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Associative Forms and Second Cohomologies of Lie Superalgebras *HO* and *KO*

Jixia Yuan, Wende Liu*, and Wei Bai

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Abstract. We consider two families of finite-dimensional simple Lie superalgebras of Cartan type, denoted by HO and KO, over an algebraically closed field of characteristic $p > 3$. Using the weight space decompositions and the principal gradings we first show that neither HO nor KO possesses a nondegenerate associative form. Then, by means of computing the superderivations from the Lie superalgebras under consideration into their dual modules, the second cohomology groups with coefficients in the trivial modules are proved to be vanishing.
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1. Introduction

The theory of modular Lie superalgebras has undergone an evolution during the last ten years, especially in the classification of classical simple modular Lie superalgebras (see, e.g., [1, 2]) and the structures and representations of simple modular Lie superalgebras of Cartan type (see, e.g., [6, 8, 9, 10, 11, 12]). Recently, one can also find the work on the representations of the classical modular Lie superalgebras. In the present paper we study the second cohomology groups of modular Lie superalgebras of Cartan type. The classical cohomology vanishing results of Lie (super)algebras depend on the characteristic of the underlying field. For simple Lie superalgebras, even in the characteristic zero case, the complete reducibility and the cohomology vanishing theorems do not hold in general. In view of these general facts, in this paper we prove that the cohomology groups vanish for two families of finite-dimensional simple modular Lie superalgebras defined by odd differential forms, known as the odd Hamiltonian superalgebras and the odd contact superalgebras, respectively. That is,

- Main Theorem (Theorem 4.10): *The second cohomology groups with coefficients in the trivial modules vanish for both the odd Hamiltonian superalgebra*

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and the odd contact superalgebra over an algebraically closed field of characteristic $p > 3$.

Let us close this introduction by briefly recalling certain results on the cohomology of modular Lie (super)algebras of Cartan type. Chiu [3] and Farnsteiner [4, 5] determined the second cohomology groups of modular Lie algebras of Cartan type. Wang and Zhang [11] determined the second cohomology groups of simple modular Lie superalgebras of Cartan type H and K, in the case which possess a nondegenerate associative form and Xie and Zhang [12] determined the second cohomology groups of simple modular Lie superalgebras of Cartan type K, in the case which do not possess a nondegenerate associative form.

2. Basics

Hereafter $\mathbb{F}$ is an algebraically closed filed of characteristic $p > 3$, $\mathbb{Z}_2 := \{\bar{0}, \bar{1}\}$ is the additive group of order 2, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ are the sets of integers, nonnegative integers and positive integers, respectively. Throughout this paper $L = \oplus_{i=-r}^{q} L_i$ is a finite dimensional $\mathbb{Z}$-graded Lie superalgebra over $\mathbb{F}$. We should mention that once the symbol $\mathrm{p}(x)$ appears in an expression, it implies that x is a $\mathbb{Z}_2$-homogeneous element of parity $\mathrm{p}(x)$. Write L^* for the dual module of L and $U(L)$ for the universal enveloping algebra of L. Then L^* inherits a $\mathbb{Z}$-graded $U(L)$-module structure:

$$\begin{aligned} &L^* = \oplus_{i=-q}^{r} (L^*)_i; \\ &(u \cdot f) = (-1)^{\mathrm{p}(u)\mathrm{p}(f)} f \circ \Theta(u)|_L \quad \text{for all } u \in U(L),\ f \in L^*, \end{aligned}$$

where Θ is the principal anti-automorphism of $U(L)$, that is, the linear even mapping satisfying that

$$\begin{aligned} &\Theta(1) = 1; \\ &\Theta(x) = -x \quad \text{for all } x \in L; \\ &\Theta(uv) = (-1)^{\mathrm{p}(u)\mathrm{p}(v)} \Theta(v)\Theta(u) \quad \text{for all } u, v \in U(L). \end{aligned}$$

A superderivation from L into L-module L^* is by definition a linear mapping $\psi : L \longrightarrow L^*$ such that

$$\psi([x, y]) = (-1)^{\mathrm{p}(\psi)\mathrm{p}(x)} x \cdot \psi(y) - (-1)^{(\mathrm{p}(\psi)+\mathrm{p}(x))\mathrm{p}(y)} y \cdot \psi(x) \quad \text{for all } x, y \in L;$$

ψ is said to be inner if there is some $f \in L^*$ such that

$$\psi(x) = (-1)^{\mathrm{p}(x)\mathrm{p}(f)} x \cdot f \quad \text{for all } x \in L;$$

ψ is said to be skew if

$$\psi(x)(y) = -(-1)^{\mathrm{p}(x)\mathrm{p}(y)} \psi(y)(x) \quad \text{for all } x, y \in L.$$

Let H be a nilpotent subalgebra of $L_0 \cap L_{\bar{0}}$ with weight space decomposition: $L = \oplus_{\alpha \in \Delta} L_{(\alpha)}$. We write θ for zero weight of L. Viewing L_i as H-modules by means of the adjoint representation, one can find subsets $\Delta_i \subset \Delta$ such that $L_i = \oplus_{\alpha\in\Delta_i} L_i \cap L_{(\alpha)}$. Thus L has a $\mathbb{Z}\times \mathrm{Map}(H,\mathbb{F})$-grading structure, which induces a $\mathbb{Z}\times \mathrm{Map}(H,\mathbb{F})$-grading structure on the dual module L^*.

Let $\mathrm{Der}_{\mathbb{F}}(L,L^*)$ be the space of superderivations from L into L^* and $\mathrm{Inn}_{\mathbb{F}}(L,L^*)$ the subspace consisting of inner superderivations. Then $\mathrm{Der}_{\mathbb{F}}(L,L^*)$ inherits a $\mathbb{Z}\times \mathrm{Map}(H,\mathbb{F})$-grading structure from L and L^* in the usual way.

3. Associative Forms and Weight Space Decompositions

Let m be a positive integer and suppose we are given two m-tuples of positive integers, $\underline{t} := (t_1,t_2,\ldots,t_m)$ and $\pi := (\pi_1,\pi_2,\ldots,\pi_m)$, where $\pi_i := p^{t_i}-1$. Let $\mathcal{O}(m;\underline{t})$ be the divided power algebra over $\mathbb{F}$ with basis $\{x^{(\alpha)} \mid \alpha\in\mathbb{A}(m;\underline{t})\}$, where $\mathbb{A}(m;\underline{t}) := \{\alpha\in\mathbb{N}^m \mid \alpha_i\le\pi_i\}$. For $\varepsilon_i := (\delta_{i1},\ldots,\delta_{im})$, we abbreviate $x^{(\varepsilon_i)}$ to x_i, $i=1,\ldots,m$. Let $\Lambda(n)$ be the exterior superalgebra over $\mathbb{F}$ with n variables $x_{m+1},\ldots,x_{m+n}$. The tensor product $\mathcal{O}(m,n;\underline{t}) := \mathcal{O}(m;\underline{t})\otimes_{\mathbb{F}}\Lambda(n)$ is an associative super-commutative superalgebra with a $\mathbb{Z}_2$-grading structure induced by the trivial $\mathbb{Z}_2$-grading of $\mathcal{O}(m;\underline{t})$ and the standard $\mathbb{Z}_2$-grading of $\Lambda(n)$. For $g\in\mathcal{O}(m,\underline{t})$, $f\in\Lambda(n)$, write gf for $g\otimes f$. Note that $x^{(\alpha)}x^{(\beta)} = \binom{\alpha+\beta}{\alpha}x^{(\alpha+\beta)}$ for $\alpha,\beta\in\mathbb{N}^m$, where $\binom{\alpha+\beta}{\alpha} := \prod_{i=1}^m\binom{\alpha_i+\beta_i}{\alpha_i}$. Let

$$\mathbb{B}(n) := \{\langle i_1,i_2,\ldots,i_k\rangle \mid m+1\le i_1<i_2<\cdots<i_k\le m+n;\ 0\le k\le n\}.$$

For $u := \langle i_1,i_2,\ldots,i_k\rangle\in\mathbb{B}(n)$, set $|u|=k$ and write $x^u = x_{i_1}x_{i_2}\cdots x_{i_k}$. Notice that we also denote the set $\{i_1,i_2,\ldots,i_k\}$ by u itself. For $u\in\mathbb{B}(n+1)$, put

$$\delta_{i\in u} = \begin{cases} 0 & \text{if } i\in u \\ 1 & \text{if } i\notin u, \end{cases} \qquad \|u\| = \begin{cases} |u|+1 & \text{if } 2n+1\in u \\ |u| & \text{if } 2n+1\notin u. \end{cases}$$

For $u,v\in\mathbb{B}(n)$ with $u\cap v=\emptyset$, define $u+v$ to be $w\in\mathbb{B}(n)$ such that $w=u\cup v$. If $v\subset u$, define $u-v$ to be $w\in\mathbb{B}(n)$ such that $w=u\setminus v$. Note that $\mathcal{O}(m,n;\underline{t})$ has a standard $\mathbb{F}$-basis $\{x^{(\alpha)}x^u \mid (\alpha,u)\in\mathbb{A}(m;\underline{t})\times\mathbb{B}(n)\}$. Let ∂_r be the superderivation of $\mathcal{O}(m,n;\underline{t})$ such that $\partial_r(x^{(\alpha)}) = x^{(\alpha-\varepsilon_r)}$ for $r\in\overline{1,m}$ and $\partial_r(x_s)=\delta_{rs}$ for $r,s\in\overline{m+1,m+n}$. The generalized Witt superalgebra $W(m,n;\underline{t})$ is spanned by all $f_r\partial_r$, where $f_r\in\mathcal{O}(m,n;\underline{t})$, $r\in\overline{1,m+n}$. Note that $W(m,n;\underline{t})$ is a free $\mathcal{O}(m,n;\underline{t})$-module with basis $\{\partial_r \mid r\in\overline{1,m+n}\}$. In particular, $W(m,n;\underline{t})$ has a standard $\mathbb{F}$-basis $\{x^{(\alpha)}x^u\partial_r \mid (\alpha,u,r)\in\mathbb{A}(m;\underline{t})\times\mathbb{B}(n)\times\overline{1,m+n}\}$. Put $\xi=|\pi|+n$. Recall the standard $\mathbb{Z}$-grading, $\mathcal{O}(m,n;\underline{t}) = \oplus_{i=0}^{\xi}\mathcal{O}(m,n;\underline{t})_{\mathbf{s},i}$, where

$$\mathcal{O}(m,n;\underline{t})_{\mathbf{s},i} := \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)}x^u \mid |\alpha|+|u|=i, \alpha\in\mathbb{A}(m;\underline{t}), u\in\mathbb{B}(n)\}.$$

It induces naturally the standard grading $W(m,n;\underline{t}) = \oplus_{i=-1}^{\xi-1}W(m,n;\underline{t})_{\mathbf{s},i}$, where

$$W(m,n;\underline{t})_{\mathbf{s},i} := \mathrm{span}_{\mathbb{F}}\{f\partial_j \mid f\in\mathcal{O}(m,n;\underline{t})_{\mathbf{s},i+1}, j\in\overline{1,m+n}\}.$$

The standard gradings of $\mathcal{O}(m,n;\underline{t})$ and $W(m,n;\underline{t})$ are also said to be of type $(1,\ldots,1 \mid 1,\ldots,1)$.

We also use the principal grading $\mathcal{O}(n,n+1;\underline{t}) = \oplus_{i=0}^{\xi+2}\mathcal{O}(n,n+1;\underline{t})_{\mathbf{p},i}$, where

$$\mathcal{O}(n,n+1;\underline{t})_{\mathbf{p},i} := \mathrm{span}_{\mathbb{F}}\{x^{(\alpha)}x^u \mid |\alpha|+\|u\|=i,\ \alpha\in\mathbb{A}(n;\underline{t}),\ u\in\mathbb{B}(n+1)\},$$

and the principal grading $W(n,n+1;\underline{t}) = \oplus_{i=-2}^{\xi+1} W(n,n+1;\underline{t})_{\mathbf{p},i}$, where

$$W(n,n+1;\underline{t})_{\mathbf{p},i} := \mathrm{span}_{\mathbb{F}}\{f\partial_j \mid f \in \mathcal{O}_{\mathbf{p},i+1+\delta_{j,2n+1}}, j \in \overline{1,2n+1}\}.$$

The principal gradings of $\mathcal{O}(n,n+1;\underline{t})$ and $W(n,n+1;\underline{t})$ are also said to be of type $(1,\ldots,1 \mid 1,\ldots,1,2)$ (see [7]).

Consider the linear operator $\mathrm{T_H} : \mathcal{O}(n,n;\underline{t}) \to W(n,n;\underline{t})$ given by

$$\mathrm{T_H}(a) := \sum_{i\in\overline{1,2n}} (-1)^{\mathrm{p}(\partial_i)\mathrm{p}(a)} \partial_i(a)\partial_{i'} \quad \text{for } a \in \mathcal{O}(n,n;\underline{t}),$$

where

$$i' := \begin{cases} 2n+1 & \text{if } i = 0 \\ i+n & \text{if } i \in \overline{1,n} \\ i-n & \text{if } i \in \overline{n+1,2n}. \end{cases}$$

Note that $\mathrm{T_H}$ is odd and that

$$[\mathrm{T_H}(a), \mathrm{T_H}(b)] = \mathrm{T_H}(\mathrm{T_H}(a)(b)) \quad \text{for all } a,b \in \mathcal{O}(n,n;\underline{t}).$$

Then

$$HO(n,n;\underline{t}) := \{\mathrm{T_H}(a) \mid a \in \mathcal{O}(n,n;\underline{t})\}$$

is a finite dimensional simple Lie superalgebra, called the odd Hamiltonian superalgebra (cf. [7, 10]). The standard $\mathbb{Z}$-grading is defined as follows:

$$HO(n,n;\underline{t}) = \oplus_{i=-1}^{\xi-2} HO(n,n;\underline{t})_i,$$

where

$$HO(n,n;\underline{t})_i := HO(n,n;\underline{t}) \cap W(n,n;\underline{t})_{\mathbf{s},i}.$$

Recall the finite dimensional odd contact superalgebra, which is a simple Lie superalgebra contained in $W(n,n+1;\underline{t})$, defined as follows (cf. [6, 7]):

$$KO(n,n+1;\underline{t}) = \{\mathrm{T_K}(a) \mid a \in \mathcal{O}(n,n;\underline{t})\},$$

where

$$\mathrm{T_K}(a) := \mathrm{T_H}(a) + (-1)^{\mathrm{p}(a)}\partial_{2n+1}(a)\mathfrak{D} + (\mathfrak{D}(a) - 2a)\partial_{2n+1}$$

and

$$\mathfrak{D} := \sum_{i=1}^{2n} x_i\partial_i, \qquad \mu(i) := \begin{cases} \bar{0} & \text{if } i \in \overline{1,n} \\ \bar{1} & \text{if } i \in \overline{n+1,2n+1}. \end{cases}$$

We have the following formula (see [6, 7]): for $a,b \in \mathcal{O}(n,n+1;\underline{t})$,

$$[\mathrm{T_K}(a), \mathrm{T_K}(b)] = \mathrm{T_K}(\mathrm{T_K}(a)(b) - (-1)^{\mathrm{p}(a)} 2\partial_{2n+1}(a)b). \tag{1}$$

The principal $\mathbb{Z}$-grading is defined as follows:

$$KO(n,n+1;\underline{t}) = \oplus_{i=-2}^{\xi} KO(n,n+1;\underline{t})_i,$$

where

$$KO(n,n+1;\underline{t})_i := KO(n,n+1;\underline{t}) \cap W(n,n+1;\underline{t})_{\mathbf{p},i}.$$

In particular,

$$KO(n,n+1;\underline{t})_{-2} = \mathbb{F}\mathrm{T_K}(1), \quad KO(n,n+1;\underline{t})_{\xi} = \mathbb{F}\mathrm{T_K}(x^{(\pi)}x^{\langle n+1,\dots,2n+1\rangle}).$$

For simplicity, we write HO and KO for $HO(n,n;\underline{t})$ and $KO(n,n+1;\underline{t})$, respectively.

For $i \in \overline{1,n}$, set

$$H_{HO} = \sum_{i\in\overline{1,n}} \mathbb{F}\mathrm{T_H}(x_i x_{i'}), \quad H_{KO} = \sum_{i\in\overline{1,n}} \mathbb{F}\mathrm{T_K}(x_i x_{i'}).$$

Write X for $= HO$ or KO. Then H_X is a torus of $X_{\bar{0}} \bigcap X_0$, known as standard. We consider the weight space decomposition of X with respect to the torus H_X,

$$X = \oplus_{\gamma\in\Delta_X} X_\gamma.$$

For $\alpha \in \mathbb{A}(n;\underline{t})$ and $u \in \mathbb{B}(n)$, define $\alpha+u$ to be the linear function on H_X,

$$\alpha + u : H_X \longrightarrow \mathbb{F}$$

such that for $i \in \overline{1,n}$,

$$\begin{aligned}(\alpha+u)(\mathrm{T_H}(x_i x_{i'})) &= \delta_{i'\in u} - \alpha_i \quad \text{if } X = HO,\\ (\alpha+u)(\mathrm{T_K}(x_i x_{i'})) &= \delta_{i'\in u} - \alpha_i \quad \text{if } X = KO.\end{aligned}$$

Since

$$[\mathrm{T_H}(x_i x_{i'}), \mathrm{T_H}(x^{(\alpha)}x^u)] = (\delta_{i'\in u} - \alpha_i)\mathrm{T_H}(x^{(\alpha)}x^u),$$

one sees that $\mathrm{T_H}(x^{(\alpha)}x^u) \in HO$ is a weight vector belonging to the weight $\alpha+u$. Similarly, $\mathrm{T_K}(x^{(\alpha)}x^u)$ and $\mathrm{T_K}(x^{(\alpha)}x^u x_{2n+1}) \in KO$ is a weight vector belonging to the weight $\alpha+u$.

For $\alpha,\beta \in \mathbb{A}(n;\underline{t})$, write $\alpha \equiv \beta \pmod p$ provided that $\alpha_i \equiv \beta_i \pmod p$ for all $i \in \overline{1,n}$.

Proposition 3.1. *For $\alpha \in \mathbb{A}(n;\underline{t})$ and $u \in \mathbb{B}(n)$, we have*

$$HO_{(\alpha+u)} = \sum_{\substack{\beta\equiv\alpha(\mathrm{mod}p), v_1\subset u,\ v_2\subset\overline{n+1,2n}\setminus u,\\ \beta-\sum_{j'\in v_1}\varepsilon_j+\sum_{j'\in v_2}\varepsilon_j\in\mathbb{A}(n;\underline{t})}} \mathbb{F}\mathrm{T_H}(x^{(\beta-\sum_{j'\in v_1}\varepsilon_j+\sum_{j'\in v_2}\varepsilon_j)}x^{u-v_1+v_2}),$$

$$KO_{(\alpha+u)} = \sum_{\substack{\beta\equiv\alpha(\mathrm{mod}p), v_1\subset u,\ v_2\subset\overline{n+1,2n+1}\setminus u\\ \beta-\sum_{j'\in v_1}\varepsilon_j+\sum_{2n+1\neq j'\in v_2}\varepsilon_j\in\mathbb{A}(n;\underline{t})}} \mathbb{F}\mathrm{T_K}(x^{(\beta-\sum_{j'\in v_1}\varepsilon_j+\sum_{2n+1\neq j'\in v_2}\varepsilon_j)}x^{u-v_1+v_2}).$$

Proof. (i) For HO, the inclusion " $\supset$ " is straightforward. To show the converse, let $\mathrm{T_H}(x^{(\beta)}x^v) \in HO_{(\alpha+u)}$, where $\beta \in \mathbb{A}(n,\underline{t})$, $v \in \mathbb{B}(n)$. Then, for $i \in \overline{1,n}$, one has

$$\begin{aligned}(\delta_{i'\in v} - \beta_i)\mathrm{T_H}(x^{(\beta)}x^v) &= (\beta+v)(\mathrm{T_H}(x_i x_{i'}))\mathrm{T_H}(x^{(\beta)}x^v)\\ &= [\mathrm{T_H}(x_i x_{i'}), \mathrm{T_H}(x^{(\beta)}x^v)]\\ &= (\alpha+u)(\mathrm{T_H}(x_i x_{i'}))\mathrm{T_H}(x^{(\beta)}x^v)\\ &= (\delta_{i'\in u} - \alpha_i)\mathrm{T_H}(x^{(\beta)}x^v).\end{aligned}$$

Consequently, $\delta_{i'\in v} - \beta_i = \delta_{i'\in u} - \alpha_i$ holds in $\mathbb{F}$, that is,

$$\alpha_i - \beta_i = \delta_{i'\in u} - \delta_{i'\in v} = \begin{cases} 1 \pmod p & \text{if } i' \in u \text{ and } i' \notin v \\ -1 \pmod p & \text{if } i' \notin u \text{ and } i' \in v \\ 0 \pmod p & \text{if } i' \in u \text{ and } i' \in v \\ 0 \pmod p & \text{if } i' \notin u \text{ and } i' \notin v. \end{cases}$$

The assertion follows.

(ii) In (i), replacing HO and $\mathrm{T_H}$ by KO and $\mathrm{T_K}$ respectively, one checks that all the arguments above still hold. Thus, for KO it can be proved in the same manner. ■

Write $\omega_0 = \langle n+1, \ldots, 2n\rangle$ and $\omega_1 = \langle n+1, \ldots, 2n+1\rangle$. By Proposition 3.1, we list the following for later use:

$$KO_\theta = \sum_{\alpha\equiv 0(\mathrm{mod}p),\, u\in\mathbb{B}(n+1)} \mathbb{F}\mathrm{T_K}(x^{(\alpha+\sum_{2n+1\neq j'\in u}\varepsilon_j)}x^u), \tag{2}$$

$$\Delta_{HO,0} = \Delta_{KO,0} = \big\{\theta, 2\varepsilon_i, \varepsilon_i+\varepsilon_j, \varepsilon_i + \langle j'\rangle, \langle i', j'\rangle \mid i \neq j \in \overline{1,n}\big\}, \tag{3}$$

$$\begin{aligned}\Delta_{HO,1} = \Delta_{KO,1} = \big\{&\varepsilon_i, \langle i'\rangle, 2\varepsilon_i+\varepsilon_j, 2\varepsilon_i + \langle j\rangle, \varepsilon_i+\varepsilon_j+\varepsilon_k, \varepsilon_i+\varepsilon_j+\langle k'\rangle,\\ &3\varepsilon_i, \varepsilon_i + \langle j', k'\rangle, \langle i', j', k'\rangle \mid i,j,k \in \overline{1,n}\,\text{distinct}\big\},\end{aligned} \tag{4}$$

$$\begin{aligned}\Delta_{HO,2} = \Delta_{KO,2} = \Big\{&\theta, \alpha+u, \beta+v \mid |\alpha|+|u| = 4, \sum_{i=1}^n \partial_i\partial_{i'}(x^{(\alpha)}x^u) = 0,\\ &|\beta|+|v| = 2, \sum_{i=1}^n \partial_i\partial_{i'}(x^{(\beta)}x^v) = 0\Big\},\end{aligned} \tag{5}$$

$$\Delta_{HO,\xi-2} = \Delta_{KO,\xi} = \{\pi+\omega_0\}. \tag{6}$$

As in the Lie algebra case, one may easily prove the following

Lemma 3.2. *Let $L = \oplus_{i=-r}^q L_i$ be a finite dimensional simple $\mathbb{Z}$-graded Lie superalgebra and H a nilpotent subalgebra of $L_{\bar 0}\cap L_0$ with weight space decomposition $L = \oplus_{\alpha\in\Delta} L_{(\alpha)}$. If L has a nondegenerate associative form, then*

$$\dim_{\mathbb{F}} L_k = \dim_{\mathbb{F}} L_{q-r-k} \quad \textit{and} \quad \dim_{\mathbb{F}} L_k \cap L_\gamma = \dim_{\mathbb{F}} L_{q-r-k}\cap L_{-\gamma}$$

for all $-r \leq k \leq q$ and $\gamma \in \Delta$.

Theorem 3.3. *Let $X = HO$ or KO. Then X does not possess any nondegenerate associative form.*

Proof. For $X = HO$, we have $X = \sum_{i=-1}^{\xi-2} X_i$. Note that

$$X_{\xi-2} = \mathrm{span}_{\mathbb{F}}\mathrm{T_H}(x^{(\pi)}x^{\omega_0}), \quad X_{-1} = \mathrm{span}_{\mathbb{F}}\{\mathrm{T_H}(x_i) \mid i \in \overline{1,2n}\}.$$

This shows that

$$1 = \dim X_{\xi-2} \neq \dim X_{-1} = 2n.$$

For $X = KO$, we have $X = \sum_{i=-2}^{\xi} X_i$. Consider the weight space decomposition with respect to the torus H_X defined above, $X = \oplus_{\alpha\in\Delta_X} X_{(\alpha)}$. ¿From (2) we have

$$1 = \dim X_{-2} \cap X_\theta \neq \dim X_\xi \cap X_\theta = 0.$$

Now our conclusion follows from Lemma 3.2. ■

4. Second Cohomology Groups

As in the Lie algebra case [5], if a finite dimensional simple Lie algebra L does not possess any nondegenerate associative form, then $H^2(L,\mathbb{F}) \cong H^1(L,L^*)$ and all the $\mathbb{Z}_2$-homogeneous superderivations from L into L^* are skew. We thereby obtain by Lemma 3.2 and Theorem 3.3 the following

Theorem 4.1. *$H^2(X,\mathbb{F})$ is isomorphic to $H^1(X,X^*)$ and $\mathrm{Der}(X,X^*)$ consists of skew superderivations for $X = HO$ or KO.*

In view of this theorem, in order to determine the second cohomology groups, it is enough to compute $H^1(X,X^*)$. To this aim, we want to establish a reduction lemma relative to the superderivations from X into X^* (see Lemma 4.5). Before doing that, we first recall a general result (Lemma 4.2), which is analogous to the Lie algebra case [5].

Let L be a finite dimensional $\mathbb{Z}$-graded Lie superalgebra over $\mathbb{F}$, $L = \oplus_{i=-r}^{q} L_i$. Write $L^- = \oplus_{i=-r}^{-1} L_i$ and $L^+ = \oplus_{i=1}^{q} L_i$. Recall the canonical mapping

$$\Phi_1 : H^1(L,L^*) \longrightarrow H^1(L^-,L^*)$$

which is naturally induced by the restriction mapping

$$\mathrm{Der}_{\mathbb{F}}(L,L^*) \longrightarrow \mathrm{Der}_{\mathbb{F}}(L^-,L^*).$$

Write $U(L)^+$ for the two-sided ideal of $U(L)$ generated by L and put

$$\mathrm{Ann}_{U(L^-)^+}(L) = \{u \in U(L^-)^+ \mid u\cdot L = 0\}.$$

Lemma 4.2. *[12] Let V be a $\mathbb{Z}_2$-graded subspace of L. Suppose there are a $\mathbb{Z}_2$-homogeneous basis $\{e_1, e_2, \ldots, e_n\}$ of L^- and a $\mathbb{Z}_2$-homogeneous basis $\{v_1, v_2, \ldots, v_m\}$ of V such that $\{e^a \cdot v_j \mid 1 \le j \le m, a \in T\}$ is a basis of L over $\mathbb{F}$ for some subset $T \subset \mathbb{N}^n$ and that $\mathrm{Ann}_{U(L^-)^+}(L) = \mathrm{span}_{\mathbb{F}}\{e^b \mid b \in \mathbb{N}^n \setminus T\}$. Then the following statements hold:*

1. *Every superderivation $\psi : L \longrightarrow L^*$ satisfying that $\ker(\mathrm{ad}e_i) \subset \ker\psi(e_i)$ for all $i \in \overline{1,n}$ defines an element of $\ker\Phi_1$, that is, $\psi|_{L^-} \in \mathrm{Inn}_{\mathbb{F}}(L^-,L^*)$.*
2. *Suppose there is $\mu \in \mathbb{N}^n$ such that $T = \{b \in \mathbb{N}^n \mid b \le \mu\}$. Then $\ker(\mathrm{ad}e_i) \subset \ker\psi(e_i)$ if and only if $e_i^{\mu_i} \cdot \psi(e_i) = 0$.*

We mention a standard result that for a Lie superalgebra L, every homogeneous superderivation $\psi : L \longrightarrow L^*$ can be extended to one and only one $U(L)$-module homomorphism $\Psi : U(L)^+ \longrightarrow L^*$ so that $\Psi(x) = \psi(x)$ for all $x \in L$.

Lemma 4.3. *If $\psi : HO \longrightarrow HO^*$ is a superderivation, then*

$$\mathrm{T_H}(x_i)\cdot\psi(\mathrm{T_H}(x_i)) = \mathrm{T_H}(x_j)^{\pi_j}\cdot\psi(\mathrm{T_H}(x_j)) = 0 \quad \text{for all } \ i\in\overline{1,n},\ j\in\overline{n+1,2n}.$$

Proof. Letting Ψ be as indicated above, we have

$$\begin{aligned}\mathrm{T_H}(x_i)\cdot\psi(\mathrm{T_H}(x_i)) &= \mathrm{T_H}(x_i)\cdot\Psi(\mathrm{T_H}(x_i)) = \Psi(\mathrm{T_H}(x_i)^2) = 0,\\ \mathrm{T_H}(x_j)^{\pi_j}\cdot\psi(\mathrm{T_H}(x_j)) &= \mathrm{T_H}(x_j)^{\pi_j}\cdot\Psi(\mathrm{T_H}(x_j)) = \Psi(\mathrm{T_H}(x_j)^{p^{t_j}})\end{aligned}$$

for all $i\in\overline{1,n}$, $j\in\overline{n+1,2n}$. Let $G\in HO$. As $\mathrm{T_H}(x_j)^{p^{t_j}}$ lies in the center $C(U(HO)^+)$, where $j\in\overline{n+1,2n}$, we have

$$G\cdot\Psi(\mathrm{T_H}(x_j)^{p^{t_j}}) = \Psi(G\mathrm{T_H}(x_j)^{p^{t_j}}) = \Psi(\mathrm{T_H}(x_j)^{p^{t_j}}G) = \mathrm{T_H}(x_j)^{p^{t_j}}\cdot\Psi(G).$$

Note that

$$(\mathrm{T_H}(x_j)^{p^{t_j}}\cdot\Psi(G))(y) = \pm\Psi(G)(\mathrm{T_H}(x_j)^{p^{t_j}}\cdot y) = 0$$

for $y\in HO$. Consequently, $\mathrm{T_H}(x_j)^{\pi_j}\cdot\psi(\mathrm{T_H}(x_j))$ lies in

$$\{f\in HO^* \mid HO\cdot f = 0\} = \{f\in HO^* \mid f([HO,HO]) = 0\},$$

which is zero, since HO is simple. The proof is complete. ■

Lemma 4.4. *For $\mu := (1,\dots,1,\pi_1,\dots,\pi_n)\in\mathbb{N}^{2n}$, we have*

$$\ker(\mathrm{ad}\mathrm{T_H}(x_i)) = \mathrm{T_H}(x_i)^{\mu_i}\cdot HO + \mathbb{F}\mathrm{T_H}(x_{i'}),\quad i\in\overline{1,2n}.$$

Proof. The inclusion "$\supset$" is clear. For $b := (b_1,b_2,\dots,b_{2n})\in\mathbb{N}^{2n}$ with $b_i = 0$ or 1 for all $i\in\overline{1,n}$, put

$$\begin{aligned}\mathrm{T_H}^b &= \mathrm{T_H}(x_1)^{b_1}\mathrm{T_H}(x_2)^{b_2}\cdots\mathrm{T_H}(x_{2n})^{b_{2n}},\\ U(HO_{-1})_{(k)} &= \mathrm{span}_{\mathbb{F}}\Big\{\mathrm{T_H}^b \mid \sum_{i=1}^{2n} b_i\le k\Big\},\\ HO_{(k)} &= U(HO_{-1})_{(k)}\cdot\mathrm{T_H}(x^{(\pi)}x^{\omega_0}).\end{aligned}$$

Then $HO = \sum_{k=0}^{|\mu|-1} HO_{(k)}$. By induction one gets

$$\ker(\mathrm{ad}\mathrm{T_H}(x_i))\cap HO_{(k)}\subset\mathrm{T_H}(x_i)^{\mu_i}\cdot HO,$$

where $0\le k\le|\mu|-2$. Now we want to show that

$$\ker(\mathrm{ad}\mathrm{T_H}(x_i))\cap HO_{(|\mu|-1)}\subset\mathrm{T_H}(x_i)^{\mu_i}\cdot HO + \mathbb{F}\mathrm{T_H}(x_{i'}).$$

Let

$$x = \sum_{0\le a<\mu}\lambda_a\mathrm{T_H}^a\cdot\mathrm{T_H}(x^{(\pi)}x^{\omega_0})\in\ker(\mathrm{ad}\mathrm{T_H}(x_i))\cap HO_{(|\mu|-1)}.$$

Then

$$\begin{aligned}x &\equiv \sum_{\substack{0\le a<\mu,a_i=\mu_i\\ |a|=|\mu|-1}}\lambda_a\mathrm{T_H}^a\cdot\mathrm{T_H}(x^{(\pi)}x^{\omega_0}) + \lambda_{\mu-\varepsilon_i}\mathrm{T_H}^{\mu-\varepsilon_i}\cdot\mathrm{T_H}(x^{(\pi)}x^{\omega_0})\\ &\equiv \sum_{\substack{0\le a<\mu,a_i=\mu_i\\ |a|=|\mu|-1}}\pm\lambda_a\mathrm{T_H}(x_i)^{\mu_i}\mathrm{T_H}^{a-\mu_i\varepsilon_i}\cdot\mathrm{T_H}(x^{(\pi)}x^{\omega_0}) \pm \lambda_{\mu-\varepsilon_i}\mathrm{T_H}(x_{i'})\end{aligned}$$

modulo $HO_{(|\mu|-2)}$. Therefore, there is $y \in HO$ such that

$$x - \mathrm{T_H}(x_i)^{\mu_i} \cdot y \pm \lambda_{\mu-\varepsilon_i}\mathrm{T_H}(x_{i'}) \in HO_{(|\mu|-2)}$$

and

$$\mathrm{ad}\mathrm{T_H}(x_i)(x - \mathrm{T_H}(x_i)^{\mu_i} \cdot y \pm \lambda_{\mu-\varepsilon_i}\mathrm{T_H}(x_{i'})) = 0.$$

Then

$$x - \mathrm{T_H}(x_i)^{\mu_i} \cdot y \pm \lambda_{\mu-\varepsilon_i}\mathrm{T_H}(x_{i'}) \in \ker(\mathrm{ad}\mathrm{T_H}(x_i)) \cap HO_{(|\mu|-2)} \subset \mathrm{T_H}(x_i)^{\mu_i} \cdot HO,$$

that is, $x \in \mathrm{T_H}(x_i)^{\mu_i} \cdot HO + \mathbb{F}\mathrm{T_H}(x_{i'})$. The proof is complete. ∎

Recall that H_X is a torus of $X_{\bar{0}} \bigcap X_0$. Consider the weight space decomposition of X with respect to H_X, $X = \oplus_{\gamma\in\Delta_X} X_\gamma$. We would like to mention a standard fact that every $\mathrm{Map}(H_X,\mathbb{F})$-homogeneous nonzero-degree superderivation from X into X^* must be inner. So we are only concerned with those superderivations of $\mathrm{Map}(H_X,\mathbb{F})$-degree θ.

Lemma 4.5. *Let X be HO or KO. If $\psi : X \longrightarrow X^*$ is a superderivation of* $\mathrm{Map}(H_X,\mathbb{F})$*-degree θ, then there exists $f \in X^*$ such that $\psi(x) = (-1)^{\mathrm{d}(x)\mathrm{d}(f)} x \cdot f$ for all $x \in X^-$.*

Proof. (i) First consider the case $X = HO$. The general assumption of Lemma 4.2 is valid for $V := \mathbb{F} \cdot \mathrm{T_H}(x^{(\pi)}x^{\omega_0})$. Put ${\mathrm{T_H}}^a = \mathrm{T_H}(x_1)^{a_1}\mathrm{T_H}(x_2)^{a_2} \cdots \mathrm{T_H}(x_{2n})^{a_{2n}}$ for $a := (a_1, a_2, \ldots, a_{2n}) \in \mathbb{N}^{2n}$ with $a_i = 0$ or 1 for all $i \in \overline{1,n}$. Suppose $a_{i_1} = a_{i_2} = \cdots = a_{i_k} = 0$, $1 \le i_1 < i_2 < \cdots i_k \le n$. Write

$$x^u = x_{i'_1}x_{i'_2}\cdots x_{i'_k}, \quad b = a_{1'}\varepsilon_1 + a_{2'}\varepsilon_2 + \cdots a_{n'}\varepsilon_n.$$

Since

$${\mathrm{T_H}}^a \cdot \mathrm{T_H}(x^{(\pi)}x^{\omega_0}) = \pm\mathrm{T_H}(x^{(\pi-b)}x^u),$$

$\{{\mathrm{T_H}}^a \cdot \mathrm{T_H}(x^{(\pi)}x^{\omega_0}) \mid a \in T\}$ is an $\mathbb{F}$-basis of HO and

$$\mathrm{Ann}_{U(HO^-)^+}(HO) = \{{\mathrm{T_H}}^a \mid a \notin T\},$$

where

$$T := \{a \in \mathbb{N}^{2n} \mid a < \mu\}, \quad \mu := (1, \ldots, 1, \pi_1, \ldots, \pi_n) \in \mathbb{N}^{2n}.$$

By Lemmas 4.2(1) and 4.4, it suffices to show that $\mathrm{T_H}(x_i)^{\mu_i} \cdot HO + \mathbb{F}\mathrm{T_H}(x_{i'}) \subset \ker\psi(\mathrm{T_H}(x_i))$ for all $i \in \overline{1,2n}$. By Lemma 4.3,

$$\begin{aligned} 0 = \mathrm{T_H}(x_i)^{\mu_i}\psi(\mathrm{T_H}(x_i))(HO) &= \pm\psi(\mathrm{T_H}(x_i))(\Theta(\mathrm{T_H}(x_i)^{\mu_i}) \cdot HO) \\ &= \pm\psi(\mathrm{T_H}(x_i))(\mathrm{T_H}(x_i)^{\mu_i} \cdot HO), \end{aligned}$$

that is, $\mathrm{T_H}(x_i)^{\mu_i} \cdot HO \subset \ker\psi(\mathrm{T_H}(x_i))$. Since ψ is a superderivation of degree θ, it is clear that $\mathbb{F}\mathrm{T_H}(x_{i'}) \subset \ker\psi(\mathrm{T_H}(x_i))$.

(ii) Consider the case $X = KO$. The general assumption of Lemma 4.2 is valid for

$$V := \mathbb{F} \cdot \mathrm{T_H}(x^{(\pi)}x^{\omega_1}).$$

Put

$$\mathrm{T_K}^a = \mathrm{T_K}(1)^{a_0}\mathrm{T_K}(x_1)^{a_1}\mathrm{T_K}(x_2)^{a_2}\cdots\mathrm{T_K}(x_{2n})^{a_{2n}}$$

for $a := (a_0, a_1, a_2, \dots, a_{2n}) \in \mathbb{N}^{2n+1}$ with $a_i = 0$ or 1 for all $i \in \overline{0, n}$. Suppose $a_{i_1} = a_{i_2} = \cdots = a_{i_k} = 0,\ 0 \le i_1 < i_2 < \cdots i_k \le n$. Write

$$x^u = x_{i_1'}x_{i_2'}\cdots x_{i_k'}, \quad b = a_{1'}\varepsilon_1 + a_{2'}\varepsilon_2 + \cdots + a_{n'}\varepsilon_n.$$

Since

$$\mathrm{T_K}^a \cdot \mathrm{T_K}(x^{(\pi)}x^{\omega_1}) = \pm\lambda\mathrm{T_K}(x^{(\pi-b)}x^u), \quad \text{where } \lambda = 1 \text{ or } 2,$$

one sees that $\{\mathrm{T_K}^a \cdot \mathrm{T_K}(x^{(\pi)}x^{\omega_1}) \mid a \in T\}$ is an $\mathbb{F}$-basis of KO and

$$\mathrm{Ann}_{U(KO^-)^+}(KO) = \{\mathrm{T_K}^a \mid a \notin T\},$$

where

$$T := \{a \in \mathbb{N}^{2n+1} \mid a \le \mu\}, \quad \mu := (1, \dots, 1, \pi_1, \dots, \pi_n) \in \mathbb{N}^{2n+1}.$$

As in Lemma 4.3, one easily gets, for $i \in \overline{1, 2n}$,

$$\mathrm{T_K}(1)\psi(\mathrm{T_K}(1)) = \mathrm{T_K}(x_i)^{\mu_{i+1}}\psi(\mathrm{T_K}(x_i)) = 0.$$

Then the desired result follows from Lemma 4.2(1) and (2). ■

As before, $L = \oplus_{i=-r}^{q} L_i$ is a finite dimensional simple $\mathbb{Z}$-graded Lie superalgebra and H a nilpotent subalgebra of $L_{\bar{0}} \cap L_0$ with weight space decomposition $L = \oplus_{\alpha\in\Delta}L_{(\alpha)}$. We consider the subalgebra

$$M(L) := [L^+, L^+].$$

Note that $M(L)$ is a graded subalgebra of L, on which H operates. Hence for $h \ge 1$ there is $\phi_h \subset \Delta_h$ such that $L_h = M(L)_h + \oplus_{\alpha\in\phi_h}L_{(\alpha)} \cap L_h$.

Lemma 4.6. *Suppose $h \ge 3$. Then*

$$HO_h = M(HO)_h + \sum_{\substack{\alpha_i \equiv 0,1 (\mathrm{mod} p)\ \forall i \in \overline{1,n} \\ u\in\mathbb{B}(n), |\alpha|+|u|-2=h}} \mathbb{F}\mathrm{T_H}(x^{(\alpha)}x^u). \tag{7}$$

Proof. The inclusion " $\supset$ " is clear. Let us consider the converse inclusion. Let $\mathrm{T_H}(x^{(\alpha)}x^u) \in HO_h$ with $h \ge 3$, $\alpha \in \mathbb{A}(n, \underline{t})$ and $u \in \mathbb{B}(n)$. If $\alpha_i \equiv 0, 1 (\mathrm{mod} p)$ for all $i \in \overline{1, n}$, then $\mathrm{T_H}(x^{(\alpha)}x^u)$ lies in the second summand of (7). Thereby we suppose $\alpha_j \not\equiv 0, 1 \pmod p$ for some $j \in \overline{1, n}$. We distinguish two cases:

Case 1. Suppose $j' \notin u$. We have

$$0 \ne \binom{\alpha}{2\varepsilon_j}\mathrm{T_H}(x^{(\alpha)}x^u) = -[\mathrm{T_H}(x^{(2\varepsilon_j)}x_{j'}), \mathrm{T_H}(x^{(\alpha-\varepsilon_j)}x^u)] \in M(HO)_h.$$

Case 2. Suppose $j' \in u$. We have

$$\frac{\alpha_j(3-\alpha_j)}{2}\mathrm{T_H}(x^{(\alpha)}x^u) = \left[\mathrm{T_H}(x^{(2\varepsilon_j)}x_{j'}), \mathrm{T_H}(x^{(\alpha-\varepsilon_j)}x^u)\right] \in M(HO)_h \tag{8}$$

and

$$\frac{\alpha_j(\alpha_j-1)(5-\alpha_j)}{6}\mathrm{T_H}(x^{(\alpha)}x^u) = \left[\mathrm{T_H}(x^{(3\varepsilon_j)}x_{j'}), \mathrm{T_H}(x^{(\alpha-2\varepsilon_j)}x^u)\right] \in M(HO)_h. \quad (9)$$

Since $\alpha_j \not\equiv 0, 1 \pmod p$, $\frac{\alpha_j(3-\alpha_j)}{2}$ and $\frac{\alpha_j(\alpha_j-1)(5-\alpha_j)}{6}$ cannot be all zero modulo p. Thus, it follows from (8) or (9) that $\mathrm{T_H}(x^{(\alpha)}x^u) \in M(HO)_h$. ■

Remark 4.7. Let $\mathcal{K}(n,\underline{t})$ be the subspace spanned by the elements $\mathrm{T_K}(a)$ with $a \in \oplus_{i\geq 1}\mathcal{O}(n,n;\underline{t})_{\mathbf{s},i}$. If $a,b \in \oplus_{i\geq 1}\mathcal{O}(n,n;\underline{t})_{\mathbf{s},i}$, by (1), we have $[\mathrm{T_K}(a), \mathrm{T_K}(b)] = \mathrm{T_K}(\mathrm{T_H}(a)(b))$. Note that $[\mathrm{T_H}(a), \mathrm{T_H}(b)] = \mathrm{T_H}(\mathrm{T_H}(a)(b))$. It follows that $\mathcal{K}(n,\underline{t})$ is a subalgebra of $KO(n,n+1,\underline{t})$. Moreover, the mapping

$$\rho : \mathcal{K}(n,\underline{t}) \longrightarrow HO(n,n;\underline{t}), \quad \mathrm{T_K}(a) \longmapsto \mathrm{T_H}(a)$$

is an isomorphism of Lie superalgebras.

Lemma 4.8. *Suppose $h \geq 3$. Then*

$$\begin{aligned} KO_h &= M(KO)_h + \sum_{\substack{\alpha_i\equiv 0,1(\mathrm{mod}p)\ \forall\ i\in\overline{1,n} \\ u\in\mathbb{B}(n), |\alpha|+|u|-2=h}} \mathbb{F}\mathrm{T_K}(x^{(\alpha)}x^u) \\ &\quad + \sum_{\substack{\alpha_j-\delta_{j'\in u}\equiv 0(\mathrm{mod}p)\ \forall j\in\overline{1,n}, \\ u\in\mathbb{B}(n), |\alpha|+|u|=h}} \mathbb{F}\mathrm{T_K}(x^{(\alpha)}x^u x_{2n+1}). \end{aligned}$$

Proof. It is sufficient to show the inclusion " $\subset$ ". Let $\mathrm{T_K}(x^{(\alpha)}x^u x_{2n+1}) \in KO_h \pmod{\mathcal{K}(n,\underline{t})}$, where $h \geq 3$, $\alpha \in \mathbb{A}(n,\underline{t})$ and $u \in \mathbb{B}(n)$. Note that for $j \in \overline{1,n}$,

$$\begin{aligned} &(\alpha_j - \delta_{j'\in u})\mathrm{T_K}(x^{(\alpha)}x^u x_{2n+1}) \\ = &\left[\mathrm{T_K}(x^{(\alpha)}x^u), \mathrm{T_K}(x_j x_{j'} x_{2n+1})\right] \in M(KO)_h \pmod{\mathcal{K}(n,\underline{t})}. \quad (10) \end{aligned}$$

If there is $j \in \overline{1,n}$ such that $\alpha_j - \delta_{j'\in u} \not\equiv 0 \pmod p$, it follows from (10) that

$$\mathrm{T_K}(x^{(\alpha)}x^u x_{2n+1}) \in M(KO)_h \pmod{\mathcal{K}(n,\underline{t})}.$$

By Remark 4.7 and Lemma 4.6, one easily sees that " $\subset$ " holds. ■

Recall that $L = \oplus_{i=-r}^{q} L_i$ denotes a finite dimensional simple $\mathbb{Z}$-graded Lie superalgebra and H a nilpotent subalgebra of $L_{\bar{0}} \cap L_0$ with weight space decomposition $L = \oplus_{\alpha\in\Delta} L_{(\alpha)}$. We record a general fact, which is completely analogous to the Lie algebra case (see [5, 12]):

Lemma 4.9. *Let $\psi : L \longrightarrow L^*$ be a $\mathbb{Z} \times \mathrm{Map}(H,\mathbb{F})$-homogeneous skew superderivation of degree (l,θ). Suppose $\psi\mid_{L^-} \in \mathrm{Inn}_{\mathbb{F}}(L^-, L^*)$.*

(i) *If $l > -q$ then ψ is inner.*

(ii) *If $l = -q$ and $\Delta_q \cap -\Delta_0 = \emptyset$ then ψ is inner.*

(iii) *If $-2q \leq l \leq -q-1$ and $-\Delta_q \not\subset \phi_{-(q+l)}$ then $\psi = 0$.*

We are now in a position to prove the main result of this paper:

Theorem 4.10. *The second cohomology group $H^2(X,\mathbb{F})$ vanishes for $X = HO$ or KO.*

Proof. By Theorem 4.1, it is sufficient to show that all the $\mathbb{Z}$-homogeneous skew superderivations from X into X^* are inner. On the other hand, as mentioned above, superderivations from X into X^* of nonzero $\mathrm{Map}(H_X,\mathbb{F})$-degrees must be inner. Hence it is sufficient to show that if $\psi : X \longrightarrow X^*$ is a skew superderivation of $\mathbb{Z}\times \mathrm{Map}(H_X,\mathbb{F})$-degree (l,θ) then ψ is inner. Hence, in the below we suppose $\psi : X \longrightarrow X^*$ is such a superderivation. By Lemma 4.5, $\psi\,|_{X^-}\in \mathrm{Inn}_{\mathbb{F}}(X^-,X^*)$.

(1) If $l > -q$, by Lemma 4.9(i), ψ is inner.

(2) Suppose $l = -q$. By (3) and (6), we have $-\Delta_{X,0}\cap\Delta_{X,q} = \emptyset$. Then by Lemma 4.9(ii), ψ is inner.

(3) Suppose $-2q \le l \le -q-1$. If $l \le -q-3$, then $-(q+l) \ge 3$. By Lemma 4.6, for HO we have

$$\phi_{-(q+l)} \subset \{\alpha+u \quad | \quad \alpha\in\mathbb{A}(n), u\in\mathbb{B}(n), \alpha_i \equiv 0,1 \pmod p,\ \forall i\in\overline{1,n}, |\alpha|+|u| = -(q+l-2)\}.$$

Combining this with (4), (5) and (6), we have

$$-\Delta_{HO,q}\not\subset\phi_{-(q+l)},\quad -\Delta_{HO,q}\not\subset\phi_2\subset\Delta_{HO,2}\quad\text{and}\quad -\Delta_{HO,q}\not\subset\Delta_{HO,1}=\phi_1.$$

By Lemma 4.8, for KO we have

$$\phi_{-(q+l)} \subset \{\theta,\alpha+u \quad | \quad \alpha\in\mathbb{A}(n), u\in\mathbb{B}(n);\ \alpha_i \equiv 0,1 \pmod p,\ \forall i\in\overline{1,n}, |\alpha|+|u| = -(q+l-2)\}.$$

Then by (4), (5) and (6), one easily sees

$$-\Delta_{KO,q}\not\subset\phi_{-(q+l)},\quad -\Delta_{KO,q}\not\subset\phi_2\subset\Delta_{KO,2}\quad\text{and}\quad -\Delta_{KO,q}\not\subset\Delta_{KO,1}=\phi_1.$$

By Lemma 4.9(iii), $\psi = 0$. The proof is complete. ■

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Jixia Yuan
School of Mathematical Sciences
Heilongjiang University
Harbin 150080, China

Wende Liu
School of Mathematical Sciences
Harbin Normal University
Harbin 150025, China
wendeliu@ustc.edu.cn

Wei Bai
School of Mathematical Sciences
Harbin Normal University
Harbin 150025, China

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