Lie algebra of differential operators: there are Lie algebra isomorphisms $\mathcal{D}(\mathcal{M}_1) \to \mathcal{D}(\mathcal{M}_2)$ not respecting the canonical filtration of differential operators $\mathcal{D}^k(\mathcal{M}) \subset \mathcal{D}^{k+1}(\mathcal{M})$, $k \geq 0$. This is because, in the pure odd situation, any linear operator on superfunctions is a differential operator. For instance, let $\mathcal{M}$ be ΠV for a finite-dimensional vector space V. Then $\mathcal{D}(\mathcal{M}) = \underline{\mathrm{End}}(\Lambda^*(V))$ where $\Lambda^* V$ is viewed as a supervector space with the canonical $\mathbb{Z}_2-$grading. Indeed, let ξ^i be linear coordinates on V counted as odd variables. It is clear that $\mathcal{D}(\mathcal{M})$ is generated as a associative algebra by ξ^i and ∂_{ξ^i} subject to the Clifford relations

$$\xi^i \partial_{\xi^j} + \partial_{\xi^j} \xi^i = \delta^i_j\,.$$

Thus $\mathcal{D}(\mathcal{M})$ is isomorphic as an associative algebra (correspondingly, as a Lie algebra) to the Clifford algebra (correspondingly, the underlying Lie algebra) of $V \oplus V^*$ supplied with the canonical pairing. The latter coincides up to an isomorphism with the algebra of all linear endomorphisms of the spinor module $\Lambda^* V$. On the other hand, V can be replaced with any maximal isotropic subspace of $V \oplus V^*$. For instance, one can interchange V and V^*; this will produce a Lie algebra automorphism of $\mathcal{D}(\mathcal{M})$ which apparently breaks the filtration. Therefore the automorphism do not have the nice geometrical description that is available for the Lie algebra of first-order differential operators.

3. Algebraic characterization of functions

This section provides a Lie algebraic characterization of superfunctions inside first-order superdifferential operators. In what follows, $\mathcal{A}$, $\mathcal{X}$, $\mathcal{D}^1$ denote the algebras of functions, vector fields, first-order differential operators of a smooth manifold $\mathcal{M}$ of dimension $p|q$.

Remark 3.1. In this paper we can assume that the odd dimension q of $\mathcal{M}$ is at least 1 – otherwise we investigate the already studied purely even situation [GP04].

Theorem 3.2. *If $\dim \mathcal{M}$ differs from $0|1$, the algebra $\mathcal{A} \subset \mathcal{D}^1$ is the unique maximal super Lie ideal of $\mathcal{D}^1$ consisting of* ad*-nilpotent elements. In particular, any automorphism of the Lie algebra $\mathcal{D}^1$ preserves $\mathcal{A}$, thus induces an automorphism of the Lie algebra $\mathcal{X} = \mathcal{D}^1/\mathcal{A}$.*

Proof. Clearly, $\mathcal{A} \subset \mathcal{D}^1$ is an ideal made up by ad-nilpotent elements. It thus suffices to prove that any another ideal with the same property is contained in $\mathcal{A}$.

First we identify $\mathcal{D}^1/\mathcal{A}$ as a Lie algebra with $\mathcal{X}$ via the canonical isomorphism $\mathcal{D}^1/\mathcal{A} \ni [D] \to D - D1 \in \mathcal{X}$. Further, we choose any diffeomorphism of supermanifolds between $\mathcal{M}$ and ΠV, where V is a certain vector bundle $V \to M$ of rank q. This choice provides a non canonical isomorphism between the algebras of functions, $\mathcal{A}$ and $\Gamma(\wedge V^*)$, which implements a $\mathbb{Z}_2$-compatible $\mathbb{N}$-grading on $\mathcal{A}$. The adjoint action of the corresponding even derivation or Euler vector field ϵ supplies $\mathcal{X}$ with a $\mathbb{Z}_2$-compatible $\mathbb{Z}$-grading $\mathcal{X}^k = \{X \in \mathcal{X} : [\epsilon, X] = kX\}$, $k \in \mathbb{Z}$, $k \geq -1$ [GKP09].

Suppose now that $\mathcal{J}$ is a super Lie ideal of $\mathcal{D}^1$ made up by ad-nilpotent elements and denote by $p : \mathcal{D}^1 \to \mathcal{D}^1/\mathcal{A} \simeq \mathcal{X}$ the canonical surjective Lie algebra morphism. Then $\mathcal{I} := p(\mathcal{J})$ is a Lie ideal of $\mathcal{X}$ whose elements are ad-nilpotent as well and that is moreover $\mathbb{Z}$-graded with respect to ϵ. To explain the last claim it suffices to prove that, if an element $X \in \mathcal{I} \subset \mathcal{X}$ decomposes as $X = \sum_{j=1}^{s} X_{k_j}$, $X_{k_j} \in \mathcal{X}^{k_j}$, $i > j \Rightarrow k_i > k_j$, then $X_{k_j} \in \mathcal{I}$ for all j. As $Y^m := ad_\epsilon^m X \in \mathcal{I}$ for all m, we obtain

$$\sum_{j=1}^{s} k_j^m X_{k_j} = Y^m \in \mathcal{I}, \quad m = 0, \ldots, s-1\,.$$

This linear system can be solved, since the corresponding Vandermonde matrix is nondegenerate, so that all X_{kj} are actually in $\mathcal{I}$. Furthermore, it follows from the proof of [GKP09, Proposition 2] that $\mathcal{I} \cap \mathcal{X}^0 = \{0\}$. Indeed, if $X \in \mathcal{I} \cap \mathcal{X}^0$, then $X \in \mathcal{I}_0$, which is a Lie ideal of $\mathcal{X}_0$ that is made up by ad-nilpotent elements. The mentioned proof then implies that $X \in \mathcal{X}^0 \cap \oplus_{i>0} \mathcal{X}^{2i}$, so that $X = 0$.

Let now X be a nonvanishing element of $\mathcal{I}$ and let $X_k \in \mathcal{X}^k \cap \mathcal{I}$ be one of its nonvanishing $\mathbb{Z}$-homogeneous terms. It is always possible to find $Y_1, \ldots, Y_r \in \mathcal{X}$ such that

$$[Y_1, \ldots, [Y_r, X_k] \ldots] \in \mathcal{X}^0 - \{0\}. \tag{3.1}$$

Indeed, it is easily seen that X_k must be 0, if the preceding multibracket vanishes, for $X_k \in \mathcal{X}^k$, $k \geq 1$, (resp. for $X_k \in \mathcal{X}^{-1}$) and for all $Y_i \in \mathcal{X}^{-1}$ (resp. $Y_i \in \mathcal{X}^1$) (a problem arises only if $\dim \mathcal{M} = 0|1$ and the degree of X is -1, as then $\mathcal{X}^1 = \{0\}$). Since the iterated bracket (3.1) is again in $\mathcal{I}$, we get a contradiction with the fact that $\mathcal{I} \cap \mathcal{X}^0 = \{0\}$. Therefore $p(\mathcal{J}) = \mathcal{I} = \{0\}$, so that $\mathcal{J} \subset \mathcal{A}$. ■

Remark 3.3. The assumption that the dimension of $\mathcal{M}$ differs from $0|1$ is crucial. Indeed, for a manifold of dimension $0|1$ there are automorphisms of the Lie algebra $\mathcal{D}^1$ not preserving $\mathcal{A}$, For, let ξ be an odd coordinate on $\mathcal{M}$. Then, $\mathcal{D}^1$ is spanned by functions $1, \xi$ and vector fields $\partial_\xi, \xi\partial_\xi$. It is easy to see that the liner map on $\mathcal{D}^1$ for which

$$\xi\partial_\xi \mapsto -\xi\partial_\xi\,, \quad \partial_\xi \leftrightarrow \xi\,, \quad 1 \mapsto 1$$

is an automorphism.

4. Reduction of the problem of automorphisms

In this section we fix a manifold $\mathcal{M}$ of dimension different from $0|1$ and we show that the quest for the automorphisms of the Lie algebra of first-order differential operators reduces to the computation of the even 1-cocycles of the Lie algebra of vector fields with values in the left $\mathcal{X}-$module of functions.

Clearly, any diffeomorphism φ of the manifold $\mathcal{M}$ induces an automorphism of the Lie algebra of first-order differential operators of $\mathcal{M}$. Indeed, any $\varphi \in \mathrm{Diff}(\mathcal{M})$ defines in particular an associative algebra automorphism $\varphi^* : \mathcal{A} \to \mathcal{A}$ and the induced Lie algebra automorphism is given by

$$\varphi_* : \mathcal{D}^1 \ni D \mapsto \varphi^{*-1} \circ D \circ \varphi^* \in \mathcal{D}^1\,.$$

In the following we use the canonical splitting $\mathcal{D}^1 = \mathcal{A} \oplus \mathcal{X}$ and we denote the projection $\mathcal{D}^1 \ni D \mapsto D - D1 \in \mathcal{X}$ by ρ. Using the fact that, for $D \in \mathcal{D}^1$ and $f \in \mathcal{A}$, we have $[D, f] = Df - (D1)f$, we immediately check that ρ is a representation by derivations of the Lie algebra $\mathcal{D}^1$ on the vector space $\mathcal{A}$.

Proposition 4.1. *Any Lie algebra automorphism $\phi\colon \mathcal{D}^1 \to \mathcal{D}^1$ splits into a product $\varphi_* \circ \phi_c$, where φ_* is the automorphism induced by a diffeomorphism $\varphi\colon \mathcal{M} \to \mathcal{M}$, and where the automorphism ϕ_c has the form $\phi_c = \mathrm{id} + c$, with c an even 1-cocycle on $\mathcal{D}^1$ with values in $\mathcal{A}$.*

Proof. It follows from Theorem 3.2 that any automorphism $\phi\colon \mathcal{D}^1 \to \mathcal{D}^1$ preserves $\mathcal{A}$. Thus ϕ induces a Lie algebra automorphism

$$\tilde{\phi} : \mathcal{X} \ni X \mapsto \phi(X) - \phi(X)1 \in \mathcal{X}. \tag{4.1}$$

All automorphisms of $\mathcal{X}$ are classified in [GKP09], see Theorem 2 and Proposition 5. It follows from the proof of Proposition 5 that they are implemented by a diffeomorphism of $\mathcal{M}$, unless $\dim \mathcal{M} = 1|1$ or $0|2$. In the latter case there exist additional automorphisms. Let us assume that $\dim \mathcal{M} = 1|1$ or $0|2$. Then the Euler vector field $\epsilon \in \mathcal{X}_0$ is well-defined up to a sign, such that any automorphism of $\mathcal{X}$, which is not coming from a diffeomorphism of

$\mathcal{M}$, interchanges ϵ with $-\epsilon$. Now we can make a canonical choice of the Euler vector field ϵ such that ϵ acts on $\mathcal{A}_1$ as the identity. Any automorphism of $\mathcal{X}$, which is induced by an automorphism of $\mathcal{D}^1$, must preserve the canonically chosen ϵ. Thus, in both exceptional (low-dimensional) cases an automorphism of $\mathcal{X}$ which exchanges ϵ and $-\epsilon$ does not admit an extension to first-order differential operators. Therefore, each $\tilde{\phi}$ is always induced by a diffeomorphism $\varphi\colon \mathcal{M} \to \mathcal{M}$.

Let us take a diffeomorphism φ of $\mathcal{M}$, such that

$$\tilde{\phi} : \mathcal{X} \ni X \mapsto \varphi^{*-1} \circ X \circ \varphi^* \in \mathcal{X}.$$

The automorphism $\tilde{\psi} = \tilde{\varphi}_*^{-1} \circ \tilde{\phi}$ of $\mathcal{X}$, induced by the automorphism $\psi := \varphi_*^{-1} \circ \phi$ of $\mathcal{D}^1$, is the identity map. Hence, for $X \in \mathcal{X}$, we have $\psi(X) = X + \psi(X)1$ and, for $D = f + X \in \mathcal{D}^1$, we thus get $\psi(D) = f + X + \psi(X)1 + \psi(f) - f = D + c(D)$, where $c : \mathcal{D}^1 \to \mathcal{A}$ is an even linear map. The injectivity and surjectivity of ψ are equivalent to the corresponding property of $(\mathrm{id} + c)|_{\mathcal{A}}$. To prove that the surjectivity of this restriction implies that of ψ, it suffices, if $\Delta = g + Y \in \mathcal{D}^1$, to set $g - c(Y) = f + c(f)$, for some $f \in \mathcal{A}$, and to observe that $\psi(f + Y) = \Delta$. Eventually, the Lie algebra homomorphism property of $\psi = \mathrm{id} + c$ is clearly equivalent with the 1-cocycle condition of $(\mathcal{D}^1, \rho)$ for c. ■

Theorem 4.2. *The automorphism ϕ_c can be uniquely written in the form*

$$\phi_c(f + X) = (\kappa f + \gamma(X)) + X\,,$$

where κ is a non-zero constant and $\gamma : \mathcal{X} \to \mathcal{A}$ is an even 1-cocycle of the Lie algebra $\mathcal{X}$ canonically represented on $\mathcal{A}$.

Proof. Note first that, in view of the preceding proof, any even 1-cocycle c on the Lie algebra $\mathcal{D}^1$ with values in $\mathcal{A}$, such that $(\mathrm{id} + c)|_{\mathcal{A}}$ is bijective, defines an automorphism $\phi_c = \mathrm{id} + c$. The 1-cocycle condition obviously splits into the intertwining condition

$$c(Xf) = X(c(f)),\ \forall X \in \mathcal{X}, f \in \mathcal{A}, \tag{4.2}$$

and the 1-cocycle condition of the Lie algebra $\mathcal{X}$ canonically represented upon $\mathcal{A}$. We prove below in Lemma 4.3 that the intertwining condition means that $c|_{\mathcal{A}} = \lambda \cdot \mathrm{id}$, $\lambda \neq -1$, where the exclusion of the value -1 is due to the condition that $(\mathrm{id} + c)|_{\mathcal{A}}$ be a bijection. Thus,

$$\phi_c(f + X) = f + X + c(f) + c(X) = ((1 + \lambda)f + c(X)) + X$$

and it is easy to see that $\gamma : \mathcal{X} \to \mathcal{A}$, $\gamma = c_{|\mathcal{X}}$, is a 1-cocycle with coefficients in the canonical representation. ■

Lemma 4.3. *Any even linear map $c : \mathcal{A} \to \mathcal{A}$ that satisfies the intertwining condition*

$$c(Xf) = X(c(f))$$

for all $X \in \mathcal{X}$ and $f \in \mathcal{A}$ is of the form $c(f) = \lambda f$, $\lambda \in \mathbb{R}$.

Proof. As in the proof of Theorem 2.7, we can assume that $\mathcal{M} = \Pi V$, where V is a vector bundle over M, and consider vector fields $Z_i \in \mathcal{X}(\Pi V)$ and functions $f_i \in C^\infty(M)$, such that $\sum_i Z_i(f_i) = 1$. Since

$$f = f \cdot 1 = \sum_i (fZ_i)(f_i),$$

we see that $c(f) = f \sum_i Z_i(c(f_i))$. Let us put $\lambda = \sum_i Z_i(c(f_i)) \in \mathcal{A}$. As $\lambda = c(1)$ and for each $X \in \mathcal{X}$ we have $X(\lambda) = X(c(1)) = c(X(1)) = 0$, the function λ is actually a constant. ■

Remark 4.4. The additional automorphisms of the Lie algebra $\mathcal{X}(\mathcal{M})$ in the exceptional cases can be described as follows. In the first case, when $\dim \mathcal{M} = 1|1$, the manifold $\mathcal{M}$ is isomorphic to ΠL for a real line bundle $L \to M$ and $\mathcal{A}_1 \simeq \Gamma(L^*)$. Taking into account that the structure group L can be reduced to $O(1) = \mathbb{Z}_2$, one can always choose a trivialization σ_0 of $L^{\otimes 2}$ and a flat connection ∇ on L (and thus a flat connection on $L^{\otimes 2}$, denoted by the same letter) such that $\nabla(\sigma_0) = 0$. On the other hand, M is either $\mathbb{R}$ or S^1, so M is always orientable. Let μ be a volume form on M, then $\sigma_0 \otimes \mu$ determines a bundle isomorphism $L^* \otimes TM \to L$, which gives rise to a bundle automorphism χ of $L \oplus L^* \otimes TM$, interchanging L and $L^* \otimes TM$ such that $\chi^2 = \mathrm{Id}$. Taking into account that $\mathcal{X}_1$ is isomorphic to $\Gamma(L \oplus L^* \otimes TM)$ as a vector space, we obtain an invertible linear map $\mathcal{X}_1 \to \mathcal{X}_1$. The choice of a flat connection ∇ determines a Lie algebra isomorphism $\mathcal{X}_0 \simeq D^1(M)$, where $D^1(M)$ is the Lie algebra of first-order differential operators on M. The commutator relations in $\mathcal{X}$ are given by the following formulas:

$$\begin{aligned} [v+f, s] &= \nabla_v(s) - fs \\ [v+f, \theta \otimes v'] &= \nabla_v(\theta) \otimes v' + \theta \otimes [v, v'] + f\theta \otimes v', \\ [s, \theta \otimes v'] &= \langle \theta, s\rangle v' + \langle \theta, \nabla_{v'} s\rangle, \end{aligned}$$

where $v, v' \in \Gamma(TM)$, $s \in \Gamma(L)$, $\theta \in \Gamma(L^*)$, and $f \in C^\infty(M)$; $\mathcal{X}_1$ is a faithful $\mathcal{X}_0-$module. The linear invertible map χ determines a Lie algebra automorphism of $\mathcal{X}_0$ of the form $v + f \mapsto v + div_\mu(v) - f$, where $div_\mu(v) = L_v(\mu)\mu^{-1}$. Using appropriate local coordinates (t, ξ), such that $\mu = \mathrm{d}t$, $\nabla(\xi) = 0$, and $\sigma_0 = \xi^{\otimes(-2)}$, we can write the whole transformation in the following form (here h and f are arbitrary local smooth functions):

$$\begin{aligned} h(t)\partial_\xi &\mapsto h(t)\xi\partial_t, \\ h(t)\xi\partial_t &\mapsto h(t)\partial_\xi, \\ h(t)\partial_t + f(t)\xi\partial_\xi &\mapsto h(t)\partial_t + (\partial_t h(t) - f(t))\xi\partial_\xi. \end{aligned}$$

It is easy to verify that such a transformation is a Lie algebra automorphism. In fact, the group of supermanifold diffeomorphisms of $\mathcal{M}$ acts freely and transitively on the set of automorphisms which interchange ϵ and $-\epsilon$, so a particular choice of χ is nothing but the choice of "an origin".

In the second case, when $\dim \mathcal{M} = 0|2$, the manifold $\mathcal{M}$ is isomorphic to ΠV for a 2-dimensional vector space V. The even part of $\mathcal{X}$ is naturally isomorphic to $\mathfrak{gl}(V)$, while the odd part is isomorphic to $V \oplus \Lambda^2 V^* \otimes V$. Let us fix a constant volume form $c \in \Lambda^2 V^*$. The Lie algebra $\mathfrak{gl}(V)$ is a direct sum of $\mathfrak{sl}(V)$ and the center of $\mathfrak{gl}(V)$ spanned by Id. The subalgebra $\mathfrak{sl}(V)$ preserves c, which makes V and $\Lambda^2 V^* \otimes V$ into isomorphic $\mathfrak{sl}(V)$-modules. Let us combine the isomorphism $V \xrightarrow{c\otimes} \Lambda^2 V^* \otimes V$ with the Lie algebra isomorphism of $\mathfrak{gl}(V)$ which preserves all elements of $\mathfrak{sl}(V)$ and interchanges Id and $-$Id (such an automorphism is a composition of a conjugation and the opposite to a transposition). One can easily check that the obtained linear map $\mathcal{X} \to \mathcal{X}$ is a Lie algebra automorphism.

5. Cohomology of supervector fields represented on functions

Let $\gamma : \mathcal{X}(\mathcal{M}) \to \mathcal{A}(\mathcal{M})$ be an even 1-cocycle, so that, for any $X, Y \in \mathcal{X}(\mathcal{M})$,

$$\gamma([X,Y]) = X(\gamma(Y)) - (-1)^{|X||Y|} Y(\gamma(X)) . \tag{5.1}$$

Proposition 5.1. *Every even 1-cocycle on the Lie algebra $\mathcal{X}(\mathcal{M})$ represented upon $\mathcal{A}(\mathcal{M})$ is a local operator.*

Proof. Let $X \in \mathcal{X}(\mathcal{M})$ such that $X|_U = 0$, $U \subset M$ open, and let $x_0 \in U$. According to Theorem 2.7, the vector field X reads $X = \sum_i [X_i, Y_i]$, for some $X_i \in \mathcal{X}(\mathcal{M})$ and some $Y_i \in \mathcal{X}_0(\mathcal{M})$ which are 0 in a neighbourhood of x_0 as well. Hence, in view of the cocycle condition,

$$\gamma(X) = \sum_i \left(X_i(\gamma(Y_i)) - (-1)^{|X_i||Y_i|} Y_i(\gamma(X_i)) \right) ,$$

$\gamma(X) = 0$ in a neighbourhood of x_0. ■

The next result gives the local form of the even 1-cocycles of $\mathcal{X}(\mathcal{M})$ with values in $\mathcal{A}(\mathcal{M})$. We will denote by $\Omega^1(U)$, $U \subset M$ open, the $\mathcal{A}(U)$-module of differential 1-forms over U and $\Omega^1_0(U)$ will refer to its even part.

Theorem 5.2. *Let $(U, u = (u^1, \ldots, u^{p+q}))$ be any coordinate chart of $\mathcal{M}$. The restriction to U of any even 1-cocycle $\gamma : \mathcal{X}(\mathcal{M}) \to \mathcal{A}(\mathcal{M})$ is of the form*

$$\gamma|_U \left(\sum_k \partial_k \cdot g^k \right) = \sum_k \left(a\, \partial_k g^k + \omega_k\, g^k \right) , \tag{5.2}$$

where $\partial_k = \partial_{u^k}$, $g^k \in \mathcal{A}(U)$, $a \in \mathbb{R}$, and $\omega = \sum_k du^k\, \omega_k$ is a closed even 1-form.

Remark 5.3. As in this text we use the Deligne sign convention for the wedge product, the super de Rham operator d has parity 0.

Proof. Due to locality, the restriction $\gamma|_U$ is an even 1-cocycle on $\mathcal{X}(U)$ with values in $\mathcal{A}(U)$. In the following we often omit the restriction to U and write simply γ, $\mathcal{X}$, $\mathcal{A}$, etc. We will of course show that γ is a differential operator.

When looking at the cocycle condition for $X = u^i\partial_j$ and $Y = g\partial_k$, where $g \in \mathcal{A}$, we are naturally led to introduce the map

$$\gamma_k : \mathcal{A} \ni g \mapsto (-1)^{|u^k||g|}\gamma(g\partial_k) = \gamma(\partial_k \cdot g) \in \mathcal{A}\,.$$

The cocycle equation then means that the map

$$g \mapsto \gamma_k(u^i\partial_j g) - \delta^i_k\gamma_j g - (-1)^{|u^k|(|u^i|+|u^j|)}u^i\partial_j(\gamma_k g) \tag{5.3}$$

is a differential operator of order 0 and parity $|u^i| + |u^j| + |u^k|$. Similarly, taking $X = \partial_j$ and $Y = g\partial_k$, we get that the map

$$g \mapsto \gamma_k(\partial_j g) - (-1)^{|u^k||u^j|}\partial_j(\gamma_k g) \tag{5.4}$$

is a differential operator of order 0 and parity $|u^j| + |u^k|$. Thus, subtracting from operator (5.3) the operator (5.4) multiplied from the left by $(-1)^{|u^k||u^i|}u^i$, we obtain that

$$T_{k,i,j} = [\gamma_k, u^i] \circ \partial_j - \delta^i_k\gamma_j \tag{5.5}$$

is a differential operator of order 0 and parity $|u^i| + |u^j| + |u^k|$.

If $i \neq k$, then $T_{k,i,j}$ reduces to $[\gamma_k, u^i] \circ \partial_j$. The latter is of order 0 and vanishes on constants, so $[\gamma_k, u^i] \circ \partial_j = 0$, for all $i \neq k$ and all j. This in turn implies that

$$[\gamma_k, u^i] = 0 \quad \text{for} \quad i \neq k\,. \tag{5.6}$$

Indeed, if there exists an even coordinate u^j, we can integrate classical functions with respect to u^j and the result is obvious. In the pure odd case, we have in particular $[\gamma_k, u^i] \circ \partial_i = 0$, $i \neq k$, so that

$$0 = [\gamma_k, u^i](\partial_i(u^i g)) = [\gamma_k, u^i](g) + (-1)^{|u^i|}[\gamma_k, u^i](u^i\partial_i g), \tag{5.7}$$

for all $g \in \mathcal{A}$. But, as easily checked, for odd u^i, the commutator $[\gamma_k, u^i]$ commutes with the multiplication by u^i, so that

$$[\gamma_k, u^i] \circ (u^i\partial_i) = (-1)^{|u^i|(|u^k|+|u^i|)}u^i[\gamma_k, u^i] \circ \partial_i = 0,$$

and, according to Equation (5.7), $[\gamma_k, u^i] = 0$, which completes the proof of Equation (5.6).

For $i = k$, the differential operator $T_{i,i,j}$ reads

$$T_{i,i,j} = [\gamma_i, u^i] \circ \partial_j - \gamma_j$$

and, since it is of order 0, we have $[T_{i,i,j}, u^j] = 0$, i.e.,

$$[\gamma_i, u^i] + (-1)^{|u^j|}[[\gamma_i, u^i], u^j] \circ \partial_j - [\gamma_j, u^j] = 0\,. \tag{5.8}$$

As for $i \neq j$, the Jacobi identity and Equation (5.6) entail

$$[[\gamma_i, u^i], u^j] = (-1)^{|u^i||u^j|}[[\gamma_i, u^j], u^i] = 0\,,$$

we get

$$[\gamma_i, u^i] = [\gamma_j, u^j]\,. \tag{5.9}$$

Choosing $i = j$ in Equation (5.8), we obtain $[[\gamma_i, u^i], u^i] \circ \partial_i = 0$, which implies

$$[[\gamma_i, u^i], u^i] = 0\,, \tag{5.10}$$

if u^i is even. However, it suffices to develop the left-hand side of Equation (5.10) to conclude that the claim holds true for odd u^i as well. Therefore, when taking into account Equation (5.6), we finally get

$$[[\gamma_i, u^j], u^k] = 0\,, \tag{5.11}$$

for all i, j, k. The latter equation means that γ_k are first-order differential operators (see Remark 2.3).

According to Equation (5.6), γ_k commutes with multiplication by u^i, $i \neq k$, so they are of the form $\gamma_k = a_k \partial_k + \omega_k$, $a_k, \omega_k \in \mathcal{A}$. Since, for any first-order operator and any function, we have $[D, f] = Df - D1 \cdot f$, Equation (5.9) shows that $a_i = a_j =: a$. Finally,

$$\gamma\left(\sum_k \partial_k \cdot g^k\right) = \gamma\left(\sum_k (-1)^{|u^k||g^k|} g^k \partial_k\right) = \sum_k \left(a\,\partial_k g^k + \omega_k\, g^k\right)\,. \tag{5.12}$$

Note now that $|a| = 0$ and $|\omega_k| = |u^k|$, since γ_k has parity $|u^k|$. Starting from

$$[f\partial_i, g\partial_j] = f\partial_i g\,\partial_j - (-1)^{(|u^i|+|f|)(|u^j|+|g|)} g\partial_j f\,\partial_i\,,$$

we straightforwardly see that the corresponding cocycle condition provides, after simplification, an identically vanishing bidifferential operator in f and g. When writing that its coefficients vanish, we get

$$\partial_i a = 0\,, \tag{5.13}$$

$$\partial_i \omega_j - (-1)^{|u^i||u^j|} \partial_j \omega_i = 0\,, \tag{5.14}$$

for all i, j. We thus conclude that the differential operator γ defined by (5.12) is a 1-cocycle if and only if $a \in \mathbb{R}$ and the even differential 1-form $\omega = \sum_k du^k \omega_k$ is closed. ■

It is easily checked that $\gamma = \gamma|_U$ defined by Equation (5.12) has the following property with respect to the right module structure of $\mathcal{X} = \mathcal{X}(U)$ over $\mathcal{A} = \mathcal{A}(U)$:

$$\gamma(X \cdot f) = \gamma(X) \cdot f + a\,Xf\,. \tag{5.15}$$

Proposition 5.4. *Any even 1-cocycle $\gamma : \mathcal{X}(\mathcal{M}) \to \mathcal{A}(\mathcal{M})$ verifies Equation (5.15), for all $X \in \mathcal{X}(\mathcal{M})$, all $f \in \mathcal{A}(\mathcal{M})$, and some $a \in \mathbb{R}$.*

Proof. Indeed, the restriction $\gamma|_U$ to any chart domain U is of the form (5.12) and thus satisfies the local equation (5.15) for some $a_U \in \mathbb{R}$. It follows that, for any global X and f, we have

$$a_U(Xf)|_{U\cap V} = (\gamma(X\cdot f) - \gamma(X)\cdot f)|_{U\cap V} = a_V(Xf)|_{U\cap V},$$

where V denotes a chart domain that intersects U. Since the base manifold M of $\mathcal{M}$ is connected, all a_U coincide, which proves the claim. ■

Some authors, see e.g. [KSM02], define a divergence operator in $\mathcal{M}$ as an operator $\gamma : \mathcal{X}(\mathcal{M}) \to \mathcal{A}(\mathcal{M})$ that satisfies only (5.15) with $a = 1$. In this paper, we assume also the cocycle condition, which can be understood as a vanishing curvature condition for γ.

Definition 5.5. A *divergence* in a manifold $\mathcal{M}$ (or a *superdivergence*) is a 1-cocycle on $\mathcal{X}(\mathcal{M})$ with values in $\mathcal{A}(\mathcal{M})$ that satisfies Equation (5.15) with $a = 1$. Any even 1-cocycle will be called a *generalized divergence.*

Remark 5.6. We prove in the last section that in any smooth supermanifold there exists a divergence operator γ_0.

We are now able to describe the even part of the first cohomology group of the Lie algebra $\mathcal{X}(\mathcal{M})$ with values in $\mathcal{A}(\mathcal{M})$.

Theorem 5.7. *Let $\mathcal{M}$ be a manifold with a fixed divergence γ_0. Any even 1-cocycle $\gamma : \mathcal{X}(\mathcal{M}) \to \mathcal{A}(\mathcal{M})$, i.e. any generalized divergence, can be uniquely written as*

$$\gamma = a\,\gamma_0 + i_\omega\,,$$

where $a \in \mathbb{R}$ and ω is a closed even 1-form on $\mathcal{M}$. The cocycle γ is a coboundary if and only if $a = 0$ and $\omega = df$, $f \in \mathcal{A}_0(\mathcal{M})$. In other words,

$$H^1_0(\mathcal{X}(\mathcal{M}), \mathcal{A}(\mathcal{M})) = \mathbb{R}\,\gamma_0 \oplus H^1_{\mathrm{DR},0}(\mathcal{M}),$$

where $H^1_{\mathrm{DR},0}(\mathcal{M})$ is the even part of the first super de Rham cohomology group of $\mathcal{M}$.

Proof. If γ is a generalized divergence, it verifies Equation (5.15) for some $a \in \mathbb{R}$. Therefore, the difference $\gamma - a\gamma_0$ is a right $\mathcal{A}(\mathcal{M})$-module morphism from $\mathcal{X}(\mathcal{M})$ to $\mathcal{A}(\mathcal{M})$, so an even 1-form $\omega \in \Omega^1_0(\mathcal{M})$. The cocycle equation for $\omega = \gamma - a\gamma_0$ and Cartan's formula for the de Rham operator d show that $d\omega = 0$. Furthermore, $a\gamma_0 + \omega = \gamma = df$, $f \in \mathcal{A}_0(\mathcal{M})$, if and only if $a = 0$ and $\omega = df$. ■

It follows from Proposition 4.1 and Theorems 4.2 and 5.7 that for supermanifolds we get an analog of Theorem 8 from [GP04].

Theorem 5.8. *Let $\mathcal{M}$ be a smooth manifold of dimension different from $0|1$ and let γ_0 be a fixed divergence in $\mathcal{M}$. Then, an even linear map $\phi : \mathcal{D}^1(\mathcal{M}) \to$*

$\mathcal{D}^1(\mathcal{M})$ is an automorphism of the Lie algebra $\mathcal{D}^1(\mathcal{M}) = \mathcal{A}(\mathcal{M}) \oplus \mathcal{X}(\mathcal{M})$ of first-order differential operators of $\mathcal{M}$ if and only if it can be written in the form

$$\phi(f + X) = \varphi_*(X) + (\varphi^{-1})^*(\kappa f + a\,\gamma_0(X) + i_\omega(X)), \tag{5.16}$$

where φ is a diffeomorphism of $\mathcal{M}$, $a, \kappa \in \mathbb{R}$, $\kappa \neq 0$, and ω is a closed even 1-form. All the objects φ, a, κ, and ω are uniquely determined by ϕ.

6. Existence of superdivergences

Our aim in this section is to prove the existence of a divergence on each supermanifold. This will be done similarly to the proof of existence of a divergence on any standard (even) manifold with the use of a nowhere-vanishing 1-density understood as a class of volume forms up to a sign. The sheaf of top forms Ω^{top} of a classical differential manifold has to be replaced, in the case of a manifold $\mathcal{M} = (M, \mathcal{A})$, by the *Berezinian sheaf* $\mathrm{Ber} = \mathrm{Ber}(\mathcal{M})$ whose nowhere-vanishing sections are *Berezinian volumes.* A homogeneous Berezinian volume s defines a divergence γ_s of a homogeneous vector field X by the formula (see [KSM02])

$$\mathcal{L}_X s = (-1)^{|X||s|} s \cdot \gamma_s(X)\,.$$

Here $\mathcal{L}_X s$ is the Lie derivative of the Berezinian volume understood as a differential operator on $\mathcal{A}$ defined by $\mathcal{L}_X s = -(-1)^{|X||s|} s \circ X$.

Let us fix an atlas of supercharts of $\mathcal{M}$. Over a domain U with coordinates $u = (x^1, \ldots, x^p, \xi^1, \ldots, \xi^q)$, the (right) $\mathcal{A}(U)$-module $\mathrm{Ber}(U)$ is given by

$$\mathrm{Ber}(U) = \Gamma(U, \mathrm{Ber}) := d^{p|q}u\,\mathcal{A}(U) \simeq \left(dx^1 \wedge \ldots \wedge dx^p \otimes \partial_{\xi^q} \circ \ldots \circ \partial_{\xi^1}\right) \mathcal{A}(U)\,.$$

Since every supermanifold is diffeomorphic to ΠV for a vector bundle V of rank q over M, we can assume that $\mathcal{M} = \Pi V$, so that $\mathcal{A}$ is an algebra isomorphic with the Grassmann algebra of sections of $\Lambda^\bullet V^* = \bigoplus_{k=0}^\infty \Lambda^k V^*$ and local odd coordinates $(\xi^1, \ldots, \xi^q)$ are represented by sections of V^*. One can easily see that with any section v of the line bundle $\mathrm{Vol}(V) = \Lambda^p T^*M \otimes_M \Lambda^q V$ one can associate a Berezinian volume s_v. In other words, we have an embedding S of the space of sections of $\mathrm{Vol}(V)$ into the space of Berezinian densities. In local coordinates this embedding reads

$$dx^1 \wedge \ldots \wedge dx^p \otimes e_q \wedge \ldots \wedge e_1 \;\mapsto\; dx^1 \wedge \ldots \wedge dx^p \otimes \partial_{\xi^q} \circ \ldots \circ \partial_{\xi^1}\,,$$

where $e_1, \ldots, e_q$ is a basis of local sections of V and ∂_{ξ_i} is the derivation of $\mathcal{A}$ being the contraction of a section of $\Lambda^\bullet V^*$ with e_i. A linear change of coordinates in the vector bundle V, say $(x, \xi) \mapsto (y(x), \eta)$ with $\eta_j = \sum_i a^i_j(x)\xi_i$, results in multiplying both sides by

$$\det\left(\frac{\partial y^a}{\partial x^b}(x)\right) \det\left(a^i_j(x)\right)^{-1}\,,$$

so the embedding is well defined globally. Since the line bundles are classified by $H^1(M, \mathbb{Z}_2)$, we can find an open covering (U_α) on M and local nowhere-vanishing sections v_α of $\mathrm{Vol}(V)_{|U_\alpha}$ such that $v_\alpha = \pm v_{\alpha'}$ on $U_\alpha \bigcap U_{\alpha'}$. Hence,

$S(v_\alpha) = \pm S(v_{\alpha'})$ over $U_\alpha \bigcap U_{\alpha'}$. But the divergence γ_s does not depend on the sign of s, so the collection σ of local nowhere-vanishing Berezinian volumes $S(v_\alpha)$ gives rise to a (divergence) γ_σ on $\mathcal{M}$. Note that the collection of local nowhere-vanishing Berezinian volumes $S(v_\alpha)$ can be viewed as a nowhere-vanishing section of the 1-*density sheaf* $\mathfrak{D}_1 = \mathrm{Ber} \otimes \mathrm{or}(M)$ of $\mathcal{M}$, defined as the Berezinian sheaf twisted by the orientation sheaf of the body. In the literature, the sections of $\mathfrak{D}_1$ are sometimes referred to as *nonoriented Berezinian sections*. If $\sigma' = f\sigma$, with an even nowhere-vanishing function f, is another such section, then $\gamma_{\sigma'}(X) = \gamma_\sigma(X) + X(\ln f)$. Thus we get the following.

Theorem 6.1. *Let $\mathcal{M} = (M, \mathcal{A})$ be a smooth manifold. Then, there is a nondegenerate global section σ of $\mathfrak{D}_1$ with even coefficients. Moreover, for any such σ,*

$$\gamma_\sigma : \mathcal{X} \ni X \mapsto (\mathcal{L}_X \sigma)\sigma^{-1} \in \mathcal{A}$$

is a divergence of $\mathcal{M}$. Any other such section of $\mathfrak{D}_1$ implements a divergence in the same cohomology class.

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