Journal of Lie Theory
Volume **23** (2013) 1129–1160

Castling Transformations of Projective Structures

Hironao Kato*

Communicated by B. Ørsted

Abstract. We construct an infinite sequence of projectively flat manifolds by using castling transformations of prehomogeneous vector spaces. We also give a classification of manifolds equipped with a flat projective structure obtained by a finite number of castling transformations, and describe these flat projective structures by atlases.
Mathematics Subject Classification 2010: Primary 53B10, 11S90; Secondary 53C10.
Key Words and Phrases: Projective structure, Grassmannian structure, prehomogeneous vector space.

1. Introduction

A flat Grassmannian structure of type (β, α) on a manifold M is a maximal atlas $\{(U_a, \varphi_a)\}_{a \in A}$ of M whose charts φ_a take values in the Grassmannian manifold $Gr_{\alpha,\alpha+\beta}$ and coordinate changes $\varphi_b \circ \varphi_a^{-1}$ belong to the projective linear group $PL(\alpha+\beta) := GL(\alpha+\beta)/R^*I$. When $\alpha = 1$, this notion gives a definition of flat projective structures on M. Obviously the projective spaces admit a flat projective structure. The classification of manifolds admitting a flat projective structure is still widely open (cf. [OT, chapter 6]) and active area. Indeed, recently in [GC] it has been proved that a connected sum $\boldsymbol{R}P^3 \# \boldsymbol{R}P^3$ does not admit a flat projective structure. In our last paper [Kat], we proved that invariant flat complex projective structures on complex Lie groups correspond to certain infinitesimal prehomogeneous vector spaces.

In the theory of prehomogeneous vector spaces there is a notion of castling transformations, which is a certain transformation of linear representations of algebraic groups preserving the prehomogeneity. In this paper we establish a transformation of manifolds equipped with a projective structure as a generalization of castling transformations. As castling transformations preserve the prehomogeneity of representations, our castling transformations of projective structures preserve the projectively flatness. Moreover, since we can repeat a castling transformation, we can construct a sequence of projectively flat manifolds from a given projec-

* The author is supported by JSPS and JSPS Strategic Young Researcher Overseas Visits Program for Accelerating Brain Circulation.

ISSN 0949–5932 / $2.50

tively flat manifold. In fact we prove the following: Let $\{(U_a, \varphi_a)\}_{a\in A}$ be a flat Grassmannian structure of type (β, α) on M. Assume $\alpha + \beta \geq 3$ and $\alpha \leq \beta$.

Theorem 1.1. *By a finite number of castling transformations from $\{(U_a, \varphi_a)\}_{a\in A}$ we obtain a projectively flat manifold N, which is a principal fiber bundle over M. There is a one-to-one correspondence between the set of structure groups $\prod_{i=1}^{j} PL(k_i)$ of N and the set of solutions $(k_1, \ldots, k_j)$ of the Grassmannian type equation*

$$(*)\quad \alpha\beta + k_1^2 + \cdots + k_j^2 - j - (\alpha+\beta)k_1 \cdots k_j + 1 = 0$$

satisfying $k_i \geq \alpha$ $(1 \leq i \leq j)$ and $j \geq 1$.

The projectively flat manifold N is described by using atlases in the last section. The case $\alpha = 1$ corresponds to the assumption that M admits a flat projective structure. Thus from any projectively flat manifold M, we can obtain a projectively flat principal fiber bundle N over M with group $\prod_{i=1}^{j} PL(k_i)$ satisfying the equation $(*)$. Furthermore the theorem yields a sequence of projectively flat manifolds, which are connected by manifolds equipped with a flat Grassmannian structure. Each flat projective structure on N is right invariant under the action of $\prod_{i=1}^{j} PL(k_i)$.

We note that a flat projective structure exists on M iff a projectively flat affine connection exists on M. Thus a flat affine connection induces a flat projective structure. However about the existence problem there is the following obstruction: A simply connected compact manifold admitting a flat projective structure is diffeomorphic to the sphere S^n (see [KN]). Thus the manifold $\prod_{i=1}^{j} S^{n_i}$ $(n_i, j \geq 2)$ does not admit any flat projective structure. This point distinguishes flat projective structures from flat affine connections and flat Riemannian metrics as any product of flat affine (resp. Riemannian) manifolds is a flat affine (resp. Riemannian) manifold again. However, in [Kat] we obtained a real Lie algebra $\mathfrak{sl}(k_1) \times \cdots \times \mathfrak{sl}(k_j)$ with a certain condition whose corresponding real Lie group admits a invariant flat real projective structure. Another aim of this paper is to generalize these examples from the view point of Grassmannian structures.

The paper is organized as follows. First of all we review the Grassmannian structures in § 1 and establish a castling transformation of projective structures by using Cartan connections in § 2 and § 3. In § 4 we investigate the base spaces obtained by successive castling transformations and describe a relation between base spaces. § 5 is devoted to some examples of base spaces. In § 6 we investigate the positive integer solutions of the Grassmannian type equation $(*)$, and give one conjecture. In § 7 we describe flat projective structures constructed in Theorem 1.1 by using atlases.

2. Preliminaries

2.1 Grassmannian structures and projective structures

Throughout this paper by a manifold we mean a C^∞ real manifold. We recall the notion of Grassmannian structures and projective structures to establish castling transformations in the differential geometry. Let M be a real manifold of dimension r. Denote by $\mathcal{L}(M)$ a bundle of linear flames of M and we regard an element of $\mathcal{L}(M)$ as a linear isomorphism $\boldsymbol{R}^r \to T_pM$. We identify $\boldsymbol{R}^r$ with $\boldsymbol{R}^n \otimes \boldsymbol{R}^m$ and consider a $GL(n) \otimes GL(m)$-structure P_tM, i.e. a subbundle of $\mathcal{L}(M)$ with structure group $GL(n) \otimes GL(m)$. If we have $n, m \geq 2$, we call P_tM a Grassmannian structure of type (n, m) on M in this paper. Note that if $n = 1$ or $m = 1$, then $P_tM = \mathcal{L}(M)$. Put $l := m + n$. There are various names and definitions. In [Han] and [Ish] a $GL(n) \otimes GL(m)$-structure is called a tensor product structure. On the other hand in [MS], an isomorphism $\sigma : TM \to V \otimes W$ itself is called a Grassmannian structure, where V and W are vector bundles with rank n and m over M $(n, m \geq 2)$. Such an isomorphism σ gives a $GL(n) \otimes GL(m)$-structure in a natural manner, however the author does not know whether the converse is true. Typical examples admitting a Grassmannian structure are Grassmannian manifolds (see [MS] for other examples). Denote by $Gr_{m,l}$ a Grassmannian manifold consisting of m-dimensional subspaces in the l-dimensional real vector space W. The real projective transformation group $PL(W)$ acts on $Gr_{m,l}$ transitively. Let $\{a_1, \ldots, a_l\}$ be a linear basis of W. With respect to the basis $\{a_i\}_{i=1}^l$ the group $PL(W)$ is expressed as the quotient $GL(l)/\boldsymbol{R}^* I_l$, which we denote by $PL(l)$. Note that now the basis $\{a_1, \ldots, a_l\}$ is identified with the natural basis of $\boldsymbol{R}^l$. Let v be a linear frame $(a_1, \cdots, a_m)$. We denote by $<v>$ the m-dimensional subspace spanned by v. Let $PL(l)_{<v>}$ be the isotropy subgroup at $<v>$. Then we have $Gr_{m,l} = PL(l)/PL(l)_{<v>}$. The Lie algebra of $PL(l)$ is isomorphic to $\mathfrak{sl}(l)$, which has the graded decomposition $\mathfrak{sl}(l) = \mathfrak{g}_{-1} + \mathfrak{g}_0 + \mathfrak{g}_1$ given by

$$\begin{aligned}
\mathfrak{g}_{-1} &= \left\{ \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \middle| \, C \in M(n,m) \right\}, \\
\mathfrak{g}_0 &= \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \, \begin{array}{l} A \in \mathfrak{gl}(m), \ B \in \mathfrak{gl}(n) \\ \operatorname{tr}(A+B) = 0 \end{array} \right\}, \\
\mathfrak{g}_1 &= \left\{ \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \middle| \, D \in M(m,n) \right\}.
\end{aligned}$$

The vector space $\boldsymbol{R}^n \otimes \boldsymbol{R}^m$ is naturally identified with $\mathfrak{g}_{-1}$ and the isotropy representation $\rho : PL(l)_{<v>} \to GL(g_{-1})$ is given by

$$\rho : \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} \mapsto B \otimes {}^tA^{-1}.$$

Thus the image of ρ is the group $GL(n) \otimes GL(m)$. The isotropy representation ρ enables us to identify $GL(n) \otimes GL(m)$ with the subgroup G_0 of $PL(l)$;

$$G_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \, A \in GL(m), B \in GL(n) \right\}.$$

Thus we obtain the imbedding $\imath : GL(n) \otimes GL(m) \to PL(l)_{<v>}$, which is defined by $A \otimes B \mapsto \begin{pmatrix} {}^tB^{-1} & 0 \\ 0 & A \end{pmatrix}$. Moreover the Lie algebra $gl(n) \otimes I_m + I_n \otimes gl(m)$ is identified with $\mathfrak{g}_0$.

Here we recall the notion of $(PL(l),\ Gr_{m,l})$-structures on M. A $(PL(l),\ Gr_{m,l})$-structure on M is a maximal atlas $\{(U_a, \varphi_a)\}_{a \in A}$ of M satisfying the following condition (cf. [Gol1], [Kat]):

(1) $\{U_a\}_{a\in A}$ is an open covering of M,
(2) φ_a maps U_a diffeomorphically onto an open subset of $Gr_{m,l}$,
(3) for every pair (b,a) with $U_a\cap U_b\neq\emptyset$ and each connected component C of $U_a\cap U_b$, $\varphi_b\circ\varphi_a^{-1}|_{\varphi_a(C)}$ is given by an element of $PL(l)$.

We call a $(PL(l),Gr_{m,l})$-structure a flat Grassmannian structure of type (n,m). A flat Grassmannian structure of type $(n,1)$ is nothing but a flat projective structure.

Now we introduce the notion of Grassmannian Cartan connections. Let Q be a principal $PL(l)_{<v>}$-bundle over M and ω be a $\mathfrak{sl}(l)$-valued 1-form on Q. Then the pair (Q,ω) is called a Grassmannian Cartan connection of type (n,m) on M if the following conditions are satisfied:

(1) $\omega\colon T_uQ\to\mathfrak{sl}(l)$ gives a linear isomorphism,
(2) $R_g^*\omega=Ad(g^{-1})\omega$ for $g\in PL(l)_{<v>}$,
(3) $\omega(A^*)=A$ $(A\in\mathfrak{sl}(l)_{<v>})$, where A^* is the fundamental vector field.

A $\mathfrak{sl}(l)$-valued 2-form Ω on Q defined by $\Omega=d\omega+\frac{1}{2}[\omega,\omega]$ is called a curvature form. A Grassmannian Cartan connection (Q,ω) is said to be flat if $\Omega=0$. Now we recall there is the following one-to-one correspondence (cf. [Kat]):

$$\{(PL(l),Gr_{m,l})\text{-structures on }M\}$$
$$\to\{\text{flat Grassmannian Cartan connections of type }(n,m)\text{ on }M\}/_{\sim}.$$

The equivalence relation of the latter set denotes the isomorphisms of Cartan connections. Generally a Grassmannian Cartan connection (Q,ω) of type (n,m) over M induces a $GL(n)\otimes GL(m)$-structure of M as follows (cf. [Tan3, p.135]). Let $\rho : PL(l)_{<v>}\to GL(g_{-1})$ be the isotropy representation. We denote the kernel of ρ by $ker\rho$. Then $PL(l)_{<v>}/ker\rho\cong GL(n)\otimes GL(m)$. Thus the quotient manifold $\widetilde{Q}:=Q/ker\rho$ is regarded as a principal fiber bundle over M with structure group $GL(n)\otimes GL(m)$. Let ω_{-1} (resp. ω_0) be the $\mathfrak{g}_{-1}$ (resp. $\mathfrak{g}_0$) component of the 1-form ω. By using the natural projection $\rho : Q\to\widetilde{Q}$, we obtain the $\mathfrak{g}_{-1}$-valued 1-form θ on $\widetilde{Q}$ defined by $\rho^*\theta=\omega_{-1}$. Then $(\widetilde{Q},\ \theta)$ can be regarded as a $GL(n)\otimes GL(m)$-structure and its canonical form. We define an injection $\iota : GL(n)\otimes GL(m)\hookrightarrow PL(l)_{<v>}$ by

$$\iota(A\otimes B)=\begin{pmatrix}{}^tB^{-1} & 0\\ 0 & A\end{pmatrix}.$$

The homogeneous space $PL(l)_{<v>}/\iota(GL(n)\otimes GL(m))$ is homeomorphic to $\mathfrak{g}_1$, and hence there exists a bundle homomorphism $h:\widetilde{Q}\hookrightarrow Q$ corresponding to ι such that $\rho\circ h=id$. Then by Proposition 7.3 of [Tan2] we can obtain the connection form χ defined by

$$\chi(X)=\omega(h_*X)_0\quad(X\in TP_tM).$$

Thus we obtain the following map

$$\Phi:\{\text{Grassmannian Cartan connections of type }(n,m)\text{ on }M\}/_{\sim}\to\{(P_tM,[\chi])\},$$

where $[\chi]$ denotes a certain equivalence class of χ defined in [Tan2, p.128]. Especially when $m\neq 1$ and $n\neq 1$, there is a one-to-one correspondence between the set of the isomorphism classes of normal Grassmannian Cartan connections over M and the set of $GL(n)\otimes GL(m)$-structures on M (see section 9 and Theorem 10.2 of [Tan2]).

Now we consider the case $m = 1$. A Grassmannian Cartan connection (Q,ω) of type $(n,1)$ is called a projective Cartan connection. Especially a normal projective Cartan connection induces a projective equivalence class of torsion-free linear connections $(\mathcal{L}(M),[\chi])$, which we call a projective structure. Now assume $n > 1$. By the restriction of Φ to the normal case with $m = 1$ gives the following one-to-one correspondence:

$$\begin{aligned}\Phi_{m=1} : \ &\{\text{normal projective Cartan connections on } M\}/_\sim \\ &\to \{\text{projective structures on } M\}.\end{aligned}$$

For more details of projective structures we refer the reader to [Tan1], [Tan2], [NS] and [Aga]. A projective structure $[\chi]$ on M is said to be projectively flat if χ is locally projectively equivalent to a flat affine connection. The map $\Phi_{m=1}$ is restricted to the bijective between the set of flat projective Cartan connections and the set of projective structures which are projectively flat. (cf. Theorem 9.2 in [Tan2] and Proposition 1.5.2 in [ČS]). The existence of a flat Grassmannian structure of type (n,m) $(n,m \geq 2)$ on M should be also described by the terminology of $GL(n)\otimes GL(m)$-structure. However the author does not know it.

2.2 Subgeometry

For the later argument, we introduce the notion of subgeometry, following [Gol2]. Let A/B and A'/B' be real homogeneous spaces. We say that A/B is a subgeometry of A'/B' if there exists a Lie group homomorphism $F : A \to A'$ satisfying the following conditions:

(1) $F(B) \subset B'$,

(2) the induced map $\hat{F} : A/B \to A'/B'$ is a local diffeomorphism.

Let us denote the Lie algebra of A by $\mathfrak{a}$, the one of B by $\mathfrak{b}$. Likewise we define $\mathfrak{a}'$ and $\mathfrak{b}'$ for A' and B' respectively. Let Λ and Λ' be Maurer-Cartan forms of A and A'. Then F gives a bundle homomorphism corresponding to $F|_B : B \to B'$ and satisfies $F^*\Lambda' = dF\circ\Lambda$.

Proposition 2.1. *Let (Q,ω) be a Cartan connection of type A/B on M. Then there exists a Cartan connection (Q',ω') of type A'/B' on M.*

Proof. The proof of this Proposition is same as the one of Theorem 1.5.15 of [ČS]. Thus we only explain the construction of a Cartan connection (Q',ω') of type A'/B' on M. Since B acts on B' via F, from the given principal bundle Q we obtain the extended bundle $Q' = Q\times_B B'$. The bundle homomorphism $\widetilde{F} : Q \to Q'$ is defined by $u \mapsto [u,e]$, which corresponds to the restriction of F to B. Next we define a $\mathfrak{a}'$-valued 1-form ω' on $\widetilde{F}(Q)$ by

$$\omega'_{[u,e]}(\widetilde{F}_*X + Z^*) = dF\circ\omega(X) + Z \quad (X \in T_uQ, Z \in \mathfrak{b}').$$

We enlarge this definition to the whole of Q' by

$$\omega'_{[u,c]} = R^*_{c^{-1}}Ad(c^{-1})\omega_{[u,e]} \quad (c \in B').$$

This definition is well defined and we can verify (Q', ω') gives a Cartan connection of type A'/B' on M. ■

Definition 2.2. *Let (Q, ω) and (Q', ω') be Cartan connections of type A/B and A'/B' respectively on M. Then we call (Q, ω) a subgeometry of (Q', ω') if there exists a bundle homomorphism $\iota : Q \to Q'$ corresponding to $F|_B : B \to B'$ such that ι induces the identity map between the base spaces and $\iota^*\omega' = dF \circ \omega$.*

In Proposition 2.1 a given Cartan connection (Q, ω) of type A/B induces (Q', ω') of type A'/B', and (Q, ω) is a subgeometry of (Q', ω').

Proposition 2.3. *Assume that a Cartan connection (Q, ω) is a subgeometry of (Q', ω'). If (Q, ω) is flat, then (Q', ω') is also flat. Moreover when the differential $dF : \mathfrak{a} \to \mathfrak{b}$ is an injective homomorphism, the converse is also true.*

Proof. We compute the curvature form Ω' of (Q', ω'). Pulling back Ω' by ι yields

$$\begin{aligned} \iota^*\Omega' &= \iota^*(d\omega' + \frac{1}{2}[\omega', \omega']) \\ &= dF(d\omega + \frac{1}{2}[\omega, \omega]) \\ &= dF(\Omega). \end{aligned}$$

Hence the assertion of the proposition follows. ■

We fix the complementary subspace $\mathfrak{m}$ of $\mathfrak{b}$ and $\mathfrak{m}'$ of $\mathfrak{b}'$. Let ρ be the linear isotropy representation of B on the tangent space to A/B at the origin o. By identifying T_oA/B with $\mathfrak{m}$, ρ is given by $\rho(b)X = Ad(b)X + \mathfrak{b}$ for $b \in B$ and $X \in \mathfrak{m}$. Thus we obtain the two linear isotropy representations $\rho : B \to GL(\mathfrak{m})$ and $\rho' : B' \to GL(\mathfrak{m}')$. We denote the kernel of ρ by C and the one of ρ' by C'. Since we assume that A/B is a subgeometry of A'/B', there is a homomorphism $F : A \to A'$ whose differential dF induces the linear isomorphism $\widehat{dF} : \mathfrak{m} \to \mathfrak{m}'$.

Lemma 2.4. *There exists an injective homomorphism $\overline{F} : \rho(B) \to \rho'(B')$ defined by $\overline{F} : \rho(b) \mapsto \widehat{dF} \circ \rho(b) \circ \widehat{dF}^{-1}$, and we have the commutative diagram:*

$$\begin{array}{ccc} B & \xrightarrow{F} & B' \\ \rho\downarrow & \circlearrowleft & \downarrow\rho' \\ \rho(B) & \xhookrightarrow{\overline{F}} & \rho'(B') \end{array}$$

Moreover F is regarded as a bundle homomorphism corresponding to $F : C \to C'$.

Proof. Firstly we verify $F(C) \subset C'$. Assume that $b \in C$. Then $\rho(b)(X + \mathfrak{b}) = X + \mathfrak{b}$ for $X \in \mathfrak{m}$. Then $\rho'(F(b))(\widehat{dF}(X)) = Ad(F(b)./nextstellen)(dF(X) + \mathfrak{b}') = dF(Ad(b)X) + \mathfrak{b}' = dF(X) + \mathfrak{b}' = \widehat{dF}(X)$. Thus $\rho'(F(b)) = id_{\mathfrak{m}'}$, and $F(b) \in C'$. We define $\overline{F} : \rho(B) \to \rho'(B')$ by $\bar{F} : \rho(b) \mapsto \rho'(F(b))$ for $b \in B$. Since $F(C) \subset C'$, this is well defined, moreover we have $\overline{F}(\rho(b)) = \widehat{dF} \circ \rho(b) \circ \widehat{dF}^{-1}$. It follows that $\overline{F}$ is injective. ■

Let (Q,ω) and (Q',ω') be Cartan connections of type A/B and A'/B' respectively. Assume that (Q,ω) is a subgeometry of (Q',ω'). We denote the quotient manifold Q/C by $\widetilde{Q}$ and Q'/C' by $\widetilde{Q}'$. Then we obtain $\rho(B)$-structure $(\widetilde{Q},\theta)$ and $\rho'(B')$-structure $(\widetilde{Q}',\theta')$ (see [Tan3, p.136]). The projection $\rho : Q \to \widetilde{Q}$ is corresponding to $\rho : B \to \rho(B)$. Recall that $(\widetilde{Q},\theta)$ gives a $\rho(B)$-structure on M as follows: concerning each point $\rho(u) \in \widetilde{Q}$, $\rho(u)^{-1}$ is regarded as a linear isomorphism $T_{\pi_{Q/C}(\rho(u))}M \to \mathfrak{m}$ by $\rho(u)^{-1} : \pi_{Q/C*}\rho_* X \mapsto \theta(\rho_* X) = \omega_{\mathfrak{m}}(X)$. Hence we obtain the bundle homomorphism $\widetilde{Q} \hookrightarrow L(M)$ corresponding to the inclusion $\rho(B) \to GL(\mathfrak{m})$. Likewise we obtain the map $\widetilde{Q}' \hookrightarrow L(M)$, where $L(M)$ is regarded as the set of all linear isomorphisms $y : \mathfrak{m}' \to T_pM (p \in M)$.

The bundle homomorphism $t : L(M) \to L(M)$ is defined by $t : x \mapsto x \circ \widehat{dF}^{-1}$, which is corresponding to $GL(\mathfrak{m}) \ni A \mapsto \widehat{dF} \circ A \circ \widehat{dF}^{-1} \in GL(\mathfrak{m}')$.

Proposition 2.5. *The $\rho(B)$-structure $\widetilde{Q}$ is a reduction of $\rho'(B')$-structure $\widetilde{Q}'$ i.e.* 1) *there exists a bundle homomorphism $\bar{\iota} : \widetilde{Q} \hookrightarrow \widetilde{Q'}$ corresponding to $\bar{F} : \rho(B) \to \rho'(B')$, and* 2) *$\bar{\iota}^*\theta' = \widehat{dF} \circ \theta$. The injection $\bar{\iota}$ is given by the restriction of $t : L(M) \to L(M)$.*

Proof. From assumption (Q,ω) is a subgeometry of (Q',ω'), thus we have a bundle homomorphism $\iota : Q \to Q'$ corresponding to $F|_B : B \to B'$. We define a map $\bar{\iota} : \widetilde{Q} \to \widetilde{Q}'$ by $\bar{\iota} : \rho(u) \to \rho' \circ \iota(u)$. Since $F|_B$ is a bundle homomorphism corresponding to $F : C \to C'$, $\bar{\iota}$ is well-defined and gives a bundle homomorphism corresponding to $\bar{F} : \rho(B) \to \rho'(B')$. Hence we obtain the commutative diagram:

$$\begin{array}{ccc} Q & \xrightarrow[i]{} & Q' \\ {\scriptstyle\rho}\downarrow & \circlearrowleft & \downarrow{\scriptstyle\rho'} \\ \widetilde{Q} & \xhookrightarrow[\bar{\iota}]{} & \widetilde{Q}' \\ {\scriptstyle\pi_{\widetilde{Q}}}\downarrow & \circlearrowleft & \downarrow{\scriptstyle\pi_{\widetilde{Q}'}} \\ M & \xrightarrow[id]{} & M. \end{array} \tag{2.1}$$

Since $\bar{\iota}$ induces the identity of base spaces and $\bar{F}$ is injective, $\bar{\iota}$ is injective. Now we show that $\bar{\iota}^*\theta' = \widehat{dF} \circ \theta$. Since $\bar{\iota} \circ \rho = \rho' \circ \iota$, pulling back $\bar{\iota}^*\theta'$ by $\rho : Q \to \widetilde{Q}$ yields $\rho^*(\bar{\iota}^*\theta') = (\rho' \circ \iota)^*\theta' = \iota * \omega'_{\mathfrak{m}'} = \widehat{dF} \circ \omega_{\mathfrak{m}}$. Hence $\bar{\iota}^*\theta' = \widehat{dF} \circ \theta$.

By using the inclusion $\widetilde{Q} \hookrightarrow L(M)$ and $\widetilde{Q}' \hookrightarrow L(M)$, t and $\bar{\iota}$, we obtain the following diagram, which will be shown commutative as follows.

$$\begin{array}{ccc} L(M) & \xrightarrow[t]{} & L(M) \\ \uparrow & \circlearrowleft & \uparrow \\ \widetilde{Q} & \xhookrightarrow[\bar{\iota}]{} & \widetilde{Q}' \end{array} \tag{2.2}$$

From the equality $\bar{\iota}^*\theta' = \widehat{dF} \circ \theta$, for $\rho(u) \in \widetilde{Q}$ we have $\theta'_{\bar{\iota}(\rho(u))}(\bar{\iota}_*\rho_* X) = \widehat{dF} \circ (\rho_* X) = \widehat{dF} \circ \omega_{\mathfrak{m}}(X)$. Thus $\bar{\iota}(\rho(u))$ gives a linear isomorphism $T_{\pi_{\widetilde{Q}'}(\bar{\iota}(\rho(u))}M \to \mathfrak{m}'$ by $\bar{\iota}(\rho(u)) : \pi_{\widetilde{Q}'*}\bar{\iota}_*\rho_* X \mapsto \widehat{dF} \circ \omega_{\mathfrak{m}}(X)$. Since the diagram (2.1) is commutative, we have $\pi_{\tilde{Q}'_*}\bar{\iota}_*\rho_*X = \pi_{\tilde{Q}_*}\rho_*X$. Therefore $\bar{\iota}(\rho(u)) = \rho(u)\circ\widehat{dF}^{-1}$. On the other hand $t\circ\rho(u) = \rho(u)\circ\widehat{dF}^{-1}$, and hence the diagram (2.2) is commutative. ■

Let X' be a homogeneous space of A'. We choose B' as the isotropy subgroup at a point v in X'. If we are given a subgroup $A\subset A'$, we can consider the isotropy subgroup A_v of A at v. Then A/A_v gives a subgeometry of A'/B'. Henceforth we say that A/B is a subgeometry of A'/B' if this condition is satisfied: A is a subgroup of A' and B is the isotropy subgroup at v.

3. Castling transformations

In this section we establish a castling transformation of projective structures. Let G be a Lie subgroup of $PL(l)$. We consider the homomorphism $F: G\times PL(m)\hookrightarrow PL(\boldsymbol{R}^l\otimes\boldsymbol{R}^m)$ defined by $(g,A)\mapsto g\otimes A$. By F we regard $G\times PL(m)$ as a subgroup of $PL(\boldsymbol{R}^l\otimes\boldsymbol{R}^m)$. When we identify $\boldsymbol{R}^l\otimes\boldsymbol{R}^m$ with $\underbrace{\boldsymbol{R}^l\oplus\cdots\oplus\boldsymbol{R}^l}_{m}$, $G\times PL(m)$ acts on $P(\boldsymbol{R}^l\otimes\boldsymbol{R}^m)$ by $(g,A).v := gv^tA$ for $v=(v_1,\ldots,v_m)\in P(\boldsymbol{R}^l\otimes\boldsymbol{R}^m)$. Assume that $n=l-m\geq 0$. Denote by $V_{m,l}$ a projective Stiefel manifold, which consists of projective frames of m-dimensional subspaces of $\boldsymbol{R}^l$.

Proposition 3.1. *Let v be a point in $P(\boldsymbol{R}^l\otimes\boldsymbol{R}^m)$. Then the rank of $v=(v_1,\ldots,v_m)$ is m and the homogeneous space $G/G_{<v>}$ is a subgeometry of $Gr_{m,l}=PL(l)/PL(l)_{<v>}$ if and only if $G\times PL(m)/G\times PL(m)_v$ is a subgeometry of the projective space $P(\boldsymbol{R}^l\otimes\boldsymbol{R}^m)=PL(\boldsymbol{R}^l\otimes\boldsymbol{R}^m)/PL(\boldsymbol{R}^l\otimes\boldsymbol{R}^m)_v$ defined by $F: G\times PL(m)\hookrightarrow PL(\boldsymbol{R}^l\otimes\boldsymbol{R}^m)$.*

Proof. The proof follows the idea of Proposition 6 of section 2 in [SK]. We can prove this proposition by showing that the following four assertions are equivalent.

(1) $G/G_{<v>}$ is a subgeometry of the Grassmannian manifold $PL(l)/PL(l)_{<v>}$.

(2) $G.<v>$ gives an open orbit in $G_{m,l}$.

(3) $G\times PL(m).v$ gives an open orbit in $V_{m,l}$.

(4) $G\times PL(m)/G\times PL(m)_v$ is a subgeometry of the projective space
$$PL(\boldsymbol{R}^l\otimes\boldsymbol{R}^m)/PL(\boldsymbol{R}^l\otimes\boldsymbol{R}^m)_v.$$

The proof of (1) ⇔ (2) is easy. To prove (2) ⇔ (3), we consider the fiber bundle

$$\begin{array}{ccc} V_{m,l} & \longleftarrow & PL(m) \\ \big\downarrow{\scriptstyle\pi} & & \\ G_{m,l} & & \end{array}$$

The natural projection π is continuous and open map. It follows (2) ⇔ (3). To prove (3) ⇔ (4), we observe that the manifold $V_{m,l}$ is naturally imbedded into $P(\boldsymbol{R}^l\otimes\boldsymbol{R}^m)$ with respect to the relative topology, indeed $V_{m,l}$ is an open submanifold in $P(\boldsymbol{R}^l\otimes\boldsymbol{R}^m)$. Next we consider the assertion (3)′: $G\times PL(m).v$

gives an open orbit in $P(\boldsymbol{R}^l \otimes \boldsymbol{R}^m)$. If $(3)'$ holds true, then v must belong to $V_{m,l}$. Hence we have $(3) \Leftrightarrow (3)'$. The proof of the equivalence $(3)' \Leftrightarrow (4)$ is same as the one of $(1) \Leftrightarrow (2)$. ∎

Now assume that $l-m \geq 1$. We fix a point v of $V_{m,l}$ and one $v^{\perp}$ of $V_{l-m,l}$ such that the subspace $< v^{\perp} >$ spanned by $v^{\perp}$ is orthogonal to the one $< v >$ spanned by v. Let us define the isomorphism $* : PL(l) \to PL(l)$ by $g \mapsto {}^t g^{-1}$. Then $*$ gives the isomorphism between $PL(l)_{<v>}$ and $PL(l)_{<v^{\perp}>}$. We denote the differential of $*$ by the same symbol, and we have $*(A) = -{}^t A$ for $A \in \mathfrak{sl}(l)$. Then $G. < v >$ gives an open orbit in $Gr_{m,l}$ if and only if $*G. < v^{\perp} >$ gives an open orbit in $Gr_{l-m,l}$. Hence we obtain the following trivial fact:

Proposition 3.2. *$G/G_{<v>}$ is a subgeometry of $Gr_{m,l}$ iff $*G/*G_{<v^{\perp}>}$ is a subgeometry of $Gr_{l-m,l}$.*

Let $\rho : H \to GL(\boldsymbol{C}^l)$ be a rational representation of a complex linear algebraic group H. Then originally the transformation

$$\rho \otimes id : H \times GL(m) \to GL(\boldsymbol{C}^l \otimes \boldsymbol{C}^m) \Leftrightarrow \rho^* \otimes id : H \times GL(l-m) \to GL(\boldsymbol{C}^l \otimes \boldsymbol{C}^{l-m})$$

is called a castling transformation in [SK]. It has been proved that ρ gives a prehomogeneous vector space iff ρ^* gives a prehomogeneous vector space.

Now we define the castling transformation of Cartan connections. Denote by $\mathfrak{g}$ the Lie algebra of G. Assume $v \in V_{m,l}$ and that $G/G_{<v>}$ is a subgeometry of $Gr_{m,l}$. We denote by Λ_1 the Maurer-Cartan form of $PL(m)$.

Proposition 3.3. *Denote by Q a principal fiber bundle over M with structure group $G_{<v>}$ and by ω a $\mathfrak{g}$-valued 1-form on Q. Then the following are equivalent.*

1. *(Q, ω) is a Cartan connection of type $G/G_{<v>}$ on M.*
2. *$(Q \times PL(m), \omega \times \Lambda_1)$ is a Cartan connection of type $G \times PL(m)/G \times PL(m)_v$ on a manifold N.*

(Q, ω) is flat iff $(Q \times PL(m), \omega \times \Lambda_1)$ is flat.

Proof. *1* $\Leftrightarrow$ *2*: Assume the assertion 1. Then $Q \times PL(m)$ is regarded as a principal fiber bundle over M with structure group $G_{<v>} \times PL(m)$. Since $G \times PL(m)_v$ is a closed subgroup of $G_{<v>} \times PL(m)$, we have the quotient $Q \times PL(m)/G \times PL(m)_v$ over which $Q \times PL(m)$ is regarded as a principal fiber bundle with structure group $G \times PL(m)_v$. Then we can directly check $(Q \times PL(m), \omega \times \Lambda_1)$ gives a Cartan connection of type $G \times PL(m)/G \times PL(m)_v$ on $Q \times PL(m)/G \times PL(m)_v$. Conversely we assume the assertion *2*. We can directly check (Q, ω) gives a Cartan connection of type $G/G_{<v>}$ on M.

Now we prove the equivalence of flatness between *1* and *2*. We first observe that $(d\omega + \frac{1}{2}[\omega, \omega], d\Lambda_1 + \frac{1}{2}[\Lambda_1, \Lambda_1])$ gives $\mathfrak{g} \times \mathfrak{sl}(m)$-valued 2-form on $Q \times PL(m)$. We can directly verify

$$d(\omega \times \Lambda_1) + \frac{1}{2}[\omega \times \Lambda_1, \omega \times \Lambda_1] = (d\omega + \frac{1}{2}[\omega, \omega], d\Lambda_1 + \frac{1}{2}[\Lambda_1, \Lambda_1]).$$

Since Λ_1 is the Maurer-Cartan form of $PL(m)$, we have $d\Lambda_1 + \frac{1}{2}[\Lambda_1, \Lambda_1] = 0$. Hence (Q, ω) is flat if and only if $(Q \times PL(m), \omega \times \Lambda_1)$ is flat. ∎

Proposition 3.4. *Let (Q, ω) be a Cartan connection of type $G/G_{<v>}$ on M. Then $(Q, *\omega)$ gives a Cartan connection of type $*G/*G_{<v^\perp>}$ on M. (Q, ω) is flat iff $(Q, *\omega)$ is flat.*

Proof. From the assumption $G_{<v>}$ acts on Q on the right. We define the action of $*G_{<v^\perp>}$ on Q by $u \cdot g := u * g$ $(u \in Q,\ g \in *G_{<v^\perp>})$. Then the bundle Q is regarded also as a principal fiber bundle over M with structure group $*G_{<v^\perp>}$. Moreover we define a one-form $*\omega$ by the composite of $* : \mathfrak{sl}(l) \to \mathfrak{sl}(l)$ and ω. On the other hand since $G/G_{<v>}$ is a subgeometry of $*G/*G_{<v^\perp>}$, the Cartan connection (Q, ω) induces a Cartan connection (Q_c, ω_c) of type $*G/*G_{<v^\perp>}$ by Proposition 2.1. This induced Cartan connection is isomorphic with $(Q, *\omega)$. Hence by Proposition 2.3 (Q, ω) is flat if and only if $(Q, *\omega)$ is flat. ∎

Now we assume the same assumption as Proposition 3.3: Denote by Q a principal fiber bundle over M with structure group $G_{<v>}$ and by ω a $\mathfrak{g}$-valued 1-form on Q. Then combining Propositions 3.1, 3.2, 3.3 and 3.4 yields the next theorem.

Theorem 3.5. *The following are equivalent.*

1. *$(Q \times PL(m), \omega \times \Lambda_1)$ is a Cartan connection of type $G \times PL(m)/G \times PL(m)_v$.*

2. *(Q, ω) is a Cartan connection of type $G/G_{<v>}$ on M.*

3. *$(Q, *\omega)$ is a Cartan connection of type $*G/*G_{<v^\perp>}$ on M.*

4. *$(Q \times PL(l-m), *\omega \times \Lambda_1)$ is a Cartan connection of type $*G \times PL(l-m)/$ $*G \times PL(l-m)_{v^\perp}$.*

Moreover the flatness of the above Cartan connections are equivalent. The Cartan connections 1 and 4 (resp. 2 and 3) are subgeometries of projective (resp. Grassmannian) Cartan connections.

In this theorem we omit the base space N of the Cartan connection $(Q \times PL(m), \omega \times \Lambda_1)$ since N is diffeomorphic to the quotient $Q \times PL(m)/G \times PL(m)_v$. This quotient is described as follows by using a Grassmannian structure on M.

Proposition 3.6. *Suppose that (Q, ω) gives a Cartan connection of type $G/G_{<v>}$ on M. Then the base space N of $(Q \times PL(m), \omega \times \Lambda_1)$ is a principal fiber bundle over M with group $PL(m)$. Moreover (Q, ω) induces a $GL(n) \otimes GL(m)$-structure P_tM on M, and N is diffeomorphic to the quotient of P_tM by $GL(n) \otimes GL(1)$.*

Proof. We fix the complementary subspace $\mathfrak{m}$ of $\mathfrak{g}_{<v>}$ in $\mathfrak{g}$, then the natural inclusion $\iota : G \to PL(l)$ gives the linear isomorphism $\hat{d\iota} : \mathfrak{m} \to M(n, m)$. We denote the isotropy representation of $G/G_{<v>}$ by $\rho : G_{<v>} \hookrightarrow GL(\mathfrak{m})$. Then from the assumption $(Q/ker\rho, \theta)$ gives a $\rho(G_{<v>})$-structure $\widetilde{Q} \subset \mathcal{L}(M)$. From Proposition 2.1 we obtain the induced Grassmannian Cartan connection (Q', ω'), which induces a $GL(n) \otimes GL(m)$-structure P_tM on M. Now by Proposition 2.5

the natural inclusion $\iota : Q \to Q'$ induces the injective $\bar\iota : \widetilde{Q} \to P_tM$. Consequently we obtain the following diagram.

$$\begin{array}{ccc} Q & \overset{}{\underset{i}{\hookrightarrow}} & Q' \\ \rho\downarrow & \circlearrowleft & \downarrow\tilde\rho \\ \widetilde{Q} & \underset{\bar\iota}{\hookrightarrow} & P_tM. \end{array}$$

Thus we obtain the map $\bar\iota \circ \rho : Q \to P_tM$ corresponding to the restriction $\tilde\rho|_{G_{<v>}}$. For a matrix $A \in GL(m)$ denote by $\bar{A}$ the image of the homomorphism $GL(m) \to PL(m)$. We define the map $\Phi : Q \times PL(m)/G \times PL(m)_v \to P_tM/GL(n) \otimes GL(1)$ by

$$\Phi : (u, \bar{A})G \times PL(m)_v \mapsto \bar\iota \circ \rho(u) I_n \otimes A^{-1} GL(n) \otimes GL(1).$$

This definition is well defined. Moreover the map Φ is a diffeomorphism. Now we observe that $PL(m)$ acts on $Q \times PL(m)/G \times PL(m)_v$ and $P_tM/GL(n) \otimes GL(1)$ on the right as follows: $(u, \bar{A})G \times PL(m)_v \cdot \bar{B} := (u, \overline{B^{-1}A})G \times PL(m)_v$ and $x\ GL(n) \otimes GL(1) \cdot \bar{B} := x\ I_n \otimes B\ GL(n) \otimes GL(1)$. We can check that by these actions both $Q \times PL(m)/G \times PL(m)_v$ and $P_tM/GL(n) \otimes GL(1)$ can be regarded as principal fiber bundles over M with structure group $PL(m)$. Then Φ gives a bundle isomorphism. If we have $m = 1$, the base space of $(Q \times PL(m), \omega \times \Lambda_1)$ is same as M, and the base space of $(Q \times PL(n), *\omega \times \Lambda_1)$ is isomorphic to $\mathcal{L}(M)/GL(1)$. ■

Corollary 3.7. *We call the transformation*

$$\text{(I) } (Q \times PL(m), \omega \times \Lambda_1) \leftrightarrow \text{(II) } (Q \times PL(l-m), *\omega \times \Lambda_1)$$

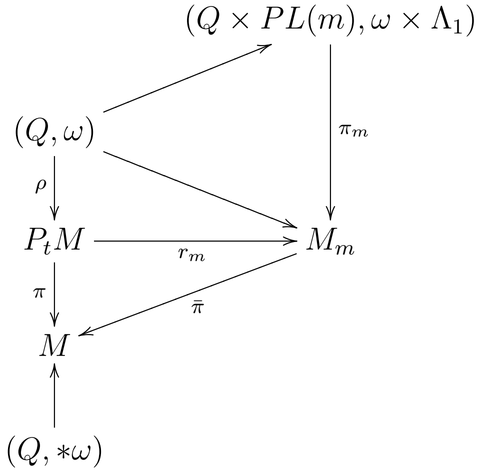
a castling transformation of projective structures. (I) *is a subgeometry of a* (*resp. flat*) *projective Cartan connection iff* (II) *is a subgeometry of a* (*resp. flat*) *projective Cartan connection.*

When $m = 1$, the Cartan connection (Q, ω) itself gives a projective Cartan connection, whose model space is $PL(l)/PL(l)_v$ and v belongs to $V_{1,l}$. By the castling transformation of (Q, ω), we obtain the Cartan connection

$$(Q \times PL(l-1), *\omega \times \Lambda_1),$$

which is a subgeometry of a projective Cartan connection.

Theorem 3.5 is described by the following commutative diagram. We assume (Q,ω) is a Grassmannian Cartan connection on M.



The manifold M_m denotes a quotient manifold $P_tM/GL(n) \otimes GL(1)$, and we denote by r_m the projection $P_tM \to P_tM/GL(n) \otimes GL(1)$. The manifold M_m is naturally isomorphic with $Q \times_{PL(l)_{<v>}} PL(m)$. Thus we obtain a natural projection from Q to M_m and there exists a natural inclusion $Q \to Q \times PL(m)$. Now denote by π_m a projection from $Q \times PL(m)$ to the quotient manifold $Q \times PL(m)/PL(l) \times PL(m)_v$. From the proof of Proposition 3.7 the quotient space $Q \times PL(m)/PL(l) \times PL(m)_v$ is identified with M_m by the bundle isomorphism given by $\pi_m(z,g) \leftrightarrow r_m \circ \rho(z)g^{-1}$.

4. Successive castling transformations

In this section we give two fundamental procedures to do castling transformations successively. The product group $PL(l) \times \prod_{i=1}^{j} PL(k_i)$ naturally acts on $P(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j} \boldsymbol{R}^{k_i})$ by the tensor product, namely by means of the inclusion $\imath : PL(l) \times \prod_{i=1}^{j} PL(k_i) \hookrightarrow PL(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j} \boldsymbol{R}^{k_i})$ given by $\imath(\bar{g}, \bar{A}_1, \ldots, \bar{A}_j) = g \otimes A_1 \otimes \cdots \otimes A_j$. We denote the natural bases of

$$\boldsymbol{R}^l, \boldsymbol{R}^{k_1}, \ldots, \boldsymbol{R}^{k_j} \text{ by } \{e_{i_0}\}_{i_0=1}^{l}, \{e_{i_1}\}_{i_1=1}^{k_1}, \ldots, \{e_{i_j}\}_{i_j=1}^{k_j}.$$

Then a point w in $\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j} \boldsymbol{R}^{k_i}$ is written by $w = \sum_{i_0,i_1,\ldots,i_j} C_{i_0 i_1 \cdots i_j} e_{i_0} \otimes e_{i_1} \otimes \cdots \otimes e_{i_j}$, where $C_{i_0 i_1 \cdots i_j}$ is a coefficient. Now let σ be an element of symmetric group of $\{1, 2, \cdots j\}$. Then σ induces a natural linear isomorphism $\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j} \boldsymbol{R}^{k_i} \to \boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j} \boldsymbol{R}^{\sigma(k_i)}$, which is defined by

$$\sigma(w) = \sum_{i_0,i_1,\ldots,i_j} C_{i_0 i_1 \cdots i_j} e_{i_0} \otimes e_{i_{\sigma(1)}} \otimes \cdots \otimes e_{i_{\sigma(j)}}.$$

The map σ induces a diffeomorphism $P(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j} \boldsymbol{R}^{k_i}) \to P(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j} \boldsymbol{R}^{\sigma(k_i)})$. The group $PL(l) \times \prod_{i=1}^{j} PL(k_{\sigma(i)})$ naturally acts on $P(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j} \boldsymbol{R}^{\sigma(k_i)})$, and the action satisfies the condition $(g, A_{\sigma(1)}, \ldots, A_{\sigma(j)}) \cdot \sigma(w) = \sigma((g, A_1, \ldots, A_j) \cdot w)$. It is easy to prove the following.

Proposition 4.1. *Assume that a point w in $P(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j} \boldsymbol{R}^{k_i})$ gives an open orbit of $PL(l) \times \prod_{i=1}^{j} PL(k_i)$. Then we have the following:*

1. *For any permutation σ of $\{1, 2, \cdots j\}$ the product group $PL(l) \times \prod_{i=1}^{j} PL(k_{\sigma(i)})$ admits an open orbit given by $\sigma(w)$. Isotropy subgroups $PL(l) \times \prod_{i=1}^{j} PL(k_i)_w$ and $PL(l) \times \prod_{i=1}^{j} PL(k_{\sigma(i)})_{\sigma(w)}$ are isomorphic.*

2. *The product group* $PL(l)\times\prod_{i=1}^{j}PL(k_i)\times PL(1)$ *acts on* $P(\boldsymbol{R}^l\otimes\bigotimes_{i=1}^{j}\boldsymbol{R}^{k_i}\otimes\boldsymbol{R})$, *which is identified with* $P(\boldsymbol{R}^l\otimes\bigotimes_{i=1}^{j}\boldsymbol{R}^{k_i})$ *naturally. Via this identification* w *gives an open orbit of* $PL(l)\times\prod_{i=1}^{j}PL(k_i)\times PL(1)$, *and isotropy subgroups* $PL(l)\times\prod_{i=1}^{j}PL(k_i)_w$ *and* $PL(l)\times\prod_{i=1}^{j}PL(k_i)\times PL(1)_w$ *are isomorphic.*

From the Lie group $PL(l)\times\prod_{i=1}^{j}PL(k_i)$ we can obtain several new Lie groups by Propositions 4.1 and castling transformations. For example let L be a Lie subgroup of $PL(3)$ such that L admits an open orbit $L.x$ in $P(\boldsymbol{R}^3)$. Then we obtain the sequence of new groups $*L\times PL(2)$, $L\times PL(2)\times PL(5)$, $*L\times PL(5)\times PL(13)$, $L\times PL(5)\times PL(13)\times PL(194)$, which admit open orbits in projective spaces. Note that $*L\times PL(2)$ is regarded as a subgroup $*L\otimes PL(2)$ of $PL(6)$. Next we apply Proposition 4.1 to Cartan connections.

Proposition 4.2. *Let* Q *be a manifold equipped with a* $\mathfrak{g}$*-valued 1-from* ω. *Assume that* $(Q\times\prod_{i=1}^{j}PL(k_i),\omega\times\prod_{i=1}^{j}\Lambda_1)$ *gives a Cartan connection over a manifold* N *of type*

$$PL(l)\times\prod_{i=1}^{j}PL(k_i)/PL(l)\times\prod_{i=1}^{j}PL(k_i)_w,$$

which is a subgeometry of $PL(\boldsymbol{R}^l\otimes\bigotimes_{i=1}^{j}\boldsymbol{R}^{k_i})/PL(\boldsymbol{R}^l\otimes\bigotimes_{i=1}^{j}\boldsymbol{R}^{k_i})_w$ *given by* $\imath:PL(l)\times\prod_{i=1}^{j}PL(k_i)\hookrightarrow PL(\boldsymbol{R}^l\otimes\bigotimes_{i=1}^{j}\boldsymbol{R}^{k_i})$. *Then we have the following:*

1. *For any permutation* σ *of* $\{1,2,\cdots,j\}$, *the pair*
$$(Q\times\prod_{i=1}^{j}PL(k_{\sigma(i)}),\omega\times\prod_{i=1}^{j}\Lambda_1)$$
gives a Cartan connection over N *of type*
$$PL(l)\times\prod_{i=1}^{j}PL(k_{\sigma(i)})/PL(l)\times\prod_{i=1}^{j}PL(k_{\sigma(i)})_{\sigma(w)},$$
which is a subgeometry of $PL(\boldsymbol{R}^l\otimes\bigotimes_{i=1}^{j}\boldsymbol{R}^{k_{\sigma(i)}})/PL(\boldsymbol{R}^l\otimes\bigotimes_{i=1}^{j}\boldsymbol{R}^{k_{\sigma(i)}})_{\sigma(w)}$.

2. $(Q\times\prod_{i=1}^{j}PL(k_i)\times PL(1),\omega\times\prod_{i=1}^{j}\Lambda_1\times\Lambda_1)$ *gives a Cartan connection over a manifold* N *of type*
$$PL(l)\times\prod_{i=1}^{j}PL(k_i)\times PL(1)/PL(l)\times\prod_{i=1}^{j}PL(k_i)\times PL(1)_w,$$
which is a subgeometry of $PL(\boldsymbol{R}^l\otimes\bigotimes_{i=1}^{j}\boldsymbol{R}^{k_i}\otimes\boldsymbol{R})/PL(\boldsymbol{R}^l\otimes\bigotimes_{i=1}^{j}\boldsymbol{R}^{k_i}\otimes\boldsymbol{R})_w$.

We consider the castling transformation of $(Q\times\prod_{i=1}^{j}PL(k_i),\omega\times\prod_{i=1}^{j}\Lambda_1)$. The group $PL(l)\times\prod_{i=1}^{j}PL(k_i)$ can be identified with a subgroup of
$PL(\boldsymbol{R}^l\otimes\bigotimes_{i=1}^{j}\boldsymbol{R}^{k_i})$ by the map $F:(g,A_1,\ldots,A_j)\mapsto g\otimes A_1\otimes\cdots\otimes A_j$. Then one form $\omega\times\prod_{i=1}^{j}\Lambda_1$ can be identified with the 1-form $dF\circ\omega\times\prod_{i=1}^{j}\Lambda_1$, which is computed as follows:

$$\begin{aligned} dF\circ\omega\times\prod_{i=1}^{j}\Lambda_1(X,Y_1,\cdots,Y_j) &= \omega(X)\otimes I\otimes\cdots\otimes I+I\otimes Y_1\otimes\cdots\otimes I \\ &\quad +\cdots+I\otimes\cdots\otimes I\otimes Y_j. \end{aligned}$$

Thus the group isomorphism $*$ of $PL(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j} \boldsymbol{R}^{k_i})$ is restricted to the subgroup $PL(l) \times \prod_{i=1}^{j} PL(k_i)$ by

$$* : (g, A_1, \ldots, A_j) \mapsto (*g, *A_1, \ldots, *A_j).$$

The Lie algebra isomorphism $*$ of $\mathfrak{sl}(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j} \boldsymbol{R}^{k_i})$ is restricted to $\mathfrak{sl}(l) \times \prod_{i=1}^{j} \mathfrak{sl}(k_i)$ and we have

$$*(\omega \times \prod_{i=1}^{j} \Lambda_1) = *\omega \times \prod_{i=1}^{j} *\Lambda_1.$$

By using Propositions 3.7 and 4.2, we can apply castling transformations successively. For example let us consider a projective Cartan connection (Q, ω) over 2-dimensional manifold M. The bundle Q is a principal fiber bundle over M with structure group $PL(3)_v$, where v is an element of $P(\boldsymbol{R}^3)$ and ω is a $\mathfrak{sl}(3)$-valued 1-form. Then for example we obtain the following sequence by successive castling transformations:

$$\begin{aligned} &(1)\ (Q, \omega) \longrightarrow (2)\ (Q, *\omega) \longrightarrow (3)\ (Q \times PL(2), *\omega \times \Lambda_1) \\ &\longrightarrow (4)\ (Q \times PL(2), \omega \times *\Lambda_1) \longrightarrow (5)\ (Q \times PL(2) \times PL(5), \omega \times *\Lambda_1 \times \Lambda_1). \end{aligned}$$

In this process we fix a point $v^\perp \in V_{2,3}$ and identify $v^\perp$ with a point w of $P(\boldsymbol{R}^3 \otimes \boldsymbol{R}^2)$, and fix a point $w^\perp \in V_{5,3\cdot 2}$. Now consider the Grassmannian manifold $PL(l)/PL(l)_{<v>}$, and we denote by ρ the isotropy representation of $PL(l)_{<v>}$. Here $PL(l)_{<v>}$ is expressed with respect to a basis obtained from $(v, v^\perp)$. Then ρ takes values in $GL(n) \otimes GL(m)$. When $\rho(g)$ is expressed as $A \otimes B$, we define a projective linear representation $\rho_m : PL(l)_{<v>} \to PL(m)$ by $\rho_m(g) = B$. The isotropy group $PL(l) \times PL(m)_v$ is equal to the set $\{(g, \rho_m(g)) \mid g \in PL(l)_{<v>}\}$. Thus the two isotropy groups $PL(l)_{<v>}$ and $PL(l) \times PL(m)_v$ are isomorphic.

Here we omitted the process of 1 and 2 in Proposition 4.2 as we omitted the Cartan connections $(Q \times PL(1), \omega \times \Lambda_1)$ and $(Q \times PL(2) \times PL(1), *\omega \times \Lambda_1 \times \Lambda_1)$. The structure group of each Cartan connection is given by

$$\begin{aligned} &(1)\quad PL(3)_v, \\ &(2)\quad PL(3)_{<v^\perp>} = *PL(3)_v, \\ &(3)\quad PL(3) \times PL(2)_{v^\perp} = \{(*g, \rho_2(*g)) \mid g \in PL(3)_v\}, \\ &(4)\quad PL(3) \times PL(2)_{<w^\perp>} = \{(g, *\rho_2(*g))\}, \\ &(5)\quad PL(3) \times PL(2) \times PL(5)_{w^\perp} = \{(g, *\rho_2(*g), \rho_5(g \otimes *\rho_2(*g)))\}. \end{aligned}$$

We express the action of $PL(2)_v$ on Q as $u \cdot g = ug$ for $u \in Q$ and $g \in PL(2)_v$. Then each structure group acts on the bundle of each Cartan connection as follows:

$$\begin{aligned} &(2)\quad u \cdot *g = ug, \\ &(3)\quad (u, A) \cdot (*g, \rho_2(*g)) = (ug, A\rho_2(*g)), \\ &(4)\quad (u, A) \cdot (g, *\rho_2(*g)) = (ug, A\rho_2(*g)), \\ &(5)\quad (u, A, B) \cdot (g, *\rho_2(*g), \rho_5(g \otimes *\rho_2(*g))) = (ug, A\rho_2(*g), B\rho_5(g \otimes *\rho_2(*g))). \end{aligned}$$

Successive castling transformations yields a sequence of manifolds admitting a projective structure or a Grassmannian structure. From now on we characterize those manifolds.

Lemma 4.3. *Let v be a point in $P(\boldsymbol{R}^l \otimes \boldsymbol{R}^\alpha)$ and assume that $PL(l) \times PL(\alpha) \cdot v$ gives an open orbit in $P(\boldsymbol{R}^l \otimes \boldsymbol{R}^\alpha)$. Suppose that a Lie group G is obtained by successive castling transformations from $PL(l) \times PL(\alpha)$, and that a point w in $P(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^j \boldsymbol{R}^{k_i})$ is obtained from v. Then G can be written in the form $PL(l) \times \prod_{i=1}^j PL(k_i)$ and $G \cdot w$ gives an open orbit in $P(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^j \boldsymbol{R}^{k_i})$ again. Moreover there exists a projective linear representation $f_i\colon PL(l)_{<v>} \to PL(k_i)$ $(0 \le i \le j)$ such that $G_w = \{(f_0(g), f_1(g), \dots, f_j(g)) \mid g \in PL(l)_{<v>}\}$, where $f_0 = id$ or $*$.*

Proof. First about the group $PL(l) \times PL(\alpha)$, the element of the isotropy subgroup $PL(l) \times PL(\alpha)_v$ is expressed as $(g, \rho_\alpha(g))$, where ρ_α is the projective linear representation of $PL(l)$ introduced after Proposition 3.1. We now proceed by induction. Assume that there exists $w \in P(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^j \boldsymbol{R}^{k_i})$ such that the group $PL(l) \times \prod_{i=1}^j PL(k_i)$ admits an open orbit given by w, and an element of the isotropy group $PL(l) \times \prod_{i=1}^j PL(k_i)_w$ is expressed as $(f_0(g), f_1(g), \dots, f_j(g))$ for some $g \in PL(l)_{<v>}$. Then the point w must belong to $V_{k_j, lk_1 \cdots k_{j-1}}$. A group obtained by using the assertion (1) or (2) in Proposition 4.1 also admits an open orbit and its isotropy group is described by using the projective linear representations of $PL(l)_{<v>}$. Since successive castling transformation consists of Propositions 4.1 and 3.1, it is enough to consider the effect of castling transformation described in Proposition 3.1. By castling transformation of the group $PL(l) \times \prod_{i=1}^j PL(k_i)$, we obtain the group

$$PL(l) \times \prod_{i=1}^{j-1} PL(k_i) \times PL(lk_1 \cdots k_{j-1} - k_j)$$

and a fixed point $w^\perp \in V_{lk_1 \cdots k_{j-1} - k_j, lk_1 \cdots k_{j-1}}$, which gives an open orbit in

$$P(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j-1} \boldsymbol{R}^{k_i} \otimes \boldsymbol{R}^{lk_1 \cdots k_{j-1} - k_j}).$$

Here we regard $PL(l) \times \prod_{i=1}^{j-1} PL(k_i)$ as the subgroup of $PL(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j-1} \boldsymbol{R}^{k_i})$. Then about the isotropy group $PL(l) \times \prod_{i=1}^{j-1} PL(k_i) \times PL(lk_1 \cdots k_{j-1} - k_j)_{w^\perp}$ its element is expressed as

$$(*f_0(g), *f_1(g), \dots, *f_{j-1}(g), \rho_{lk_1 \cdots k_{j-1} - k_j}(*f_0(g) \otimes *f_1(g) \otimes \cdots \otimes *f_{j-1}(g)))$$

for some $g \in PL(l)_{<v>}$, where $\rho_{lk_1 \cdots k_{j-1} - k_j}$ is the projective linear representation of $PL(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j-1} \boldsymbol{R}^{k_i})_{<w^\perp>}$. Thus we completes the induction step. ∎

This Lemma also shows the fact that G_w is isomorphic to $PL(l)_{<v>}$.

Now let (Q, ω) be a Grassmannian Cartan connection of type $(l - \alpha, \alpha)$ over M. Thus the model space is $PL(l)/PL(l)_{<v>}$, where $<v>$ is an element of $Gr_{\alpha, l}$. By successive castling transformations from (Q, ω) we obtain a Cartan connection

$$(Q \times \prod_{i=1}^j PL(k_i), \omega' \times \prod_{i=1}^j (\Lambda_1)_i) \text{ of type}$$
$$PL(l) \times \prod_{i=1}^j PL(k_i) / \ PL(l) \times \prod_{i=1}^j PL(k_i)_w,$$

where $(\Lambda_1)_i = \Lambda_1$ or $*\Lambda_1$ and $\omega' = \omega$ or $\omega' = *\omega$. We denote by N the base space of $(Q \times \prod_{i=1}^j PL(k_i), \omega' \times \prod_{i=1}^j (\Lambda_1)_i)$. Then the Cartan connection $(Q \times \prod_{i=1}^j PL(k_i), \omega' \times \prod_{i=1}^j (\Lambda_1)_i)$ over N induces a projective Cartan connection over N. The next proposition determines the relation of base spaces obtained by successive castling transformations. We assume that $k_i \neq 1$. Remove the s-th component from $(Q \times \prod_{i=1}^j PL(k_i), \omega' \times \prod_{i=1}^j (\Lambda_1)_i)$ and denote it by $(Q \times \prod_{i \neq s}^j PL(k_i), \omega' \times \prod_{i \neq s}^j (\Lambda_1)_i)$.

Proposition 4.4. *Choose s and t satisfying $1 \leq s < t \leq j$.*
(1) *The base space N is a principal fiber bundle over M with group $\prod_{i=1}^{j} PL(k_i)$.*
(2) *The pair $(Q \times \prod_{i\neq s}^{j} PL(k_i), \omega' \times \prod_{i\neq s}^{j} (\Lambda_1)_i)$*
(resp. $(Q \times \prod_{i\neq t}^{j} PL(k_i), \omega' \times \prod_{i\neq t}^{j} (\Lambda_1)_i)$)
is a Cartan connection on a manifold L (resp. K), which is a subgeometry of a Grassmannian Cartan connection of type

$$(lk_1 \cdots k_{s-1}k_{s+1} \cdots k_j - k_s, k_s) \ (\text{resp. } (lk_1 \cdots k_{t-1}k_{t+1} \cdots k_j - k_t, k_t)).$$

The base space L is a $\prod_{i\neq s}^{j} PL(k_i)$-bundle over M.
(3) *The base space N is regarded as a principal fiber bundle over L (resp. K) with group $PL(k_s)$ (resp. $PL(k_t)$), and $PL(k_t)$ acts on N and L. Thus we obtain following diagram:*

$$\begin{array}{ccccccc} & & PL(k_s) & \longrightarrow N \longleftarrow & PL(k_t) & & \\ & \swarrow & & & & \searrow & \\ L & \longleftarrow PL(k_t) & & & & PL(k_s) \longrightarrow & K. \end{array}$$

Moreover N is isomorphic to the bundle

$$P_t L / GL(lk_1 \cdots k_{s-1}k_{s+1} \cdots k_j - k_s) \otimes GL(1)$$

and the action of $PL(k_t)$ on L induces the action on N by the differential. The quotient $N/PL(k_t)$ is isomorphic to K.

Proof. We can assume that the number t is equal to j without loss of generality. Firstly we show that N is a principal fiber bundle over M. From the assumption N is diffeomorphic to the quotient manifold

$$Q \times \prod_{i=1}^{j} PL(k_i) / PL(l) \times \prod_{i=1}^{j} PL(k_i)_w,$$

thus we identify N with this quotient manifold. By using this identification the group $\prod_{i=1}^{j} PL(k_i)$ naturally acts on N as follows: let $[z, A_1, \ldots, A_j]$ be an element of N and $(B_1, \ldots, B_j)$ be an element of $\prod_{i=1}^{j} PL(k_i)$. We define the action by $[z, A_1, \ldots, A_j] \cdot (B_1, \ldots, B_j) := [z, B_1^{-1}A_1, \ldots, B_j^{-1}A_j]$. This action is free. By using the projection $\pi : Q \to M$ we define the projection $\pi_N : N \to M$ by $[z, A_1, \ldots, A_j] \mapsto \pi(z)$. Moreover $\pi : Q \to M$ is a principal fiber bundle, thus for each open neighborhood U of M there is a local trivialization $\pi^{-1}(U) \to U \times PL(l)_{<v>}$ mapping z to $(\pi(z), \phi(z))$. Now by Lemma 4.3 the isotropy subgroup $PL(l) \times \prod_{i=1}^{j} PL(k_i)_w$ is given by the set $\{(f_0(g), f_1(g), \ldots, f_j(g)) \mid g \in PL(l)_{<v>}\}$. We define the map $\Phi : \pi_N^{-1}(U) \to U \times \prod_{i=1}^{j} PL(k_i)$ by

$$[z, A_1, \ldots, A_j] \mapsto (\pi(z), {f_1}'(\phi(z))A_1^{-1}, \ldots, {f_j}'(\phi(z))A_j^{-1}),$$

where ${f_i}' = f_i$ when $(\Lambda_1)_i = \Lambda_1$ or ${f_i}' = *f_i$ when $(\Lambda_1)_i = *\Lambda_1$. The map Φ is well-defined and diffeomorphism, moreover preserving the action of $\prod_{i=1}^{j} PL(k_i)$. By using π_N and the local trivializations Φ, it is shown that N is a principal fiber bundle over M with group $\prod_{i=1}^{j} PL(k_i)$.

Since $\prod_{i=1}^{j-1} \{e\} \times PL(k_j)$ is a normal closed subgroup of the structure group $\prod_{i=1}^{j} PL(k_i)$ of N, the quotient $N/PL(k_j)$ is again a principal fiber bundle over M with group $\prod_{i=1}^{j-1} PL(k_i)$. Put $H = \prod_{i=1}^{j-1} PL(k_i)$. Then the manifold $Q \times H$ is regarded as a principal fiber bundle over M with group $PL(l)'_{<v>} \times H$, where

$PL(l)'_{<v>} = PL(l)_{<v>}$ if $\omega' = \omega$ or $PL(l)'_{<v>} = *PL(l)_{<v>}$ if $\omega' = *\omega$. The group $PL(l)'_{<v>} \times H$ contains the closed subgroup $PL(l) \times H_{<w>}$. Hence we obtain the fiber bundle $Q \times H$ over the quotient manifold $Q \times H/PL(l) \times H_{<w>}$ with structure group $PL(l) \times H_{<w>}$. There is a diffeomorphism from

$$Q \times H/PL(l) \times H_{<w>} \text{ to } N/PL(k_j) \text{ defined by}$$
$$[u, A_1, \ldots, A_{j-1}] \mapsto [u, A_1, \ldots, A_{j-1}, e]PL(k_j).$$

By Proposition 3.7 $(Q \times H, \omega \times \prod_{i=1}^{j-1} \Lambda_1)$ is a Cartan connection over $N/PL(k_j)$ of type $PL(l) \times H/PL(l) \times H_{<w>}$. From the assumption the model space $PL(l) \times H/PL(l) \times H_{<w>}$ is a subgeometry of the Grassmannian manifold $PL(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j-1} \boldsymbol{R}^i)/PL(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j-1} \boldsymbol{R}^i)_{<w>}$ according to the map $F : PL(l) \times H \hookrightarrow PL(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j-1} \boldsymbol{R}^i)$ defined by the tensor product. Note that a point w is included in $V_{k_j, lk_1 \cdots k_{j-1}}$. Therefore $N/PL(k_j)$ admits a Grassmannian Cartan connection of type $(lk_1 \cdots k_{j-1} - k_j, k_j)$.

We observe naturally $PL(k_s)$ acts freely on $Q \times H$ by $(u, A_1, \ldots, A_s, \ldots, A_{j-1}) \cdot B_s := (u, A_1, \ldots, B_s^{-1} A_s, \ldots, A_{j-1})$ for $B_s \in PL(k_s)$. We denote this right action by R_{B_s}. Put $L = N/PL(k_j)$. The action of $PL(k_s)$ on $Q \times \prod_{i=1}^{j-1} PL(k_i)$ induces the action of $PL(k_s)$ on L by using the bundle isomorphism between L and $Q \times H/PL(l) \times H_{<w>}$.

We fix the complementary subspace $\mathfrak{m}$ of $\mathfrak{sl}(l)_{<v>}$ in $\mathfrak{sl}(l)$, thus we have $\mathfrak{sl}(l) = \mathfrak{m} \oplus \mathfrak{sl}(l)_{<v>}$. Then $\mathfrak{m}' \times \prod_{i=1}^{j-1} \mathfrak{sl}(k_i)$ gives a complementary subspace of $\mathfrak{sl}(l) \times \prod_{i=1}^{j-1} \mathfrak{sl}(k_i)_{<w>}$ in $\mathfrak{sl}(l) \times \prod_{i=1}^{j-1} \mathfrak{sl}(k_i)$, where $\mathfrak{m}' = \mathfrak{m}$ when $\omega' = \omega$ or $\mathfrak{m}' = *\mathfrak{m}$ when $\omega' = *\omega$. Furthermore the differential dF of F induces a linear isomorphism $\hat{dF}$ from $\mathfrak{m}' \times \prod_{i=1}^{j-1} \mathfrak{sl}(k_i)$ to $M(lk_1 \cdots k_{j-1} - k_j, k_j)$.

We denote by ρ the isotropy representation of the model space $PL(l) \times H/PL(l) \times H_{<w>}$. Denote $Q \times H$ by P, and the natural projection $P \to P/\ker \rho$ by ρ. We denote by $\widetilde{P}$ the quotient space $P/\ker \rho$. The action of $PL(k_s)$ on P induces the action on $\widetilde{P}$. Then there exists a unique 1-form θ on $\widetilde{P}$ such that $\rho^*\theta = \omega' \times \prod_{i=1}^{j-1} \Lambda_{1\,\mathfrak{m}' \times \prod_{i=1}^{j-1} \mathfrak{sl}(k_i)}$. The pair $(\widetilde{P}, \theta)$ gives a $\rho(PL(l) \times H_{<w>})$-structure over L, which is a subbundle of a $GL(lk_1 \cdots k_{j-1} - k_j) \otimes GL(k_j)$-structure $P_t L$. By Proposition 2.5 the imbedding ι of $\widetilde{P}$ into $P_t L$ is given by the restriction of the bundle isomorphism $t : \mathcal{L}(L) \to \mathcal{L}(L)$, which is defined by $t : x \mapsto x \circ \hat{dF}^{-1}$.

The equality ${R_{B_s}}^*\omega \times \prod_{i=1}^{j-1} \Lambda_1 = \omega \times \prod_{i=1}^{j-1} \Lambda_1$ yields ${R_{B_s}}^*\rho^*\theta = \rho^*\theta$. Since we have the commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{R_{B_s}} & P \\ \rho\downarrow & \circlearrowleft & \downarrow\rho \\ \widetilde{P} & \xrightarrow{R_{B_s}} & \widetilde{P} \\ \downarrow & \circlearrowleft & \downarrow \\ L & \xrightarrow{R_{B_s}} & L, \end{array} \tag{4.1}$$

it follows that ${R_{B_s}}^*\rho^*\theta = \rho^*{R_{B_s}}^*\theta$. Thus ${R_{B_s}}^*\theta = \theta$. Therefore the action $R_{B_s} : \widetilde{P} \to \widetilde{P}$ is induced by the differential of the action $R_{B_s} : L \to L$. Moreover

$R_{B_s}: \tilde{P} \to \tilde{P}$ is uniquely extended to the action $R_{B_s}: P_tL \to P_tL$ by

$$\iota \circ \rho(z, A_1, \ldots, A_s, \ldots, A_{j-1})B \otimes C \cdot D = \iota \circ \rho(z, A_1, \ldots, D^{-1}A_s, \ldots, A_{j-1})B \otimes C,$$

where $(z, A_1, \ldots, A_s, \ldots, A_{j-1}) \in Q'$, $B \otimes C \in GL(lk_1 \cdots k_{j-1} - k_j) \otimes GL(k_j)$ and $D \in PL(k_s)$. This action naturally induces the action of $PL(k_s)$ on $P_tL/GL(lk_1 \cdots k_{j-1} - k_j) \otimes GL(1)$. We describe the process that the action of $PL(k_s)$ on P induces the actions on other manifolds by the following diagram.

$$\begin{array}{ccccccc} PL(k_s) & & PL(k_s) & & PL(k_s) & & PL(k_s) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P & \dashrightarrow & \widetilde{P} & \dashrightarrow & P_tL & \dashrightarrow & P_tL/GL(lk_1 \cdots k_{j-1} - k_j) \otimes GL(1) \end{array}$$

By the proof of Proposition 3.7 there is a $PL(k_j)$-bundle isomorphism from $N = P \times PL(k_j)/H \times \prod_{i=1}^{j} PL(k_i)_w$ to $P_tL/GL(lk_1 \cdots k_{j-1} - k_j) \otimes GL(1)$. Via this isomorphism we see that the action of $PL(k_s)$ on N coincides with the one on $P_tL/GL(lk_1 \cdots k_{j-1} - k_j) \otimes GL(1)$. Thus the induced action of $PL(k_s)$ on N is given by $[u, A_1, \ldots, A_s, \ldots, A_j] \cdot B_s = [u, A_1, \ldots, B_s^{-1}A_s, \ldots, A_j]$ for $B_s \in PL(k_s)$. Consequently the action of $PL(k_s)$ on L induces the action on N by the differential. By this action we have the quotient $K = N/PL(k_s)$. On the other hand let σ be a permutation defined by

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & s-1 & s & s+1 & \cdots & j-1 & j \\ 1 & 2 & \cdots & s-1 & s+1 & \cdots & j-1 & j & s \end{pmatrix}.$$

Then by Proposition 4.2 the given Cartan connection

$$(Q \times \prod_{i=1}^{j} PL(k_i), \omega' \times \prod_{i=1}^{j} (\Lambda_1)_i)$$

over N can be regarded as a Cartan connection over N of type

$$PL(l) \times \prod_{i=1}^{s-1} PL(k_i) \times \prod_{i=s+1}^{j} PL(k_i) \times PL(k_s)/PL(l) \times$$
$$\times \prod_{i=1}^{s-1} PL(k_i) \times \prod_{i=s+1}^{j} PL(k_i) \times PL(k_s)_{\sigma(w)},$$

which is a subgeometry of the projective space

$$P(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{s-1} \boldsymbol{R}^{k_i} \otimes \bigotimes_{i=s+1}^{j} \boldsymbol{R}^{k_i} \otimes \boldsymbol{R}^s).$$

By Proposition 3.7 it follows that K admits a Grassmannian Cartan connection of type $(lk_1 \cdots k_{s-1}k_{s+1} \cdots k_j - k_s, k_s)$. ■

5. Examples of successive castling transformations

Let (Q, ω) be a projective Cartan connection over M. As we demonstrate it after Proposition 4.2 by successive castling transformations we can obtain the following Cartan connections:

$$\begin{aligned} &(Q, \omega) \\ &\longrightarrow (Q \times PL(n), *\omega \times \Lambda_1) \longrightarrow (Q \times PL(n) \times PL(n^2+n-1), \omega \times *\Lambda_1 \times \Lambda_1) \\ &\longrightarrow (Q \times PL(n^2+n-1), \omega \times \Lambda_1). \end{aligned}$$

We denote the base space of (Q, ω) by M_1, the one of $(Q \times PL(n), *\omega \times \Lambda_1)$ by M_n and so on. Generally $M_{k_1 \times \cdots \times k_j}$ denotes a principal fiber bundle over M

with structure group $\prod_{i=1}^{j} PL(k_i)$. Then following the above successive Cartan connections from (Q,ω) we obtain the sequence of base spaces: $M_1 \longrightarrow M_2 \longrightarrow M_{2\times 5} \longrightarrow M_5$. The Cartan connection $(Q \times PL(n), *\omega \times \Lambda_1)$ over M_n and $(Q \times PL(n) \times PL(n^2+n-1), \omega \times *\Lambda_1 \times \Lambda_1)$ over $M_{n\times(n^2+n-1)}$ induce projective Cartan connections, and the Cartan connection $(Q \times PL(n^2+n-1), \omega \times \Lambda_1)$ over M_{n^2+n-1} induces a Grassmannian Cartan connection. Now we describe those base spaces more explicitly:

Proposition 5.1. *M_n is isomorphic to the projective frame bundle of M, and $M_{n\times(n^2+n-1)}$ and M_{n^2+n-1} are isomorphic to the following bundles respectively:*

$$\begin{aligned}\widetilde{\mathcal{L}}_{n\times(n^2+n-1)} := \{&(u_1,u_2) \mid u_1 : \textit{projective frame of } T_pM, \\ &u_2 : \textit{projective frame of } T_pM \rtimes \mathfrak{sl}(T_pM), p \in M\}, \\ \widetilde{\mathcal{L}}_{n^2+n-1} := \{&u_2 \mid u_2 : \textit{projective frame of } T_pM \rtimes \mathfrak{sl}(T_pM), p \in M\}.\end{aligned}$$

Proof. By definition Q is a principal fiber bundle over M with structure group $PL(n)_v$, where $v = (1,0,\cdots,0)$ is an element of $V_{1,n}$. From the argument of § 2.1 we have the injection $h : \mathcal{L}(M) \hookrightarrow Q$ corresponding to the injection $\iota : GL(n) \hookrightarrow PL(n+1)_v$, which is defined by

$$\iota : A \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

By castling transformation of (Q,ω) we obtain the Cartan connection $(Q \times PL(n), *\omega \times \Lambda_1)$ over M_n whose structure group is $PL(n+1) \times PL(n)_{<v^\perp>}$, where $v^\perp = \begin{pmatrix} 0 \\ I_n \end{pmatrix}$ is an element of $V_{n,n+1}$. Then $\iota(GL(n)) \times PL(n)_{v^\perp}$ is given by the set $\{(*\iota(A), A)\}$. The manifold $h(\mathcal{L}(M)) \times PL(n)$ is a principal fiber bundle over M_n with structure group $\iota(GL(n)) \times PL(n)_{v^\perp}$, and gives a reduction of $Q \times PL(n)$. Then we have the following bundle isomorphism $h(\mathcal{L}(M)) \times PL(n)/\iota(GL(n)) \times PL(n)_{v^\perp} \to \mathcal{L}(M)/GL(1)$ defined by

$$[h(x), g] \mapsto q(x)g^{-1},$$

where q is the projection $\mathcal{L}(M) \to \mathcal{L}(M)/GL(1)$. Note that the action of $\iota(GL(n)) \times PL(n)_{v^\perp}$ on $h(\mathcal{L}(M)) \times PL(n)$ is given by $(h(x), g) \cdot (*\iota(A), A) = (h(x)\iota(A), gA)$. Thus the base space M_n is isomorphic to the projective frame bundle of M. Concerning the Cartan connection $(Q \times PL(n) \times PL(n^2+n-1), \omega \times *\Lambda_1 \times \Lambda_1)$ over $M_{n\times(n^2+n-1)}$, the structure group of $Q \times PL(n) \times PL(n^2+n-1)$ is $PL(n+1) \times PL(n) \times PL(n^2+n-1)_{w^\perp}$, where $w^\perp$ is a fixed projective frame of the vector space $\begin{pmatrix} M(1,n) \\ \hline \mathfrak{sl}(n) \end{pmatrix}$. When we are given a base x of a vector space V, we denote by $q(x)$ the projective frame, then q gives the projection from the linear Stiefel manifold to the projective Stiefel manifold corresponding to the natural projection $GL(n) \to PL(n)$. Then the subgroup $\iota(GL(n)) \times PL(n) \times PL(n^2+n-1)_{w^\perp}$ is given by the set $\{(\iota(A), *A, \rho_{n^2+n-1}(\iota(A) \otimes *A))\}$, where $\rho_{n^2+n-1}(\iota(A) \otimes *A)$ is expressed as the matrix

$$\begin{pmatrix} A & 0 \\ 0 & Ad(A) \end{pmatrix}$$

with respect to the basis

$$\{\begin{pmatrix} e_i \\ \hline 0 \end{pmatrix} (1 \le i \le n), \begin{pmatrix} 0 \\ \hline E^k_j - \delta^k_j E^n_n \end{pmatrix} (1 \le i, j \le n)\}.$$

By using a base $x = \{X_1, \cdots, X_n\}$ of T_pM and $Y = \sum Y^i_j E^j_i \in \mathfrak{sl}(n)$ we define an element $Y_x \in \mathfrak{sl}(T_pM)$ by $Y_x := x \circ Y \circ x^{-1}$, where we regard x as a linear isomorphism. That is to say Y_x is the map $Y_x : X_k \mapsto \sum Y^i_k X_i$. Then for $A \in GL(n)$ we have $Y_{xA} = Ad(A)(Y_x)$. We denote by ${E'}^k_j$ the element $E^k_j - \delta^k_j E^n_n$ of $\mathfrak{sl}(n)$. Now we define the map $\phi : h(\mathcal{L}(M)) \times PL(n) \times PL(n^2+n-1) \to \widetilde{\mathcal{L}}_{n\times(n^2+n-1)}$ by

$$[h(x), g_1, g_2] \mapsto \{q(x){g_1}^{-1}, q(x, ({E'}^k_j)_x){g_2}^{-1})\}$$

We show that the map ϕ is well defined. We put

$$(h(x'), g'_1, g'_2) := (h(x), g_1, g_2) \cdot (\iota(A), *A, \rho_{n^2+n-1}(\iota(A) \otimes *A)),$$

which is equal to $(h(xA), g_1A, g_2\rho_{n^2+n-1}(\iota(A) \otimes *A))$. Then we have

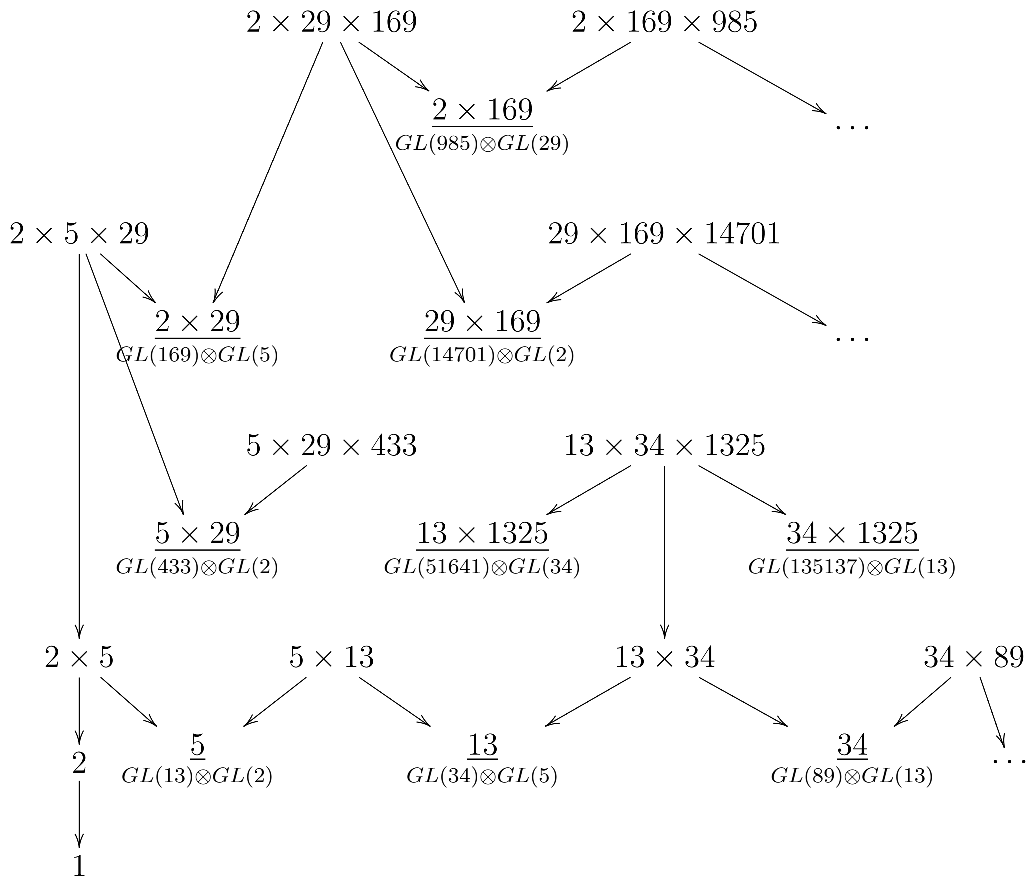
$$\begin{aligned} & \{q(x'){g'_1}^{-1}, q(x', ({E'}^k_j)_{x'}){g'_2}^{-1})\} \\ = \ & \{q(xA)(g_1A)^{-1}, q(xA, ({E'}^k_j)_{xA})(g_2\rho_{n^2+n-1}(\iota(A) \otimes *A))^{-1}\}. \end{aligned}$$

The last expression is equal to $\{q(x){g_1}^{-1}, q(x, ({E'}^k_j)_x){g_2}^{-1})\}$ since we have

$$q(xA, ({E'}^k_j)_{xA}) = q(x, ({E'}^k_j)_x)\rho_{n^2+n-1}(\iota(A) \otimes *A).$$

Moreover ϕ is a bundle isomorphism. Likewise we can show that M_{n^2+n-1} is isomorphic to $\widetilde{\mathcal{L}}_{n^2+n-1}$. ■

Now we explain how these manifolds $M_{k_1\times\cdots\times k_j}$ are related in the case of $n = 2$. From Propositions 5.1 and 3.7 M_2 is a projective frame bundle of M_1 and $M_{2\times5}$ is a projective frame bundle of M_2. Thus M_2 has the right action of $PL(2)$ and this action gives rise to the action of $PL(2)$ on the frame bundle of M_2 by the differential. Thus $PL(2)$ naturally acts on the projective frame bundle $M_{2\times5}$ of M_2. From Proposition 4.4 the quotient $M_{2\times5}/PL(2)$ is equal to M_5. Furthermore Cartan connections (Q, ω), $(Q \times PL(2), *\omega \times \Lambda_1)$ and $(Q \times PL(2) \times PL(5), \omega \times *\Lambda_1 \times \Lambda_1)$ induce projective Cartan connections and $(Q \times PL(5), \omega \times \Lambda_1)$ induces Grassmannian Cartan connections. The same result holds true about the general dimension n. If we continue the successive castling transformations we can obtain the following tree, where we only describe the base spaces. We abbreviate $M_{k_1\times\cdots\times k_j}$ to $k_1 \times \cdots \times k_j$. If a Cartan connection over a base space induces a Grassmannian structure of type (β, α) then we write $GL(\beta) \otimes GL(\alpha)$ under the base space. If a Cartan connection induces a projective structure, then we write nothing.



The relation of the base spaces of the tree is completely described by Proposition 4.4. The all underlined manifolds admit a Grassmannian structure. If the given projective structure on M is projectively flat, then manifolds which are not underlined admit a flat projective structure and underlined manifolds admit a flat Grassmannian structure. Especially a underlined manifold $M_{k_1\times\cdots\times k_{j-1}}$ admits a Grassmannian structure of type (β,α) which is given by an extension of the bundle $\mathcal{L}(M)\times\prod_{i=1}^{j-1}PL(k_i)$ over $M_{k_1\times\cdots\times k_{j-1}}$. The structure group of $\mathcal{L}(M)\times\prod_{i=1}^{j-1}PL(k_i)$ is isomorphic to $GL(2)$. This $GL(2)$-bundle gives a reduction of $GL(3k_1\cdots k_{j-1}-k_j)\otimes GL(k_j)$-structure on $M_{k_1\times\cdots\times k_{j-1}}$. We can prove this assertion generally as follows: we use the same notations in Proposition 4.4, thus M admits a Grassmannian Cartan connection of type $(l-\alpha,\alpha)$, and N is a fiber bundle $M_{k_1\times\cdots\times k_j}$ equipped with a projective structure. Denote by L the quotient $N/PL(k_j)$ and by P the bundle $Q\times H$ over L with structure group $PL(l)\times H_{<w>}$. Now we assume that $lk_1\cdots k_{j-1}-k_j\neq 1$, hence L admits a Grassmannian structure P_tL. The Lie group homomorphism $\rho: PL(l)\times H_{<w>}\to GL(M(l-\alpha,\alpha)\times\prod_{i=1}^{j-1}\mathfrak{sl}(k_i))$ is the isotropy representation of $PL(l)\times H/PL(l)\times H_{<w>}$. The quotient space $(P/ker\rho,\theta)$ can be considered as a subbundle of P_tL, and we have the natural projection $\rho: P\to\widetilde{P}$. The group $PL(l)_{<v>}$ has the subgroup G_0 and the restriction of the isotropy representation ρ to $G_0\times H_{<w>}$ is injective. The bundle P has the subbundle $h(P_tM)\times H$ with structure group $G_0\times H_{<w>}$. Hence the restriction of $\rho: P\to\widetilde{P}$ to

$h(P_tM)\times H$ is injective. Thus we obtain the sequence of reduction $(h(P_tM)\times H,\theta)$ $\subset$ $(\widetilde{P},\theta)$ $\subset$ P_tL corresponding to the sequence $G_0 \times H_{<w>}$ $\subset$ $\rho(PL(l)\times H_{<w>})$ $\subset$ $GL(lk_1\cdots k_{j-1}-k_j)\otimes GL(k_j)$. Hence L admits a reduction $h(P_tM)\times H$ of the Grassmannian structure P_tL of L.

Case of Lie groups

In the case of Lie groups the base spaces obtained by successive castling transformations are described more explicitly. Let $(\mathcal{L}(L),[\chi])$ be a projective structure on a n-dimensional Lie group L. Then we can construct a projective Cartan connection (Q,ω), where Q is a principal fiber bundle over M with structure group $PL(n+1)_v$, and we denote by h a injective bundle map from P_L to Q. Then by a successive castling transformations of (Q,ω) we obtain a Cartan connection $(Q\times\prod_{i=1}^{j}PL(k_i),\omega'\times\prod_{i=1}^{j}(\Lambda_1)_i)$ over a manifold N whose type is a subgeometry of the projective space $P(\boldsymbol{R}^{n+1}\otimes\bigotimes_{i=1}^{j}\boldsymbol{R}^{k_i})$. Then the base space is described as follows:

Proposition 5.2. *N is isomorphic to the product $L\times\prod_{i=1}^{j}PL(k_i)$.*

Proof. The proof is similar to the one of Proposition 4.4. The base space N is diffeomorphic to the quotient manifold $Q\times\prod_{i=1}^{j}PL(k_i)/PL(l)\times\prod_{i=1}^{j}PL(k_i)_w$ and by Proposition 4.3 the isotropy group $PL(l)\times\prod_{i=1}^{j}PL(k_i)_w$ is given by the set $\{(f_0(g),f_1(g),\ldots,f_j(g))\mid g\in PL(n+1)_v\}$, where $f_0=id$ or $*$. Any element of Q is written by the form $h(x)g_0$ for some $x\in\mathcal{L}(L)$ and $g_0\in PL(n+1)_v$. We denote by ρ a projection from Q to $\mathcal{L}(L)$. The frame bundle $\mathcal{L}(L)$ is isomorphic to the product $L\times GL(n)$ and an element $x\in\mathcal{L}(L)$ is written as $(a(x),A(x))$. Now we construct the map $N\to L\times\prod_{i=1}^{j}PL(k_i)$ defined by

$$[h(x)g_0,g_1,\ldots,g_j]\mapsto(a(x),g_1f_1'(g_0)^{-1}f_1'\circ\imath(A(x))^{-1},\ldots,g_jf_j'(g_0)^{-1}f_j'\circ\imath(A(x))^{-1}),$$

where $f_i'=f_i$ when $(\Lambda_1)_i=\Lambda_1$ and $f_i'=*f_i$ when $(\Lambda_1)_i=*\Lambda_1$. Then this map is well defined and a bundle isomorphism. ■

The Cartan connection $(Q\times\prod_{i=1}^{j-1}PL(k_i),\omega'\times\prod_{i=1}^{j-1}(\Lambda_1)_i)$ can be extended to Grassmannian Cartan connection, and from the proof of 4.4 its base space K is also given by the product $L\times\prod_{i=1}^{j-1}PL(k_i)$. Thus the base space obtained by successive castling transformations admitting a Grassmannian structure is also given by a product Lie group. Furthermore if the given projective structure $(\mathcal{L}(L),[\chi])$ on L is left invariant (resp. flat) under the group action of L, then a projective structure on N is also left invariant (resp. flat), and the Grassmannian Cartan connection over K is also left invariant (resp. flat).

6. A classification of manifolds obtained by successive castling transformations

Let j,l and α be positive natural numbers such that $\alpha\leq l-\alpha$. We consider the equation $(*):(l-\alpha)\alpha+k_1^2+k_2^2+\cdots+k_j^2-j=lk_1\cdots k_j-1$. We define a castling transformation for a set of positive natural numbers.

Definition 6.1. Let l be a fixed integer such that $l \geq 3$. Let $(k_1, k_2, \ldots, k_j)$ be a set of integers. We define an integer by $k_i' := lk_1 \cdots k_{i-1}k_{i+1} \cdots k_j - k_i$ for $j \geq 2$, $1 \leq i \leq j$, and by $k_{j+1}' := lk_1 \cdot \cdots \cdot k_j - 1$. If j = 1, we define k_1' to be $l - k_1$. We call the set $(k_1, \ldots, k_{i-1}, k_i', k_{i+1}, \ldots, k_j)$ *a castling transform of* $(k_1, k_2, \ldots, k_j)$ *at* i*-th position*, and $(k_1, \ldots, k_j, k_{j+1}')$ *a castling transform at* $(j+1)$*-th position.* We call each k_i' and k_{j+1}' *a number obtained by castling transform.*

If k_i is a positive natural number for $1 \leq i \leq j$ and $(k_1, \ldots, k_j)$ satisfies the equation $(*) : (l - \alpha)\alpha + k_1^2 + k_2^2 + \cdots + k_j^2 - j = lk_1 \cdots k_j - 1$, then castling transform at any position gives another solution $(k_1, \ldots, k_{i-1}, k_i', k_{i+1}, \ldots, k_j)$ for the equation $(*)$ and k_i' is a positive natural number again.

We observe that when $j = 1$, α gives a solution for the equation $(*)$: $(l - \alpha)\alpha + k_1^2 - lk_1 = 0$. We investigate the whole solutions of $(*)$ given by the successive castling transformations from α.

If we repeat a castling transformation for $(k_1, k_2, \ldots, k_j)$ at the same position twice, then we obtain the same set as $(k_1, k_2, \ldots, k_j)$. From now on we assume that successive castling transformations does not include this repetition. Namely if a set $\theta := (k_1, k_2, \ldots, k_j)$ is a castling transform of a set θ' at i-th position, then we only consider a castling transform of θ at m-th position with $m \neq i$. A sequence $\theta_1 \to \cdots \to \theta_n$ obtained by successive castling transformations from θ_1 is said to be reduced if the sequence does not contain any repetition of castling transformation.

Lemma 6.2. *Let $\theta_1 \to \cdots \to \theta_n$ be a reduced sequence obtained by successive castling transformations and assume that $\theta_n = (k_1, k_2, \ldots, k_j)$ and $\theta_1 = \alpha$. If k_j is a number obtained in the castling transformation $\theta_{n-1} \to \theta_n$, then k_j is the unique largest number in $(k_1, k_2, \ldots, k_j)$.*

Proof. The proof is by induction on m $(n \geq m \geq 2)$. We can express θ_m as $(h_1, h_2, \ldots, h_{N_m})$, where $1 \leq N_m \leq m$ and $h_i \geq 2$ $(1 \leq i \leq N_m)$. We assume that h_{N_m} is a number obtained by a castling transform of θ_{m-1} and h_{N_m} is the largest number in $\{h_i\}_{1 \leq i \leq N_m}$. Then θ_{m+1} is a castling transform of θ_m at some i-th position with the condition $i \neq N_m$. A new number κ of θ_{m+1} obtained by castling transform is written as (1) $lh_1 \cdots h_{i-1}h_{i+1} \cdots h_{N_m} - h_i$ or (2) $lh_1 \cdots h_{N_m} - 1$, and in both cases $\kappa > h_{N_m}$. The set θ_{m+1} can be written as $(h_1, \ldots, h_{i-1}, \kappa, h_{i+1}, \ldots, h_{N_m})$ in the case (1) and $(h_1, \ldots, h_{N_m}, \kappa)$ in the case (2) . Hence κ is the unique largest number in θ_{m+1}. When $m = 2$, θ_2 can be $(\alpha, l\alpha - 1)$ or $l - \alpha$. In both cases a number obtained by castling transform is the unique largest number in θ_2. Hence the induction proves the lemma. ∎

Proposition 6.3. *Let $(k_1, k_2, \ldots, k_j)$ be a set obtained by successive castling transformations from α. Then we have $k_i \geq \alpha$ for $1 \leq i \leq j$.*

Proof. Let $\theta_1 \to \cdots \to \theta_n$ be a reduced sequence obtained by successive castling transformations from $\theta_1 = \alpha$. Denote by κ_i $(i \geq 2)$ a new number of θ_i obtained by the castling transform of θ_{i-1}. Set $\kappa_1 := \alpha$. For instance $\theta_2 = (\kappa_1, \kappa_2) = (\alpha, l\kappa_1 - 1)$, and $\theta_3 = (\kappa_1, \kappa_2, \kappa_3) = (\alpha, l\kappa_1 - 1, l\kappa_1\kappa_2 - 1)$ and $\theta_4 = (\kappa_1, \kappa_4, \kappa_3) = (\alpha, l\kappa_1\kappa_3 - \kappa_2, l\kappa_1\kappa_2 - 1)$. Thus we see that θ_m $(m \geq 2)$ can be written as $(\kappa_{i_1}, \kappa_{i_2}, \cdots, \kappa_{i_{N_m}})$ where $N_m \leq m$ and some κ_{i_l} is equal to κ_m. Then

κ_{m+1} $(m \geq 2)$ is equal to $l\kappa_{i_1} \cdots \kappa_{i_{s-1}} \kappa_{i_{s+1}} \cdots \kappa_{i_{N_m}} - \kappa_{i_s}$ for some s $(1 \leq s \leq N_m)$ satisfying $i_s \neq m$ or $l\kappa_{i_1} \cdots \kappa_{i_{N_m}} - 1$ since successive castling transformations do not include a repetition. By Lemma 6.2 κ_m is the largest number in θ_m, thus $\kappa_{m+1} > \kappa_m$ for $m \geq 2$. By definition κ_2 can be $l - \alpha$ or $l\alpha - 1$. From assumption $l - \alpha \geq \alpha$. Thus $\kappa_2 \geq \alpha$. Hence $\alpha = \kappa_1 \leq \kappa_2 < \kappa_3 < \cdots < \kappa_n$, which proves the proposition. ■

From the proof of this proposition, we see that for any given reduced sequence $\theta_1 \to \cdots \to \theta_n$ of successive castling transformations from $\theta_1 = \alpha$, we have $\theta_i \neq \theta_j$ if $i \neq j$ and $i, j \geq 2$. The following is important in this section.

Proposition 6.4. *Let j, l and α be positive natural numbers such that $j \geq 2$, $l \geq 3$ and $\alpha \leq l - \alpha$. Let k_i $(1 \leq i \leq j)$ be natural numbers such that $2 \leq k_1 \leq k_2 \leq \cdots \leq k_j$. Assume that we have the equality $(*)$: $(l-\alpha)\alpha + k_1^2 + k_2^2 + \cdots + k_j^2 - j = lk_1 \cdots k_j - 1$. Moreover assume that $k_j \geq \alpha$. Then we have $0 < lk_1 k_2 \cdots k_{j-1} - k_j < k_j$.*

Proof. Put $h_j := lk_1k_2 \cdots k_{j-1} - k_j$. Suppose that we have the equality $(*)$: $(l-\alpha)\alpha + k_1^2 + k_2^2 + \cdots + k_j^2 - j = lk_1 \cdots k_j - 1$ and the inequality $2 \leq k_1 \leq k_2 \leq \cdots \leq k_j$. Now we assume $lk_1k_2 \cdots k_{j-1} - k_j \leq 0$. Then $lk_1k_2 \cdots k_{j-1}k_j \leq k_j^2$. From $(*)$ we have $(l-\alpha)\alpha + 1 + \sum_{i=1}^{j} k_i^2 - j - k_j \leq 0$. Thus

$$\begin{aligned} 0 &\geq (l-\alpha)\alpha + 1 - j + k_1^2 + \cdots k_{j-1}^2 \\ &\geq (l-\alpha)\alpha + 1 - j + 4(j-1) \\ &= 3j + (l-\alpha)\alpha - 3 \geq 3j - 1. \end{aligned}$$

Since $j \geq 2$, the last expression is greater than or equal to 0. This is a contradiction. Hence $0 < lk_1k_2 \cdots k_{j-1} - k_j$.

Next we divide the proof into the two cases: $\alpha = 1$ and $\alpha \geq 2$. Firstly we consider the case $\alpha \geq 2$. Then l must satisfy $l \geq 4$. Now we prove that if $\alpha \leq k_j \leq l2^{j-1} - \alpha$, and $2 \leq k_i \leq l2^{j-1} - \alpha$ for $1 \leq i \leq j-1$, then we have $\sum_{i=1}^{j} {k_i}^2 - l\prod_{i=1}^{j} k_i \leq 4(j-1) + \alpha^2 - l2^{j-1}\alpha$. We prove this by using the idea and technique of the proof of [SK, Lemma 2 in p.42]. Put $b := l2^{j-1} - \alpha$, and assume $\alpha \leq k_j \leq b$ and $2 \leq k_i \leq b$ for $1 \leq i \leq j-1$. We put $f(k_1, \ldots, k_j) := \sum_{i=1}^{j} {k_i}^2 - l\prod_{i=1}^{j} k_i$. For $1 \leq \mu \leq j-1$ we set

$$M_\mu^\alpha := f(\underbrace{2, \ldots, 2}_{\mu}, \underbrace{b, \cdots, b}_{j-\mu-1}, \alpha) \text{ and } M_\mu^b := f(\underbrace{2, \ldots, 2}_{\mu}, \underbrace{b, \cdots, b}_{j-\mu-1}, l2^{j-1} - \alpha).$$

Then we have $M_\mu^\alpha = 2^2\mu + (j-\mu-1)b^2 + \alpha^2 - l2^\mu b^{j-\mu-1}\alpha$, and $M_\mu^b = 2^2\mu + (j-\mu)b^2 - l2^\mu b^{j-\mu}$. Since $M_{j-1}^\alpha - M_\mu^\alpha = (j-\mu-1)(2^2 - b^2) - l2^\mu\alpha(2^{j-\mu-1} - b^{j-\mu-1})$, we obtain

$$\begin{aligned} \frac{M_{j-1}^\alpha - M_\mu^\alpha}{b-2} &\geq -(j-\mu-1)(b+2) + l\alpha(j-\mu-1)2^{j-2} \\ &= (j-\mu-1)\{-b-2+l\alpha 2^{j-2}\} \geq 0. \end{aligned}$$

On the other hand since $M_\mu^\alpha - M_\mu^b = \alpha^2 - b^2 + l2^\mu b^{j-\mu-1}\{b-\alpha\}$, we obtain

$$\frac{M_\mu^\alpha - M_\mu^b}{b-\alpha} = -(\alpha+b) + l2^\mu b^{j-\mu-1}.$$

When $\mu = j-1$, this value is equal to zero. Thus we have $M_\mu^\alpha - M_\mu^b \geq 0$. Since f attains the maximum at the boundary points, we obtain the desired assertion $f \leq M_{j-1}^\alpha$.

Now from the equality $(*)$ we have

$$\begin{aligned} 0 &= (l-\alpha)\alpha - (j-1) + k_1^2 + k_2^2 + \cdots + k_j^2 - lk_1 \cdots k_j \\ &\leq (l-\alpha)\alpha - (j-1) + 4(j-1) + \alpha^2 - l2^{j-1}\alpha \\ &= (l-\alpha)\alpha + 3(j-1) + \alpha(\alpha - l2^{j-1}) \\ &= (l-\alpha)\alpha + 3(j-1) + \alpha(\alpha - l) - \alpha l(2^{j-1}-1) \end{aligned}$$

Since $j \geq 2, l \geq 4, \alpha \geq 2$ the last expression is negative. However this is a contradiction. It follows that from the assumption $k_j \geq \alpha$ we obtain $k_j > l2^{j-1} - \alpha$.

From now on we consider the case $j = 2$. Firstly we show that $k_2 > k_1 + \frac{l}{2}$. Assume that $k_2 \leq k_1 + \frac{l}{2}$. Then we have

$$\begin{aligned} 0 &= (l-\alpha)\alpha - 1 + k_1^2 + k_2^2 - lk_1k_2 \\ &\leq (l-\alpha)\alpha - 1 + 2k_2^2 - l(k_2 - \frac{l}{2})k_2. \end{aligned}$$

Since $k_2 > 2l - \alpha$, we have

$$\begin{aligned} 0 &< (l-\alpha)\alpha - 1 + (2-l)(2l-\alpha)^2 + \frac{l^2}{2}(2l-\alpha) \\ &= (l-\alpha)\alpha - 1 + (2l-\alpha)\{-\frac{3}{2}l^2 - 2\alpha + l(4+\alpha)\}. \end{aligned}$$

About the part of this expression we have

$$-\frac{3}{2}l^2 - 2\alpha + l(4+\alpha) = l(-\frac{3}{2}l + 4) + \alpha(l-2).$$

Since $\alpha \leq \frac{l}{2}$, the last expression is less than or equal to $l(-l+3) \leq -l$. It follows that $(l-\alpha)\alpha + (2l-\alpha)(-l) < 0$. This is a contradiction. Therefore $k_2 > k_1 + \frac{l}{2}$.

Finally assume that $h_2 \geq k_2$. This condition is equivalent to $lk_1 \geq 2k_2$. Then from the equation $(*)$ we have

$$\begin{aligned} 0 &= (l-\alpha)\alpha - 1 + k_1^2 + k_2^2 - lk_1k_2 \\ &\leq (l-\alpha)\alpha - 1 + k_1^2 + k_2^2 - 2k_2^2 \\ &= (l-\alpha)\alpha - 1 + k_1^2 - k_2^2 \\ &< (l-\alpha)\alpha - 1 + k_1^2 - (k_1 + \frac{l}{2})^2 \\ &\leq (l-\alpha)\alpha - 1 - (lk_1 + \frac{l^2}{4}) \\ &\leq \frac{l^2}{4} - \frac{l^2}{4} - 1 - lk_1 < 0. \end{aligned}$$

This is a contradiction. Hence $h_2 < k_2$, which is our assertion in this Proposition.

Next we consider the case $j \geq 3$. Now assume that $k_j \leq \frac{l}{4}(j-1)k_{j-1}$. Then we have

$$
\begin{aligned}
0 &= (l-\alpha)\alpha-(j-1)+k_1^2+\cdots+k_j^2-lk_1\cdots k_j \\
&\le (l-\alpha)\alpha-(j-1)+k_1^2+\cdots+k_j^2-\frac{4}{j-1}k_1\cdots k_{j-2}k_j^2.
\end{aligned}
$$

Since we have $\frac{4}{j-1}k_1\cdots k_{j-2}\ge\frac{4}{j-1}2^{j-2}\ge j+1$, this yields

$$
(l-\alpha)\alpha-(j-1)+k_1^2+\cdots+k_j^2-\frac{4}{j-1}k_1\cdots k_{j-2}k_j^2\le(l-\alpha)\alpha-(j-1)-k_j^2.
$$

The inequality $k_j>l2^{j-1}-\alpha$ gives

$$
\begin{aligned}
0 &< (l-\alpha)\alpha-(j-1)-(l2^{j-1}-\alpha)^2 \\
&\le \frac{l^2}{4}-(j-1)-(3l)^2<0.
\end{aligned}
$$

This is a contradiction. Hence we obtain $k_j>\frac{l}{4}(j-1)k_{j-1}$.

Finally suppose that $h_j\ge k_j$. This condition is equivalent to $k_j\le\frac{l}{2}k_1\cdots k_{j-1}$. Then combining this assumption with the equation $(*)$ yields

$$
\begin{aligned}
0 &= (l-\alpha)\alpha-(j-1)+k_1^2+\cdots+k_j^2-lk_1\cdots k_j \\
&\le (l-\alpha)\alpha-(j-1)+k_1^2+\cdots+k_j^2-2k_j^2 \\
&< (l-\alpha)\alpha-(j-1)+k_1^2+\cdots+k_{j-1}^2-(\frac{l}{4})^2(j-1)^2k_{j-1}^2 \\
&\le (l-\alpha)\alpha-(j-1)-(j-1)(\frac{l^2}{4^2}(j-1)-1)k_{j-1}^2.
\end{aligned}
$$

Since $j\ge 3$ and $k_{j-1}\ge 2$, the last expression is less than or equal to

$$
\begin{aligned}
& (l-\alpha)\alpha-(j-1)-(l^2-8) \\
&= (l-\alpha)\alpha-(j-1)-(l^2-8) \\
&\le (l-\alpha)\alpha-l^2+6 \\
&= -l(l-\alpha)-\alpha^2+6<0.
\end{aligned}
$$

This is a contradiction, which concludes $h_j<k_j$.

Now we consider the case $\alpha=1$. In this case we have to also consider the case $l=3$. We can prove the inequality $h_j<k_j$ by almost the same way as the case $\alpha\ge 2$. In the following we give the outline of the proof with emphasizing the difference between the cases $\alpha=1$ and $\alpha\ge 2$.

Firstly if we assume $2\le k_i\le l2^{j-1}-2$ for $1\le i\le j$, then from Lemma 2 of [SK, p. 42] we can directly obtain $\sum_{i=1}^{j}{k_i}^2-l\prod_{i=1}^{j}{k_i}^2\le 4j-l2^j$. Combining this inequality with the equation $(*)$ implies a contradiction by a similar argument to the case $\alpha\ge 2$. Hence we obtain $k_j>l2^{j-1}-2$. Now we divide the proof into the two cases $j=2$ and $j\ge 3$. When $j=2$, by using the inequality $k_j>2l-2$ we can prove $k_2>k_1+\frac{l}{3}$ similarly to the case $\alpha\ge 2$, but not $k_2>k_1+\frac{l}{2}$. Moreover by using $k_2>k_1+\frac{l}{3}$, we can obtain $h_2<k_2$. When $j\ge 3$, by using the inequality $k_j>l2^{j-1}-2$, we can obtain $k_j>\frac{l}{4}(j-1)k_{j-1}$. Moreover if we suppose $h_j\ge k_j$, then combining $k_j>\frac{l}{4}(j-1)k_{j-1}$ with the equation $(*)$ yields a contradiction. Thus we obtain $h_j<k_j$. ∎

The proof of Proposition 6.4 is a generalization of the one of [Kat, Lemma 7.3]. By Proposition 6.3 and 6.4 we obtain the following:

Theorem 6.5. *Let l and α be positive natural numbers such that $l \geq 3$ and $\alpha \leq l-\alpha$. Let $\theta = (k_1, k_2, \dots, k_j)$ be a set of positive natural numbers. Then θ is obtained by a finite number of castling transformations from α if and only if θ gives a solution of the equation $(*)$: $(l-\alpha)\alpha-(j-1)+k_1^2+\cdots+k_j^2-lk_1\cdots k_j = 0$ and satisfies $k_i \geq \alpha$ for $1 \leq i \leq j$.*

Furthermore Proposition 6.4 implies a stronger result: suppose that a set of positive natural numbers $\theta = (k_1,\ k_2,\ \dots,\ k_j)$ gives a solution of the equation $(*)$ and θ is not contained in the cube $C_\alpha^j = \{(x_1, \dots, x_j) \mid |x_i| \leq \alpha - 1\}$. Then θ is obtained by a finite number of castling transformations from the solution α of $(*)$ with $j = 1$. Thus we obtain the following.

Proposition 6.6. *Assume that there exists no positive integer solutions $(k_1, \dots, k_j)$ of the equation $(*)$ satisfying $k_i \leq \alpha - 1$. Then any positive integer solution of $(*)$ is obtained by a finite number of castling transformations from the solution α.*

From many computations of the equation $(*)$, it seems that there exists no positive integer solution $\theta = (k_1, k_2, \dots, k_j)$ such that θ is contained in the cube C_α^j. Hence we conjecture the following.

Conjecture. *There exists no positive integer solutions $(k_1, \dots, k_j)$ of the equation $(*)$ satisfying $k_i \leq \alpha - 1$ for $1 \leq i \leq j$.*

This conjecture is true for $\alpha = 1, 2, 3$ or $j = 1$.

Now let (Q, ω) be a Grassmannian Cartan connection of type (β, α) over a manifold M, where we assume $l = \alpha + \beta \geq 3$ and $\alpha \leq \beta$. Denote by $S(l, \alpha)$ the set of positive natural number solutions of the equation $(*)$. We consider the tree T of Cartan connections obtained by successive castling transformations from (Q, ω). Each node is written as $(Q \times \prod_{i=1}^j PL(k_i), \omega' \times \prod_{i=1}^j (\Lambda_1)_i)$, where $\omega' = \omega$ or $*\omega$ and $(\Lambda_1)_i = \Lambda_1$ or $(\Lambda_1)_i = *\Lambda_1$. The model space of $(Q \times \prod_{i=}^j PL(k_i), \omega' \times \prod_{i=1}^j (\Lambda_1)_i)$ is $PL(l) \times \prod_{i=1}^j PL(k_i)/PL(l) \times \prod_{i=1}^j PL(k_i)_w$, where w is a point of $P(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^j \boldsymbol{R}^{k_i})$. We define the map $\Phi : T \to S(l, \alpha)$ by $(Q \times \prod_{i=1}^j PL(k_i), \omega' \times \prod_{i=1}^j (\Lambda_1)_i) \mapsto (k_1, \dots, k_j)$. Each node of T induces a base space $M_{k_1 \times \cdots \times k_j}$ by Proposition 4.4. Thus we obtain the map $\overline{\Phi}$ from the set of the base spaces of nodes in T to $S(l, \alpha)$ defined by $\overline{\Phi} : M_{k_1 \times \cdots \times k_j} \mapsto (k_1, \dots, k_j)$. Moreover $\overline{\Phi}$ induces the map Ψ from the set of the fibers $\prod_{i=1}^j PL(k_i)$ of base spaces of nodes in T to $S(l, \alpha)$. The map Ψ is bijective from Theorem 6.5. Thus we obtain the following.

Theorem 6.7. *There is a one-to-one correspondence between the set of structure groups $\prod_{i=1}^j PL(k_i)$ of the base spaces obtained by a finite number of castling transformations from (Q, ω) and the set of solutions $(k_1, \dots, k_j)$ of the equation*

$$(*)\ \alpha\beta + k_1^2 + \cdots + k_j^2 - (j-1) - (\alpha+\beta)k_1 \cdots k_j = 0.$$

satisfying $k_i \geq \alpha$ $(1 \leq i \leq j)$ and $j \geq 1$.

Each solution $(k_1, \dots, k_j)$ corresponds to a manifold equipped with a projective structure, which is projectively flat if (Q, ω) is flat.

From this theorem we obtain Theorem 1.1.

Remark 6.8. For a principal fiber bundle N over M corresponding to the solution $(k_1, \dots, k_j)$, a bundle $L = N/PL(k_j)$ is equipped with a projective structure again if $lk_1 \cdots k_{j-1} - k_j = 1$ and this manifold corresponds to a solution $(k_1, \cdots, k_{j-1})$ of the equation $(*)$: $\alpha\beta - (j-2) + k_1^2 + \cdots + k_{j-1}^2 - lk_1 \cdots k_{j-1} = 0$. Indeed the set of numbers $(k_1, \dots, k_{j-1}, lk_1 \cdots k_{j-1} - k_j)$ also gives a solution of $(*)$. If we have $lk_1 \cdots k_{j-1} - k_j = 1$, then $(k_1, \cdots, k_{j-1}, 1)$ is a solution of $(*)$. Thus $(k_1, \cdots, k_{j-1})$ is a solution of $(*)$. If we have $lk_1 \cdots k_{j-1} - k_j \neq 1$, then L admits a Grassmannian structure of type $(lk_1 \cdots k_{j-1} - k_j, k_j)$, whose corresponding Grassmannian Cartan connection is flat if (Q, ω) is flat.

Remark 6.9. The equation $(*)$ $\alpha\beta + k_1^2 + \cdots + k_j^2 - (j-1) - (\alpha+\beta)k_1 \cdots k_j = 0$ is a generalization of the equation $(**)$ $a^2 + k_1^2 + \cdots k_j^2 - j - 2ak_1 \cdots k_j = 0$ ($a = 2$, 3 or 5) which we obtained in our preceding paper [Kat, Theorem1.1]. Indeed put $\alpha := a - 1$ and $\beta := a + 1$ in $(*)$. Then we obtain $(**)$.

7. Description of flat projective structures

Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be a flat Grassmannian structure on M, and (Q, ω) be the corresponding flat Grassmannian Cartan connection on M. Then by a finite number of castling transformations, we obtain a manifold N corresponding to the solution $(k_1, \dots, k_j)$ in Theorem 6.7. The manifold N is equipped with a Cartan connection

$$(Q \times \textstyle\prod_{i=1}^j PL(k_i), \omega' \times \prod_{i=1}^j (\Lambda_1)_i) \text{ of type}$$
$$PL(l) \times \textstyle\prod_{i=1}^j PL(k_i)/PL(l) \times \prod_{i=1}^j PL(k_i)_w,$$

where w is an element of $V_{k_j, lk_1 \cdots k_{j-1}} \subset P(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^j \boldsymbol{R}^{k_i})$ obtained in the process of successive castling transformations. Then N admits a flat projective structure. We also showed that N is a principal fiber bundle over M with group $\prod_{i=1}^j PL(k_i)$. Furthermore in Proposition 4.4 we described the relation of the base spaces corresponding to the solutions of $(*)$.

Now finally we shall describe the flat projective structure on N by using the flat Grassmannian structure $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ on M. For each connected component C of the nonempty intersection $U_\alpha \cap U_\beta$, the coordinate change $\varphi_\beta \circ \varphi_\alpha^{-1}|_C$ is given by an element $\tau(C; \beta, \alpha)$ of $PL(l)$. Let $\pi_Q : Q \to M$ and $\pi : PL(l) \to Gr_{m,l}$ be the projections. Denote by $\widetilde{U}_\alpha$ the open subset $\pi^{-1}(\phi_\alpha(U_\alpha))$ of $PL(l)$. Then $\widetilde{U}_\alpha$ is naturally regarded as a principal fiber bundle over U_α and we denote the projection $\widetilde{U}_\alpha \to U_\alpha$ by π_α. Put $\widetilde{U}'_\alpha := \widetilde{U}_\alpha$, $\pi'_\alpha := \pi_\alpha$ and $\tau(C; \beta, \alpha)' = \tau(C; \beta, \alpha)$ if $\omega' = \omega$ or $\widetilde{U}'_\alpha := *\widetilde{U}_\alpha$, $\pi'_\alpha := \pi_\alpha \circ *$ and $\tau(C; \beta, \alpha)' = *\tau(C; \beta, \alpha)$ if $\omega' = *\omega$.

Theorem 7.1. *The manifold N is diffeomorphic to a patchwork of the open submanifolds $\widetilde{U}'_\alpha \otimes \bigotimes_{i=1}^j PL(k_i).w$ of the projective space $P(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^j \boldsymbol{R}^{k_i})$:*

$$N \simeq \bigsqcup_{\alpha \in A} \widetilde{U}'_\alpha \otimes \bigotimes_{i=1}^{j} PL(k_i).w/_\sim.$$

The elements $\tilde{g}_\alpha = g \otimes A_1 \otimes \cdots \otimes A_j.w$ and $\tilde{h}_\beta = h \otimes B_1 \otimes \cdots \otimes B_j.w$ of the open submanifolds $\widetilde{U}'_\gamma \otimes \bigotimes_{i=1}^j PL(k_i).w$ ($\gamma = \alpha,\ \beta$) are identified iff

(i) $\pi'_\alpha(g) = \pi'_\beta(h)$,

(ii) $\tilde{h}_\beta = \tau(C;\beta,\alpha)' \otimes id \otimes \cdots \otimes id \cdot \tilde{g}_\alpha, \qquad (\pi'_\alpha(g) \in C)$.

Thus N admits an atlas inducing a flat projective structure, whose coordinate changes are the same as ones of the flat Grassmannian structure on M.

Proof. Let $\pi_Q : Q \to M$ and $\pi : PL(l) \to Gr_{m,l}$ be the projections. We denote by ω_G the Maurer-Cartan form of $PL(l)$. Since the Cartan connection (Q,ω) is constructed from the atlas $\{(U_\alpha,\varphi_\alpha)\}_{\alpha\in A}$, the Cartan connection $(\pi_Q^{-1}(U_\alpha),\omega)$ is isomorphic to the one $(\widetilde{U}_\alpha,\omega_G)$ via an isomorphism $\widetilde{\varphi}_\alpha$. Let $\widetilde{\varphi}_\beta$ be an isomorphism between $(\pi_Q^{-1}(U_\beta),\omega)$ and $(\widetilde{U}_\beta,\omega)$. Then the transition function between $\widetilde{\varphi}_\alpha$ and $\widetilde{\varphi}_\beta$ over a connected component C of $U_\alpha \cap U_\beta$ is given as follows: for an element z of $\pi_Q^{-1}(C)$, $\widetilde{\varphi}_\beta(z) = \tau(C;\beta,\alpha)\widetilde{\varphi}_\alpha(z)$.

The manifold $\pi_Q^{-1}(U_\alpha) \times \prod_{i=1}^j PL(k_i)$ itself is an open submanifold of $Q \times \prod_{i=1}^j PL(k_i)$ and moreover is the bundle over the open submanifold $\pi_N^{-1}(U_\alpha)$ of N, where π_N is the projection $N \to M$. we define a map $\psi_\alpha : \pi_Q^{-1}(U_\alpha) \times \prod_{i=1}^j PL(k_i) \to PL(l) \times \prod_{i=1}^j PL(k_i)$ by $(z, A_1,\ldots,A_j) \mapsto (\widetilde{\varphi}_\alpha{}'(z), A'_1,\ldots,A'_j)$, where $\widetilde{\varphi}_\alpha{}'(z) := \widetilde{\varphi}_\alpha(z)$ and $A'_i = A_i$ if $\omega' = \omega$ or $\widetilde{\varphi}_\alpha{}'(z) := *\widetilde{\varphi}_\alpha(z)$ and $A'_i = *A_i$ if $\omega' = *\omega$. Then since $\widetilde{\varphi}_\alpha^*\omega_G = \omega_\alpha$, the pullback of the Maurer-Cartan form of $PL(l) \times \prod_{i=1}^j PL(k_i)$ by ψ_α is equal to $\omega'_\alpha \times \prod_{i=1}^j(\Lambda_1)_i$. The map ψ_α is compatible with the action $PL(l) \times \prod_{i=1}^j PL(k_i)_w$. Since the homogeneous space $PL(l) \times \prod_{i=1}^j PL(k_i)/PL(l) \times \prod_{i=1}^j PL(k_i)_w$ is identified with the orbit $PL(l) \otimes \bigotimes_{i=1}^j PL(k_i).w$, ψ_α induces a map $\bar{\psi}_\alpha$ of the base spaces, which is a diffeomorphism between the open submanifold $\pi_N^{-1}(U_\alpha)$ of N and the open submanifold $\widetilde{U}'_\alpha \otimes \bigotimes_{i=1}^j PL(k_i).w$ of $P(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^j \boldsymbol{R}^{k_i})$.

On the intersection $\pi_Q^{-1}(C) \times \prod_{i=1}^j PL(k_i)$ the transition function between ψ_α and ψ_β is given as follows: $\psi_\beta \circ \psi_\alpha^{-1} = \tau(C;\beta,\alpha)' \times id \times \cdots \times id$. Thus concerning the base spaces the coordinate change $\bar{\psi}_\beta \circ \bar{\psi}_\alpha^{-1}$ over $\pi_N^{-1}(C)$ is given by $\tau(C;\beta,\alpha)' \otimes id \otimes \cdots \otimes id$. Hence the atlas $\{(\pi_N^{-1}(U_\alpha),\bar{\psi}_\alpha)\}_{\alpha\in A}$ of N, which is induced form the Cartan connection $(Q \times \prod_{i=1}^j PL(k_i), \omega' \times \prod_{i=1}^j(\Lambda_1)_i)$, naturally gives an atlas of a flat projective structure on N via the inclusion $PL(l) \times \prod_{i=1}^j PL(k_i) \to PL(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^j \boldsymbol{R}^{k_i})$. ∎

We constructed a flat projective structure $\mathscr{A}$ on N in Theorem 7.1, which is induced from the flat Cartan connection $(Q \times \prod_{i=1}^j PL(k_i), \omega' \times \prod_{i=1}^j(\Lambda_1)_i)$. On the other hand $(Q \times \prod_{i=1}^j PL(k_i), \omega' \times \prod_{i=1}^j(\Lambda_1)_i)$ induces a flat projective Cartan connection (P,ξ) by Proposition 2.1. The Cartan connection (P,ξ) induces a flat projective structure $\mathscr{B}$, which is the same as the one stated in Theorem 6.7 and thus Theorem 1.1. We say that two atlases are equivalent if they are compatible. Finally we shall prove the following:

Proposition 7.2. *The flat projective structures $\mathscr{A}$ and $\mathscr{B}$ on N are equivalent.*

Proof. Firstly we note that the model space

$$PL(l) \times \prod_{i=1}^j PL(k_i)/PL(l) \times \prod_{i=1}^j PL(k_i)_w$$

is a subgeometry of

$$PL(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^j \boldsymbol{R}^{k_i})/PL(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^j \boldsymbol{R}^{k_i})_w.$$

We now generally consider a homogeneous space A/B which is a subgeometry of A'/B'. Thus there is the inclusion F of A into A' and F induces a diffeomorphism $\hat{F}$ of A/B onto an open subset of A'/B'. Denote by ω_A the Maurer-Cartan form of A, then $(\pi : A \to A/B, \omega_A)$ gives the standard flat Cartan connection on A/B. Likewise we obtain the flat Cartan connection $(\pi' : A' \to A'/B', \omega_{A'})$. Then we have $dF \circ \omega_A = F^* \omega_{A'}$. Let (Q, ω) (resp. (Q', ω')) be a flat Cartan connection of type A/B (resp. A'/B') on a manifold N, and assume that (Q, ω) is a subgeometry of (Q', ω') defined by a bundle homomorphism $\tilde{F} : Q \to Q'$. We consider an atlas $\mathscr{C}$ of $(A, A/B)$-structure on N induced from (Q, ω) and an atlas $\mathscr{D}$ of $(A', A'/B')$-structure on N induced from (Q', ω') (See [Kat] for the terminology). Hence any chart of $\mathscr{C}$ is constructed as follows: For arbitrary point $p \in N$, there exists a neighbourhood U of p in N and a bundle isomorphism $f : \pi_Q^{-1}(U) \to V$, where V is an open subset of A, such that $f^* \omega_A = \omega$. Then f induces a diffeomorphism $\bar{f} : U \to \pi(V)$ and we obtain the chart $(U, \bar{f})$ belonging to $\mathscr{C}$. Likewise there exists an isomorphism $f' : \pi_{Q'}^{-1}(U') \to V'$ around p, where V' is an open subset of A', and f' induces a diffeomorphism $\bar{f}' : U' \to \pi'(V')$. Note that $\hat{F} \circ \bar{f}$ gives a diffeomorphism of U onto the open subset $\hat{F} \circ \pi(V)$ of A'/B'. Thus we obtain two charts $(U, \hat{F} \circ \bar{f})$ and $(U', \bar{f}')$ of type A'/B'.

Now we compare the composite $f' \circ \widetilde{F} \circ f^{-1}$ and F. Since (Q, ω) is a subgeometry of (Q', ω'), we have the equality

$$(f' \circ \widetilde{F} \circ f^{-1})^* \omega_{A'} = dF \circ \omega_A = F^* \omega_{A'}. \tag{7.1}$$

Let C be a connected component of $U \cap U'$. Then $\bar{f}(C)$ is a connected open subset of A/B. We consider the inverse image $\pi^{-1}(\bar{f}(C))$ and decompose it into the connected components $\{D_\gamma\}_{\gamma \in \Gamma}$. Since there exists a connection in the principal fiber bundle $\pi : A \to A/B$, it can be shown that each D_γ is mapped onto $\bar{f}(C)$ by π. Moreover from the equality (7.1) there exists a unique element a_γ of A' for each γ such that $f' \circ \widetilde{F} \circ f^{-1} = a_\gamma \cdot F$ on D_γ (cf. Theorem 1.2.4 of [ČS]). We can prove that all the elements of $\{a_\gamma\}_{\gamma \in \Gamma}$ are the same, and we put $a' := a_\gamma$. Thus on $\pi^{-1}(\bar{f}(C))$ we have $f' \circ \widetilde{F} \circ f^{-1} = a' \cdot F$. Hence concerning two coordinates $\hat{F} \circ \bar{f}$ and $\bar{f}'$ of type A'/B', the coordinate change $\bar{f}' \circ (\hat{F} \circ \bar{f})^{-1}$ on C is given by the translation of a'. Let $\mathscr{C}'$ be an atlas of $(A', A'/B')$-structure on N given rise to by $\mathscr{C}$ via F. Then from the above discussion it follows that $\mathscr{C}'$ is equivalent to $\mathscr{D}$.

By applying this result to the case that

$$PL(l) \times \prod_{i=1}^{j} PL(k_i) / PL(l) \times \prod_{i=1}^{j} PL(k_i)_w$$

is a subgeometry of

$$PL(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j} \boldsymbol{R}^{k_i}) / PL(\boldsymbol{R}^l \otimes \bigotimes_{i=1}^{j} \boldsymbol{R}^{k_i})_w,$$

we obtain the assertion of this proposition. ∎

Acknowledgments. The author expresses his gratitude to Thomas Bruun Madsen for his encouragement. Thanks are due to the referee for the helpful comments, especially Theorem 7.1 was made from the referee's suggestion. The author wishes to thank the Osaka City University Advanced Mathematical Institute and the King's College London for financial support and hospitality.

References

[Aga] Agaoka, Y., *Invariant flat projective structures on homogeneous spaces*, Hokkaido Math. J. **11** (1982), 125–172.

[ČS] Čap, A., and J. Slovák, "Parabolic Geometries I, Background and General Theory," Math. Surveys Monogr. **154**, Amer. Math. Soc. (2009).

[Gol1] Goldman, W. M., *Geometric structures on manifolds and varieties of representations*, Geometry of Group Representation, Contemporary Math. **74** (1998), 169–198.

[Gol2] —, *Locally homogeneous geometric manifolds*, International Congress of Mathematicians **II** (2010), 717–744.

[GC] Goldman, W. M., and D. Cooper, *A 3-manifold with no Real Projective Structure*, arXiv:1207.2007v1 (2012).

[Han] Hangan, T., *Tensor-product tangent bundles*, Arch. Math. **19** (1968), 436–440.

[Ish] Ishihara, T., *On tensor-product structures and Grassmannian structures*, J. Math. Tokushima Univ. **4** (1970), 1–17.

[Kat] Kato, H., *Left invariant flat projective structures on Lie groups and prehomogeneous vector spaces*, Hiroshima Math. J. **42** (2012), 1–35.

[KN] Kobayashi, S. and T. Nagano, *On projective connections*, J. Math. Mech. **13** (1964), 215–235.

[MS] Machida, Y. and H. Sato, *Twistor theory of manifolds with Grassmannian structures*, Nagoya Math. J. **160** (2000), 17–102.

[NS] Nomizu, K. and T. Sasaki, "Affine differential geometry. Geometry of affine immersions," Cambridge University Press, 1994.

[OT] Ovsienko, V., and S. Tabachnikov, "Projective differential Geometry Old and New," Cambridge University Press, 2005.

[SK] Sato, M., and T. Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–155.

[Tan1] Tanaka, N., *Projective connections and projective transformations*, Nagoya Math. J. **12** (1957), 1–24.

[Tan2] —, *On the equivalence problems associated with a certain class of homogeneous spaces*, J. Math. Soc. Japan **17** (1965), 103–139.

[Tan3] —, *On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections*, Japan. J. Math. (N.S.) **2** (1976), 131–190.

Hironao Kato
Osaka City University
Advanced Mathematical Institute
3-3-138 Sugimoto, Sumiyoshi-ku
Osaka 558-8585 Japan
katohiro@sci.osaka-cu.ac.jp
and
Department of Mathematics
King's College London
Strand, London WC2R 2LS, United Kingdom
hironao.kato@kcl.ac.uk

Received January 23, 2013
and in final form April 23, 2013