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Unitary Harish-Chandra Modules over Block Type Lie Algebras $\mathcal{B}(q)$

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Abstract. For any nonzero complex number q, there is a Lie algebra of Block type, denoted by $\mathcal{B}(q)$. In this paper, we classify the unitary Harish-Chandra modules for these algebras. We first give a description of conjugate-linear anti-involutions on $\mathcal{B}(q)$. Then we obtain the sufficient and necessary conditions for irreducible uniformly bounded unitary Harish-Chandra modules and irreducible unitary highest (lowest) weight Harish-Chandra modules.
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1. Introduction

Because of wide applications in many mathematics and physics branches, the representation theory of the Virasoro algebra has been extensively studied ([CP], [KR]). Recently, many authors studied a class of Lie algebras closely related to the Virasoro algebra, namely, the so called Block type algebras $\mathcal{B}(q)$, where q a nonzero complex number.

In particular, [SXX] divided the irreducible Harish-Chandra modules over $\mathcal{B}(q)$ into two classes: highest or lowest weight modules and uniformly bounded modules; the same result was obtained by [WT] for the algebra $\mathcal{B}(1)$. Recently, [CGZ] completely classified all irreducible Harish-Chandra modules over $\mathcal{B}(q)$, which turn out to be highest weight modules, lowest weight modules and modules of intermediate series. In [XZ], the unitarity was determined for the highest weight Harish-Chandra modules over $\mathcal{B}(q)$ with respect to a special conjugate-linear anti-involution on $\mathcal{B}(q)$. In the present paper, we determined the unitarity of any Harish-Chandra module over $\mathcal{B}(q)$ with respect to any conjugate-linear anti-involution on $\mathcal{B}(q)$. Let us first recall the definition of the Lie algebras $\mathcal{B}(q)$.

Denote by $\mathbb{Z}$, $\mathbb{N}$, $\mathbb{Z}_+$, $\mathbb{R}$ and $\mathbb{C}$ the sets of integers, positive integers, nonnegative integers, real numbers and complex numbers respectively. For any complex number q, the Lie algebra $\mathcal{B}(q)$ has a basis $\{L_{m,i}, C \mid m \in \mathbb{Z}, i \in \mathbb{Z}_+\}$ over $\mathbb{C}$ subject to the following Lie

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brackets

$$
\begin{aligned}
[L_{m,i}, L_{n,j}] &= (n(i+q) - m(j+q))L_{m+n,i+j} + \delta_{m+n,0}\delta_{i+j,0}\frac{m^3-m}{12}C, \\
[C, L_{m,i}] &= 0, \quad \forall\ m,n \in \mathbb{Z},\ i,j \in \mathbb{Z}_+.
\end{aligned}
\tag{1}
$$

Note that the Lie algebras $\mathcal{B}(q)$ are in fact subalgebras of some special cases of generalized Block algebras studied in [DZ], and the Lie algebra $\mathcal{B}(0)$ is a half part of the well-known Virasoro-like algebra. The most important fact is that for any $q \neq 0$, $\mathcal{B}(q)$ contains a subalgebra isomorphic to the Virasoro algebra, that is, the subalgebra spanned by C and $L_{m,0}, m \in \mathbb{Z}$.

The paper is organized as follows. In Section 2, we first recall some known results for the algebras $\mathcal{B}(q)$ and the Virasoro algebra. In Section 3, we give a description of conjugate-linear anti-involutions on $\mathcal{B}(q)$ for $q \neq 0$, then we obtain the conditions for irreducible Harish-Chandra modules to be unitary.

For any subset S in $\mathbb{C}$, denote $S^* = S \setminus \{0\}$. Throughout this paper, we always assume $q \in \mathbb{C}^*$. All vector spaces and (Lie) algebras are over $\mathbb{C}$. For a Lie algebra $\mathcal{G}$, we denote its universal enveloping algebra by $U(\mathcal{G})$. For $a \in \mathbb{C}$ and $S \subseteq \mathbb{C}$, $\delta_{a,S} = 1$ if $a \in S$ and 0 otherwise.

2. Preliminaries

Let $\mathcal{B}_0$ be the canonical Virasoro subalgebra of $\mathcal{B}(q)$, i.e.,

$$
\mathcal{B}_0 = \bigoplus_{m\in\mathbb{Z}} \mathbb{C}L_{m,0} \oplus \mathbb{C}C.
$$

Let $\mathcal{G} = \mathcal{B}(q)$ or $\mathcal{B}_0$, then we have the standard Cartan subalgebra $Z + \mathbb{C}L_{0,0}$, where Z is the center of $\mathcal{G}$. It is not hard to see that $Z = \mathbb{C}C \oplus \mathbb{C}L_{0,-q}$ if $\mathcal{G} = \mathcal{B}(q)$ if $-q \in \mathbb{N}$ and $Z = \mathbb{C}C$ if $\mathcal{G} = \mathcal{B}_0$ or $\mathcal{G} = \mathcal{B}(q)$ for $-q \notin \mathbb{N}$. Let $\mathcal{G}_\pm$ be the span of vectors in $\mathcal{G}$ with positive/negative eigenvalues of $\mathrm{ad}(q^{-1}L_{0,0})$ respectively.

We can define weight modules for $\mathcal{G}$ with respect to the Cartan subalgebra $Z+\mathbb{C}L_{0,0}$, that is, $\mathcal{G}$-modules with the action of $Z + \mathbb{C}L_{0,0}$ being diagonalizable. Since our objects in this paper are irreducible modules, where the central elements act as scalars, so we can make the convention that **all weight modules are referred to the weight modules with central elements acting as scalars** in this paper. In this sense any weight $\mathcal{G}$-module V can be decomposed as

$$
V = \bigoplus_{n\in\mathbb{Z}} V_n, \quad \text{where } V_n = \{v \in V \mid L_{0,0}v = (\lambda_0 + qn)v\}
\tag{2}
$$

for some complex number λ_0 depending on V.

A weight $\mathcal{G}$-module V is called *highest/lowest weight* if $V = U(\mathcal{G})v$ for some nonzero v with $\mathcal{G}_\pm v = 0$ respectively, is called *Harish-Chandra* (or *quasifinite*) if all its weight spaces are finitely dimensional, is called *uniformly bounded* if all its weight spaces have dimension less than a fixed number, and is called *a module of the intermediate series* if all its weight spaces are no more than 1-dimensional.

By the representation theory of the Virasoro algebra, we have that any irreducible uniformly bounded $\mathcal{B}_0$-module is isomorphic to an irreducible submodule of $V(\alpha,\beta)$ for some $\alpha,\beta\in\mathbb{C}$. The module $V(\alpha,\beta)$ has a basis $\{v_n \mid n\in\mathbb{Z}\}$ with trivial central action and

$$L_{m,0}v_n = q(\alpha+n+m\beta)v_{m+n}.$$

The module $V(\alpha,\beta)$ is reducible if and only if $\alpha\in\mathbb{Z}$ and $\beta=0,1$.

Now we can define some $\mathcal{B}(q)$-modules of intermediate series:

(1). $V=V(\alpha,\beta,0,0)$ for all $q\neq 0$: $L_{m,i}V=0$ for all $i\geq 1$ and $V\cong V(\alpha,\beta)$ as $\mathcal{B}_0$-modules;

(2). $V=V(\alpha,\beta,K,0)$ for $-2q\in\mathbb{N}$: $L_{m,i}V=0$ for all $i\geq 1$ and $(m,i)\neq(0,-2q)$, $V\cong V(\alpha,\beta)$ as $\mathcal{B}_0$-modules and $L_{0,-2q}$ acts as a scalar K;

(3). $V=V(\alpha,\beta,K,F)$ for $q=-1$: $L_{m,i}V=0$ for all $i\geq 2$ and $(m,i)\neq(0,2)$, $L_{m,1}v_n=Fv_{m+n}$, $V\cong V(\alpha,\beta)$ as $\mathcal{B}_0$-modules and $L_{0,2}$ acts as a scalar K.

It is easy to see that $V(\alpha,\beta,K,F)$ is reducible if and only if $F=0,\alpha\in\mathbb{Z}$ and $\beta=0,1$. We denote the unique infinite-dimensional irreducible subquotient by $V'(\alpha,\beta,K,F)$. For any $\alpha\in\mathbb{Z}$, $\beta\in\{0,1\}$, $V(\alpha,\beta,K,0)$ has a unique 1-dimensional subquotient independent of the choice of α and β, which we denote by $T(K)$.

Recently all irreducible Harish-Chandra modules over $\mathcal{B}(q)$ were classified in [CGZ].

Theorem 2.1. *Let $q\in\mathbb{C}^*$. Any irreducible Harish-Chandra module over $\mathcal{B}(q)$ is isomorphic to an irreducible highest weight module or an irreducible lowest weight module, or a module of the form $T(K)$ or $V'(\alpha,\beta,K,F)$ for suitable $\alpha,\beta,K,F\in\mathbb{C}$.*

3. Unitary modules

In this section we study the unitarity of Harish-Chandra modules over $\mathcal{B}(q)$. As a result, we give a complete classification of irreducible unitary Harish-Chandra modules over $\mathcal{B}(q)$.

Definition 3.1. Let $\mathcal{G}$ be a Lie algebra. A map $\theta:\ \mathcal{G}\to\mathcal{G}$ is called a *conjugate-linear anti-involution* of $\mathcal{G}$ if it satisfies

$$\theta(ax)=\bar{a}\theta(x),\ \ \theta(x+y)=\theta(x)+\theta(y),\ \ \theta([x,y])=[\theta(y),\theta(x)],\ \ \theta^2(x)=x$$

for all $x,y\in\mathcal{G}$ and $a\in\mathbb{C}$.

Definition 3.2. A module V over $\mathcal{G}$ is called *unitary* for some conjugate-linear anti-involution θ of $\mathcal{G}$ if there exists a positive definite Hermitian form $\langle\ \cdot\,,\cdot\,\rangle$ which is contravariant with respect to θ, namely,

$$\langle\, xv\,,\,w\,\rangle=\langle\, v\,,\,\theta(x)w\,\rangle,\ \forall\, x\in\mathcal{G},v,w\in V.$$

Since $\mathcal{B}(q)$ contains a subalgebra $\mathcal{B}_0$ isomorphic to the Virasoro algebra, we first recall some known results for conjugate-linear anti-involutions on $\mathcal{B}_0$ and unitary modules over $\mathcal{B}_0$. It is easy to check that the following formulas define conjugate-linear anti-involutions $\theta_a^{\pm}$ for $\mathcal{B}_0$:

(1). $\theta_a^+(q^{-1}L_{m,0}) = a^m q^{-1}L_{-m,0}$, $\theta_a^+(q^{-2}C) = q^{-2}C$, for some $a \in \mathbb{R}^*$;

(2). $\theta_a^-(q^{-1}L_{m,0}) = -a^m q^{-1}L_{m,0}$, $\theta_a^-(q^{-2}C) = -q^{-2}C$, for some $a \in S^1$,

where S^1 is the set of all complex numbers with norm one.

Proposition 3.3 ([CP]). *Any conjugate-linear anti-involution of $\mathcal{B}_0$ is one of $\theta_a^{\pm}$ defined above. Let V be a nontrivial irreducible Harish-Chandra $\mathcal{B}_0$-module.*

(1) If V is unitary for some conjugate-linear anti-involution θ of $\mathcal{B}_0$, then $\theta = \theta_a^+$ for some $a > 0$.

(2) If V is unitary for θ_a^+ for some $a > 0$, then V is unitary for θ_1^+.

(3) If V is unitary, then V is either highest or lowest weight, or isomorphic to $V(\alpha,\beta)$ for some $\alpha \in \mathbb{R}$, $\beta \in \frac{1}{2} + \sqrt{-1}\mathbb{R}$.

Now we can give a description of all conjugate-linear anti-involutions on $\mathcal{B}(q)$.

Proposition 3.4. *Suppose θ is a conjugate-linear anti-involution of $\mathcal{B}(q)$, then $q \in \mathbb{R}^*$ and θ is one of the following types:*

(1) $\theta^+_{a,b,c,d}(L_{m,i}) = a^m b^{2i} L_{-m,i} + \delta_{m,0}\delta_{i,-2q}\big(cC + \delta_{-q,\mathbb{N}} dL_{0,-q}\big)$, $\theta^+_{a,b,c,d}(C) = C$, for some $a \in \mathbb{R}^*$, $b \in S^1$ and $c, d \in \mathbb{C}$ with $b^{2q}c \in \sqrt{-1}\mathbb{R}$ and $b^{3q}d \in \sqrt{-1}\mathbb{R}$;

(2) $\theta^-_{a,b,c,d}(L_{m,i}) = -a^m b^{2i} L_{m,i} + \delta_{m,0}\delta_{i,-2q}\big(cC + \delta_{-q,\mathbb{N}} dL_{0,-q}\big)$, $\theta^-_{a,b,c,d}(C) = -C$, for some $a, b \in S^1$ and $c, d \in \mathbb{C}$ with $b^{2q}c \in \sqrt{-1}\mathbb{R}$ and $b^{3q}d \in \sqrt{-1}\mathbb{R}$.

Proof. Assume that θ is a conjugate-linear anti-involution of $\mathcal{B}(q)$. For convenience, assume

$$\theta(L_{m,i}) = \sum_{(n,j)\in\mathbb{Z}\times\mathbb{Z}_+} a_{m,i}^{n,j} L_{n,j} + b_{m,i}C,$$

for some $a_{m,i}^{n,j}, b_{m,i} \in \mathbb{C}$, of which only finitely many are nonzero for fixed $(m,i) \in \mathbb{Z}\times\mathbb{Z}_+$.

Since Z is the center of $\mathcal{B}(q)$, we have $\theta(Z) = Z$. Recall that $Z = \mathbb{C}C$ if $-q \notin \mathbb{N}$ and $Z = \mathbb{C}C \oplus \mathbb{C}L_{0,-q}$ if $-q \in \mathbb{N}$, so we may assume that

$$\theta(C) = \epsilon_1 C + \delta_{-q,\mathbb{N}}\epsilon_2 L_{0,-q},\ \theta(L_{0,-q}) = a_{0,-q}^{0,-q} L_{0,-q} + b_{0,-q}C \ \text{ (when } -q \in \mathbb{N}).$$

Note also that $H = Z \oplus \mathbb{C}L_{0,0}$ is the unique maximal abelian subalgebra of $\mathcal{B}(q)$ which acts semisimply on $\mathcal{B}(q)$ in the adjoint representation. We have $\theta(H) = H$ and may assume that

$$\theta(L_{0,0}) = a_{0,0}^{0,0} L_{0,0} + \delta_{-q,\mathbb{N}} a_{0,0}^{0,-q} L_{0,-q} + b_{0,0}C.$$

Since $\theta^2(L_{0,0}) = L_{0,0}$, we have $|a_{0,0}^{0,0}| = 1$. For convenience, we denote $\eta' = a_{0,0}^{0,0}$ and $\eta = \eta' q/\bar{q}$, then $|\eta| = |\eta'| = 1$. Applying θ to $[L_{0,0}, L_{m,i}] = mqL_{m,i}$, we have

$$-\eta' q \sum_{(n,j)\in(\mathbb{Z},\mathbb{Z}_+)} n a_{m,i}^{n,j} L_{n,j} = m\bar{q} \sum_{(n,j)\in(\mathbb{Z},\mathbb{Z}_+)} a_{m,i}^{n,j} L_{n,j} + m\bar{q} b_{m,i} C.$$

This implies that $\eta = \pm 1$, $b_{m,i} = 0$ for $m \neq 0$ and

$$\theta(L_{m,i}) = \sum_{j\in\mathbb{Z}_+} a_{m,i}^{-\eta m,j} L_{-\eta m,j} + \delta_{m,0} b_{m,i} C, \ \forall\, m \in \mathbb{Z}.$$

Claim 1. $\theta(\mathcal{B}_0) = \mathcal{B}_0$.

It suffices to prove that $a_{m,0}^{-\eta m,j} = 0$ for all $m \in \mathbb{Z}$ and $j \in \mathbb{N}$ and $\theta(C) = \epsilon_1 C$. Apply θ to $[L_{0,i}, L_{m,0}] = m(q+i)L_{m,i}$, hence we get

$$(\bar{q}+i)\theta(L_{m,i}) = \eta \sum_{j,k\in\mathbb{Z}_+} (q+k) a_{m,0}^{-\eta m,j} a_{0,i}^{0,k} L_{-\eta m,j+k}, \ \forall\, m \neq 0. \tag{3}$$

Then apply θ to $[L_{-m,0}, L_{m,i}] = m(2q+i)L_{0,i} + \delta_{i,0}\frac{m-m^3}{12}C$, we get

$$\begin{aligned}\sum_{j,k,l\in\mathbb{Z}_+} &(q+k)(2q+j+k+l) a_{m,0}^{-\eta m,j} a_{0,i}^{0,k} a_{-m,0}^{\eta m,l} L_{0,j+k+l} \\ &= (\bar{q}+i)(2\bar{q}+i) \sum_{k\in\mathbb{Z}_+} a_{0,i}^{0,k} L_{0,k} + \delta_{i,0}\delta_{-q,\mathbb{N}} \frac{1-m^2}{12} \bar{q}\epsilon_2 L_{0,-q}, \ \forall m \in \mathbb{Z}^*.\end{aligned} \tag{4}$$

Noticing that $a_{0,0}^{0,k} = \delta_{k,0}\eta'$ for $k \neq -q$, we have particularly for $i = 0$ that

$$\sum_{j,l\in\mathbb{Z}_+} \eta(2q+j+l) a_{m,0}^{-\eta m,j} a_{-m,0}^{\eta m,l} L_{0,j+l} = 2\bar{q}\eta' L_{0,0} + \delta_{-q,\mathbb{N}}\Big(2\bar{q} a_{0,0}^{0,-q} + \frac{1-m^2}{12}\epsilon_2\Big) L_{0,-q}, \ \forall m \in \mathbb{Z}^*. \tag{5}$$

For any $i \in \mathbb{N}$, we take the maximal $j, k, l \in \mathbb{Z}_+$ such that $a_{m,0}^{-\eta m,j} a_{0,i}^{0,k} a_{-m,0}^{\eta m,l} \neq 0$. If $j+l > 0$, then we can deduce $(q+k)(2q+j+k+l) = 0$ or $j+k+l = -q$ from (4) and $2q+j+l = 0$ or $j+l = -q$ from (5). Consequently $k(q+k)^2 = 0$ and

$$\theta(L_{0,i}) = \sum_{k'\in\mathbb{Z}, k'\leq|q|} a_{0,i}^{0,k'} L_{0,k'} + b_{0,i} C,$$

and hence $\dim(\theta(\sum_{i\in\mathbb{Z}} \mathbb{C}L_{0,i})) < +\infty$, contradicting the fact that $\theta^2 = 1$. Thus $j+l = 0$ and $a_{m,0}^{-\eta m,j} = 0$ for all $m \in \mathbb{Z}^*, j \in \mathbb{N}$. Finally

$$2\bar{q}\theta(L_{0,0}) = \theta([L_{-1,0}, L_{1,0}]) = [\theta(L_{1,0}), \theta(L_{-1,0})] \in \mathcal{B}_0$$

and

$$\theta(C) = \theta(8qL_{0,0} + 2[L_{2,0}, L_{-2,0}]) \in \mathcal{B}_0 \cap Z$$

complete the proof of the claim.

Now the restriction $\theta|_{\mathcal{B}_0}$ is a conjugate-linear anti-involution on $\mathcal{B}_0$ which is isomorphic to the Virasoro algebra. By Proposition 3.3, we have $\theta|_{\mathcal{B}_0} = \theta_a^\eta$, i.e., $a_{m,0}^{-\eta m,j} = \delta_{j,0}\eta' a^m$ or

$$\theta(L_{m,0}) = \eta a^m \bar{q} q^{-1} L_{-\eta m,0}, \ \ \theta(C) = \eta \bar{q}^2 q^{-2} C, \ \ \forall\ m \in \mathbb{Z},$$

for some $a \in \mathbb{R}^*$ if $\eta = 1$ and for some $a \in S^1$ if $\eta = -1$. Here we make the convention that η indicates the symbol $\pm$ when it appears as a superscript of θ, and otherwise η indicates the number ± 1. Taking sufficient large i in (4), we can deduce $q \in \mathbb{R}^*$ and $\eta = \eta'$.

By (3) we get

$$(q+i)\theta(L_{m,i}) = a^m \sum_{k\in\mathbb{Z}_+} (q+k) a_{0,i}^{0,k} L_{-\eta m,k}, \ \ \forall\ m \in \mathbb{Z}^*,\ i \in \mathbb{Z}_+. \tag{6}$$

Then applying θ on $[L_{m,i}, L_{n,0}] = (nq + ni - mq)L_{m+n,i} + \delta_{m+n,0}\delta_{i,0}\frac{m^3-m}{12}C$, we can deduce

$$\sum_{k\in\mathbb{Z}_+} a_{0,i}^{0,k}(q+k)(nq+nk-mq)L_{-\eta(m+n),k} = (nq+ni-mq)\sum_{k\in\mathbb{Z}_+} a_{0,i}^{0,k}(q+k)L_{-\eta(m+n),k},$$

for all $m(m+n) \neq 0$ and

$$q a_{0,i}^{0,0}(m^2-1)C + 12 b_{0,i}(q+i)(2q+i)C - \delta_{i,0}\eta q(m^2-1)C = 0, \ \forall\ m \neq 0,$$

which imply $a_{0,i}^{0,k} = 0$ for all $k \neq -q$, $i \neq k$ and $b_{0,i} = 0$ for all $i \neq -q, -2q$.

Setting $b_i = \eta a_{0,i}^{0,i}$, we deduce from (6) that

$$\theta(L_{m,i}) = \eta a^m b_i L_{-\eta m,i}, \ \forall\ m \neq 0,\ i \neq -q.$$

Apply θ to the Lie bracket (1) with $n(m+n)(j+q)(i+j+q) \neq 0$, we get

$$\sum_{k\in\mathbb{Z}} \big(m(q+j) - n(q+k)\big)\big(a_{m,i}^{-\eta m,k} - \delta_{k,i}\eta a^m b_{i+j} b_j^{-1}\big) L_{-\eta(m+n),j+k} = 0,$$

which yields that

$$a_{m,i}^{-\eta m,k} = \delta_{k,i}\eta a^m b_{i+j} b_j^{-1}, \ \forall\ m(j+q)(i+j+q) \neq 0$$

or in other words

$$\theta(L_{m,i}) = \eta a^m b_{i+j} b_j^{-1} L_{-\eta m,i}, \ \forall\ m(j+q)(i+j+q) \neq 0. \tag{7}$$

From (6) and (7), we deduce $b_i b_j = b_{i+j}$ for all $i, j, i+j \neq -q$ and hence there exists $b \in \mathbb{C}^*$ such that $b_i = b^{2i}$ for all $i \neq -q$. Indeed, take any $k > |q|$ and choose $b \in \mathbb{C}$ such that $b^2 = b_k^{-1} b_{k+1}$, then we can deduce that $b_i = b^{2i}$ for all $i \neq -q$. Now (7) becomes

$$\theta(L_{m,i}) = \eta a^m b^{2i} L_{-\eta m,i}, \ \forall\ m \neq 0. \tag{8}$$

At last apply θ to $[L_{m,i}, L_{-m,0}] = -m(2q+i)L_{0,i}$ for $m, i \in \mathbb{N}$, we obtain

$$\theta(L_{0,i}) = \eta b^{2i} L_{0,i}, \ \forall\ i \neq -2q. \tag{9}$$

Note that $a_{0,-2q}^{0,k} = 0$ unless $k = -q$ or $k = -2q$. Denote $c = \delta_{-2q,\mathbb{N}} b_{0,-2q}$ and $d = \delta_{-q,\mathbb{N}} a_{0,-2q}^{0,-q}$, we obtain

$$\theta(L_{m,i}) = \eta a^m b^{2i} L_{-\eta m,i} + \delta_{m,0}\delta_{i,-2q} cC + \delta_{m,0}\delta_{i,-2q}\delta_{-q,\mathbb{N}} dL_{0,-q}, \quad \forall\, m \in \mathbb{Z}, i \in \mathbb{Z}_+. \tag{10}$$

Finally by considering $\theta^2 = 1$ and the case $-2q \in \mathbb{N}$, we get that $b \in S^1$, $b^{2q}c \in \sqrt{-1}\mathbb{R}$ if $-2q \in \mathbb{N}$ and $b^{3q}d \in \sqrt{-1}\mathbb{R}$ if $-q \in \mathbb{N}$. That is, $\theta = \theta_{a,b,c,d}^{\eta}$. ∎

By Claim 1 in the proof of Proposition 3.4, we can get

Corollary 3.5. *Let θ be a conjugate-linear anti-involution of $\mathcal{B}(q)$, then $\theta|_{\mathcal{B}_0}$ is a conjugate-linear anti-involution on $\mathcal{B}_0$, that is, $\theta|_{\mathcal{B}_0} = \theta_a^\eta$ for appropriate η and a.*

Note that in Proposition 3.4 we reproduce one result in [XZ] that $\mathcal{B}(q)$ does not admit any conjugate-linear anti-involution if $q \notin \mathbb{R}$. From now on, we always assume that q is a nonzero real number.

It is easy to show if $-2q \in \mathbb{N}$ then the 1-dimensional $\mathcal{B}(q)$-module $T(K)$ is unitary with respect to $\theta_{a,b,c,d}^{\eta}$ if and only if $\overline{K} = \eta b^{-4q} K$. In what follows, we suppose V is an irreducible Harish-Chandra $\mathcal{B}(q)$-module not isomorphic to $T(K)$ for any $K \in \mathbb{C}$.

We can easily deduce their counterparts of (1) and (2) of Proposition 3.3 for $\mathcal{B}(q)$.

Lemma 3.6. *For an irreducible Harish-Chandra $\mathcal{B}(q)$-module V, we have*

(1) If V is unitary for some conjugate-linear anti-involution θ of $\mathcal{B}(q)$, then $\theta = \theta_{a,b,c,d}^{+}$ for some $a > 0, b \in S^1$ and $c, d \in \mathbb{C}$ with $b^{2q}c \in \sqrt{-1}\mathbb{R}$ and $b^{3q}d \in \sqrt{-1}\mathbb{R}$.

(2) If V is unitary for $\theta_{a,b,c,d}^{+}$ described in (1), then V is unitary for $\theta_{1,b,c,d}^{+}$.

Proof. Assume that $\theta = \theta_{a,b,c,d}^{\eta}$ as defined in Proposition 3.4. By Corollary 3.5, $\theta|_{\mathcal{B}_0} = \theta_a^\eta$. Noticing that V has a nontrivial irreducible Vir-subquotient module which is unitary for this θ_a^η, we have $\eta = +$ and $a > 0$ from Proposition 3.3. Thus (1) follows.

Now suppose that V is unitary with respect to $\theta = \theta_{a,b,c,d}^{+}$ and the positive definite Hermitian form $\langle\, \cdot\, , \cdot\, \rangle$ on V. Let $V = \oplus_{n\in\mathbb{Z}} V_n$ be the weight decomposition as in (2). It is easy to see that $\langle\, V_n\, , V_m\, \rangle = 0$ for $m \neq n$. Now we define a new positive definite Hermitian form $\ll \cdot, \cdot \gg$ by

$$\ll v, w \gg = a^{-n} \langle\, v\, , w\, \rangle,\ \forall\, v, w \in V_n.$$

Then with the form $\ll \cdot, \cdot \gg$, V is unitary for $\theta_{1,b,c,d}^{+}$, as desired. ∎

Theorem 3.7. *Let V be an irreducible uniformly bounded module over $\mathcal{B}(q)$ not isomorphic to $T(K')$ for any $K' \in \mathbb{C}$. If V is unitary relative to $\theta_{1,b,c,d}^{+}$ for some $b \in S^1$ and $c, d \in \mathbb{C}$ with $b^{2q}c \in \sqrt{-1}\mathbb{R}, b^{3q}d \in \sqrt{-1}\mathbb{R}$, then V is isomorphic to $V'(\alpha, \beta, K, F)$ for some $\alpha, \beta, K, F \in \mathbb{C}$ such that $\alpha \in \mathbb{R}$, $(\beta - \frac{1}{2}) \in \sqrt{-1}\mathbb{R}$, $bF \in \mathbb{R}$ and $\overline{b^{-2q}K} = b^{-2q}K + b^{-2}dF$.*

Proof. By Theorem 2.1, it suffices to determine the nontrivial irreducible unitary modules of the intermediate series. It is straightforward to check that $V'(\alpha, \beta, F, K)$ is unitary relative to $\theta = \theta_{1,b,c,d}^{+}$ provided it satisfies the conditions stated in the theorem.

Now suppose $V = V'(\alpha, \beta, F, K)$ is unitary relative to θ, then $\alpha \in \mathbb{R}$ and $(\beta - \frac{1}{2}) \in \sqrt{-1}\mathbb{R}$ by Proposition 3.3. If $-2q \notin \mathbb{N}$, there is nothing to prove. Now suppose that $-2q \in \mathbb{N}$ and take any nonzero $v \in V$. Consider

$$\langle L_{0,-2q}v , v \rangle = \langle v , \theta(L_{0,-2q})v \rangle \tag{11}$$

and

$$\langle L_{0,1}v , v \rangle = \langle v , \theta(L_{0,1})v \rangle \tag{12}$$

If $F = 0$, then (11) implies that $\overline{b^{-2q}K} = b^{-2q}K$. If $F \neq 0$, then $q = -1$ and (11) gives $\overline{b^{-2q}K} = b^{-2q}K + b^{-2}dF$ and (12) implies $bF \in \mathbb{R}$. ■

Now let us discuss the unitary highest (lowest) weight modules. Let $V = \bigoplus_{n\in\mathbb{Z}_+} V_{\lambda_0 - n}$ be an irreducible highest weight $\mathcal{B}(q)$-module with $V_{\lambda_0} \neq 0$. By PBW Theorem, we have V_{λ_0} is irreducible as an $\widetilde{H} = \mathrm{span}_{\mathbb{C}}\{C, L_{0,i} \mid i \in \mathbb{Z}_+\}$ module, hence $\dim V_{\lambda_0} = 1$. Then there exists a linear function Λ on $\widetilde{H}$ such that

$$xv = \Lambda(x)v, \ \forall\ x \in \widetilde{H}, v \in V_{\lambda_0}.$$

Denote $\xi = \Lambda(C)$ and $\lambda_i = \Lambda(L_{0,i})$ for $i \in \mathbb{Z}_+$. Since V is uniquely determined by this Λ, we denote it by $L(\Lambda)$. Let $L^\star(\Lambda)$ be the restricted dual space of $L(\Lambda)$. We can make $L^\star(\Lambda)$ into a $\mathcal{B}(q)$-module in a canonical way. Obviously $L^\star(\Lambda)$ is an irreducible lowest weight $\mathcal{B}(q)$-module and $L^\star(\Lambda)$ is unitary if and only if $L(\Lambda)$ is unitary.

[XZ] gave the following description of unitary highest weight Harish-Chandra modules over $\mathcal{B}(q)$ with respect to $\theta_{1,1,0,0}$.

Theorem 3.8. *For $q \in \mathbb{R}^*$, the irreducible highest weight Harish-Chandra $\mathcal{B}(q)$-module $L(\Lambda)$ is unitary with respect to $\theta_{1,1,0,0}$ if and only if $\lambda_i = \delta_{i,-2q}\lambda$ with $\lambda \in \mathbb{R}$ for $i \geq 1$ and one of the following conditions holds:*

(i) $\xi \geq q^2$, and $q\lambda_0 \leq 0$, or

(ii) there exists integer $m \geq 2$, and $r, s \in \mathbb{Z}$ with $1 \leq s \leq r < m$ such that

$$\xi = c_{m,q} = q^2 - \frac{6q^2}{m(m+1)}, \ \lambda_0 = \lambda_{m,q}^{r,s} = \frac{q - q((m+1)r - ms)^2}{4m(m+1)}.$$

Now consider an arbitrary conjugate-linear anti-involution $\theta = \theta^+_{1,b,c,d}$ with $b \in S^1$ and $c, d \in \mathbb{C}$ with $b^{2q}c \in \sqrt{-1}\mathbb{R}$ and $b^{3q}d \in \sqrt{-1}\mathbb{R}$. Let $L(\Lambda)$ be an irreducible highest weight Harish-Chandra module over $\mathcal{B}(q)$. We denote

$$H_{m,i} = b^i L_{m,i} + \frac{b^{2q}}{2}\delta_{m,0}\delta_{i,-2q}\big(cC + \delta_{-q,\mathbb{N}}dL_{0,-q}\big),$$

then we have

$$[H_{m,i}, H_{n,j}] = (n(i{+}q) - m(j + q))H_{m+n,i+j} + \delta_{m+n,0}\delta_{i+j,0}\frac{m^3 - m}{12}C$$

and $\theta(H_{m,i}) = H_{-m,i}$ for all $(m, i), (n, j) \in \mathbb{Z} \times \mathbb{Z}_+$. By replacing $L_{m,i}$ with $H_{m,i}$, we can apply Theorem 3.8 to $L(\Lambda)$. In summary, we have

Corollary 3.9. *For $q \in \mathbb{R}^*$, the irreducible highest weight Harish-Chandra $\mathcal{B}(q)$-module $L(\Lambda)$ and its dual lowest weight module $L^\star(\Lambda)$ are unitary with respect to $\theta^+_{1,b,c,d}$ as described above if and only if $\lambda_i = \delta_{i,-2q} b^{2q}\big(\lambda - \frac{b^{2q}c}{2}\xi\big)$ with $\lambda \in \mathbb{R}$ for $i \geq 1$ and one of the following conditions holds:*

(i) $\xi \geq q^2$, and $q\lambda_0 \leq 0$, or

(ii) there exists integer $m \geq 2$, and $r, s \in \mathbb{Z}$ with $1 \leq s \leq r < m$ such that

$$\xi = c_{m,q} = q^2 - \frac{6q^2}{m(m+1)}, \ \lambda_0 = \lambda^{r,s}_{m,q} = \frac{q - q((m+1)r - ms)^2}{4m(m+1)}.$$

Remark 3.10. Note that by Theorem 3.7 and Corollary 3.9 there are modules $V'(\alpha, \beta, F, K)$ and highest (lowest) modules which are unitary relative to $\theta^+_{1,1,0,0}$ but not unitary relative to $\theta^+_{1,b,0,0}$ for some $b \neq 1$.

Now we have a complete classification of the unitary Harish-Chandra modules over $\mathcal{B}(q)$.

Theorem 3.11. *Let $q \in \mathbb{R}^*$ and $\theta = \theta^\eta_{a,b,c,d}$ a conjugate-linear anti-involution on $\mathcal{B}(q)$.*

(1) If $\eta = -$ or a is not positive, then the only irreducible unitary Harish-Chandra modules over $\mathcal{B}(q)$ are $T(K)$ with $\overline{K} = -b^{-4q}K$ up to isomorphism.

(2) If $\eta = +$ and $a > 0$, then any irreducible unitary Harish-Chandra module over $\mathcal{B}(q)$ is isomorphic to $T(K)$ with $\overline{K} = b^{-4q}K$, or one of $V'(\alpha, \beta, K, F)$ described in Theorem 3.7, or a highest (or lowest) weight module described in Corollary 3.9.

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