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Ricci Yang-Mills Solitons on Nilpotent Lie Groups

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Abstract. The purpose of this paper is to introduce the Ricci Yang-Mills soliton equations on nilpotent Lie groups N. As in the case of Ricci solitons, we demonstrate that such metrics arise from automorphisms of N/Z, where Z is the center of N. Additionally, using techniques from Geometric Invariant Theory, we produce a characterization of Ricci Yang-Mills solitons on 2-step nilpotent Lie groups as critical points of a natural functional.

Applying our work on nilpotent Lie groups, we study compact torus bundles over tori with locally (nilpotent) homogeneous metrics. On such spaces, we prove that Ricci Yang-Mills solitons are precisely the metrics whose Ricci tensor is invariant under the geodesic flow.

We finish this note by producing examples of Lie groups that do not admit Ricci soliton metrics but that do admit Ricci Yang-Mills soliton metrics.
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1. Introduction

This work addresses the classical question of finding a preferred or distinguished metric on a manifold. The manifolds of interest are nilpotent Lie groups and their compact quotients, called compact nilmanifolds. In the case of nilpotent Lie groups, we restrict ourselves to the class of left-invariant metrics. In the case of compact nilmanifolds, we restrict to the class of locally homogeneous metrics.

We approach the question of a distinguished metric by looking for metrics which are either critical points of a functional or generalized fixed points of a geometric evolution. Here we consider the Ricci Yang-Mills flow. The generalized fixed points that we are interested in are the so-called *Ricci Yang-Mills soliton metrics.*

The Ricci Yang-Mills flow is a geometric evolution designed specifically for principal bundles. It is a coupling of the Ricci flow on the base of the bundle with the Yang-Mills heat flow on the set of connections of the bundle. Such coupled flows are natural and have gained attention in recent literature. For example, in

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[Lo07], Lott studied the long-time behavior of Ricci flow on flat $\mathbb{R}^n$-vector bundles. In this setting, the evolution can be interpreted as a coupling of the Ricci flow with the harmonic map flow. See also [Kn09] and [Wil10] for more results concerning these equations. Additionally, in [BCP10], the authors studied the heat equation coupled with Ricci flow, and obtained gradient estimates for solutions to the heat equation.

1.1. Special and Distinguished Metrics. A natural class of metrics to consider as distinguished are those metrics with nice curvature properties. For example, metrics of constant sectional or constant Ricci curvature, so-called Einstein metrics, have been researched extensively. It is well-known that nilpotent Lie groups cannot admit left-invariant Einstein metrics [Jen71]. It is also well-known that compact nilmanifolds cannot admit Einstein metrics; this is true even if one extends the search beyond the set of locally homogeneous metrics.

In recent years, a natural generalization of Einstein metrics called Ricci soliton metrics has been explored as a potential preferred metric on our spaces of interest. On nilpotent Lie groups, Ricci soliton metrics, when they exist, are now considered by many to be a preferred choice of metrics. This preference is due to the facts that Ricci solitons on nilpotent Lie groups are unique when they exist, they are 'as close as possible' to being Einstein among all left-invariant metrics [La01], and these metrics have maximal isometry groups among all left-invariant metrics [Jab11b]. However, not all nilpotent Lie groups are able to admit left-invariant Ricci soliton metrics.

Question 1.1. Given a nilpotent Lie group which cannot admit a left-invariant Ricci soliton metric, are there other special metrics with nice properties?

In this work, we investigate left-invariant Ricci Yang-Mills solitons as a potential alternative to Ricci soliton metrics. Our results are described in more detail in the following section.

On compact nilmanifolds, the situation is somewhat different. In addition to not admitting Einstein metrics, compact nilmanifolds cannot even admit Ricci soliton metrics (even among metrics which are not locally homogeneous). This follows from the works [Iv93, Wy08].

Question 1.2. Are there natural classes of metrics on compact nilmanifolds which are preferred or distinguished?

One notion that has been considered is that of a *local nilsoliton.* These are the metrics which lift to simply-connected Ricci solitons on nilpotent groups. While these metrics do have some nice geometric properties, local nilsolitons are often not unique, they do not always exist, and their definition is not intrinsic to the given manifold. In contrast, the definition of a Ricci Yang-Mills soliton on a compact nilmanifold is intrinsic to the manifold.

In the following section we introduce Ricci Yang-Mills solitons and give examples of nice geometric properties that these metrics have.

1.2. Ricci Yang-Mills Solitons. We will now summarize the main results of the paper. To analyze Ricci Yang-Mills solitons on compact nilmanifolds, we study nilpotent Lie groups endowed with left-invariant metrics as they are the simply connected covers of compact nilmanifolds. These spaces admit a natural principal bundle structure, where the structure group is the central subgroup. We look for Ricci Yang-Mills solitons on these spaces (see Definition 2.1).

Our first result shows that Ricci Yang-Mills solitons on nilmanifolds satisfy rigid structural constraints. The following reduces the definition and search for Ricci Yang-Mills solitons on nilmanifolds to an algebra problem, see also Prop. 4.9. This reduction is an analogue of a well-known result of Lauret for Ricci solitons on nilmanifolds [La01].

Proposition 4.2. *Let (g_t, ω_t) be a Ricci Yang-Mills soliton on a nilmanifold N; that is, $h_0 = \pi^* g_0 + k\omega_0$ is a left-invariant metric on N and there exist $\sigma(t) \in \mathbb{R}$, $\psi_t \in \mathfrak{Diff}(N/Z)$ such that $(g_t, \omega_t) = (\sigma(t)\psi_t^* g_0, \psi_t^* \omega_0)$ is a solution to the Ricci Yang-Mills flow.*

Then, there exists a derivation $D \in Der(\mathfrak{N}/\mathfrak{Z})$ such that ψ_t may be chosen to be the family of automorphisms $exp(tD)$ of N/Z.

As compact nilmanifolds have a natural principal bundle structure, we may search for such soliton metrics on them. Of particular interest are compact 2-step nilmanifolds, which are precisely principal torus bundles over tori [PS61].

Among locally homogeneous metrics, we characterize these metrics on compact 2-step nilmanifolds as follows. Recall, if $\mathfrak{G}_t : TM \to TM$ denotes the geodesic flow on the tangent bundle of M, then the Ricci $(2,0)$-tensor $ric_g(\cdot,\cdot)$ is said to be *geodesically flow invariant* if $ric_g(\mathfrak{G}_t v, \mathfrak{G}_t w) = ric_g(v, w)$ for all $v, w \in TM$, $t \in \mathbb{R}$.

Theorem 5.1. *The locally homogeneous Ricci Yang-Mills solitons on a torus bundle over a torus (i.e. on compact 2-step nilmanifolds) are precisely the metrics whose Ricci tensor is invariant under the geodesic flow.*

To study Ricci Yang-Mills solitons on 2-step nilmanifolds, we rephrase our questions in the language of metric Lie algebras and study the moment map for a particular representation of $GL_q\mathbb{R}$. A similar implementation of Geometric Invariant Theory has been carried out by Lauret, Eberlein, Jablonski, et. al, in the study of Ricci solitons on nilpotent Lie groups (see, for example, [La01], [Eb08], [Jab08]). Motivated by the study of Ricci solitons on nilmanifolds, we obtain the following extremal characterization of Ricci Yang-Mills solitons on 2-step nilpotent Lie groups.

Before stating the next result, we set up some notation. A given 2-step nilpotent Lie group with left-invariant metric (N, g) can naturally be identified with a certain tuple, C, of skew-symmetric matrices to be precisely defined later. The nilpotent Lie group with left-invariant metric associated to C is denoted N_C.

Theorem 6.2. *The Riemannian Lie group N_C is a Ricci Yang-Mills soliton if and only if N_C satisfies one of the following equivalent statements:*

1. *C is a critical point of the functional $F(B) = \frac{tr(Ric_B^{\mathcal{H}})^2}{sc(B)^2}$ on $\mathfrak{so}(q)^p$,*
2. *$tr(Ric_C^{\mathcal{H}})^2 = \inf\{tr(Ric_B^{\mathcal{H}})^2 \mid B = \phi C \phi^t$, for $\phi \in GL_q\mathbb{R}$, and $sc(B) = sc(C)\}$,*

where $Ric_B^{\mathcal{H}}$ is the horizontal component of the Ricci tensor Ric_B, and $sc(B)$ is the scalar curvature, corresponding to N_B.

We compare this characterization of Ricci Yang-Mills solitons with other similar characterizations of different, and geometrically interesting, metrics on 2-step nilmanifolds in the following table.

Table 1: Comparison of metrics on two-step nilmanifolds

Metric of interest	Functional	Characterization
Ricci soliton	$G(\mu) = \frac{tr\ Ric_\mu^2}{sc(\mu)^2}$	critical points of G
Metric with geodesically flow invariant Ricci tensor	$F(\mu) = \frac{tr\ (Ric_\mu^{\mathcal{H}})^2}{sc(\mu)^2}$	absolute minimum $\frac{(tr\ Ric_\mu^{\mathcal{H}})^2}{q\ sc(\mu)^2}$ of F
Ricci Yang-Mills soliton	$F(\mu) = \frac{tr\ (Ric_\mu^{\mathcal{H}})^2}{sc(\mu)^2}$	critical points of F

In the above, $Ric^{\mathcal{H}}$ represents the horizontal component of the Ricci tensor. It is from this perspective that Ricci Yang-Mills solitons may be seen as natural generalizations of metrics with geodesic flow invariant Ricci tensor.

As stated above, on compact nilmanifolds there is a potential alternative preference of metric: local nilsolitons. However, it is known that these metrics do not always exist on a given compact nilmanifold.

Theorem 7.1. *There are examples of Ricci Yang-Mills solitons on compact 2-step nilmanifolds that cannot admit local nilsolitons.*

The rest of the paper is organized as follows. In §2, we describe the Ricci Yang-Mills flow and define the notion of Ricci Yang-Mills solitons. In §3, we provide background information about the geometry of 2-step nilpotent Lie groups. We also recall some useful facts from Geometric Invariant Theory. We study the Ricci Yang-Mills soliton equations on 2-step nilpotent Lie groups in §4, and in §5, we characterize these metrics on compact 2-step nilmanifolds. In §6, we realize Ricci Yang-Mills solitons as critical points of a geometric functional on 2-step nilpotent Lie groups. Finally, in §7, we provide examples of nilpotent Lie groups that cannot admit Ricci solitons but that do admit Ricci Yang-Mills solitons.

2. Ricci Yang-Mills flow

The Ricci Yang-Mills flow is a natural coupling of the Ricci flow and the Yang-Mills heat flow. Let $\pi : P \to M$ be a principal bundle with structure group G. Let g be a metric on M, k an $\mathfrak{Ad}$-invariant metric on $\mathfrak{g}$, and ω the connection 1-form on P. We will consider so-called *bundle metrics* on P of the form

$$h = \pi^* g + k\omega,$$

where $k\omega$ is defined as $k\omega(Y, Z) = k(\omega(Y), \omega(Z))$, for vector fields X, Y on P.

Writing the Ricci flow equations for a metric of this form with the additional hypothesis that the size of the fiber remains fixed, one can define the Ricci Yang-Mills flow to be

$$\frac{\partial h}{\partial t} = -2(Rc - Rc^V), \tag{1}$$

where Rc^V is the projection of the Ricci tensor onto its vertical component. This flow was defined independently in [St07a] and [Yo08]. Existence and uniqueness of solutions to the Ricci Yang-Mills flow have been studied in [St07a, Yo08, Yo10], long-time behavior of the flow has been studied in [St10], and stability properties have been considered in [Yo10].

Remark. The Ricci Yang-Mills flow arises as a renormalization group flow, although not a strictly physical one. Additionally, in [Jan08], it was discovered that the Ricci Yang-Mills flow is an ideal candidate for studying magnetic flows.

If G is abelian, using the definition of h and the structure of $Rc(h)$, one can show that Eq. 1 is equivalent to the following system of equations:

$$\frac{\partial g}{\partial t} = -2Rc(g) + \tilde{\Omega}^2, \tag{2a}$$

$$\frac{\partial \tilde{\omega}}{\partial t} = -\delta\tilde{\Omega}. \tag{2b}$$

Here $\tilde{\omega}$ and $\tilde{\Omega}$ are the pullbacks under a local section of the connection 1-form and the bundle curvature, respectively. Recall that when G is abelian, $\tilde{\Omega}$ is a well-defined Lie algebra-valued 2-form on the base. In coordinates, $\tilde{\Omega}^2_{ij} = g^{kl}k^{\alpha\beta}\tilde{\Omega}_{\alpha ki}\tilde{\Omega}_{\beta lj}$, where the Greek indices are the Lie algebra indices and the Roman indices correspond to quantities measured with respect to g. See Appendix A for more details.

Remark. When one views the Ricci Yang-Mills flow in the form of Eq. 2, one sees immediately that this flow is a coupling of the Ricci flow on the base manifold to the Yang-Mills heat flow on connections.

In Appendix B, we show that the set of left-invariant metrics is preserved under the Ricci Yang-Mills flow, so we can interpret the Ricci Yang-Mills flow as an evolution of the metric on a single tangent space. More precisely, we are able to reduce this flow to a system of ODE on the space of inner products on the

Lie algebra. This is a standard approach to studying geometric evolutions on Lie groups with left-invariant metrics.

2.1. Self-similar solutions to Ricci Yang-Mills equations.

Analogous to the case of Ricci flow, we define Ricci Yang-Mills solitons to be generalized fixed points of Eq. 2.

Definition 2.1. A solution $(g_t, \tilde{\omega}_t)$ to the Ricci Yang-Mills equations is a *self similar solution* if there exists a scaling $\sigma(t)$ and a family of diffeomorphisms $\psi_t \in \mathfrak{Diff}(M)$ such that

$$g_t = \sigma(t) \cdot \psi_t^* g \quad \textit{and} \quad \tilde{\Omega}_t = \psi_t^* \tilde{\Omega}$$

with $\sigma(0) = 1$ and ψ_0 the identity.

Let $X \in \Gamma(M, TM)$ generate ψ near $t = 0$. As in the case of Ricci flow, one can show that the notion of self-similar solutions is equivalent to $(g, \tilde{\omega})$ satisfying

$$(\tilde{\Omega}^2 - 2Rc)(g) = \sigma'(0)g + \mathcal{L}_X g, \tag{3a}$$

$$\Delta_d \tilde{\Omega} = \mathcal{L}_X \tilde{\Omega}, \tag{3b}$$

where Δ_d is the Hodge Laplacian. We will call solutions satisfying Eq. 3 *Ricci Yang-Mills solitons.*

We refer to an Einstein Yang-Mills metric as being one such that the metric on the base is Einstein and the connection is Yang-Mills; i.e. $\delta\tilde{\Omega} = 0$. Ricci Yang-Mills solitons are not direct generalizations of Einstein Yang-Mills metrics in the same way that Ricci solitons are generalizations of Einstein metrics. Recall that Ricci solitons are fixed points of the volume-normalized Ricci flow, which differs from the Ricci flow only by a change of scale in space and time. Einstein Yang-Mills metrics are fixed points of a certain volume normalized Ricci Yang-Mills flow. However, due to the lack of scale invariance of this equation, the volume normalized Ricci Yang-Mills flow does not differ only by a change of scale (see [Yo10]).

Remark. One should note that referring to the above metrics as Einstein Yang-Mills is not totally standard. For example, these metrics are not necessarily solutions of the Einstein Yang-Mills equations in general relativity.

3. 2-step nilpotent Lie groups and Geometric Invariant Theory

We would like to study Ricci Yang-Mills solitons on 2-step nilpotent Lie groups. To do so, we will need some facts about these Lie groups and about Geometric Invariant Theory. We include this section as a review of background material. The new results of this paper will be presented in subsequent sections.

A natural test case in the search for Ricci Yang-Mills solitons on principal bundles is the case of a torus bundle over a torus. These compact manifolds are precisely the locally homogeneous manifolds which are modeled on 2-step nilpotent Lie groups (see [PS61]). More precisely, these spaces are quotients of nilpotent

Lie groups by cocompact lattices. As in the case of Ricci flow, to understand the dynamics of this geometric evolution on a compact manifold, we study the evolution on the simply connected cover, a nilpotent Lie group with a left-invariant metric. This cover is also a principal bundle and the covering map is a morphism of bundles. We recall some basic facts for the convenience of the reader.

Definition 3.1. Let $\mathfrak{N}$ be a finite dimensional Lie algebra, and for $i \geq 1$, let $\mathfrak{N}^i = [\mathfrak{N}, \mathfrak{N}^{i-1}]$, where $\mathfrak{N}^0 = \mathfrak{N}$. Then $\mathfrak{N}$ is said to be nilpotent if $\mathfrak{N}^i = \{0\}$ for some i. A nilpotent Lie algebra is k-step if $\mathfrak{N}^k = \{0\}$ but $\mathfrak{N}^{k-1} \neq \{0\}$. A Lie group is said to be (k-step) nilpotent if its Lie algebra is (k-step) nilpotent.

Definition 3.2. A 2-step nilpotent Lie group N or Lie algebra $\mathfrak{N}$ is said to be of type (p, q) if dim $[\mathfrak{N}, \mathfrak{N}] = p$ and codim $[\mathfrak{N}, \mathfrak{N}] = q$.

Observe that p above satisfies $1 \leq p \leq \frac{1}{2}q(q-1) = \dim \mathfrak{so}(q)$. Stratifying the space of 2-step nilpotent Lie algebras into types (p, q) is very convenient in terms of phrasing generic results.

3.1. Principal bundle structure on 2-step nilpotent Lie groups.

Every 2-step nilpotent Lie group N can be viewed as a non-trivial principal bundle. The total space will be $P = N$, and the group G will be the commutator subgroup $[N, N]$ of N acting on the right. We describe this construction in detail.

Let N be a 2-step nilpotent Lie group with Lie algebra $\mathfrak{N}$. We endow N with a left-invariant metric h; this is equivalent to endowing the Lie algebra $\mathfrak{N}$ with an inner product. Consider the commutator subgroup $[N, N]$, with Lie algebra $[\mathfrak{N}, \mathfrak{N}]$, and let $\mathcal{H} = [\mathfrak{N}, \mathfrak{N}]^\perp$ the orthogonal complement relative to the given metric on $\mathfrak{N}$. We point out that $[N, N]$ is a central subgroup since N is 2-step nilpotent. Note, in the work that follows, one could choose to work with the full center $Z(N)$ instead of $[N, N]$. Our choice is made in anticipation of notational convenience.

Let $\{X_1, \ldots, X_q\} \cup \{Z_1, \ldots, Z_p\}$ be an orthonormal basis of $\mathfrak{N} = \mathcal{H} \oplus [\mathfrak{N}, \mathfrak{N}]$. Here $q = \dim \mathcal{H}$, $p = \dim[\mathfrak{N}, \mathfrak{N}]$, and $n = q + p = \dim N$; that is, our 2-step nilpotent algebra is of type (p, q). By left-translating, we can treat this basis of $\mathfrak{N}$ as a left-invariant frame on N. Notice that $\pi : N \to N/[N, N]$ is naturally a (non-trivial) principal G-bundle where $G = \mathbb{R}^p \simeq [N, N]$. The action of $G \simeq [N, N]$ will be given by first injecting $[N, N]$ into N and then multiplying on the right. To distinguish between $[\mathfrak{N}, \mathfrak{N}]$ abstractly versus embedded in $\mathfrak{N}$, we will use lower case letters to denote elements of $\mathfrak{g}$ and upper case letters to denote elements of $[\mathfrak{N}, \mathfrak{N}] \subset \mathfrak{N}$; that is, given $z \in \mathfrak{g}$, $z \to Z \in [\mathfrak{N}, \mathfrak{N}] \subset \mathfrak{N}$.

A choice of a horizontal subspace $\mathcal{H}$ yields a connection 1-form ω which vanishes on $\mathcal{H}$ and takes values in the Lie algebra $\mathfrak{g}$. More precisely, we define our connection so that $\omega(Z) = z$ and $\omega(X) = 0$ for $Z \in [\mathfrak{N}, \mathfrak{N}]$, $X \in \mathcal{H}$. Thus far, we have defined our connection on $\mathfrak{N} = T_e N$. We extend the definition of the connection to the rest of N by imposing ω be left-invariant under N.

The geometry of a simply-connected nilpotent group N can be completely encoded by a tuple of structure matrices $(C^1, \ldots, C^p)$ which is an element of $\mathfrak{so}(q)^p$. We establish this perspective below. Our main references are [Eb08] and

[Jab08].

Recall that a simply-connected nilpotent Lie group N with left-invariant metric $\langle\cdot,\cdot\rangle$ is equivalent to a nilpotent Lie algebra $\mathfrak{N}$ with inner product, also denoted $\langle\cdot,\cdot\rangle$. Let $\{X_1,\dots,X_q\}\cup\{Z_1,\dots,Z_p\}$ be the orthonormal basis of $\mathfrak{N}$ described above. Relative to this basis we may compute the structure constants $\{c_{ij}^k\}$ defined via

$$[X_i,X_j]=\sum_k c_{ij}^k Z_k.$$

Thus we may associate to our basis a p-tuple of matrices $(C^1,\dots,C^p)$ where $(C^k)_{ij}=c_{ij}^k$. Notice that different p-tuples of matrices can be associated to a given $\mathfrak{N}$. We describe below how these different tuples of matrices are related to each other.

Conversely, given a tuple $C=(C^1,\dots,C^p)$ in $\mathfrak{so}(q)^p$, we can naturally associate to it a metric 2-step nilpotent Lie algebra. This construction is dual to the construction of p-tuples above. We require the C^k to be linearly independent in $\mathfrak{so}(q)$ so that the commutator of the constructed nilpotent algebra has dimension p.

Let $\{e_1,\dots,e_q,e_{q+1},\dots,e_{q+p}\}$ be the standard basis of $\mathbb{R}^q\oplus\mathbb{R}^p$. Endow $\mathbb{R}^q\oplus\mathbb{R}^p$ with the standard inner product so that this basis is orthonormal. Define the Lie bracket as

$$[e_i,e_j]=\sum_k C_{ij}^k e_{q+k}$$

for $1\le i,j\le q$, and let all other brackets be trivial. This Lie algebra is clearly 2-step nilpotent with commutator equal to $\mathbb{R}^p=span\langle e_{q+1},\dots,e_{q+p}\rangle$. We denote the metric 2-step nilpotent Lie algebra associated to C by $\mathfrak{N}_C$ and the simply-connected 2-step nilpotent Lie group with left-invariant metric associated to $\mathfrak{N}_C$ by N_C.

3.2. Geometric Invariant Theory and 2-step nilpotent Lie algebras. Using tuples of matrices, we may study 2-step nilpotent Lie algebras not just individually but as elements of the much larger space $\mathfrak{so}(q)^p$. This perspective will allow us to use techniques and results from Geometric Invariant Theory.

On the vector space $\mathfrak{so}(q)^p$ there is a natural action of $GL_q\mathbb{R}\times GL_p\mathbb{R}$ which is a linear representation.

Definition 3.3. Given $g\in GL_q\mathbb{R}$ and $C=(C^1,\dots,C^p)$, we define the *action of $GL_q\mathbb{R}$ on $\mathfrak{so}(q)^p$* by

$$g\cdot C=(gC^1g^t,\dots,gC^pg^t).$$

It is clear that $g\cdot C^k=gC^kg^t\in\mathfrak{so}(q)$ given that $C^k\in\mathfrak{so}(q)$. Notice that this action acts on each coordinate individually. On the other hand, the action of $GL_p\mathbb{R}$ takes linear combinations of the coordinates of $C=(C^1,\dots,C^p)$.

Definition 3.4. Given $h\in GL_p\mathbb{R}$ we define the *action of $GL_p\mathbb{R}$ on $\mathfrak{so}(q)^p$* by

$$h\cdot C=D=(D^1,\dots,D^p)\quad\text{with}\quad D^k=\sum_l h_{lk}C^l.$$

One can compute that these actions commute, and thus we have an action of $GL_q\mathbb{R} \times GL_p\mathbb{R}$ on $\mathfrak{so}(q)^p$.

Using this action, we can easily describe when two different sets of structure matrices produce the same nilpotent Lie group. For proofs of the next two theorems we refer the reader to [Eb08], see Propositions 1.1 & 2.2.

Proposition 3.5 (Eberlein). *Let $C, D \in \mathfrak{so}(q)^p$ correspond to simply connected 2-step nilpotent Lie groups N_C, N_D, respectively. Then N_C and N_D are isomorphic Lie groups if and only if $D \in GL_q\mathbb{R} \times GL_p\mathbb{R} \cdot C$, the orbit of C under the group action of $GL_q\mathbb{R} \times GL_p\mathbb{R}$.*

Here we were only concerned with the underlying Lie group structure of N_C and N_D. The following proposition considers the metric structures as well.

Proposition 3.6 (Wilson). *Let $C, D \in \mathfrak{so}(q)^p$ correspond to simply connected 2-step nilpotent Lie groups N_C, N_D, respectively, with left-invariant metrics. Then N_C and N_D are isometric as Riemannian manifolds if and only if $D \in O(q) \times O(p) \cdot C$, the orbit of C under the compact subgroup $O(q) \times O(p) \subset GL_q\mathbb{R} \times GL_p\mathbb{R}$.*

The above result is due to Wilson [Wi82] and translated to this setting in Section 2 of [Eb08].

This representation of $GL_q\mathbb{R} \times GL_p\mathbb{R}$ on $\mathfrak{so}(q)^p$ has even more structure from the view point of Geometric Invariant Theory. Once translated, these extra structures on the representation space have strong consequences on the Riemannian geometry of associated nilpotent Lie groups. We present a brief discussion below and refer the reader to [Jab10] for a more thorough treatment.

Associated to the representation of $GL_q\mathbb{R} \times GL_p\mathbb{R}$ on $\mathfrak{so}(q)^p$, we have a Lie algebra representation of $\mathfrak{gl}_q\mathbb{R} \times \mathfrak{gl}_p\mathbb{R}$ on $\mathfrak{so}(q)^p$. This is obtained in the usual way via differentiation; that is, given $(X, Y) \in \mathfrak{gl}_q\mathbb{R} \times \mathfrak{gl}_p\mathbb{R}$ and $C \in \mathfrak{so}(q)^p$ we have

$$(X, Y) \cdot C = X \cdot C + Y \cdot C,$$

where $X \cdot C = (XC^1 + C^1X^t, \dots, XC^p + C^pX^t)$ and $Y \cdot C = D$ with $D^k = \sum_l Y_{lk}C^l$.

The space $\mathfrak{so}(q)$ has the inner product $\langle C, D \rangle = tr(CD^t) = -tr(CD)$. This inner product extends to $\mathfrak{so}(q)^p$ by making the coordinates of the tuple orthogonal; that is, for $C = (C^1, \dots, C^p)$ and $D = (D^1, \dots, D^p)$ we have

$$\langle C, D \rangle = \langle (C^1, \dots, C^p), (D^1, \dots, D^p) \rangle = \sum_\alpha \langle C^\alpha, D^\alpha \rangle = \sum_\alpha -tr(C^\alpha D^\alpha).$$

We define two polynomials, m_1 and m_2, which are associated to our representation of $GL_q\mathbb{R} \times GL_p\mathbb{R}$.

Definition 3.7. We define $m_1 : \mathfrak{so}(q)^p \to symm_q$ to be

$$m_1(C) = -2 \sum_\alpha (C^\alpha)^2, \tag{4}$$

where $symm_q$ is the space of symmetric $q \times q$ matrices. Here m_1 is the *moment map for the action of $GL(q, \mathbb{R})$ on $\mathfrak{so}(q)^p$.*

Definition 3.8. We define $m_2 : \mathfrak{so}(q)^p \to symm_p$ to be

$$m_2(C)_{ij} = \langle C^i, C^j \rangle, \tag{5}$$

where $symm_p$ is the space of symmetric $p \times p$ matrices. Here m_2 is the *moment map for the action of* $GL(p, \mathbb{R})$ *on* $\mathfrak{so}(q)^p$.

Adding these together one has the moment map $m = m_1 + m_2$ for the action of $GL_q\mathbb{R} \times GL_p\mathbb{R}$. This is valued in $symm_q \oplus symm_p$. For the general setting of moment maps of representations of reductive groups, we refer the reader to [Jab12].

We are interested in three different group actions on $\mathfrak{so}(q)^p$; namely, the actions of the full group $GL_q\mathbb{R} \times GL_p\mathbb{R}$ and its subgroups $GL_q\mathbb{R}$ and $SL_q\mathbb{R}$. In the following definition, G will denote one of these three groups, and m_G will denote the moment map corresponding to G.

Definition 3.9. We call a point $C \in \mathfrak{so}(q)^p$ G*-distinguished* if $m_G(C) \cdot C = rC$ for some $r \in \mathbb{R}$. We call a point G*-minimal* if $m_G(C) \cdot C = 0$.

Remark. Minimal points are obviously a special kind of distinguished point.

Here $m_G(C) \in \mathfrak{gl}_q\mathbb{R} \times \mathfrak{gl}_p\mathbb{R}$ acts via the Lie algebra action of $\mathfrak{gl}_q\mathbb{R} \times \mathfrak{gl}_p\mathbb{R}$ on $\mathfrak{so}(q)^p$. Distinguished and minimal points can be defined more generally for any representation of a reductive group on a vector space [Jab12]. We will see that distinguished points correspond to geometrically interesting metrics.

Proposition 3.10. *Let* $C \in \mathfrak{so}(q)^p$ *and* $D \in \mathfrak{gl}_q\mathbb{R}$ *such that* $D \cdot C = 0$. *Let* m_1 *denote the moment map of the action of* $GL_q\mathbb{R}$ *on* $\mathfrak{so}(q)^p$. *If* $m_1(C) = \lambda Id + D + D^t$, *then* $D^t \cdot C = 0$. *Hence, we may write* $m_1(C) = \lambda Id + D'$, *for some* $D' \in \mathfrak{gl}_q\mathbb{R}$ *with* $D' \cdot C = 0$.

Proof. This result follows quickly from the general fact that

$$\nabla ||m_1||^2(C) = m_1(C) \cdot C$$

Assume $m_1(C) = \lambda Id + D + D^t$ for some $D \in \mathfrak{gl}_q\mathbb{R}$ such that $D \cdot C = 0$. Then $m_1(C) = \lambda Id + D + X$, for some $X \in \mathfrak{so}(q)$. This shows that $\nabla ||m_1||^2(C) = m_1(C) \cdot C = 2\lambda C + X \cdot C$ is tangent to the orbit $\mathbb{R} \times SO(q) \cdot C$.

Now consider the function $G(C) = \frac{||m_1||^2(C)}{|C|^4}$, for $C \in \mathfrak{so}(q)^p$. As m_1 is homogeneous of degree 2, the function G is constant under dilations and hence its gradient is tangent to spheres in $\mathfrak{so}(q)^p$.

If C is such that $m_1(C) = \lambda Id + D + D^t$ for some D satisfying $D \cdot C = 0$, then we have $\nabla G(C)$ is tangent to $\mathbb{R} \times SO(q) \cdot C \cap \{$*sphere of radius* $|C|\}$. As the $\mathbb{R}$ factor of $\mathbb{R} \times SO(q)$ simply rescales C, we see that $\nabla G(C)$ is tangent to $SO(q) \cdot C$.

However, G is constant on the orbit $SO(q) \cdot C$ since $||m_1||^2$ and $|\cdot|^4$ do not change under orthogonal transformations. Therefore, C is a fixed point of the gradient flow of G. This implies $\nabla ||m_1||^2(C) = \lambda C$.

As $\nabla||m_1||^2(C) = m_1(C)\cdot C$, we see that $m_1(C) = \lambda Id + D'$, for some $D' \in \mathfrak{gl}_q\mathbb{R}$ such that $D'\cdot C = 0$. ∎

Remark 3.11. Although the previous proposition could be written as a general statement which holds for moment maps of linear representation of reductive groups, we do not know of it appearing in the literature. This result is essential for classifying Ricci Yang-Mills solitons on 2-step nilpotent Lie groups.

4. Ricci Yang-Mills solitons on 2-step Nilpotent lie groups

We now study the Ricci Yang-Mills soliton equations on 2-step nilpotent Lie groups. One of the main goals of this paper is to characterize the Ricci Yang-Mills soliton metrics on 2-step nilpotent Lie groups and present some nice geometric properties they exhibit which make them distinguished metrics.

As described in §3, we will view a 2-step nilpotent Lie group N as a principal bundle $N \to N/[N,N]$, where $[N,N]$ is the commutator subgroup. Recall that the Ricci Yang-Mills soliton equations are

$$\begin{aligned} -2Rc_g + \tilde{\Omega}^2 &= \mathcal{L}_X g + \lambda g \\ \Delta_d \tilde{\Omega} &= \mathcal{L}_X \tilde{\Omega}. \end{aligned}$$

Here g is the induced metric on $M = N/[N,N]$. As $[N,N]$ is a normal subgroup of N, $N/[N,N]$ is a Lie group, and the metric g is left $N/[N,N]$-invariant.

Proposition 4.1. *On a 2-step nilpotent Lie group, viewed as a principal bundle, the Ricci Yang-Mills soliton equations are*

$$\tilde{\Omega}^2 = \mathcal{L}_X g + \lambda g \tag{6a}$$

$$0 = \mathcal{L}_X \tilde{\Omega}. \tag{6b}$$

for some $\lambda \in \mathbb{R}$ and some smooth vector field X on $M = N/[N,N]$. In other words, on such manifolds, Ricci Yang-Mills solitons are precisely Einstein Yang-Mills metrics.

Proof. First, notice that the base is the abelian Lie group $N/[N,N]$ with left-invariant metric, so it is flat and hence $Rc_g = 0$.

Next, we will show that $\tilde{\omega}$ is a Yang-Mills connection; i.e. that $\delta\tilde{\Omega} = 0$. Let $U \subset M$ be an open set, and let $s : U \to N$ be a local section. We can define $\tilde{X}_i = \pi_* X_i$ to be a left-$N/[N,N]$-invariant vector field on the base. We have that $[X_i, X_j] = c_{ij}^k Z_k$. Thus

$$\begin{aligned} \tilde{\Omega}(\tilde{X}_i, \tilde{X}_j) &= s^*\Omega(\tilde{X}_i, \tilde{X}_j) \\ &= \Omega(s_*\tilde{X}_i, s_*\tilde{X}_j) \\ &= \Omega(X_i, X_j). \end{aligned}$$

So $\tilde{\Omega} = \sum_{ijk} -c_{ij}^k z_k \tilde{\sigma}^i \wedge \tilde{\sigma}^j$, where $\{\tilde{\sigma}^i\}$ are dual to $\{\tilde{X}_i\}$. By linearity, to compute $\delta\tilde{\Omega}$, we only need to compute $d\star d\star(\tilde{\sigma}^i \wedge \tilde{\sigma}^j)$, where $\star$ is the Hodge star operator.

We compute that $d\star(\tilde\sigma^i\wedge\tilde\sigma^j)=(-1)^{i+j-1}d(\tilde\sigma^1\wedge\cdots\wedge\hat{i}\hat{j}\wedge\cdots\wedge\tilde\sigma^q)$, and

$$d\tilde\sigma^k(\tilde X_i,\tilde X_j)=\tilde X_i(\tilde\sigma^k(\tilde X_j)-\tilde X_j(\tilde\sigma^k(\tilde X_i)-\tilde\sigma^k([\tilde X_i,\tilde X_j])=-\tilde\sigma^k([\tilde X_i,\tilde X_j]).$$

Again, we have used left-invariance to make two of the middle terms vanish in the above equation. Lastly, $[\tilde X_i,\tilde X_j]=0$ as $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}]$ is abelian.

Thus $\delta\tilde\Omega=(-1)^q\star d\star\tilde\Omega=0$. Notice that this implies that $\Delta_d\tilde\Omega=0$, since $d\tilde\Omega=0$ by the Bianchi identity. ∎

Now we will show that the Ricci Yang-Mills solitons are generated by automorphisms of $N/[N,N]$. Although our primary interest is in 2-step nilpotent groups, the following holds for any nilpotent group.

Proposition 4.2. *Let (g_t,ω_t) be a Ricci Yang-Mills soliton on a nilmanifold N with central subgroup Z; that is, $h_0=\pi^*g_0+k\omega_0$ is a left-invariant metric on N and there exist $\sigma(t)\in\mathbb{R}$, $\psi_t\in\mathfrak{Diff}(N/Z)$ such that $(g_t,\omega_t)=(\sigma(t)\psi_t^*g_0,\psi_t^*\omega_0)$ is a solution to the Ricci Yang-Mills flow.*

Then, there exists a derivation $D\in Der(\mathfrak{N}/\mathfrak{Z})$ such that ψ_t may be chosen to be the family of automorphisms $exp(tD)$ of N/Z.

Proof. The proof of the claim is similar to the analogous fact for Ricci solitons, cf. [La01]. Let $\varphi_t\in Diff(M)$ and $\sigma(t)\in\mathbb{R}$ be such that $g_t=\sigma(t)\varphi_t^*g_0$ is a solution to Eq. 2a. Recall, if the initial metric h is left N-invariant, then h_t is left N-invariant for all t, see Corollary B.3. Thus, g_t and $\tilde\omega_t$ are left N/Z-invariant for all t.

Next we may assume φ_t fixes the identity $e\in N/Z$. To see this, let $n_t=\varphi_t(e)$; this is a smooth curve in N/Z. Now, by left invariance of g_t and $\tilde\omega_t$, $L_{n_t}\circ\varphi_t$ is smooth and solves Eq. 2. So we may replace φ_t with $L_{n_t}\circ\varphi_t$ and just assume φ_t fixes e.

Fix t. As $\varphi_t^*g_0$ is left N/Z-invariant, we have $\varphi_t^{-1}\circ L_n\circ\varphi_t\in Isom(N/Z,g_0)$ for all $n\in N/Z$. Considering all $n\in N/Z$, we have a nilpotent group of isometries (as it is the homomorphic image of a nilpotent group) which acts transitively on $M=N/Z$. Thus, this subgroup of $Isom(N/Z,g_0)$ is equal to N/Z by Theorem 2, part 4, of [Wi82]; that is, there is a function $f:N/Z\to N/Z$ such that $\varphi_t^{-1}\circ L_n\circ\varphi_t=L_{f(n)}$.

Observe that f is an automorphism of N/Z. Evaluating at $e\in N/Z$, we have $\varphi_t^{-1}(n)=f(n)$, that is, φ_t is an automorphism of N/Z for each t. Now φ_t is a one-parameter group of automorphisms, thus there exists $D\in Lie\ Aut(N/Z)=Der(\mathfrak{N}/Z)$ such that $\varphi_t=exp(tD)$. This proves the assertion. ∎

Proposition 4.3. *Using the family of automorphisms of $N/[N,N]$ from Proposition 4.2, the Ricci Yang-Mills soliton equations on 2-step nilpotent Lie groups become*

$$\tilde\Omega^2(v,w)=g((D+D^t)v,w)+\lambda g(v,w) \tag{7a}$$

$$0=\tilde\Omega(Dv,w)+\tilde\Omega(v,Dw), \tag{7b}$$

where v,w are left-invariant vector fields on $M=N/[N,N]$.

Proof. Let $\mathcal{L}_X$ denote the Lie derivative corresponding to the 1-parameter family of automorphisms φ_t. Given a left-invariant vector field w on $N/[N,N]$, $\mathcal{L}_X w$ is also left-invariant; moreover, evaluated at the identity $e \in N/[N,N]$, $(\mathcal{L}_X w)_e = -D(w_e)$, where D is the derivation above generating φ_t. Then for a left-invariant metric g on $N/[N,N]$ evaluated at left-invariant vector fields v, w we have

$$\begin{aligned}(\mathcal{L}_X g)(v,w) &= \mathcal{L}_X(g(v,w)) - g(\mathcal{L}_X v, w) - g(v, \mathcal{L}_X w)\\ &= 0 + g(Dv,w) + g(v,Dw)\\ &= g((D+D^t)v,w).\end{aligned}$$

Similarly, we have $(\mathcal{L}_X\Omega)(v,w) = \Omega(Dv,w)+\Omega(v,Dw)$, as Ω is left-invariant. ∎

Recall, every linear invertible map of an abelian Lie algebra is an automorphism of the Lie algebra. Thus, $\mathfrak{N}$ being a 2-step nilpotent Lie algebra means $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}]$ is abelian, and we can use any linear map $D : \mathfrak{N}/[\mathfrak{N},\mathfrak{N}] \to \mathfrak{N}/[\mathfrak{N},\mathfrak{N}]$ when trying to solve Eq. 7.

Definition 4.4. We say that a left-invariant Ricci Yang-Mills soliton is *symmetric* if the derivation D, above, is symmetric with respect to the inner product g_0 on $\mathfrak{N}/[\mathfrak{N},\mathfrak{N}]$.

One obvious benefit of having a symmetric Ricci Yang-Mills soliton is a simplified presentation of Eqn. 7 (see also Eqn. 8). Moreover, we will see that symmetric Ricci Yang-Mills solitons are precisely critical points of a natural functional (see Theorem 6.2).

In the sequel, we show that 2-step Ricci Yang-Mills solitons are always symmetric (see Prop. 4.9). Presently, we have no examples of (higher-step) left-invariant Ricci Yang-Mills solitons which are not symmetric. In contrast, all Ricci solitons have D which are symmetric. It is not clear that this must be the case for Ricci Yang-Mills solitons.

4.1. Ricci Yang-Mills solitons and Geometric Invariant Theory. In this section we approach the problem of finding Ricci Yang-Mills solitons using structure matrices. Studying 2-step nilpotent Lie groups from this point of view is quite natural and has been used by Eberlein, Jablonski, and others to obtain results about Ricci solitons. Using this approach, we are able to characterize the Ricci Yang-Mills solitons on 2-step nilpotent Lie groups.

First we will recall some results obtained via Geometric Invariant Theory for Ricci solitons. We will see that Ricci Yang-Mills solitons provide a geometric interpretation to certain phenomena that arise in this setting. By nilsoliton we mean a nilpotent Lie group with left-invariant Ricci soliton metric. Also recall Definition 3.9: a point $C \in \mathfrak{so}(q)^p$ is called a distinguished point of a G-action if $m_G(C)\cdot C = rC$ for some $r \in \mathbb{R}$. If $r=0$, the distinguished point is called a minimal point.

Theorem 4.5 (Lauret). *Let $C \in \mathfrak{so}(q)^p$ correspond to a 2-step nilpotent Lie group N_C with left-invariant metric. Then N_C is a nilsoliton if and only if C is* *a distinguished point of the $GL_q\mathbb{R} \times GL_p\mathbb{R}$ action.*

This was originally proven for all nilpotent Lie groups (not just 2-step) by Jorge Lauret [La01]. In the 2-step nilpotent setting, Eberlein phrased this in the frame work of structure matrices. This result may be found in [Jab11a, Theorem 1.6].

Recall, if $\mathfrak{G}_t : TM \to TM$ denotes the geodesic flow on the tangent bundle of M, then the Ricci $(2,0)$ tensor $ric_g(\cdot,\cdot)$ is said to be *geodesically flow invariant* if $ric_g(\mathfrak{G}_t v, \mathfrak{G}_t w) = ric_g(v,w)$ for all $v, w \in TM$, $t \in \mathbb{R}$. The following is Proposition 7.3 of [Eb08].

Proposition 4.6 (Eberlein). *Let $C \in \mathfrak{so}(q)^p$ correspond to a 2-step nilpotent Lie group N_C with left-invariant metric. Then the metric on N_C has a geodesic flow invariant Ricci tensor if and only if C is a minimal point of the $SL_q\mathbb{R}$ action.*

The following question was asked to us by Pat Eberlein and will be answered below.

Question 4.7. Is there good geometric meaning to Lie groups N_C which correspond to a point C which is a distinguished point of the $GL_q\mathbb{R}$-action? In other words, are there metrics that are natural generalizations of the geodesic flow invariant metrics?

As Ricci solitons are naturally associated to distinguished points, one is lead to investigate if there is a similar connection between Ricci Yang-Mills solitons and Geometric Invariant Theory. There is a strong, and similar, relationship in the Ricci Yang-Mills setting. We will study the Ricci Yang-Mills soliton equations from the perspective of structure matrices. We begin by translating Eq. 7 into a system of equations on tuples of matrices.

In the following, $C \in \mathfrak{so}(q)^p$ corresponds to the metric Lie group N_C, and D is an element of $\mathfrak{gl}_q\mathbb{R}$. Also, m_1 is the moment map defined in Eq. 4. We present the Ricci Yang-Mills equations here as $(1,1)$ tensors as opposed to $(2,0)$ tensors, as it is more natural from the perspective of structure matrices.

Lemma 4.8. *The Ricci Yang-Mills soliton equations (Eqns. 7 a & b) on 2-step nilpotent Lie groups can be written as*

$$m_1(C) = -2\sum_\alpha (C^\alpha)^2 = 2\lambda Id + 2(D + D^t) \tag{8a}$$

$$0 = D^t \cdot C \tag{8b}$$

where $D^t \cdot C$ denotes the action of $\mathfrak{gl}_q$ on $\mathfrak{so}(q)^p$; that is,
$D^t \cdot C = (D^t \cdot C^1, \dots, D^t \cdot C^p)$ and $D^t \cdot C^i = D^t C^i + C^i D$.

Proof. We begin by raising an index on $\tilde{\Omega}^2$ and show that this is $\frac{1}{2}m_1(C)$. Recall that

$$\tilde{\Omega}^2_{ij} = \sum_{\alpha\beta kl} g^{kl} k^{\alpha\beta} \tilde{\Omega}_{\alpha ki} \tilde{\Omega}_{\beta lj}.$$

We will use the orthonormal basis $\{X_i\} \cup \{Z_\alpha\}$ of $\mathfrak{N}$ that was used to calculate our structure matrix. Thus we have

$$\begin{aligned}\tilde{\Omega}^2_{ij} = \sum_{\alpha\beta kl} g^{kl} k^{\alpha\beta} \tilde{\Omega}_{\alpha ki} \tilde{\Omega}_{\beta lj} &= \sum_{\alpha l} \tilde{\Omega}_{\alpha li} \tilde{\Omega}_{\alpha lj} \\ &= \sum_{\alpha l} (-C^\alpha)_{li} (-C^\alpha)_{lj} = -\sum_{\alpha} (C^\alpha)^2_{ij} = \frac{1}{2} m_1(C)_{ij}\end{aligned}$$

The right-hand side of Eq. 7a is easily converted to a $(1,1)$-tensor to obtain the claimed result.

For the second equation, recall that $\tilde{\Omega} = -\sum z_k c^k_{ij} \tilde{\sigma}_i \wedge \tilde{\sigma}_j$ and $c^k_{ij} = \langle C^k X_i, X_j \rangle$. Thus

$$\tilde{\Omega}(v,w) = -\sum_k z_k \langle C^k v, w \rangle,$$

where we are identifying $\mathcal{H} \simeq \mathfrak{N}/\mathfrak{Z}$ isometrically via π_*. Therefore

$$\tilde{\Omega}(Dv,w) + \tilde{\Omega}(v,Dw) =$$

$$-\sum_k z_k \langle C^k Dv, w\rangle + \langle C^k v, Dw \rangle = -\sum_k z_k \langle (C^k D + D^t C^k) v, w \rangle$$

as required. ■

Proposition 4.9. *A Ricci Yang-Mills soliton on a 2-step nilpotent Lie group must be symmetric (cf. Def. 4.4).*

Proof. This is an immediate consequence of the previous lemma and Prop. 3.10. ■

Remark 4.10. Without the tool of Geometric Invariant Theory, there does not seem to be an easy way to show that 2-step Ricci Yang-Mills solitons are symmetric.

The following is an analogue of a well-known result for Ricci solitons.

Theorem 4.11. *Let N_C be the metric 2-step nilpotent Lie group corresponding to a tuple $C \in \mathfrak{so}(q)^p$. Then the metric on N is a left-invariant Ricci Yang-Mills soliton if and only if C is a distinguished point of the action of $GL_q\mathbb{R}$ on $\mathfrak{so}(q)^p$.*

Proof. By definition, C being a distinguished point of the action of GL_q on $\mathfrak{so}(q)^p$ is equivalent to $m_1(C) \cdot C = a(C)C$ where $a(C) > 0$. This holds if and only if $m_1(C) = \frac{1}{2}a(C) + B$ where $B \in Stab_C$, that is, $B \cdot C = 0$. Since $m_1(C)$ is always a symmetric matrix, B is a symmetric matrix.

Using Eq. 8, we see that if D is symmetric, we have our equivalence using $a = 2\lambda$ and $D = -4B$. ■

At this point we are able to use general theorems from Geometric Invariant Theory to prove results about the existence of Ricci Yang-Mills solitons.

Corollary 4.12. *If the orbit $SL_q\mathbb{R}{\cdot}C$ is closed in $\mathfrak{so}(q)^p$ then the corresponding Lie group admits a (trivial) Ricci Yang-Mills soliton. Moreover, such soliton metrics are precisely the geodesic flow-invariant metrics.*

This corollary follows from the fact that if $SL_q\mathbb{R}\cdot C$ is closed, then there exists a minimal point on the orbit (assume it is C) satisfying $m_1(C) = r\ Id$. See [Jab08] for more details. Once $m_1(C)$ has this form, it is clear that C will be a (trivial) Ricci Yang-Mills soliton. This corollary provides us with a very general procedure for building examples of (trivial) Ricci Yang-Mills solitons.

Corollary 4.13. *Almost every 2-step nilpotent Lie group admits a Ricci Yang-Mills soliton when $p \leq \frac{1}{2}q(q-1)-2$.*

Proof. Consider 2-step nilpotent Lie groups of type (p,q). If $p \leq \frac{1}{2}q(q-1)-2$ then almost every $SL_q\mathbb{R}$-orbit is closed [Jab11a]. ■

In the 2-step nilpotent setting, we can make precise the sense in which Ricci Yang-Mills solitons are weaker than Ricci solitons. For this observation, we need the following result of Eberlein which may be found in [Jab08, Proposition 7.9].

Proposition 4.14 (Eberlein). *Let N_C be a 2-step nilpotent Lie group of type (p,q) with left-invariant metric corresponding to $C \in \mathfrak{so}(q)^p$. The metric nilpotent group N_C is a Ricci soliton with geodesic flow invariant Ricci tensor if and only if $m_1(C) = r\ Id_q$ and $m_2(C) = s\ Id_p$ for some $r, s \in \mathbb{R}$.*

Corollary 4.15. *If N_C admits a geodesic flow invariant Ricci soliton, then such a metric is also a Ricci Yang-Mills soliton.*

Notice that, in general, a manifold that admits a Ricci soliton will not necessarily admit a Ricci Yang-Mills soliton, as Ricci Yang-Mills solitons are only defined on manifolds that are also principal bundles.

5. Ricci Yang-Mills solitons on compact nilmanifolds

Let P be a principal T^p-bundle over a torus T^q. It is well-known that $P = \Gamma\backslash N$ where N is a 2-step nilpotent Lie group of type (p,q) and Γ is a lattice, i.e. a discrete cocompact subgroup. For information on torus bundles see [PS61], and for information on lattices in nilpotent Lie groups see [Rag72]. In this setting, we are able to characterize the Ricci Yang-Mills solitons.

Theorem 5.1. *The locally homogeneous Ricci Yang-Mills solitons on a torus bundles over a torus (i.e. on compact 2-step nilmanifolds) are precisely the metrics whose Ricci tensor is invariant under the geodesic flow.*

Proof. Observe that the simply connected cover $\overline{P} = N$ of $P = \Gamma\backslash N$ is a principal $\mathbb{R}^p$-bundle over $\mathbb{R}^q$ which fits into the following commutative diagram. In the following, $\overline{G} = \mathbb{R}^p$ is the commutator subgroup $[N,N]$ which acts on $\overline{P} = N$

on the right and $G = T^p = ([N,N] \cap \Gamma)\backslash[N,N]$ acts on $P = \Gamma\backslash N$ on the right. (Recall, G is compact by [Rag72, Corollary 2.3.1].)

$$\begin{array}{ccc} \overline{P} = N & \xrightarrow{\pi_\Gamma} & P = \Gamma\backslash N \\ \pi_{\overline{G}}\Big\downarrow & & \Big\downarrow \pi_G \\ \overline{M} = \overline{P}/\overline{G} = N/[N,N] & \xrightarrow[\pi_{\Gamma\cdot[N,N]}]{} & M = P/G = (\Gamma\backslash N)/([N,N]/[N,N]\cap\Gamma) \end{array}$$

If N is endowed with a left-invariant metric h, then $\Gamma\backslash N$ is endowed with the locally homogeneous metric $(\pi_\Gamma)_* h$. As the Ricci Yang-Mills flow commutes with the above quotient morphisms, a Ricci Yang-Mills soliton $(\pi_\Gamma)_* h$ on $\Gamma\backslash N$ lifts so that h is a Ricci Yang-Mills soliton on N. More precisely, the diffeomorphisms φ_t of M lift to diffeomorphisms $\overline{\varphi}_t$ on $\overline{M}$ which leave the lattice $\Gamma/([N,N]\cap\Gamma)$ of $\overline{M} = N/[N,N]$ stable, that is, $\overline{\varphi}_t(\gamma) \in \Gamma/([N,N]\cap\Gamma)$ for all $\gamma \in \Gamma/([N,N]\cap\Gamma)$.

As $\overline{\varphi}_t = exp(tD)$ for some $D \in Der(\mathfrak{N}/[\mathfrak{N},\mathfrak{N}])$ by Proposition 4.2, $\overline{\varphi}_t$ stabilizes $\Gamma/([N,N]\cap\Gamma)$ if and only if $D = 0$, that is, if and only if the Ricci Yang-Mills soliton corresponds to the trivial family of diffeomorphisms.

Lastly, recall that a Ricci Yang-Mills soliton with $D = 0$ corresponds to a metric whose Ricci tensor is geodesic flow invariant, see Proposition 4.6. As π_Γ is a local isometry, the metric h on N has a geodesic flow invariant Ricci tensor if and only if the metric $(\pi_\Gamma)_* h$ on $\Gamma\backslash N$ has such a Ricci tensor. ■

Question 5.2. Let P be a principal T^p-bundle over T^q. Are (inhomogenous) Ricci Yang-Mills solitons and geodesic flow invariant metrics on P equivalent?

6. Ricci Yang-Mills solitons as critical points

In this section we describe the relationship between Geometric Invariant Theory and Ricci Yang-Mills solitons from a slightly different perspective. Here we realize Ricci Yang-Mills solitons on 2-step nilpotent Lie groups as critical points of a natural functional.

Let N be a simply-connected nilpotent Lie group. Endow N with a left-invariant metric g. By left-invariance, the Ricci tensor Ric_g is completely determined by its values on $\mathfrak{N}$. Left-invariant Ricci solitons on nilpotent groups can be characterized in the following way (cf. [La01]).

As in Section 3, given a 2-step nilpotent Lie group N with left-invariant metric g, we identify (N,g) with N_C where $C \in \mathfrak{so}(q)^p$. Here $N \simeq N_C$ is of type (p,q).

Theorem 6.1 (Lauret). *Let N_C be a 2-step nilpotent Lie group with left-invariant metric corresponding to $C \in \mathfrak{so}(q)^p$. This Riemannian Lie group N_C is a Ricci soliton if and only if N_C satisfies one of the following equivalent statements:*

1. *C is a critical point of the functional $G(B) = \frac{tr(Ric_B)^2}{sc(B)^2}$ on $\mathfrak{so}(q)^p$,*

2. *$tr(Ric_C)^2 = \inf\{tr(Ric_B)^2 \mid B = \phi\cdot C$, for $\phi \in GL_q\mathbb{R} \times GL_p\mathbb{R}$, and $sc(B) = sc(C)\}$,*

where Ric_B denotes the Ricci tensor, and $sc(B)$ the scalar curvature, of the 2-step Riemannian nilpotent Lie group N_B.

Notice that the set $\phi \cdot C$, with $\phi \in GL_q\mathbb{R} \times GL_p\mathbb{R}$, gives all left-invariant metrics on the underlying Lie group of N_C. We achieve an analogous extremal characterization for Ricci Yang-Mills solitons on 2-step nilpotent Lie groups.

Let N be a 2-step nilpotent Lie group with commutator subgroup $[N, N]$ and corresponding Lie algebras $\mathfrak{N}$ and $[\mathfrak{N}, \mathfrak{N}]$. Since N is 2-step nilpotent, Ric preserves $[\mathfrak{N}, \mathfrak{N}]$ (cf. [Eb94, Proposition 2.5]) and hence Ric preserves $\mathcal{H} = [\mathfrak{N}, \mathfrak{N}]^\perp$. As Ric preserves the horizontal and vertical distributions $\mathcal{H}$ and $[\mathfrak{N}, \mathfrak{N}]$, we will write the Ricci tensor in components: $Ric = Ric^{\mathcal{H}} + Ric^{[\mathfrak{N},\mathfrak{N}]}$.

Theorem 6.2. *Let N_C be a 2-step nilpotent Lie group with left-invariant metric corresponding to $C \in \mathfrak{so}(q)^p$. This Riemannian Lie group N_C is a Ricci Yang-Mills soliton if and only if N_C satisfies one of the following equivalent statements:*

1. *C is a critical point of the functional $F(B) = \frac{tr(Ric_B^{\mathcal{H}})^2}{sc(B)^2}$ on $\mathfrak{so}(q)^p$,*

2. *$tr(Ric_C^{\mathcal{H}})^2 = \inf\{tr(Ric_B^{\mathcal{H}})^2 \mid B = \phi \cdot C$, for $\phi \in GL_q\mathbb{R}$, and $sc(B) = sc(C)\}$,*

where $Ric_B^{\mathcal{H}}$ is the horizontal component of the Ricci tensor Ric_B, and $sc(B)$ is the scalar curvature, corresponding to N_B.

Remark 6.3. Notice that the infimum above is not taken over all left-invariant metrics, rather we only vary the metric in the horizontal component.

To prove this proposition, we phrase it in terms of moment maps and use the work of §4. Let $C \in \mathfrak{so}(q)^p$ correspond to N_C, the 2-step nilpotent Lie group with left-invariant metric of interest. The Ricci tensor of N_C and the moment map of the $GL_q\mathbb{R} \times GL_p\mathbb{R}$ action at C are related as follows

$$Ric_C^{\mathcal{H}} = -\frac{1}{4}m_1(C) \quad \textit{and} \quad Ric_C^{\mathfrak{z}} = \frac{1}{4}m_2(C)$$

from which we see that $Ric_C = -\frac{1}{4}m_1(C) + \frac{1}{4}m_2(C)$ and hence $sc(C) = -\frac{1}{4}|C|^2$. Notice that the level sets of the scalar curvature function are spheres in $\mathfrak{so}(q)^p$.

Part (2) of the above proposition is equivalent to having N_C such that

$$||m_1(C)||^2 = \inf\{||m_1(\phi \cdot C)||^2 \mid \phi \in GL_q\mathbb{R} \text{ and } |\phi \cdot C| = |C|\}$$

where the norm on $\mathfrak{gl}(q, \mathbb{R})$ is the standard norm: $||A||^2 = tr(AA^t)$. In the above, the condition $\phi \in GL_q\mathbb{R}$ is equivalent to varying the metric only in the horizontal directions.

From the study of moment maps of representations of reductive groups, we have the following well-known lemma (cf. Section 5 of [Jab12]).

Lemma 6.4. *The following are equivalent*

1. $||m_1(C)||^2 = \inf\{||m_1(g \cdot C)||^2 \mid g \in GL_q\mathbb{R} \text{ and } |g \cdot C| = |C|\}$

2. *C is a critical point of $||m_1||^2$ on the set $GL_q\mathbb{R} \cdot C \cap \{$ the sphere of radius $|C|\}$*

3. *C is a distinguished point.*

Combining this lemma with Theorem 4.11, our theorem is proven.

7. Examples

We now provide examples of nilpotent Lie groups that do not admit Ricci solitons but that do admit Ricci Yang-Mills solitons. From the perspective of Geometric Invariant Theory, that such examples exist is not a surprise. Although such examples should be rare, we expect their existence as there should be plenty of points $C \in \mathfrak{so}(q)^p$ whose $SL_q\mathbb{R} \times SL_p\mathbb{R}$-orbit is not closed but whose $SL_q\mathbb{R}$-orbit is closed.

We state the following theorem and provide three families of examples as its proof.

Theorem 7.1. *There are examples of Ricci Yang-Mills solitons on compact 2-step nilmanifolds that cannot admit local nilsolitons.*

Example 7.2. The first example can be produced from the work of Cynthia Will. In [Wi10], Will constructs a curve of (pairwise) non-isomorphic nilpotent Lie groups which do not admit Ricci solitons. We use the algebra corresponding to $t = 1$ in her curve $\overline{\mu}_t$. This algebra has structure matrices

$$C^1 = \left[\begin{array}{cc|cc|cc} & a^2 & & & & \\ -a^2 & & & & & \\ \hline & & & & & 1 \\ & & & & -1 & \\ \hline & & & 1 & & \\ & & -1 & & & \end{array}\right], \qquad C^2 = \left[\begin{array}{cc|cc|cc} & & & & & a \\ & & & & -a & \\ \hline & & & 0 & & \\ & & 0 & & & \\ \hline & a & & & & \\ -a & & & & & \end{array}\right],$$

$$C^3 = \left[\begin{array}{cc|cc|cc} & & & a & & \\ & & -a & & & \\ \hline & a & & & & \\ -a & & & & & \\ \hline & & & & & 0 \\ & & & & 0 & \end{array}\right].$$

The algebra presented above is isomorphic to Will's example but has different structure matrices. The above is $g \cdot C = gCg^t$ where $g = diag\{a, a, 1, 1, 1, 1\}$ for Will's set of structure matrices C.

A simple computation shows that

$$m_1(C) = -2 \left[\begin{array}{cc:cc:cc} -a^4-2a^2 & & & & & \\ & -a^4-2a^2 & & & & \\ \hdashline & & -1-a^2 & & & \\ & & & -1-a^2 & & \\ \hdashline & & & & -1-a^2 & \\ & & & & & -1-a^2 \end{array}\right].$$

By Theorem 4.11, the above will be a Ricci Yang-Mills soliton if we can show C is a distinguished point; that is, if $m_1(C)\cdot C = rC$ for some $r \in \mathbb{R}$. This is possible for $a^2 = \frac{-1+\sqrt{5}}{2} > 0$, and we have the desired result. Notice that in this case, our soliton is generated by the trivial vector field (i.e., $D = 0$ in Eq. 8), since $m_1(C)$ is a multiple of the identity.

The 2-step nilpotent Lie group corresponding to C above admits a compact quotient since using $a = 1$ we have rational structure constants (this is Mal'cev's criterion). Moreover, the Ricci Yang-Mills soliton on N_C descends to the compact quotient as the generating derivation D is trivial.

Remark. We will provide examples below which are not generated by trivial vector fields D.

Example 7.3. The second family of examples uses the groups constructed in [Jab11a]. In particular, one can construct continuous familes of algebras of type (p,q) for $2 \le p \le 6$ that are shown to not admit Ricci soliton metrics. To do so, we must first describe a process called concatenation.

Consider $A = (A_1,\dots,A_p) \in \mathfrak{so}(q_1)^p$ and $B = (B_1,\dots,B_p) \in \mathfrak{so}(q_2)^p$ which are structure matrices associated to nilpotent Lie algebras N_A and N_B of types (p,q_i), respectively. Then we can build a new nilpotent Lie algebra N_C corresponding to the structure matrix $C \in \mathfrak{so}(q)^p$, where $q = q_1 + q_2$ and

$$C_i = \begin{pmatrix} A_i & \\ & B_i \end{pmatrix}.$$

We call this process *concatenation* and denote it by $C = A +_c B$. As A and B have linearly independent components, the same is true for C and hence C corresponds to a 2-step nilpotent Lie algebra of type (p,q). Additionally, we will abuse notation and concatenate $A \in \mathfrak{so}(q_1)^{p_1}$ and $B \in \mathfrak{so}(q_2)^{p_2}$ where $p_1 < p_2$. This is an element of $\mathfrak{so}(q_1+q_2)^{p_2}$ defined as

$$(A_1,\dots,A_{p_1},\underbrace{0,\dots,0}_{p_2-p_1}) +_c (B_1,\dots,B_{p_2}).$$

We are interested in concatenating the following structure matrices. Denote by J the 2×2 matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. Define $A_1 \in \mathfrak{so}(2k)$ to be the concatenation $A_1 = J \underbrace{+_c \cdots +_c}_{k} J$. This is just a block diagonal matrix with all blocks being copies of J. Define $B_1, B_2, \ldots, B_6 \in \mathfrak{so}(4)$ as

$$B_1 = \left[\begin{array}{cc:cc} 0 & 1 & & \\ -1 & 0 & & \\ \hdashline & & 0 & 1 \\ & & -1 & 0 \end{array}\right], \quad B_2 = \left[\begin{array}{cc:cc} & & 0 & 1 \\ & & 1 & 0 \\ \hdashline 0 & -1 & & \\ -1 & 0 & & \end{array}\right], \quad B_3 = \left[\begin{array}{cc:cc} & & 1 & 0 \\ & & 0 & 1 \\ \hdashline -1 & 0 & & \\ 0 & -1 & & \end{array}\right],$$

$$B_4 = \left[\begin{array}{cc:cc} 0 & 1 & & \\ -1 & 0 & & \\ \hdashline & & 0 & -1 \\ & & 1 & 0 \end{array}\right], \quad B_5 = \left[\begin{array}{cc:cc} & & 0 & 1 \\ & & -1 & 0 \\ \hdashline 0 & 1 & & \\ -1 & 0 & & \end{array}\right], \quad B_6 = \left[\begin{array}{cc:cc} & & 1 & 0 \\ & & 0 & -1 \\ \hdashline -1 & 0 & & \\ 0 & 1 & & \end{array}\right].$$

Now define $C = (C_1, \ldots, C_j) = a_1 A_1 +_c (b_1 B_1, c_1 B_2) +_c \cdots +_c (b_{n-1} B_1, c_{n-1} B_2) +_c (d_1 B_1, \ldots, d_j B_j)$ for $b_i, c_i, d_i \in \mathbb{R}$. As concatenations have such a simple presentation, it is easy to compute the value of m_1 at such an element. For details see [Jab11a]: $m_1(C) = -2 \sum_i C_i^2 =$

$$\begin{bmatrix} 2a_1^2 Id_{2k} & & & & \\ & 2(b_1^2 + c_1^2) Id_4 & & & \\ & & \ddots & & \\ & & & 2(b_{n-1}^2 + c_{n-1}^2) Id_4 & \\ & & & & 2(d_1^2 + \cdots + d_j^2) Id_4 \end{bmatrix}$$

Then by Theorem 4.11, an algebra of this type will admit a Ricci Yang-Mills soliton as long as

$$a_1^2 = b_1^2 + c_1^2 = \cdots = b_{n-1}^2 + c_{n-1}^2 = d_1^2 + \cdots + d_j^2.$$

Thus we have a $n-1$-parameter family of non-isomorphic algebras (by letting the b_i vary) that admit Ricci Yang-Mills solitons but that do not admit Ricci solitons.

Notice that in this example, since $m_1(w)$ is a multiple of the identity, $D \equiv 0$. Moreover, choosing the coefficients a_i, b_i, c_i, d_i to be rational numbers, we see that such nilpotent Lie groups admit compact quotients (by Mal'cev's criterion) and these compact quotients are endowed with a Ricci Yang-Mills soliton since $D = 0$.

Example 7.4. These examples of Ricci Yang-Mills solitons are generated by non-trivial vector fields and will be of types $(3,9), \ldots, (6,9)$. Again, it is shown in [Jab11a] that these algebras do not admit Ricci soliton metrics. Let C be the concatenation

$$a_1 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} +_c \lambda \left(\begin{bmatrix} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & \\ & 0 \end{bmatrix}, \begin{bmatrix} 0 & \\ & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{bmatrix} \right) +_c (b_1\ B_1, \ldots, b_j\ B_j).$$

In this case, $m_1(C) \neq rId$, so if the manifold admits a Ricci Yang-Mills soliton, it will be nontrivial.

Specifically, we compute $m_1(C)$ to be

$$m_1(C) = -2 \begin{bmatrix} 2a_1^2 Id_2 & & & & \\ & 2\lambda^2 & & & \\ & & 4\lambda^2 & & \\ & & & 2\lambda^2 & \\ & & & & 2(b_1^2 + \cdots + b_j^2) Id_4 \end{bmatrix}.$$

Then a Ricci Yang-Mills soliton is admitted if $4a_1^2 = 6\lambda^2 = 4(b_1^2 + \cdots + b_j^2) = r$. Using the notation of Theorem 4.11, we see that

$$B = \begin{bmatrix} 0 & & & & \\ & -\lambda^2 & & & \\ & & \lambda^2 & & \\ & & & -\lambda^2 & \\ & & & & 0 \end{bmatrix}.$$

A simple computation confirms that B is a stabilizer of C. Thus we obtain a $j-1$-parameter family of non-trivial Ricci Yang-Mills soliton metrics on this algebra.

Remark 7.5. Most algebras should admit many non-isometric Ricci Yang-Mills solitons.

A. Derivation of the Ricci Yang-Mills flow equations

Here we give a brief idea of how the Ricci Yang-Mills equations, Eqs. 2a and 2b, can be derived from the Ricci flow equations on a principal bundle. This computation is straightforward when done with respect to the frame described below. For precise details, see [Yo08, Chapter 1].

Let M be a closed Riemannian manifold with metric g, and let $U \subset M$ be a local coordinate chart with coordinates $\{x^i\}_{i=1}^n$. Let G be a compact Lie group with smooth $\mathfrak{Ad}$-invariant metric k on the Lie algebra $\mathfrak{g}$. Let $\{y^\theta\}_{\theta=n+1}^m$ be local coordinates on G. Then let $\pi : P \to M$ be a principal G-bundle over M, having connection ω. For vector fields X and Y on P, let $k\omega(X,Y) = k(\omega(X), \omega(Y))$.

We consider a metric h on the total space P of the form

$$h = g_{ij}dx^i dx^j + k\omega_{\theta\rho}(dy^\theta + \tilde{\omega}^\theta_k dx^k)(dy^\rho + \tilde{\omega}^\rho_l dx^l). \tag{9}$$

Here, $\tilde{\omega} = \sigma^*\omega$, where $\sigma : U \to P$ is a smooth local section. We have the following basis for one-forms: $dz^i = dx^i$ and $dz^\theta = dy^\theta + \tilde{\omega}^\theta_i dx^i$ with the corresponding frame $e_i = \frac{\partial}{\partial x^i} - \tilde{\omega}^\theta_i \frac{\partial}{\partial y^\theta}$ and $e_\theta = \frac{\partial}{\partial y^\theta}$.

The following lemma follows from a straightforward computation of structure constants and Christoffel symbols. One should note that Roman indices denote quantities on the base and Greek indices denote quantities on the fiber.

Lemma A.1. *The components of the Ricci tensor of the metric h given by Eq. 9 have the form*

- $R_{jk} = R(g)_{jk} - \frac{1}{2}\tilde{\Omega}^{\theta l}_j \tilde{\Omega}_{\theta lk}$,
- $R_{j\mu} = \frac{1}{2}\delta\tilde{\Omega}_{j\mu}$,
- $R_{\mu\rho} = \frac{1}{4}\tilde{\Omega}^{ij}_\mu \tilde{\Omega}_{\rho ij}$,

where Ω is the bundle curvature and $\tilde{\Omega} = \sigma^*\Omega$ is the pullback under a local section.

Let us now consider this metric on a principal G-bundle, where G is abelian. In this setting, we can suppress the bundle indices, as the bundle curvature is actually a 2-form on the base. If we let $\frac{\partial h}{\partial t} = -2Rc$ with the assumption that the size of the fiber remains fixed, we obtain the following natural system of coupled equations:

$$\frac{\partial g_{ij}}{\partial t} = -2R_{ij} + \tilde{\Omega}^l_j \tilde{\Omega}_{lk}, \tag{10a}$$

$$\frac{\partial \tilde{\omega}_i}{\partial t} = -\delta\tilde{\Omega}_i, \tag{10b}$$

as required.

B. Ricci Yang-Mills flow preserves left-invariant metrics

To see that left-invariant metrics are preserved under the Ricci Yang-Mills flow, we will show that the flow is invariant under diffeomorphisms which preserve the principal bundle structure.

Definition B.1. An automorphism of a principal bundle $\pi : P \to M$ is a diffeomorphism $F : P \to P$ such that $F(pg) = F(p)g$ for all $g \in G, p \in P$. We denote this set by $\mathfrak{Diff}_G$.

Let $F \in \mathfrak{Diff}_G$ be an automorphism of the principal bundle P. Then F descends to a diffeomorphism f on M satisfying $f \circ \pi = \pi \circ F$, and if ω is a connection on P, then $F^*\omega$ is also a connection.

Theorem B.2. *The Ricci Yang-Mills flow is invariant under automorphisms of P.*

Proof. Using the properties above, one immediately sees that F^*h is a bundle metric for $F \in \mathfrak{Diff}_G$. More precisely,

$$F^*h = \pi^*(f^*g) + kF^*\omega.$$

For each $z \in \mathfrak{g}$, there is a canonical vector field on P defined by $Z_p = \frac{d}{dt}\big|_{t=0} p \cdot exp(tz)$. As F preserves the G action and $\omega(\frac{d}{dt}\big|_{t=0} p \cdot exp(tz)) = z$, we see that $F_*(Z_p) = Z_{F(p)}$ and $\omega(Z_p) = \omega(Z_{F(p)}) = (F^*\omega)(Z_p) = z$. Observe that if $\{z_i\}$ is an orthonormal basis of $\mathfrak{g}$ then the induced vector fields $\{Z_i\}$ form an orthonormal frame of the vertical space relative to both metrics, h and F^*h.

We can write the Ricci Yang-Mills flow as $\frac{\partial h}{\partial t} = -2(Rc - Rc^V)$, where Rc^V is the projection of the Ricci tensor onto its vertical component. Specifically, if U is a vector field on P, then we can define the projection onto its vertical component to be $p_h(U) = \sum_i h(Z_i, U)Z_i$, where $\{Z_i\}$ is an orthonormal basis (relative to h) of the vertical space as above. In this notation, we have $Rc^V(h)(U, V) = Rc(p_h(U), p_h(V))$.

Let $F : P \to P$ be a bundle automorphism. Since F is a diffeomorphism, clearly $Rc(F^*h) = F^*Rc(h)$. It remains only to check that $Rc^V(F^*h) = F^*Rc^V(h)$. First we show that $p_{F^*h} = p_h \circ F_*$. By definition,

$$\begin{aligned}
p_{F^*h}(U) &= \sum_i (F^*h)(Z_i, U)Z_i \\
&= \sum_i (f^*g)(\pi_*Z_i, \pi_*U) + k(\omega(F_*U), \omega(F_*Z_i))Z_i \\
&= \sum_i k(\omega(F_*U), \omega(Z_i))Z_i \\
&= \sum_i h(F_*U, Z_i)Z_i \\
&= p_h(F_*U)
\end{aligned}$$

Here we have used the fact that $\{Z_i\}$ will be orthonormal in both metrics h and F^*h. Thus $p_{F^*h} = p_h \circ F_*$. Using this fact and the diffeomorphism invariance of Rc, one sees that in fact $Rc^V(F^*h) = F^*Rc^V(h)$. Thus the Ricci Yang-Mills flow is invariant under bundle automorphisms of P. ∎

Corollary B.3. *The Ricci Yang-Mills flow preserves the set of left-invariant metrics on a Lie group N.*

Proof. Left multiplication $L_g(p) = gp$ is a bundle automorphism since left and right multiplication commute. Thus the result follows from above. ∎

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