Journal of Lie Theory
Volume **23** (2013) 507–549

Generalized Multiplicity-Free Representations of Nongraded Divergence-Free Lie Algebras

Ling Chen

Communicated by K.-H. Neeb

Abstract. Divergence-free Lie algebras originate from the Lie algebras of volume-preserving transformation groups. Xu constructed a certain nongraded generalization, which do not contain any toral Cartan subalgebra in generic case. In this paper, we give a complete classification of the generalized weight modules over these algebras with weight multiplicities less than or equal to one.
Mathematics Subject Classification 2010: 17B10, 17B65.
Key Words and Phrases: Divergence-free Lie algebra, Cartan type, irreducible module, generalized weight module, classification.

1. Introduction

The Lie algebras of volume-preserving transformation groups consist of divergence-free first-order differential operators. They are called the special Lie algebras of Cartan type, or divergence-free Lie algebras (see [24]). These algebras play important roles in geometry, Lie groups and dynamics. Graded generalizations of them were constructed and studied by Kac [5], Osborn [14], Djokovic and Zhao [4], and Zhao [31]. Bergen and Passman [1] gave a certain characterization of graded special Lie algebras. Kac also investigated suppersymmetric graded generalizations of them [6, 7]. Xu [28] introduced nongraded generalizations of the Lie algebras of Cartan type and determined their simplicity. An important feature of the nongraded Lie algebras of Cartan type is that they do not contain any toral Cartan subalgebra in generic case. The aim of this work is to study the generalized weight representations of Xu's nongraded divergence-free Lie algebras. The isomorphism classes of nongraded divergence-free Lie algebras were determined by Su and Xu [24].

The representations of the Lie algebras of Cartan type were vastly studied. Shen [19–21] studied mixed product of graded modules over the graded Lie algebras of Cartan type, and obtained certain irreducible modules over a field with characteristic p. Penkov and Serganova [16] gave an explicit description of the support of an arbitrary irreducible weight module over the classical Witt algebra, as well as its subalgebra with constant divergence. Rao [17, 18] investigated the

ISSN 0949–5932 / $2.50

irreducibility of the weight modules over the derivation Lie algebra of the algebra of Laurent polynomials virtually constructed by Shen [19]. Zhao [33] determined the module structure of Shen's mixed product over Xu's nongraded Lie algebras of Witt type (cf. [28]). Moreover, she [35] obtained a composition series for a family of modules with parameters over Xu's nongraded Hamiltonian Lie algebras (cf. [28]). These modules are constructed from finite-dimensional multiplicity-free irreducible modules of symplectic Lie algebras by Shen's mixed product.

Starting from Kac's conjecture [8, 9] on irreducible representations of the Virasoro algebra (which can also be viewed as a central extension of the rank one classical Witt algebra), Kaplansky [10, 11] and Santharoubane [11] gave classifications of multiplicity-free representations of classical Virasoro algebras. Later, Mathieu [12] proved that any Harish-Chandra module over the Virasoro algebra is a highest weight module, a lowest weight module or a multiplicity-free module, which confirms Kac's conjecture. Su [22, 23] generalized Kaplansky and Santharoubane's result to multiplicity-free modules over high rank Virasoro algebras and super-Virasoro algebras. Based on the classifications of multiplicity-free representations of generalized Virasoro algebras in [27], Zhao [32] classified the multiplicity-free representations of graded generalized Witt algebras. Su and Zhou [26] then further generalized Zhao's result [32] to generalized weight modules over the nongraded Witt algebras introduced by Xu [28].

Motivated by Kaplansky and Santharoubane's works [10, 11], we [2] gave a complete classification on multiplicity-free representations of graded divergence-free Lie algebras introduced by Djokovic and Zhao [4]. Based on this result, we present in this paper a complete classification of the irreducible and indecomposable multiplicity-free generalized weight modules over nongraded divergence-free Lie algebras introduced by Xu [28].

Throughout this paper, we let $\mathbb{F}$ be an algebraically closed field of characteristic zero. All the vector spaces are assumed over $\mathbb{F}$. Denote by $\mathbb{N}$ the set of nonnegative integers $\{0, 1, 2, \cdots\}$ and by $\mathbb{Z}_+$ the set of positive integers. A linear transformation T of a vector space V is called *locally finite* if $\dim(\sum_{n=0}^{\infty} \mathbb{F}T^n(v))$ is finite for any $v \in V$. Moreover, T is called *locally nilpotent*, if for any $u \in V$, there exists a positive integer m such that $T^m(u) = 0$. A subspace U of $\mathrm{End}V$ is called *locally finite* if each element of U is locally finite.

Let $\mathscr{A}$ be an associative commutative algebra with an identity element. Let $\mathscr{D}$ be a nonzero finite-dimensional vector space of commuting locally-finite derivations of $\mathscr{A}$ such that $\bigcap_{\partial\in\mathscr{D}} \ker \partial = \mathbb{F}$. Moreover, we suppose that $\mathscr{A}$ is derivation-simple with respect to $\mathscr{D}$; that is, there does not exist a subspace $\mathscr{I}$ of $\mathscr{A}$ such that $\mathscr{I} \neq \{0\}, \mathscr{A}$ and

$$u\mathscr{I}, \partial(\mathscr{I}) \subseteq \mathscr{I} \quad \text{for } u \in \mathscr{A}, \partial \in \mathscr{D}. \tag{1}$$

Su, Xu and Zhang [25] proved that the derivation subspace $\mathscr{W} = \mathscr{A}\mathscr{D}$ with the bracket

$$[u\partial, v\partial'] = u\partial(v)\partial' - v\partial'(u)\partial \quad \text{for } u, v \in \mathscr{A},\ \partial, \partial' \in \mathscr{D}, \tag{2}$$

is isomorphic to the simple generalized Witt algebra constructed by Xu [28] (also cf. [15] for the relation between the derivation-simplicity and the simplicity of $\mathscr{W}$).

Let $\{\partial_1, \partial_2, \cdots, \partial_n\}$ be a basis of $\mathscr{D}$. Define the divergence on $\mathscr{W}$ by

$$\mathrm{div}\partial = \sum_{i=1}^{n} \partial_i(u_i) \quad \text{for } \partial = \sum_{i=1}^{n} u_i \partial_i \in \mathscr{W}. \tag{3}$$

It is clear that the divergence is independent of the choice of $\{\partial_1, \partial_2, \cdots, \partial_n\}$. The subspace

$$\mathscr{S} = \{\partial \in \mathscr{W} \mid \mathrm{div}\partial = 0\} \tag{4}$$

is a divergence-free Lie algebra studied by Xu [28].

Note that $\mathscr{D} \subset \mathscr{S}$. Let $\mathcal{V}$ be an $\mathscr{S}$-module. We call $\mathcal{V}$ a *generalized weight module* if $\mathscr{D}|_{\mathcal{V}}$ is locally finite. For a generalized weight module $\mathcal{V}$ and a linear function λ of $\mathscr{D}$, we put

$$\mathcal{V}_\lambda = \{v \in \mathcal{V} \mid (\partial - \lambda(\partial))^m(v) = 0 \text{ for } \partial \in \mathscr{D} \text{ and some } m \in \mathbb{Z}_+\}, \tag{5}$$

$$\mathcal{V}_\lambda^{(k)} = \{v \in \mathcal{V} \mid (\partial - \lambda(\partial))^{k+1}(v) = 0 \text{ for } \partial \in \mathscr{D}\},\ k \in \mathbb{N}. \tag{6}$$

Moreover, $\mathcal{V}$ is called *multiplicity-free* if $\dim \mathcal{V}_\lambda^{(0)} \leq 1$ for any linear function λ of $\mathscr{D}$. Set

$$\mathscr{D}_1 = \{\partial \in \mathscr{D} \mid \partial \text{ is locally nilpotent on } \mathscr{A}\}, \tag{7}$$

$$\mathscr{D}_2 = \{\partial \in \mathscr{D} \mid \partial \text{ is diagonalizable on } \mathscr{A}\}. \tag{8}$$

In [2], we classified all the irreducible and indecomposable multiplicity-free generalized weight $\mathscr{S}$-modules when $\mathscr{D} = \mathscr{D}_2$. The aim of this paper is to generalize that result to more general $\mathscr{S}$.

When $\mathscr{D} \neq \mathscr{D}_1 + \mathscr{D}_2$, the Lie algebras $\mathscr{W}$ and $\mathscr{S}$ do not contain any toral Cartan subalgebra (cf. [25, 28]). Thus they are not graded in the traditional sense. In order to generalize the result of [2] to general $\mathscr{S}$, we have to use extra techniques. Let μ be a linear function of $\mathscr{D}$. We define the $\mathscr{S}$-module $\mathscr{A}^\mu$ to be the same vector space as $\mathscr{A}$ with the action

$$\partial.v = \sum_{i=1}^{n} u_i(\partial_i + \mu(\partial_i))(v) \qquad \text{for } \partial = \sum_{i=1}^{n} u_i \partial_i \in \mathscr{S}. \tag{9}$$

The main result of this paper is:

Theorem 1.1. *Suppose that* $\dim \mathscr{D} \geq 3 + \dim \mathscr{D}_1, 3 + \dim \mathscr{D}_2$*. Any irreducible or indecomposable multiplicity-free generalized weight* $\mathscr{S}$*-module is either isomorphic to the 1-dimensional trivial module or isomorphic to the module* $\mathscr{A}^\mu$ *for some linear function* μ *of* $\mathscr{D}$*.*

A similar result for the multiplicity-free generalized weight modules over the nongraded Witt algebra $\mathscr{W}$ was proved by Su and Zhou [26]. Their definition of generalized weight module $\mathcal{V}$ in [26] is slightly different: it requires that $\mathscr{D}|_{\mathcal{V}}$ is locally finite and that $\mathscr{D}_1|_{\mathcal{V}}$ is locally nilpotent. However, their main result can be easily generalized to the generalized weight modules over which $\mathscr{D}$ is locally finite but $\mathscr{D}_1$ is not necessarily locally nilpotent.

This paper is organized as follows. In Section 2, we give some properties of the Lie algebra $\mathscr{S}$ based on the classification result of [25]. Moreover, we review our earlier classification theorem on the multiplicity-free representations of the graded divergence-free Lie algebras (cf. [2]), and give some primary results of the $\mathscr{S}$-module $\mathscr{A}^\mu$. In Sections 3 and 4, we prove the main theorem.

2. Some properties

In this section, we shall give a more explicit description of the divergence-free Lie algebra $\mathscr{S}$ based on the classification result of [25]. Moreover, we shall derive some properties of the Lie algebra $\mathscr{S}$ and the $\mathscr{S}$-module $\mathscr{A}^\mu$, which are useful for the proof of the main theorem.

First, we want to use the classification result of [25].

For any positive integer n, an additive subgroup G of $\mathbb{F}^n$ is called *nondegenerate* if G contains an $\mathbb{F}$-basis of $\mathbb{F}^n$. Let l_1, l_2 and l_3 be three nonnegative integers such that

$$l = l_1 + l_2 + l_3 > 0. \tag{10}$$

Take any nondegenerate additive subgroup Γ of $\mathbb{F}^{l_2+l_3}$ and let $\Gamma = \{0\}$ when $l_2 + l_3 = 0$. Let $\mathscr{A}(l_1, l_2, l_3; \Gamma)$ be the semi-group algebra $\mathbb{F}[\Gamma \times \mathbb{N}^{l_1+l_2}]$ with the basis

$$\{x^\alpha t^{\mathbf{i}} \mid \alpha \in \Gamma, \mathbf{i} \in \mathbb{N}^{l_1+l_2}\}. \tag{11}$$

Define a commutative associative algebraic operation "$\cdot$" on $\mathscr{A}(l_1, l_2, l_3; \Gamma)$ by

$$x^\alpha t^{\mathbf{i}} \cdot x^\beta t^{\mathbf{j}} = x^{\alpha+\beta} t^{\mathbf{i}+\mathbf{j}} \quad \text{for } \alpha, \beta \in \Gamma,\ \mathbf{i}, \mathbf{j} \in \mathbb{N}^{l_1+l_2}. \tag{12}$$

Note that $x^0 t^{\mathbf{0}}$ is the identity element, which is denoted by 1 for convenience. Moreover, we write $t^{\mathbf{i}}$ and x^α instead of $x^0 t^{\mathbf{i}}$ and $x^\alpha t^{\mathbf{0}}$ for short. When the context is clear, we omit the notation "$\cdot$" in any associative algebra product.

We view $\mathbb{F}^{l_2+l_3}$ as an imbedding subspace of $\mathbb{F}^l$ by writing

$$\alpha = (0, \cdots, 0, \alpha_{l_1+1}, \cdots, \alpha_l) \ \text{ for } \alpha \in \mathbb{F}^{l_2+l_3}. \tag{13}$$

Moreover, we write

$$\mathbf{i} = (i_1, i_2, \cdots, i_l) \ \text{ for } \mathbf{i} \in \mathbb{N}^l. \tag{14}$$

Elements in $\mathbb{N}^{l_1+l_2}$ will be denoted by (14) with $i_{l_1+l_2+1} = i_{l_1+l_2+2} = \cdots = i_l = 0$. For $m, n \in \mathbb{Z}$, we shall use the notation

$$\overline{m,n} = \{m, m+1, m+2, \ldots, n\} \text{ if } m \le n; \quad \overline{m,n} = \emptyset \text{ if } m > n. \tag{15}$$

Denote

$$k_{[p]} = (0, \cdots, 0, \overset{p}{k}, 0, \cdots, 0) \in \mathbb{N}^l \ \text{ for } k \in \mathbb{Z}_+,\ p \in \overline{1,l}. \tag{16}$$

We define linear transformations $\{\partial_1, \partial_2, \ldots, \partial_l\}$ on $\mathscr{A}(l_1, l_2, l_3; \Gamma)$ by

$$\partial_p(x^\alpha t^{\mathbf{i}}) = \alpha_p x^\alpha t^{\mathbf{i}} + i_p x^\alpha t^{\mathbf{i}-1_{[p]}} \quad \text{for } p \in \overline{1,l},\ \mathbf{i} \in \mathbb{N}^{l_1+l_2},\ \alpha \in \Gamma, \tag{17}$$

where the monomial $i_p x^\alpha t^{\mathbf{i}-1_{[p]}}$ is treated as zero when $\mathbf{i}-1_{[p]}\notin\mathbb{N}^{l_1+l_2}$. Then $\{\partial_1,\cdots,\partial_l\}$ are commuting locally-finite derivations of $\mathscr{A}(l_1,l_2,l_3;\Gamma)$, and they are $\mathbb{F}$-linearly independent. Let

$$\mathscr{W}(l_1,l_2,l_3;\Gamma)=\sum_{i=1}^{l}\mathscr{A}(l_1,l_2,l_3;\Gamma)\partial_i. \tag{18}$$

Then $\mathscr{W}(l_1,l_2,l_3;\Gamma)$ is the simple Lie algebra of nongraded Witt type constructed by Xu [28].

Let $\mathscr{A}$ and $\mathscr{D}$ be as in the introduction. Su, Xu and Zhang [25] proved that

$$\mathscr{A}\cong\mathscr{A}(l_1,l_2,l_3;\Gamma)\text{ and }\mathscr{D}=\sum_{i=1}^{l}\mathbb{F}\partial_i \tag{19}$$

for some data $(l_1,l_2,l_3;\Gamma)$. So we can and shall identify $\mathscr{A}$ with $\mathscr{A}(l_1,l_2,l_3;\Gamma)$, $\mathscr{W}$ with $\mathscr{W}(l_1,l_2,l_3;\Gamma)$, and the corresponding divergence-free Lie algebra $\mathscr{S}$ (cf. (4)) with $\mathscr{S}(l_1,l_2,l_3;\Gamma)$.

According to (7), (8) and (17), we have

$$\mathscr{D}_1=\sum_{i=1}^{l_1}\mathbb{F}\partial_i,\qquad \mathscr{D}_2=\sum_{i=l_1+l_2+1}^{l}\mathbb{F}\partial_i. \tag{20}$$

So the assumption $\dim\mathscr{D}\geq 3+\dim\mathscr{D}_1, 3+\dim\mathscr{D}_2$ in Theorem 1.1 is equivalent to $l_1+l_2\geq 3$ and $l_2+l_3\geq 3$. Now we briefly explain why this assumption is made.

Remark 2.1. The assumption $l_2+l_3\geq 3$ is made because our proof of the main theorem is based on the result of [2]. As for the assumption $l_1+l_2\geq 3$, the reasons are as follows. Firstly, $\mathscr{S}(l_1,l_2,l_3;\Gamma)$ is not simple when $l_1+l_2\leq 1$, but its derived subalgebra $\mathscr{S}'(l_1,l_2,l_3;\Gamma)$ is. For the subcase $l_1+l_2=0$, we have $\mathscr{S}(0,0,l_3;\Gamma)=\mathscr{D}\oplus\mathscr{S}'(0,0,l_3;\Gamma)$. Although there is no concept of weight module for $\mathscr{S}'(0,0,l_3;\Gamma)$, we [2] gave a classification of multiplicity-free graded representations of the graded Lie algebra $\mathscr{S}'(0,0,l_3;\Gamma)$, which can be extended immediately to the classification of the multiplicity-free weight modules of $\mathscr{S}(0,0,l_3;\Gamma)$. For the subcase $l_1+l_2=1$, we have $\mathscr{S}(l_1,l_2,l_3;\Gamma)=\mathbb{F}\partial_1\oplus\mathscr{S}'(l_1,l_2,l_3;\Gamma)$. Even though the derived subalgebra $\mathscr{S}'(l_1,l_2,l_3;\Gamma)$ has the toral Cartan subalgebra $\mathscr{D}_2$ when $l_1+l_2=1$, any of its irreducible or indecomposable multiplicity-free weight modules with respect to $\mathscr{D}_2$ is a 1-dimensional trivial module. Secondly, we expect that Theorem 1.1 holds for the case $l_1+l_2=1,2$ and $l_2+l_3\geq 3$,; but there are more technical difficulties in deriving the possible actions of the elements $\{t^{1_{[p]}}\partial_r, t^{1_{[p]}}\partial_p-t^{1_{[q]}}\partial_q\mid p,q\in\overline{1,l_1+l_2}, r\in\overline{1,l}\backslash\{p\}\}$. In this paper, we shall consider the case $l_1+l_2\geq 3$ and $l_2+l_3\geq 3$. The other unsolved cases will be studied later.

Throughout the rest of this paper, we always assume that $l_1+l_2\geq 3$ and $l_2+l_3\geq 3$.

Denote

$$D_{p,q}(u) = \partial_p(u)\partial_q - \partial_q(u)\partial_p \quad \text{for } p, q \in \overline{1, l},\ u \in \mathscr{A}. \tag{21}$$

Then it can be verified that

$$\mathscr{S} = \mathrm{Span}\{D_{p,q}(u) \mid p, q \in \overline{1, l},\ u \in \mathscr{A}\}. \tag{22}$$

Identify the dual vector space $\mathscr{D}^*$ with $\mathbb{F}^l$ by the pairing

$$< \beta, \partial >= \beta(\partial) = \partial(\beta) = \sum_{p=1}^{l} a_p \beta_p \tag{23}$$

for $\partial = \sum_{p=1}^{l} a_p \partial_p \in \mathscr{D}$ and $\beta = (\beta_1, \cdots, \beta_l) \in \mathbb{F}^l$. For $\mathscr{S}$-module $\mathscr{A}^\mu$ (cf. (9)), it is easy to verify:

Proposition 2.2. (1) *The module $\mathscr{A}^\mu$ is irreducible if and only if $\mu \notin \Gamma$.*

(2) *When $\mu \in \Gamma$, $\mathscr{A}^\mu \simeq \mathscr{A}^0$ is indecomposable, and it has the trivial submodule $\mathbb{F}x^{-\mu}$. The quotient module $\bar{\mathscr{A}}^\mu = \mathscr{A}^\mu / \mathbb{F}x^{-\mu}$ is simple.*

(3) *The possible isomorphisms between modules of type $\mathscr{A}^\mu$ are: $\mathscr{A}^\mu \simeq \mathscr{A}^\eta$ iff $\mu - \eta \in \Gamma$.*

In the above proposition, the simple quotient module $V = \bar{\mathscr{A}}^0 = \mathscr{A}^0/\mathbb{F}1$ is a generalized weight $\mathscr{S}$-module; and we have $\dim V_0^{(0)} = l_1 + l_2$, $\dim V_\beta^{(0)} = 1$ for all $\beta \in \Gamma \backslash \{0\}$.

We now review the classification theorem in [2]. By (17) and (23), we have

$$\partial(x^\alpha) = \partial(\alpha) x^\alpha \quad \text{for } \alpha \in \Gamma \text{ and } \partial \in \mathscr{D}. \tag{24}$$

Denote

$$\ker\beta = \{\partial \in \mathscr{D} \mid \partial(\beta) = 0\} \quad \text{for } \beta \in \mathbb{F}^l. \tag{25}$$

Let $\mathscr{S}'$ be the derived subalgebra of $\mathscr{S}$. In [2], we gave a classification of the multiplicity-free representations of the graded divergence-free Lie algebra $\mathscr{S}'(0,0,l;\Gamma)$, which is:

Proposition 2.3. *Suppose $l \geq 3$. Assume that $V = \bigoplus_{\theta\in\Gamma} V_\theta$ is a Γ-graded $\mathscr{S}'(0,0,l;\Gamma)$-module with $\dim V_\theta \leq 1$ for $\theta \in \Gamma$. If V is irreducible or indecomposable, then V is isomorphic to one of the following modules for appropriate $\zeta \in \mathbb{F}^l$ and $\eta \in \mathbb{F}^l \backslash \{0\}$:*

$$(i) \text{ the trivial module } \mathbb{F}v, \quad (ii)\ \bar{\mathcal{A}}^\zeta, \quad (iii)\ \mathcal{A}(\eta), \quad (iv)\ \mathcal{B}(\eta), \tag{26}$$

where $\mathcal{A}^\zeta$, $\mathcal{A}(\eta)$ and $\mathcal{B}(\eta)$ are $\mathscr{S}'(0,0,l;\Gamma)$-modules with basis $\{v_\alpha \mid \alpha \in \Gamma\}$ and the following actions:

$$\mathcal{A}^\zeta: \qquad x^\alpha\partial.v_\beta = \partial(\beta+\zeta)v_{\alpha+\beta} \text{ for } \alpha \in \Gamma\backslash\{0\}, \beta \in \Gamma, \partial \in \ker\alpha; \tag{27}$$

$$\mathcal{A}(\eta): \quad x^\alpha\partial.v_\beta = \partial(\beta)v_{\alpha+\beta} \textit{ for } \alpha,\beta\in\Gamma\backslash\{0\}, \partial\in \textit{ker}\,\alpha,$$
$$x^\alpha\partial.v_0 = \partial(\eta)v_\alpha \textit{ for } \alpha\in\Gamma\backslash\{0\}, \partial\in\textit{ker}\,\alpha; \tag{28}$$

$$\mathcal{B}(\eta): \quad x^\alpha\partial.v_\beta = \partial(\beta)v_{\alpha+\beta} \textit{ for } \alpha\in\Gamma\backslash\{0\}, \beta\in\Gamma\backslash\{-\alpha\}, \partial\in\textit{ker}\,\alpha,$$
$$x^\alpha\partial.v_{-\alpha} = \partial(\eta)v_0 \textit{ for } \alpha\in\Gamma\backslash\{0\}, \partial\in\textit{ker}\,\alpha; \tag{29}$$

and

$$\bar{\mathcal{A}}^\zeta = \mathcal{A}^\zeta \textit{ if } \zeta\notin\Gamma, \quad \bar{\mathcal{A}}^\zeta = \mathcal{A}^\zeta/\mathbb{F}v_{-\zeta} \textit{ if } \zeta\in\Gamma. \tag{30}$$

We would like to remark that, we do not have modules of types $\mathcal{A}(\eta)$, $\mathcal{B}(\eta)$ for $\mathscr{S}(l_1,l_2,l_3;\Gamma)$ when $l_1+l_2\geq 3$ and $l_2+l_3\geq 3$.

Next, we derive two generating properties of the Lie algebra $\mathscr{S}(l_1,l_2,l_3;\Gamma)$. The following conclusion will be frequently used:

Lemma 2.4. *Let* $\mathbf{T}$ *be a linear transformation on a vector space* U*, and let* U_1 *be a subspace of* U *such that* $\mathbf{T}(U_1)\subset U_1$*. Suppose that* $u_1,\ u_2,\ \cdots,\ u_n$ *are eigenvectors of* $\mathbf{T}$ *corresponding to different eigenvalues. If* $\sum_{j=1}^n u_j\in U_1$*, then* $u_1,\ u_2,\ \cdots,\ u_n\ \in U_1$.

Denote

$$|\mathbf{i}| = \sum_{p=1}^{l_1+l_2} i_p \quad \text{for } \mathbf{i}\in\mathbb{N}^{l_1+l_2}. \tag{31}$$

Then:

Proposition 2.5. *The Lie algebra* $\mathscr{S}(l_1,l_2,l_3;\Gamma)$ *is generated by*

$$\{D_{p,q}(x^\alpha), D_{p,q}(t^{\boldsymbol{j}}) \mid p,q\in\overline{1,l},\ \alpha\in\Gamma\backslash\{0\},\ \boldsymbol{j}\in\mathbb{N}^{l_1+l_2} \textit{ with } |\boldsymbol{j}|\leq 2\}. \tag{32}$$

Proof. Denote by $\mathscr{K}$ the Lie subalgebra of $\mathscr{S}(l_1,l_2,l_3;\Gamma)$ which is generated by the set (32). It suffices to prove $\mathscr{K}=\mathscr{S}(l_1,l_2,l_3;\Gamma)$. We give our proof in two steps.

Step 1. $D_{p,q}(x^\alpha t^{\mathbf{i}})\in\mathscr{K}$ for any $p,q\in\overline{1,l}$, $\alpha\in\Gamma\backslash\{0\}$ and $\mathbf{i}\in\mathbb{N}^{l_1+l_2}$.

We prove this step by induction on $|\mathbf{i}|$. It holds when $|\mathbf{i}|=0$ (cf. (32)). Assume that

$$D_{p,q}(x^\alpha t^{\mathbf{i}})\in\mathscr{K} \text{ for any } p,q\in\overline{1,l},\ \alpha\in\Gamma\backslash\{0\},\ \mathbf{i}\in\mathbb{N}^{l_1+l_2} \text{ with } |\mathbf{i}|\leq k, \tag{33}$$

where $k\geq 0$. We need to prove $D_{p,q}(x^\alpha t^{\mathbf{i}+1_{[r]}})\in\mathscr{K}$ for all $p,q\in\overline{1,l}$, $\alpha\in\Gamma\backslash\{0\}$, $r\in\overline{1,l_1+l_2}$ and $\mathbf{i}\in\mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}|=k$.

Fix any $\alpha\in\Gamma\backslash\{0\}$ and $\mathbf{i}\in\mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}|=k$. Choose $s\in\overline{1,l}$ such that $\alpha_s\neq 0$. Then for any $p,q\in\overline{1,l}$ and $r\in\overline{1,l_1+l_2}\backslash\{s\}$, we have

$$\begin{aligned}\mathscr{K} &\ni [D_{p,q}(x^\alpha t^{\mathbf{i}}), D_{r,s}(t^{2_{[r]}})]\\ &= 2[x^\alpha t^{\mathbf{i}}(\alpha_p\partial_q-\alpha_q\partial_p)+i_p x^\alpha t^{\mathbf{i}-1_{[p]}}\partial_q - i_q x^\alpha t^{\mathbf{i}-1_{[q]}}\partial_p, t^{1_{[r]}}\partial_s]\\ &= 2(\delta_{p,r}D_{s,q}(x^\alpha t^{\mathbf{i}})-\delta_{q,r}D_{s,p}(x^\alpha t^{\mathbf{i}})-i_sD_{p,q}(x^\alpha t^{\mathbf{i}+1_{[r]}-1_{[s]}})\\ &\quad -\alpha_sD_{p,q}(x^\alpha t^{\mathbf{i}+1_{[r]}})).\end{aligned} \tag{34}$$

Thus the induction hypothesis (33) gives

$$D_{p,q}(x^{\alpha}t^{\mathbf{i}+1_{[r]}}) \in \mathscr{K} \quad \text{for } p,q \in \overline{1,l},\ r \in \overline{1,l_1+l_2}\backslash\{s\}. \tag{35}$$

If $s \notin \overline{1,l_1+l_2}$, then it is done. Consider the case that $s \in \overline{1,l_1+l_2}$. If $\alpha_{s'} \neq 0$ for some $s' \in \overline{1,l}\backslash\{s\}$, then with s replaced by s' in the above arguments, we can get $D_{p,q}(x^{\alpha}t^{\mathbf{i}+1_{[s]}}) \in \mathscr{K}$ for $p,q \in \overline{1,l}$. Otherwise, we have $\alpha_{s'} = 0$ for all $s' \in \overline{1,l}\backslash\{s\}$. Pick $r \in \overline{1,l_1+l_2}\backslash\{s\}$. Expressions (32) and (35) give

$$\begin{aligned}\mathscr{K} &\ni [D_{p,q}(x^{\alpha}t^{\mathbf{i}+1_{[r]}}), D_{s,r}(t^{2_{[s]}})]\\ &= -2((i_r+1)D_{p,q}(x^{\alpha}t^{\mathbf{i}+1_{[s]}}) + \delta_{p,s}D_{q,r}(x^{\alpha}t^{\mathbf{i}+1_{[r]}}) - \delta_{q,s}D_{p,r}(x^{\alpha}t^{\mathbf{i}+1_{[r]}}))\end{aligned} \tag{36}$$

for $p,q \in \overline{1,l}$. So this and (35) indicate

$$D_{p,q}(x^{\alpha}t^{\mathbf{i}+1_{[s]}}) \in \mathscr{K} \quad \text{for } p,q \in \overline{1,l}. \tag{37}$$

Thus, (35) and (37) give

$$D_{p,q}(x^{\alpha}t^{\mathbf{i}+1_{[r]}}) \in \mathscr{K} \tag{38}$$

for $p,q \in \overline{1,l}$, $\alpha \in \Gamma\backslash\{0\}$, $r \in \overline{1,l_1+l_2}$ and $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k$, which completes the proof of the step.

Step 2. $D_{p,q}(t^{\mathbf{i}}) \in \mathscr{K}$ *for any* $p,q \in \overline{1,l}$ *and* $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$.

We shall prove this step by induction on $|\mathbf{i}|$. When $|\mathbf{i}| = 0$, $D_{p,q}(t^{\mathbf{i}}) = 0$ for any $p,q \in \overline{1,l}$. Assume that

$$D_{p,q}(t^{\mathbf{j}}) \in \mathscr{K} \text{ for any } p,q \in \overline{1,l},\ \mathbf{j} \in \mathbb{N}^{l_1+l_2} \text{ with } |\mathbf{j}| \leq k, \tag{39}$$

where $k \geq 0$. Then we need to prove $D_{p,q}(t^{\mathbf{i}}) \in \mathscr{K}$ for all $p,q \in \overline{1,l}$ and $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$. Suppose $p \neq q$ and $|\mathbf{i}| = k+1$ in the following. We give the proof in several cases.

Case 1. $p,q \in \overline{l_1+l_2+1,l}$.

In this case, $D_{p,q}(t^{\mathbf{i}}) = 0$.

Case 2. $p,q \in \overline{1,l_1+l_2}$.

Pick $r \in \overline{l_1+1,l}\backslash\{p,q\}$. Choose $\alpha \in \Gamma\backslash\{0\}$ such that $\alpha_r \neq 0$. Then Step 1 indicates

$$\begin{aligned}\mathscr{K} &\ni [D_{r,q}(x^{\alpha}t^{\mathbf{i}}), D_{r,p}(x^{-\alpha})]\\ &= \alpha_r^2 D_{p,q}(t^{\mathbf{i}}) + \alpha_r\alpha_p D_{q,r}(t^{\mathbf{i}}) + \alpha_r\alpha_q D_{r,p}(t^{\mathbf{i}}) + \alpha_p i_r D_{q,r}(t^{\mathbf{i}-1_{[r]}})\\ &\quad + \alpha_r i_p D_{r,q}(t^{\mathbf{i}-1_{[p]}}).\end{aligned} \tag{40}$$

So the induction hypothesis (39) gives rise to

$$\alpha_r^2 D_{p,q}(t^{\mathbf{i}}) + \alpha_r\alpha_p D_{q,r}(t^{\mathbf{i}}) + \alpha_r\alpha_q D_{r,p}(t^{\mathbf{i}}) \in \mathscr{K}. \tag{41}$$

Moreover, since

$$[D_{p,q}(t^{1_{[p]}+1_{[q]}}), D_{p,q}(t^{\mathbf{i}})] = (i_q - i_p)D_{p,q}(t^{\mathbf{i}}), \tag{42}$$

$$[D_{p,q}(t^{1_{[p]}+1_{[q]}}), D_{q,r}(t^{\mathbf{i}})] = (i_q - i_p - 1)D_{q,r}(t^{\mathbf{i}}), \tag{43}$$

$$[D_{p,q}(t^{1_{[p]}+1_{[q]}}), D_{r,p}(t^{\mathbf{i}})] = (i_q - i_p + 1)D_{r,p}(t^{\mathbf{i}}), \tag{44}$$

we have $D_{p,q}(t^{\mathbf{i}}) \in \mathscr{K}$ by (32), (41) and Lemma 2.4.

Case 3. $p \in \overline{1, l_1 + l_2}$, $q \in \overline{l_1 + l_2 + 1, l}$.

Pick $s \in \overline{1, l_1 + l_2} \backslash \{p\}$. Case 2 tells that $D_{p,s}(t^{\mathbf{i}}) \in \mathscr{K}$. So by (32), we have

$$\mathscr{K} \ni [D_{p,s}(t^{\mathbf{i}}), \frac{1}{2}D_{s,q}(t^{2_{[s]}})] = i_p t^{\mathbf{i}-1_{[p]}}\partial_q = D_{p,q}(t^{\mathbf{i}}). \tag{45}$$

Case 4. $p \in \overline{l_1 + l_2 + 1, l}$, $q \in \overline{1, l_1 + l_2}$.

We have $D_{p,q}(t^{\mathbf{i}}) = -D_{q,p}(t^{\mathbf{i}}) \in \mathscr{K}$ by Case 3. So this step holds.

Thus, from (22) we see that $\mathscr{K} = \mathscr{S}(l_1, l_2, l_3; \Gamma)$. So this proposition holds. ∎

We also have:

Corollary 2.6. *The set*

$$\{D_{p,q}(x^{\alpha}t^{\boldsymbol{i}}) \mid p, q \in \overline{1, l},\ \alpha \in \Gamma\backslash\{0\}, \boldsymbol{i} \in \mathbb{N}^{l_1+l_2}\} \tag{46}$$

generates the Lie algebra $\mathscr{S}(l_1, l_2, l_3; \Gamma)$.

Proof. Denote by $\mathscr{I}$ the Lie subalgebra of $\mathscr{S}(l_1, l_2, l_3; \Gamma)$ which is generated by the set (46). By Proposition 2.5, it suffices to prove $D_{p,q}(t^{\mathbf{i}}) \in \mathscr{I}$ for all $p, q \in \overline{1, l}$ and $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| \leq 2$. Notice that

$$\mathrm{Span}_{\mathbb{F}}\{D_{p,q}(t^{1_{[r]}}) \mid p, q \in \overline{1, l},\ r \in \overline{1, l_1 + l_2}\} = \mathscr{D}.$$

So first, we need to prove $\mathscr{D} \subseteq \mathscr{I}$.

Fix some $r \in \overline{l_1 + 1, l}$. Choose $\alpha \in \Gamma$ such that $\alpha_r \neq 0$. Pick $p \in \overline{1, l_1 + l_2} \backslash \{r\}$. Then by (46), we have

$$\mathscr{I} \ni \frac{1}{\alpha_r}[D_{r,q}(x^{\alpha}t^{1_{[p]}}), D_{r,p}(x^{-\alpha})] = \alpha_r\partial_q - \alpha_q\partial_r \quad \text{for } q \in \overline{1, l}\backslash\{r, p\}. \tag{47}$$

By changing the choice of $p \in \overline{1, l_1 + l_2} \backslash \{r\}$, we can get

$$\alpha_r\partial_q - \alpha_q\partial_r \in \mathscr{I} \quad \text{for } q \in \overline{1, l}\backslash\{r\}. \tag{48}$$

Take $r' \in \overline{l_1 + 1, l} \backslash \{r\}$. Choose $\beta \in \Gamma$ such that $\beta_r \neq 0$ and $\alpha_r\beta_{r'} - \alpha_{r'}\beta_r \neq 0$. With α replaced by β in (47)–(48), we can get that $\beta_r\partial_{r'} - \beta_{r'}\partial_r \in \mathscr{I}$. Since (48) also gives $\alpha_r\partial_{r'} - \alpha_{r'}\partial_r \in \mathscr{I}$, we have

$$\partial_r = \frac{1}{\alpha_r\beta_{r'} - \alpha_{r'}\beta_r}(\beta_r(\alpha_r\partial_{r'} - \alpha_{r'}\partial_r) - \alpha_r(\beta_r\partial_{r'} - \beta_{r'}\partial_r)) \in \mathscr{I}. \tag{49}$$

Thus, (48) and (49) indicate

$$\mathscr{D} \subseteq \mathscr{I}. \tag{50}$$

Second, we want to prove $t^{1_{[p]}}\partial_q \in \mathscr{I}$ for any $p \in \overline{1, l_1+l_2}$ and $q \in \overline{1,l}\backslash\{p\}$. Fix any $p \in \overline{1, l_1+l_2}$. Pick $r \in \overline{l_1+1, l}\backslash\{p\}$. Choose $\alpha \in \Gamma$ such that $\alpha_r \neq 0$. Then by (46), we have

$$\mathscr{I} \ni \frac{1}{2\alpha_r}[D_{r,q}(x^\alpha t^{2_{[p]}}), D_{r,p}(x^{-\alpha})] = t^{1_{[p]}}(\alpha_r\partial_q - \alpha_q\partial_r) \quad \text{for } q \in \overline{1,l}\backslash\{r,p\}. \tag{51}$$

Pick $r' \in \overline{l_1+1, l}\backslash\{r,p\}$. Choose $\beta \in \Gamma$ such that $\beta_r \neq 0$ and $\alpha_r\beta_{r'} - \alpha_{r'}\beta_r \neq 0$. Likewise, we can get $t^{1_{[p]}}(\beta_r\partial_{r'} - \beta_{r'}\partial_r) \in \mathscr{I}$. Since (51) gives $t^{1_{[p]}}(\alpha_r\partial_{r'} - \alpha_{r'}\partial_r) \in \mathscr{I}$, we have

$$t^{1_{[p]}}\partial_r = \frac{1}{\alpha_r\beta_{r'} - \alpha_{r'}\beta_r}(\beta_r t^{1_{[p]}}(\alpha_r\partial_{r'} - \alpha_{r'}\partial_r) - \alpha_r t^{1_{[p]}}(\beta_r\partial_{r'} - \beta_{r'}\partial_r)) \in \mathscr{I}. \tag{52}$$

So (51) and (52) show

$$t^{1_{[p]}}\partial_q \in \mathscr{I} \text{ for all } p \in \overline{1, l_1+l_2} \text{ and } q \in \overline{1,l}\backslash\{p\}. \tag{53}$$

Third, by (53) we have

$$D_{p,q}(t^{1_{[p]}+1_{[q]}}) = [t^{1_{[q]}}\partial_p, t^{1_{[p]}}\partial_q] \in \mathscr{I} \qquad \text{for all } p, q \in \overline{1, l_1+l_2}. \tag{54}$$

So in summary, (50), (53) and (54) give

$$\{D_{p,q}(t^{\mathbf{i}}) \mid p, q \in \overline{1,l},\ \mathbf{i} \in \mathbb{N}^{l_1+l_2} \text{ with } |\mathbf{i}| \leq 2\} \subseteq \mathscr{I}. \tag{55}$$

So the corollary follows from (46), (55) and Proposition 2.5. ■

3. Proof of Theorem 1.1 (I)

Suppose that V is an irreducible or indecomposable multiplicity-free generalized weight module of $\mathscr{S}(l_1, l_2, l_3; \Gamma)$. Since $\mathscr{D}$ act locally finitely on V, we have

$$V = \bigoplus_{\beta \in \mathbb{F}^l} V_\beta \tag{56}$$

by (5), (6) and (23). Let

$$V(\mu) = \bigoplus_{\alpha \in \Gamma} V_{\alpha+\mu} \qquad \text{for } \mu \in \mathbb{F}^l. \tag{57}$$

Then $V(\mu)$ is a submodule of V, and V is a direct sum of $V(\mu)$ for different $\mu \in \mathbb{F}^l$. Since V is irreducible or indecomposable, we must have

$$V = V(\mu) = \bigoplus_{\alpha \in \Gamma} V_{\alpha+\mu} \text{ for some } \mu \in \mathbb{F}^l. \tag{58}$$

Moreover, since $V = V(\mu)$ is multiplicity-free, we have $\dim V_{\alpha+\mu}^{(0)} \leq 1$ for all $\alpha \in \Gamma$.

To prove Theorem 1.1, we need to derive the action of $\mathscr{S}$ on $V = V(\mu)$, or more directly, we need to derive the action of the set (32). Except the trivial case in the following Lemma 3.1, we shall first determine a basis of $V = V(\mu)$, then derive the action of the set (32) on the basis. The main proof shall be divided into two cases: $\mu \in \mathbb{F}^l\backslash\Gamma$; $\mu \in \Gamma$. In this section, we first obtain some general results for general $\mu \in \mathbb{F}^l$, then prove Theorem 1.1 under the condition $\mu \in \mathbb{F}^l\backslash\Gamma$. In the next section, we shall prove Theorem 1.1 under the condition $\mu \in \Gamma$.

Firstly, we let $\mu \in \mathbb{F}^l$ be arbitrary.

Lemma 3.1. *If $V^{(0)}_{\sigma+\mu} \neq \{0\}$ for some $\sigma \in \Gamma$, and $V^{(0)}_{\rho+\mu} = \{0\}$ for all $\rho \in \Gamma\backslash\{\sigma\}$, then $\sigma+\mu = 0$, and $V = V^{(0)}_0$ is a 1-dimensional trivial $\mathscr{S}(l_1,l_2,l_3;\Gamma)$-module.*

Proof. By (6) and (57), we have $V = V_{\sigma+\mu}$. So

$$D_{p,q}(x^\gamma t^{\mathbf{i}}).V_{\sigma+\mu} = \{0\} \text{ for all } p,q \in \overline{1,l},\ \gamma \in \Gamma\backslash\{0\}, \mathbf{i} \in \mathbb{N}^{l_1+l_2} \tag{59}$$

by (5). Thus, (59) and Corollary 2.6 give

$$\mathscr{S}(l_1,l_2,l_3;\Gamma).V_{\sigma+\mu} = \{0\}, \tag{60}$$

which says $V = V_{\sigma+\mu}$ is a direct sum of trivial submodules. Since V is irreducible or indecomposable, we must have $V = V^{(0)}_{\sigma+\mu}$ as a trivial $\mathscr{S}$-module. As one result, $\partial.V^{(0)}_{\sigma+\mu} = \{0\}$ for all $\partial \in \mathscr{D}$, which implies $\sigma+\mu = 0$. So $V = V^{(0)}_0$, and it is a trivial $\mathscr{S}(l_1,l_2,l_3;\Gamma)$-module. ∎

In the rest of this paper, we consider the case

$$V^{(0)}_{\sigma+\mu} \neq \{0\} \text{ and } V^{(0)}_{\tau+\mu} \neq \{0\} \text{ for some } \sigma,\tau \in \Gamma \text{ with } \sigma \neq \tau. \tag{61}$$

Lemma 3.2. *If $V^{(0)}_{\sigma+\mu} \neq \{0\}$ and $V^{(0)}_{\tau+\mu} \neq \{0\}$ for some $\sigma,\tau \in \Gamma$ with $\sigma \neq \tau$, then $V^{(0)}_{\gamma+\mu} \neq \{0\}$ for all $\gamma \in \Gamma\backslash\{-\mu\}$. Moreover, there exist nonzero $v_{\gamma,\boldsymbol{0}} \in V^{(0)}_{\gamma+\mu}$ with $\gamma \in \Gamma\backslash\{-\mu\}$, such that*

$$D_{p,q}(x^\alpha).v_{\gamma,\boldsymbol{0}} = (\alpha_p(\gamma_q+\mu_q) - \alpha_q(\gamma_p+\mu_p))v_{\alpha+\gamma,\boldsymbol{0}} \tag{62}$$

for $p,q \in \overline{1,l}$, $\alpha \in \Gamma\backslash\{0\}$ and $\gamma \in \Gamma\backslash\{-\mu\}$.

Proof. Since

$$\mathscr{S}_0 := \mathrm{Span}_{\mathbb{F}}\{D_{p,q}(x^\alpha) \mid p,q \in \overline{l_1+1,l},\ \alpha \in \Gamma\backslash\{0\}\} \simeq \mathscr{S}'(0,0,l_2+l_3;\Gamma), \tag{63}$$

we can see $V^{(0)} = \bigoplus_{\alpha\in\Gamma} V^{(0)}_{\alpha+\mu}$ as a Γ-graded $\mathscr{S}_0$-submodule with $\dim V^{(0)} \geq 2$ and $\dim V^{(0)}_{\alpha+\mu} \leq 1$ for all $\alpha \in \Gamma$. ¿From Proposition 2.3 it can be derived that, either $V^{(0)}$ is a direct sum of some trivial $\mathscr{S}_0$-submodules, or there exists $\zeta \in \mathbb{F}^{l_2+l_3}$ such that, $V^{(0)}_{\gamma+\mu} \neq \{0\}$ for all $\gamma \in \Gamma\backslash\{-\zeta\}$, and with proper choice of nonzero $v_{\gamma,\mathbf{0}} \in V^{(0)}_{\gamma+\mu}$ for $\gamma \in \Gamma\backslash\{-\zeta\}$,

$$D_{p,q}(x^\alpha).v_{\gamma,\mathbf{0}} = x^\alpha(\alpha_p\partial_q - \alpha_q\partial_p).v_{\gamma,\mathbf{0}} = (\alpha_p(\gamma_q+\zeta_q) - \alpha_q(\gamma_p+\zeta_p))v_{\alpha+\gamma,\mathbf{0}} \tag{64}$$

for $p,q \in \overline{l_1+1,l}$, $\alpha \in \Gamma\backslash\{0\}$ and $\gamma \in \Gamma\backslash\{-\zeta,-\zeta-\alpha\}$.

First of all, we shall prove:

Claim 1. $\mathscr{S}_0.V^{(0)} \neq \{0\}$.

Suppose $\mathscr{S}_0.V^{(0)} = \{0\}$, namely, $V^{(0)}$ is a direct sum of some trivial $\mathscr{S}_0$-submodules, then this will lead to a contradiction. Let $\bar{\alpha} = \alpha + \mu$ for all $\alpha \in \Gamma$.

Since $V^{(0)}_{\sigma+\mu} \neq \{0\}$ and $V^{(0)}_{\tau+\mu} \neq \{0\}$ for some $\sigma, \tau \in \Gamma$ with $\sigma \neq \tau$, there exists $\theta \in \Gamma$ such that $V^{(0)}_{\theta+\mu} \neq \{0\}$ and that $\bar{\theta}_r \neq 0$ for some $r \in \overline{l_1+1, l}$. Fix such θ and r. Pick $s \in \overline{l_1+1, l}\backslash\{r\}$. Choose $\alpha \in \Gamma$ such that $\alpha_r \neq 0$ and $\bar{\theta}_r\alpha_s - \bar{\theta}_s\alpha_r \neq 0$. Moreover, we let

$$\partial = \bar{\theta}_r\partial_s - \bar{\theta}_s\partial_r, \quad \partial' = \alpha_r\partial_s - \alpha_s\partial_r, \quad \partial'' = (\bar{\theta}_r - \alpha_r)\partial_s - (\bar{\theta}_s - \alpha_s)\partial_r. \tag{65}$$

Pick $p \in \overline{1, l_1+l_2}\backslash\{r, s\}$. From (6) it can be deduced that

$$t^{1_{[p]}}\partial.V^{(0)}_{\theta+\mu} \subseteq V^{(0)}_{\theta+\mu}, \quad t^{1_{[p]}}\partial''.V^{(0)}_{\theta-\alpha+\mu} \subseteq V^{(0)}_{\theta-\alpha+\mu}. \tag{66}$$

Take $0 \neq v_\theta \in V^{(0)}_{\theta+\mu}$. So on one hand, we have

$$\partial'.v_\theta = \partial'(\bar{\theta})v_\theta = (\bar{\theta}_s\alpha_r - \bar{\theta}_r\alpha_s)v_\theta \neq 0 \tag{67}$$

by (6) and (65). On the other hand, (66) and $\mathscr{S}_0.V^{(0)} = \{0\}$ give

$$x^\alpha t^{1_{[p]}}\partial'.v_\theta = -\frac{1}{\partial(\alpha)}[x^\alpha\partial', t^{1_{[p]}}\partial].v_\theta = -\frac{1}{\partial(\alpha)}x^\alpha\partial'.(t^{1_{[p]}}\partial.v_\theta) = 0 \tag{68}$$

and

$$x^\alpha t^{1_{[p]}}\partial'.V^{(0)}_{\theta-\alpha+\mu} = -\frac{1}{\bar{\theta}_r\alpha_s - \bar{\theta}_s\alpha_r}[x^\alpha\partial', t^{1_{[p]}}\partial''].V^{(0)}_{\theta-\alpha+\mu} = \{0\}, \tag{69}$$

which further implies

$$\begin{aligned} \partial'.v_\theta &= -\frac{1}{\alpha_r}[x^\alpha t^{1_{[p]}}\partial', x^{-\alpha}(\alpha_r\partial_p - \alpha_p\partial_r)].v_\theta \\ &= -\frac{1}{\alpha_r}x^\alpha t^{1_{[p]}}\partial'.(x^{-\alpha}(\alpha_r\partial_p - \alpha_p\partial_r).v_\theta) = 0, \end{aligned} \tag{70}$$

where $x^{-\alpha}(\alpha_r\partial_p - \alpha_p\partial_r).v_\theta \in V^{(0)}_{\theta-\alpha+\mu}$. Since (70) contradicts (67), we must have $\mathscr{S}_0.V^{(0)} \neq \{0\}$. Thus this claim holds.

So by Claim 1, there exists $\zeta \in \mathbb{F}^{l_2+l_3}$ such that, $V^{(0)}_{\gamma+\mu} \neq \{0\}$ for all $\gamma \in \Gamma\backslash\{-\zeta\}$, and with proper choice of nonzero $v_{\gamma,\mathbf{0}} \in V^{(0)}_{\gamma+\mu}$ for $\gamma \in \Gamma\backslash\{-\zeta\}$,

$$D_{p,q}(x^\alpha).v_{\gamma,\mathbf{0}} = (\alpha_p(\gamma_q + \zeta_q) - \alpha_q(\gamma_p + \zeta_p))v_{\alpha+\gamma,\mathbf{0}} \tag{71}$$

for $p, q \in \overline{l_1+1, l}$, $\alpha \in \Gamma\backslash\{0\}$ and $\gamma \in \Gamma\backslash\{-\zeta, -\zeta-\alpha\}$.

Next, we want to prove:

Claim 2. $\zeta_p = \mu_p$ for all $p \in \overline{l_1+1, l}$.

It suffices to prove $\partial(\zeta-\mu) = 0$ for all $\partial \in \sum_{i\in\overline{l_1+1,l}}\mathbb{F}\partial_i$. Choose $v_{\gamma,\mathbf{0}} \in V^{(0)}_{\gamma+\mu}$ for $\gamma \in \Gamma\backslash\{-\zeta\}$ as in (71). We give the proof in two cases.

Case 1. $l \geq 4$.

Pick $r \in \overline{1, l_1}$ if $l_1 > 0$ or $r \in \overline{1, l_2}$ if $l_1 = 0$. Take $s \in \overline{l_1+1, l} \backslash \{r\}$. Choose $\alpha \in \Gamma \backslash \{0\}$ such that $\alpha_s \neq 0$. Moreover, we choose $\beta \in \Gamma$ such that $\beta + \zeta \neq 0 \neq \beta + \alpha + \zeta$. Then we must have

$$\{0\} \neq \ker(\beta+\mu) \bigcap \ker \alpha \bigcap \Big(\sum_{i \in \overline{l_1+1, l} \backslash \{r\}} \mathbb{F}\partial_i \Big) \subseteq \ker(\zeta - \mu). \tag{72}$$

Suppose there exists

$$\partial \in \Big(\ker(\beta+\mu) \bigcap \ker \alpha \bigcap \Big(\sum_{i \in \overline{l_1+1, l} \backslash \{r\}} \mathbb{F}\partial_i \Big) \Big) \backslash \ker(\zeta - \mu). \tag{73}$$

We will see this leads to a contradiction. By (6) and (73), we have

$$t^{1_{[r]}} \partial . v_{\beta, \mathbf{0}} \in V^{(0)}_{\beta+\mu} \text{ and } t^{1_{[r]}} \partial . v_{\beta+\alpha, \mathbf{0}} \in V^{(0)}_{\beta+\alpha+\mu}. \tag{74}$$

So $t^{1_{[r]}} \partial . v_{\beta, \mathbf{0}} = c v_{\beta, \mathbf{0}}$ for some $c \in \mathbb{F}$. Since (71) implies

$$0 = [x^\alpha \partial, t^{1_{[r]}} \partial] . v_{\beta, \mathbf{0}} = c \partial(\beta+\zeta) v_{\beta+\alpha, \mathbf{0}} - \partial(\beta+\zeta) t^{1_{[r]}} \partial . v_{\beta+\alpha, \mathbf{0}}, \tag{75}$$

and (73) indicates $\partial(\beta+\zeta) = \partial(\zeta-\mu) \neq 0$, we find $t^{1_{[r]}} \partial . v_{\beta+\alpha, \mathbf{0}} = c v_{\beta+\alpha, \mathbf{0}}$. Thus (71) and (73) give rise to

$$\begin{aligned} 0 &\neq x^\alpha \partial . v_{\beta, \mathbf{0}} \\ &= -\frac{1}{\alpha_s} [D_{r,s}(x^\alpha), t^{1_{[r]}} \partial] . v_{\beta, \mathbf{0}} \\ &= -\frac{1}{\alpha_s} (c D_{r,s}(x^\alpha) . v_{\beta, \mathbf{0}} - t^{1_{[r]}} \partial . (D_{r,s}(x^\alpha) . v_{\beta, \mathbf{0}})) \\ &= -\frac{1}{\alpha_s} (c D_{r,s}(x^\alpha) . v_{\beta, \mathbf{0}} - c D_{r,s}(x^\alpha) . v_{\beta, \mathbf{0}}) \\ &= 0, \end{aligned} \tag{76}$$

where in the third line, $D_{r,s}(x^\alpha) . v_{\beta, \mathbf{0}} \in V^{(0)}_{\beta+\alpha+\mu}$. This is a contradiction. Thus (72) holds. By changing the choice of β in (72), we can get

$$\ker \alpha \bigcap \Big(\sum_{i \in \overline{l_1+1, l} \backslash \{r\}} \mathbb{F}\partial_i \Big) \subseteq \ker(\zeta - \mu). \tag{77}$$

Moreover, by changing the choice of α, we further find

$$\sum_{i \in \overline{l_1+1, l} \backslash \{r\}} \mathbb{F}\partial_i \subseteq \ker(\zeta - \mu). \tag{78}$$

Recall that $r \in \overline{1, l_1}$ if $l_1 > 0$ and that $r \in \overline{1, l_2}$ if $l_1 = 0$. In the case that $r \in \overline{1, l_2}$, by picking another $r \in \overline{1, l_2}$, we can always get

$$\sum_{i \in \overline{l_1+1, l}} \mathbb{F}\partial_i \subseteq \ker(\zeta - \mu). \tag{79}$$

Case 2. $l=3$.

Since $l_1+l_2\geq 3$ and $l_2+l_3\geq 3$, we have $l_1=l_3=0$ and $l_2=l=3$. Pick $r\in\overline{1,3}$. Take $s\in\overline{1,3}\backslash\{r\}$. Choose $\alpha\in\Gamma\backslash\{0\}$ such that $\alpha_s\neq 0$. Then

$$\dim\big(\ker\alpha\bigcap\big(\sum_{i\in\overline{1,3}\backslash\{r\}}\mathbb{F}\partial_i\big)\big)=1. \tag{80}$$

Picking $0\neq\tilde{\partial}_1\in\ker\alpha\bigcap\big(\sum_{i\in\overline{1,3}\backslash\{r\}}\mathbb{F}\partial_i\big)$, we shall first prove that

$$\tilde{\partial}_1(\zeta-\mu)=0. \tag{81}$$

Choose $\beta\in\Gamma$ such that $\beta+\zeta\neq 0\neq\beta+\alpha+\zeta$ and

$$\tilde{\partial}_1(2\beta+\zeta+\mu)\neq 0. \tag{82}$$

If

$$\tilde{\partial}_1(\beta+\mu)=0, \tag{83}$$

then similar arguments as those in Case 1 give (81). If $\tilde{\partial}_1(\beta+\mu)\neq 0$, namely,

$$\ker(\beta+\mu)\bigcap\ker\alpha\bigcap\big(\sum_{i\in\overline{1,3}\backslash\{r\}}\mathbb{F}\partial_i\big)=\{0\}, \tag{84}$$

we pick

$$0\neq\tilde{\partial}_2\in\ker(\beta-\alpha+\mu)\bigcap\big(\sum_{i\in\overline{1,3}\backslash\{r\}}\mathbb{F}\partial_i\big). \tag{85}$$

Then $\{\tilde{\partial}_1,\tilde{\partial}_2\}$ is a basis of $\sum_{i\in\overline{1,3}\backslash\{r\}}\mathbb{F}\partial_i$. Take

$$\begin{gathered}0\neq\partial\in\ker(\beta+\mu)\bigcap\big(\textstyle\sum_{i\in\overline{1,3}\backslash\{r\}}\mathbb{F}\partial_i\big),\\ 0\neq\partial'\in\ker\alpha\bigcap\ker(\beta+\zeta),\qquad 0\neq\partial''\in\ker\alpha.\end{gathered} \tag{86}$$

It follows that $\partial=a_1\tilde{\partial}_1+a_2\tilde{\partial}_2$ for some $a_1,a_2\in\mathbb{F}\backslash\{0\}$. So on one hand, by (6) we have

$$\begin{aligned}&[x^{-\alpha}\partial'',[x^{\alpha}\partial',t^{1_{[r]}}\partial]].v_{\beta,\mathbf{0}}\\ =\;&\partial(\alpha)\cdot(\partial'(t^{1_{[r]}})\partial''-\partial''(t^{1_{[r]}})\partial').v_{\beta,\mathbf{0}}\\ =\;&\partial(\alpha-\beta-\mu)\cdot(\partial'(t^{1_{[r]}})\partial''-\partial''(t^{1_{[r]}})\partial')(\beta+\mu)v_{\beta,\mathbf{0}}\\ =\;&-a_1\tilde{\partial}_1(\beta+\mu)\cdot(\partial'(t^{1_{[r]}})\partial''-\partial''(t^{1_{[r]}})\partial')(\beta+\mu)v_{\beta,\mathbf{0}},\end{aligned} \tag{87}$$

where in the third line we use the fact that $\partial(\beta+\mu)=0$, and in the fourth line we use $\partial = a_1\tilde{\partial}_1 + a_2\tilde{\partial}_2$ and (85). On the other hand, by (71) and (86), we have

$$\begin{aligned}
& [x^{-\alpha}\partial'', [x^{\alpha}\partial', t^{1_{[r]}}\partial]].v_{\beta,\mathbf{0}} \\
= \ & x^{-\alpha}\partial''.x^{\alpha}\partial'.(t^{1_{[r]}}\partial.v_{\beta,\mathbf{0}}) - x^{-\alpha}\partial''.t^{1_{[r]}}\partial.x^{\alpha}\partial'.v_{\beta,\mathbf{0}} \\
& -x^{\alpha}\partial'.t^{1_{[r]}}\partial.x^{-\alpha}\partial''.v_{\beta,\mathbf{0}} + t^{1_{[r]}}\partial.x^{\alpha}\partial'.(x^{-\alpha}\partial''.v_{\beta,\mathbf{0}}) \\
= \ & -x^{\alpha}\partial'.t^{1_{[r]}}\partial.x^{-\alpha}\partial''.v_{\beta,\mathbf{0}} \\
= \ & -\partial''(\beta+\zeta)x^{\alpha}\partial'.t^{1_{[r]}}(a_1\tilde{\partial}_1 + a_2\tilde{\partial}_2).v_{\beta-\alpha,\mathbf{0}} \\
= \ & -a_1\partial''(\beta+\zeta)x^{\alpha}\partial'.t^{1_{[r]}}\tilde{\partial}_1.v_{\beta-\alpha,\mathbf{0}} \\
= \ & -a_1\partial''(\beta+\zeta)[x^{\alpha}\partial', t^{1_{[r]}}\tilde{\partial}_1].v_{\beta-\alpha,\mathbf{0}} \\
= \ & -a_1\partial''(\beta+\zeta)\cdot\partial'(t^{1_{[r]}})x^{\alpha}\tilde{\partial}_1.v_{\beta-\alpha,\mathbf{0}} \\
= \ & -a_1\partial'(t^{1_{[r]}})\cdot\tilde{\partial}_1(\beta+\zeta)\cdot\partial''(\beta+\zeta)v_{\beta,\mathbf{0}},
\end{aligned} \tag{88}$$

where in the second equation $t^{1_{[r]}}\partial.v_{\beta,\mathbf{0}} \in V^{(0)}_{\beta+\mu}$ and $x^{-\alpha}\partial''.v_{\beta,\mathbf{0}} \in V^{(0)}_{\beta-\alpha+\mu}$, in the fourth equation $t^{1_{[r]}}\tilde{\partial}_2.v_{\beta-\alpha,\mathbf{0}} \in V^{(0)}_{\beta-\alpha+\mu}$ and $x^{\alpha}\partial'.(t^{1_{[r]}}\tilde{\partial}_2.v_{\beta-\alpha,\mathbf{0}}) = 0$. If $\partial'(t^{1_{[r]}}) = 0$, by taking $0 \neq \partial'' \in \ker\alpha \bigcap \ker(\beta+\mu)$ and by comparing (87) with (88), we get $\partial''(t^{1_{[r]}}) \neq 0$ (cf. (84)) and $\partial'(\beta+\mu) = 0$, which together with (84) and (86) further implies

$$0 \neq \partial' \in \ker\alpha \bigcap \ker(\beta+\mu) \bigcap \Big(\sum_{i\in\overline{1,3}\setminus\{r\}} \mathbb{F}\partial_i\Big) = \{0\}. \tag{89}$$

This is absurd. So we must have $\partial'(t^{1_{[r]}}) \neq 0$. Therefore, taking $\partial'' = \tilde{\partial}_1$ and comparing (87) with (88), we get

$$(\tilde{\partial}_1(\beta+\zeta))^2 = (\tilde{\partial}_1(\beta+\mu))^2. \tag{90}$$

Thus by (82) and (90), we find

$$\tilde{\partial}_1(\zeta-\mu) = 0. \tag{91}$$

So (81) holds. In other words,

$$\ker\alpha \bigcap \Big(\sum_{i\in\overline{1,3}\setminus\{r\}} \mathbb{F}\partial_i\Big) \subseteq \ker(\zeta-\mu). \tag{92}$$

By changing the choice of α, we can get

$$\sum_{i\in\overline{1,3}\setminus\{r\}} \mathbb{F}\partial_i \subseteq \ker(\zeta-\mu). \tag{93}$$

Moreover, by picking another $r \in \overline{1,3}$, we further obtain

$$\mathscr{D} = \sum_{i\in\overline{1,3}} \mathbb{F}\partial_i \subseteq \ker(\zeta-\mu). \tag{94}$$

So this claim holds.

Choose $v_{\gamma,\mathbf{0}} \in V^{(0)}_{\gamma+\mu}$ for $\gamma \in \Gamma\backslash\{-\zeta\}$ as in (71). Claim 2 shows that

$$D_{p,q}(x^\alpha).v_{\gamma,\mathbf{0}} = (\alpha_p(\gamma_q+\mu_q) - \alpha_q(\gamma_p+\mu_p))v_{\alpha+\gamma,\mathbf{0}} \tag{95}$$

for $p, q \in \overline{l_1+1, l}$, $\alpha \in \Gamma\backslash\{0\}$ and $\gamma \in \Gamma\backslash\{-\zeta, -\zeta-\alpha\}$. Next, we want to prove:

Claim 3. If $\zeta \in \Gamma$, then for any $\alpha \in \Gamma\backslash\{0\}$ and $p, q \in \overline{l_1+1, l}$,

$$D_{p,q}(x^\alpha).v_{-\zeta-\alpha,\mathbf{0}} = 0 \text{ and } D_{p,q}(x^\alpha).V^{(0)}_{-\zeta+\mu} = \{0\}. \tag{96}$$

We only prove the first equation; the second one can be proved similarly. Suppose

$$D_{p,q}(x^\alpha).v_{-\zeta-\alpha,\mathbf{0}} \neq 0 \tag{97}$$

for some $\alpha \in \Gamma\backslash\{0\}$ and $p, q \in \overline{l_1+1, l}$, then this will lead to a contradiction. Without loss of generality, we further assume that $\alpha_p \neq 0$. Pick $r \in \overline{1, l_1+l_2}\backslash\{p, q\}$. Since (97) indicates $V^{(0)}_{-\zeta+\mu} \neq \{0\}$, we can take nonzero $v_{-\zeta,\mathbf{0}} \in V^{(0)}_{-\zeta+\mu}$. It follows from (6) and Claim 2 that

$$t^{1_{[r]}}(\alpha_p\partial_q - \alpha_q\partial_p).v_{-\zeta-\alpha,\mathbf{0}} \in V^{(0)}_{-\zeta-\alpha+\mu} \text{ and } t^{1_{[r]}}(\alpha_p\partial_q - \alpha_q\partial_p).v_{-\zeta,\mathbf{0}} \in V^{(0)}_{-\zeta+\mu}. \tag{98}$$

So $t^{1_{[r]}}(\alpha_p\partial_q - \alpha_q\partial_p).v_{-\zeta-\alpha,\mathbf{0}} = cv_{-\zeta-\alpha,\mathbf{0}}$ for some $c \in \mathbb{F}$. Since

$$\begin{aligned} & t^{1_{[r]}}(\alpha_p\partial_q - \alpha_q\partial_p).(x^\alpha(\alpha_p\partial_q - \alpha_q\partial_p).v_{-\zeta-\alpha,\mathbf{0}}) \\ = \; & x^\alpha(\alpha_p\partial_q - \alpha_q\partial_p).(t^{1_{[r]}}(\alpha_p\partial_q - \alpha_q\partial_p).v_{-\zeta-\alpha,\mathbf{0}}) \\ = \; & c(x^\alpha(\alpha_p\partial_q - \alpha_q\partial_p).v_{-\zeta-\alpha,\mathbf{0}}), \end{aligned} \tag{99}$$

we find $t^{1_{[r]}}(\alpha_p\partial_q - \alpha_q\partial_p).v_{-\zeta,\mathbf{0}} = cv_{-\zeta,\mathbf{0}}$ by (97). Thus

$$\begin{aligned} & x^\alpha(\alpha_p\partial_q - \alpha_q\partial_p).v_{-\zeta-\alpha,\mathbf{0}} \\ = \; & \frac{1}{\alpha_p}[x^\alpha(\alpha_p\partial_r - \alpha_r\partial_p), t^{1_{[r]}}(\alpha_p\partial_q - \alpha_q\partial_p)].v_{-\zeta-\alpha,\mathbf{0}} \\ = \; & \frac{1}{\alpha_p}(x^\alpha(\alpha_p\partial_r - \alpha_r\partial_p).(cv_{-\zeta-\alpha,\mathbf{0}}) - c(x^\alpha(\alpha_p\partial_r - \alpha_r\partial_p).v_{-\zeta-\alpha,\mathbf{0}})) \\ = \; & 0, \end{aligned} \tag{100}$$

which contradicts (97). So this claim holds.

Claim 4. If $l_1 > 0$, then

$$x^\alpha\partial_p.v_{\gamma,\mathbf{0}} = \mu_p v_{\gamma+\alpha,\mathbf{0}} \text{ for all } p \in \overline{1, l_1},\ \alpha \in \Gamma\backslash\{0\} \text{ and } \gamma \in \Gamma\backslash\{-\zeta, -\zeta-\alpha\}. \tag{101}$$

Let $\bar{\beta} = \beta + \mu$ for all $\beta \in \Gamma$. Fix any $\alpha \in \Gamma\backslash\{0\}$, $p \in \overline{1, l_1}$ and $\gamma \in \Gamma\backslash\{-\zeta, -\zeta-\alpha\}$. Choose $r \in \overline{l_1+1, l}$ such that $\alpha_r \neq 0$. We shall prove this claim in two cases.

Case 1. $(\ker\alpha \cap \sum_{i=l_1+1}^{l} \mathbb{F}\partial_i)\backslash \ker\bar{\gamma} \neq \emptyset$.

Pick $q \in \overline{1, l_1} \backslash \{p\}$ if $l_1 > 1$ or $q \in \overline{l_1+1, l_1+l_2} \backslash \{r\}$ if $l_1 = 1$.

Subcase 1.1. There exists $s \in \overline{l_1+1, l} \backslash \{r, q\}$ such that $\alpha_r \bar{\gamma}_s - \alpha_s \bar{\gamma}_r \neq 0$.

Choose $s \in \overline{l_1+1, l} \backslash \{r, q\}$ such that $\alpha_r \bar{\gamma}_s - \alpha_s \bar{\gamma}_r \neq 0$. Let $\partial = \alpha_r \partial_s - \alpha_s \partial_r$. Then $\partial(\bar{\gamma}) \neq 0$. By (6), we get

$$t^{1_{[q]}}(\partial(\bar{\gamma})\partial_p - \mu_p \partial).v_{\gamma,\mathbf{0}} \in V^{(0)}_{\gamma+\mu} \text{ and } t^{1_{[q]}}(\partial(\bar{\gamma})\partial_p - \mu_p \partial).v_{\gamma+\alpha,\mathbf{0}} \in V^{(0)}_{\gamma+\alpha+\mu}. \tag{102}$$

So $t^{1_{[q]}}(\partial(\bar{\gamma})\partial_p - \mu_p \partial).v_{\gamma,\mathbf{0}} = c v_{\gamma,\mathbf{0}}$ for some $c \in \mathbb{F}$. Moreover, (95) gives

$$\begin{aligned} & t^{1_{[q]}}(\partial(\bar{\gamma})\partial_p - \mu_p \partial).v_{\gamma+\alpha,\mathbf{0}} \\ = & \frac{1}{\partial(\bar{\gamma})} t^{1_{[q]}}(\partial(\bar{\gamma})\partial_p - \mu_p \partial).(x^\alpha \partial.v_{\gamma,\mathbf{0}}) \\ = & \frac{1}{\partial(\bar{\gamma})} x^\alpha \partial.(t^{1_{[q]}}(\partial(\bar{\gamma})\partial_p - \mu_p \partial).v_{\gamma,\mathbf{0}}) = c v_{\gamma+\alpha,\mathbf{0}}. \end{aligned} \tag{103}$$

Thus

$$\begin{aligned} x^\alpha(\partial(\bar{\gamma})\partial_p - \mu_p \partial).v_{\gamma,\mathbf{0}} &= \frac{1}{\alpha_r}[x^\alpha(\alpha_r \partial_q - \alpha_q \partial_r), t^{1_{[q]}}(\partial(\bar{\gamma})\partial_p - \mu_p \partial)].v_{\gamma,\mathbf{0}} \\ &= \frac{1}{\alpha_r}(x^\alpha(\alpha_r \partial_q - \alpha_q \partial_r).(c v_{\gamma,\mathbf{0}}) - c(x^\alpha(\alpha_r \partial_q - \alpha_q \partial_r).v_{\gamma,\mathbf{0}})) \\ &= 0, \end{aligned} \tag{104}$$

which combined with (95) implies

$$x^\alpha \partial_p.v_{\gamma,\mathbf{0}} = \mu_p v_{\gamma+\alpha,\mathbf{0}}. \tag{105}$$

Subcase 1.2. $\alpha_r \bar{\gamma}_s - \alpha_s \bar{\gamma}_r = 0$ for all $s \in \overline{l_1+1, l} \backslash \{r, q\}$.

Since $(\ker \alpha \cap \sum_{i=l_1+1}^{l} \mathbb{F}\partial_i) \backslash \ker \bar{\gamma} \neq \emptyset$, we have $q \in \overline{l_1+1, l_1+l_2} \backslash \{r\}$ and $\alpha_r \bar{\gamma}_q - \alpha_q \bar{\gamma}_r \neq 0$. Pick $s \in \overline{l_1+1, l} \backslash \{r, q\}$. Choose $\beta \in \Gamma \backslash \{0\}$ such that $\alpha_r \beta_s - \alpha_s \beta_r \neq 0$. Then Subcase 1.1 indicates

$$x^\alpha \partial_p.v_{\gamma-\beta,\mathbf{0}} = \mu_p v_{\gamma-\beta+\alpha,\mathbf{0}}. \tag{106}$$

Let $\partial = \alpha_r \partial_s - \alpha_s \partial_r$ and $\partial' = \alpha_r \partial_q - \alpha_q \partial_r$. Then $\partial(\beta) \neq 0$ and $\partial'(\bar{\gamma}) \neq 0$. So (95) and (106) give rise to

$$\begin{aligned} x^\alpha \partial_p.v_{\gamma,\mathbf{0}} &= \frac{1}{\partial(\beta)\partial'(\bar{\gamma})} x^\alpha \partial_p.x^\beta(\partial(\beta)\partial' - \partial'(\beta)\partial).v_{\gamma-\beta,\mathbf{0}} \\ &= \frac{1}{\partial(\beta)\partial'(\bar{\gamma})} x^\beta(\partial(\beta)\partial' - \partial'(\beta)\partial).x^\alpha \partial_p.v_{\gamma-\beta,\mathbf{0}} \\ &= \mu_p v_{\gamma+\alpha,\mathbf{0}}. \end{aligned} \tag{107}$$

Case 2. $(\ker \alpha \cap \sum_{i=l_1+1}^{l} \mathbb{F}\partial_i) \subseteq \ker \bar{\gamma}$.

Pick $s \in \overline{l_1+1, l} \backslash \{r\}$. Choose $\beta \in \Gamma \backslash \{0\}$ such that $\alpha_r \beta_s - \alpha_s \beta_r \neq 0$. Let $\partial = \beta_s \partial_r - \beta_r \partial_s$. Then $\partial(\alpha) \neq 0$. Moreover, since $(\ker \alpha \cap \sum_{i=l_1+1}^{l} \mathbb{F}\partial_i) \subseteq \ker \bar{\gamma}$

and $\gamma+\zeta\neq 0\neq\gamma+\zeta+\alpha$, we must have $\partial(\bar{\gamma})\neq 0$ and $\partial(\bar{\gamma}+\alpha)\neq 0$ by Claim 2. So Case 1 gives

$$x^{\alpha}\partial_p.v_{\gamma-\beta,\mathbf{0}}=\mu_p v_{\gamma-\beta+\alpha,\mathbf{0}},\quad x^{\alpha+\beta}\partial_p.v_{\gamma-\beta,\mathbf{0}}=\mu_p v_{\gamma+\alpha,\mathbf{0}} \tag{108}$$

because $\alpha_r\beta_s-\alpha_s\beta_r\neq 0$ and $(\alpha_r+\beta_r)(\bar{\gamma}_s-\beta_s)-(\alpha_s+\beta_s)(\bar{\gamma}_r-\beta_r)=-\partial(\bar{\gamma}+\alpha)\neq 0$. Thus (95) indicates

$$\begin{aligned}x^{\alpha}\partial_p.v_{\gamma,\mathbf{0}} &= \frac{1}{\partial(\bar{\gamma})}x^{\alpha}\partial_p.x^{\beta}\partial.v_{\gamma-\beta,\mathbf{0}}\\ &= \frac{1}{\partial(\bar{\gamma})}(x^{\beta}\partial.x^{\alpha}\partial_p.v_{\gamma-\beta,\mathbf{0}}-\partial(\alpha)x^{\alpha+\beta}\partial_p.v_{\gamma-\beta,\mathbf{0}})\\ &= \mu_p v_{\gamma+\alpha,\mathbf{0}}.\end{aligned} \tag{109}$$

So this claim holds.

Moreover, we have:

Claim 5. If $\zeta\in\Gamma$, $l_1>0$ and $\mu_q\neq 0$ for some $q\in\overline{1,l_1}$, then $V^{(0)}_{-\zeta+\mu}\neq\{0\}$ and there exists nonzero $v_{-\zeta,\mathbf{0}}\in V^{(0)}_{-\zeta+\mu}$ such that

$$x^{\alpha}\partial_p.v_{-\zeta-\alpha,\mathbf{0}}=\mu_p v_{-\zeta,\mathbf{0}} \text{ and } x^{\alpha}\partial_p.v_{-\zeta,\mathbf{0}}=\mu_p v_{-\zeta+\alpha,\mathbf{0}} \tag{110}$$

for all $p\in\overline{1,l_1},\ \alpha\in\Gamma\setminus\{0\}$.

Fix some $q\in\overline{1,l_1}$ such that $\mu_q\neq 0$. Moreover, we fix some $\beta\in\Gamma\setminus\{0\}$. By (101) we have

$$x^{-\beta}\partial_q.x^{\beta}\partial_q.v_{-\zeta-\beta,\mathbf{0}}=x^{\beta}\partial_q.x^{-\beta}\partial_q.v_{-\zeta-\beta,\mathbf{0}}=\mu_q^2 v_{-\zeta-\beta,\mathbf{0}}\neq 0. \tag{111}$$

So we can choose

$$0\neq v_{-\zeta,\mathbf{0}}=\frac{1}{\mu_q}x^{\beta}\partial_q.v_{-\zeta-\beta,\mathbf{0}}\in V^{(0)}_{-\zeta+\mu}. \tag{112}$$

It follows immediately from (111) that

$$x^{-\beta}\partial_q.v_{-\zeta,\mathbf{0}}=\mu_q v_{-\zeta-\beta,\mathbf{0}}. \tag{113}$$

For any $p\in\overline{1,l_1}$ and $\alpha\in\Gamma\setminus\{0,\beta\}$, we have

$$x^{\alpha}\partial_p.v_{-\zeta,\mathbf{0}}=\frac{1}{\mu_q}x^{\alpha}\partial_p.x^{\beta}\partial_q.v_{-\zeta-\beta,\mathbf{0}}=\frac{1}{\mu_q}x^{\beta}\partial_q.x^{\alpha}\partial_p.v_{-\zeta-\beta,\mathbf{0}}=\mu_p v_{-\zeta+\alpha,\mathbf{0}} \tag{114}$$

by (101) and (112). Moreover, (101) and (112) imply

$$x^{\alpha}\partial_p.v_{-\zeta-\alpha,\mathbf{0}}=\frac{1}{\mu_q}x^{\alpha}\partial_p.x^{\beta}\partial_q.v_{-\zeta-\alpha-\beta,\mathbf{0}}=\frac{1}{\mu_q}x^{\beta}\partial_q.x^{\alpha}\partial_p.v_{-\zeta-\alpha-\beta,\mathbf{0}}=\mu_p v_{-\zeta,\mathbf{0}} \tag{115}$$

for any $p\in\overline{1,l_1}$ and $\alpha\in\Gamma\setminus\{0,-\beta\}$. Furthermore, (101), (114) and (115) give

$$x^{\beta}\partial_p.v_{-\zeta,\mathbf{0}}=\frac{1}{\mu_q}x^{\beta}\partial_p.x^{\alpha}\partial_q.v_{-\zeta-\alpha,\mathbf{0}}=\frac{1}{\mu_q}x^{\alpha}\partial_q.x^{\beta}\partial_p.v_{-\zeta-\alpha,\mathbf{0}}=\mu_p v_{-\zeta+\beta,\mathbf{0}}, \tag{116}$$

$$x^{-\beta}\partial_p.v_{-\zeta+\beta,\mathbf{0}} = \frac{1}{\mu_q}x^{-\beta}\partial_p.x^{\alpha}\partial_q.v_{-\zeta+\beta-\alpha,\mathbf{0}} = \frac{1}{\mu_q}x^{\alpha}\partial_q.x^{-\beta}\partial_p.v_{-\zeta+\beta-\alpha,\mathbf{0}} = \mu_p v_{-\zeta,\mathbf{0}} \quad (117)$$

for any $p \in \overline{1,l_1}$, where $\alpha \in \Gamma\backslash\{0,\pm\beta\}$. So this claim follows from (114)–(117) .

Claim 6. If $\zeta \in \Gamma$, $l_1 > 0$ and $\mu_q = 0$ for all $q \in \overline{1,l_1}$, then

$$x^{\alpha}\partial_p.v_{-\zeta-\alpha,\mathbf{0}} = 0 \text{ and } x^{\alpha}\partial_p.V^{(0)}_{-\zeta+\mu} = \{0\} \quad \text{for all } p \in \overline{1,l_1},\ \alpha \in \Gamma\backslash\{0\}. \quad (118)$$

The proof of this claim is similar to that of Claim 3. We omit the details here.

When $\zeta \in \Gamma$ and $\mu_q = 0$ for all $q \in \overline{1,l_1}$, we have $\mu = \zeta \in \Gamma$ by Claim 2. So Claims 2–6 imply that $V^{(0)}_{\gamma+\mu} \neq \{0\}$ for all $\gamma \in \Gamma\backslash\{-\mu\}$, and that there exist nonzero $v_{\gamma,\mathbf{0}} \in V^{(0)}_{\gamma+\mu}$ with $\gamma \in \Gamma\backslash\{-\mu\}$ such that

$$D_{p,q}(x^{\alpha}).v_{\gamma,\mathbf{0}} = (\alpha_p(\gamma_q+\mu_q) - \alpha_q(\gamma_p+\mu_p))v_{\alpha+\gamma,\mathbf{0}} \quad (119)$$

for $p,q \in \overline{1,l}$, $\alpha \in \Gamma\backslash\{0\}$ and $\gamma \in \Gamma\backslash\{-\mu\}$. This completes the proof of the lemma. ■

Lemma 3.2 enables us to choose nonzero $v_{\gamma,\mathbf{0}} \in V^{(0)}_{\gamma+\mu}$ with $\gamma \in \Gamma\backslash\{-\mu\}$ such that

$$D_{p,q}(x^{\alpha}).v_{\gamma,\mathbf{0}} = (\alpha_p(\gamma_q+\mu_q) - \alpha_q(\gamma_p+\mu_p))v_{\alpha+\gamma,\mathbf{0}} \quad (120)$$

for $p,q \in \overline{1,l}$, $\alpha \in \Gamma\backslash\{0\}$ and $\gamma \in \Gamma\backslash\{-\mu\}$. Next, we define $v_{\gamma,\mathbf{i}}$ for $\gamma \in \Gamma\backslash\{-\mu\}$ and $\mathbf{0} \neq \mathbf{i} \in \mathbb{N}^{l_1+l_2}$ inductively as follows:

Let $\bar{\alpha} = \alpha + \mu$ for all $\alpha \in \Gamma$. For $\gamma \in \Gamma\backslash\{-\mu\}$ and $\mathbf{0} \neq \mathbf{i} \in \mathbb{N}^{l_1+l_2}$, we define

$$p_{\mathbf{i}} = \min\{r \in \overline{1,l_1+l_2} \mid i_r \neq 0\}, \quad (121)$$

$$P(\mathbf{i}) = \{r \in \overline{1,l_1+l_2}\backslash\{p_{\mathbf{i}}\} \mid i_r \neq 0\}, \quad (122)$$

$$Q(\gamma,\mathbf{i}) = \{r \in \overline{1,l}\backslash\{p_{\mathbf{i}}\} \mid \bar{\gamma}_r \neq 0\}. \quad (123)$$

Definition 3.3. We choose $v_{\gamma,\mathbf{0}}$ for $\gamma \in \Gamma\backslash\{-\mu\}$ as those in (120). Suppose that we have defined $v_{\gamma,\mathbf{i}}$ for $\gamma \in \Gamma\backslash\{-\mu\}$ and $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| \leq k$, where $k \geq 0$. We then define $v_{\gamma,\mathbf{i}}$ for $\gamma \in \Gamma\backslash\{-\mu\}$ and $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$ in three steps:

Step 1. Defining $v_{\gamma,\mathbf{i}}$ for the case that $Q(\gamma,\mathbf{i}) \neq \emptyset$.

Set $q(\gamma,\mathbf{i}) = \min Q(\gamma,\mathbf{i})$. We define

$$v_{\gamma,\mathbf{i}} = \frac{1}{\bar{\gamma}_{q(\gamma,\mathbf{i})}}\big(t^{1_{[p_{\mathbf{i}}]}}\partial_{q(\gamma,\mathbf{i})}.v_{\gamma,\mathbf{i}-1_{[p_{\mathbf{i}}]}} - i_{q(\gamma,\mathbf{i})}v_{\gamma,\mathbf{i}-1_{[q(\gamma,\mathbf{i})]}}\big). \quad (124)$$

Step 2. Defining $v_{\gamma,\mathbf{i}}$ for the case that $Q(\gamma,\mathbf{i}) = \emptyset$ and $P(\mathbf{i}) \neq \emptyset$.

Set $p(\mathbf{i}) = \min P(\mathbf{i})$. Since $\gamma + \mu \neq 0$ and $Q(\gamma, \mathbf{i}) = \emptyset$, (123) indicates $\bar{\gamma}_{p_\mathbf{i}} \neq 0$. We define

$$v_{\gamma,\mathbf{i}} = \frac{1}{\bar{\gamma}_{p_\mathbf{i}}}\big(t^{1_{[p(\mathbf{i})]}}\partial_{p_\mathbf{i}}.v_{\gamma,\mathbf{i}-1_{[p(\mathbf{i})]}} - i_{p_\mathbf{i}} v_{\gamma,\mathbf{i}-1_{[p_\mathbf{i}]}}\big). \tag{125}$$

Step 3. Defining $v_{\gamma,\mathbf{i}}$ for the case that $Q(\gamma, \mathbf{i}) = \emptyset$ and $P(\mathbf{i}) = \emptyset$.

Set $s(\mathbf{i}) = \min\{\overline{1, l_1 + l_2}\backslash\{p_\mathbf{i}\}\}$. Since $(\gamma, \mathbf{i} - 1_{[p_\mathbf{i}]} + 1_{[s(\mathbf{i})]})$ belongs to the Case in Step 1 or Step 2, $v_{\gamma,\mathbf{i}-1_{[p_\mathbf{i}]}+1_{[s(\mathbf{i})]}}$ was defined in (124) or (125). Thus we can define

$$v_{\gamma,\mathbf{i}} = t^{1_{[p_\mathbf{i}]}}\partial_{s(\mathbf{i})}.v_{\gamma,\mathbf{i}-1_{[p_\mathbf{i}]}+1_{[s(\mathbf{i})]}}. \tag{126}$$

Now we have defined $\{v_{\gamma,\mathbf{i}} \mid \gamma \in \Gamma\backslash\{-\mu\},\ \mathbf{i} \in \mathbb{N}^{l_1+l_2}\}$ inductively. Next we want to derive the action of the set (32) on $\{v_{\gamma,\mathbf{i}} \mid \gamma \in \Gamma\backslash\{-\mu\},\ \mathbf{i} \in \mathbb{N}^{l_1+l_2}\}$ in several lemmas.

Lemma 3.4. *For any $p \in \overline{1, l_1 + l_2}$, $q \in \overline{1, l}\backslash\{p\}$ and $\beta \in \Gamma\backslash\{-\mu\}$, we have*

$$t^{1_{[p]}}\partial_q.v_{\beta,\mathbf{0}} = (\beta_q + \mu_q)v_{\beta,1_{[p]}}. \tag{127}$$

Proof. Let $\bar{\alpha} = \alpha + \mu$ for all $\alpha \in \Gamma$. Define $Q(\beta, 1_{[p]})$ for $\beta \in \Gamma\backslash\{-\mu\}$ and $p \in \overline{1, l_1 + l_2}$ as in (121)–(123). We divide $\beta \in \Gamma\backslash\{-\mu\}$ and $p \in \overline{1, l_1 + l_2}$ into two cases.

Case 1. $Q(\beta, 1_{[p]}) \neq \emptyset$.

Let $r = \min Q(\beta, 1_{[p]})$. Then (124) shows

$$t^{1_{[p]}}\partial_r.v_{\beta,\mathbf{0}} = \bar{\beta}_r v_{\beta,1_{[p]}}. \tag{128}$$

Fix some $s \in \overline{1, l_1 + l_2}\backslash\{p, r\}$. Let $\tilde{\partial}_1 = \bar{\beta}_r\partial_p - \bar{\beta}_p\partial_r$ and $\tilde{\partial}_2 = \bar{\beta}_r\partial_s - \bar{\beta}_s\partial_r$. Then

$$\tilde{\partial}_1(t^{1_{[p]}+1_{[s]}})\tilde{\partial}_2 - \tilde{\partial}_2(t^{1_{[p]}+1_{[s]}})\tilde{\partial}_1 = \bar{\beta}_r(t^{1_{[s]}}\tilde{\partial}_2 - t^{1_{[p]}}\tilde{\partial}_1) \in \mathscr{S}(l_1, l_2, l_3; \Gamma). \tag{129}$$

Since (6) gives

$$(t^{1_{[s]}}\tilde{\partial}_2 - t^{1_{[p]}}\tilde{\partial}_1).v_{\beta,\mathbf{0}} = a_1 v_{\beta,\mathbf{0}}, \quad t^{1_{[p]}}\tilde{\partial}_2.v_{\beta,\mathbf{0}} = a_2 v_{\beta,\mathbf{0}} \tag{130}$$

for some $a_1, a_2 \in \mathbb{F}$, we have

$$t^{1_{[p]}}\tilde{\partial}_2.v_{\beta,\mathbf{0}} = \frac{1}{2\bar{\beta}_r}[t^{1_{[p]}}\tilde{\partial}_2, t^{1_{[s]}}\tilde{\partial}_2 - t^{1_{[p]}}\tilde{\partial}_1].v_{\beta,\mathbf{0}} = 0. \tag{131}$$

Moreover, from (6) it follows that

$$t^{1_{[s]}}(\bar{\beta}_r\partial_q - \bar{\beta}_q\partial_r).v_{\beta,\mathbf{0}} \in V^{(0)}_{\beta+\mu} \quad \text{for } q \in \overline{1, l}\backslash\{p, r, s\}. \tag{132}$$

So (131) and (132) imply

$$t^{1_{[p]}}(\bar{\beta}_r\partial_q - \bar{\beta}_q\partial_r).v_{\beta,\mathbf{0}} = \frac{1}{\bar{\beta}_r}[t^{1_{[p]}}\tilde{\partial}_2, t^{1_{[s]}}(\bar{\beta}_r\partial_q - \bar{\beta}_q\partial_r)].v_{\beta,\mathbf{0}} = 0 \tag{133}$$

for $q \in \overline{1,l}\backslash\{p,r,s\}$. Since $\bar{\beta}_r \neq 0$, (128), (131) and (133) imply

$$t^{1_{[p]}}\partial_q.v_{\beta,\mathbf{0}} = (\beta_q+\mu_q)v_{\beta,1_{[p]}} \quad \text{for } q \in \overline{1,l}\backslash\{p\}. \tag{134}$$

Case 2. $Q(\beta,1_{[p]})=\emptyset$.

Pick $s \in \overline{1,l_1+l_2}\backslash\{p\}$. For any $q \in \overline{1,l}\backslash\{p,s\}$, (6) and $Q(\beta,1_{[p]})=\emptyset$ give rise to

$$t^{1_{[p]}}\partial_s.v_{\beta,\mathbf{0}} = bv_{\beta,\mathbf{0}} \text{ and } t^{1_{[s]}}\partial_q.v_{\beta,\mathbf{0}} = b_q v_{\beta,\mathbf{0}} \tag{135}$$

with some $b, b_q \in \mathbb{F}$. So

$$t^{1_{[p]}}\partial_q.v_{\beta,\mathbf{0}} = [t^{1_{[p]}}\partial_s, t^{1_{[s]}}\partial_q].v_{\beta,\mathbf{0}} = 0 \quad \text{for } q \in \overline{1,l}\backslash\{p,s\}. \tag{136}$$

By picking another $s \in \overline{1,l_1+l_2}\backslash\{p\}$, we can get

$$t^{1_{[p]}}\partial_q.v_{\beta,\mathbf{0}} = 0 = (\beta_q+\mu_q)v_{\beta,1_{[p]}} \text{ for } q \in \overline{1,l}\backslash\{p\}. \tag{137}$$

This completes the proof of the lemma. ■

Lemma 3.5. *For any $\beta \in \Gamma\backslash\{-\mu\}$ and $p,q \in \overline{1,l_1+l_2}$ with $p \neq q$, we have*

$$D_{p,q}(t^{1_{[p]}+1_{[q]}}).v_{\beta,\boldsymbol{0}} = (\beta_q+\mu_q)v_{\beta,1_{[q]}} - (\beta_p+\mu_p)v_{\beta,1_{[p]}}. \tag{138}$$

Proof. Let $\bar{\alpha} = \alpha+\mu$ for all $\alpha \in \Gamma$. Fix any $\beta \in \Gamma\backslash\{-\mu\}$ and $p,q \in \overline{1,l_1+l_2}$ with $p \neq q$. Then we shall give the proof in two cases.

Case 1. There exists $s \in \overline{1,l}\backslash\{p,q\}$ such that $\bar{\beta}_s \neq 0$.

Choose $s \in \overline{1,l}\backslash\{p,q\}$ such that $\bar{\beta}_s \neq 0$. Let $\tilde{\partial}_1 = \bar{\beta}_s\partial_p - \bar{\beta}_p\partial_s$ and $\tilde{\partial}_2 = \bar{\beta}_s\partial_q - \bar{\beta}_q\partial_s$. Then

$$[t^{1_{[p]}}\tilde{\partial}_2, t^{1_{[q]}}\tilde{\partial}_1] = -\bar{\beta}_s^2 D_{p,q}(t^{1_{[p]}+1_{[q]}}) + \bar{\beta}_s(\bar{\beta}_q t^{1_{[q]}}\partial_s - \bar{\beta}_p t^{1_{[p]}}\partial_s) \tag{139}$$

So by (139) and Lemma 3.4, we have

$$\begin{aligned} D_{p,q}(t^{1_{[p]}+1_{[q]}}).v_{\beta,\mathbf{0}} &= \frac{1}{\bar{\beta}_s^2}(\bar{\beta}_s(\bar{\beta}_q t^{1_{[q]}}\partial_s - \bar{\beta}_p t^{1_{[p]}}\partial_s) - [t^{1_{[p]}}\tilde{\partial}_2, t^{1_{[q]}}\tilde{\partial}_1]).v_{\beta,\mathbf{0}} \\ &= \bar{\beta}_q v_{\beta,1_{[q]}} - \bar{\beta}_p v_{\beta,1_{[p]}}, \end{aligned} \tag{140}$$

which coincides with (138).

Case 2. $\bar{\beta}_s = 0$ for all $s \in \overline{1,l}\backslash\{p,q\}$.

Subcase 2.1. $\bar{\beta}_p \neq 0$ and $\bar{\beta}_q = 0$.

Take $r = \min\{\overline{1,l_1+l_2}\backslash\{p\}\}$. By (126) we know that

$$v_{\beta,1_{[p]}} = t^{1_{[p]}}\partial_r.v_{\beta,1_{[r]}}. \tag{141}$$

Since $\bar{\beta}_q = 0$ and $\bar{\beta}_r = 0$, Lemma 3.4 tells that

$$D_{r,q}(t^{1_{[q]}+1_{[r]}}).v_{\beta,\mathbf{0}} = (t^{1_{[q]}}\partial_q - t^{1_{[r]}}\partial_r).v_{\beta,\mathbf{0}} = [t^{1_{[q]}}\partial_r, t^{1_{[r]}}\partial_q].v_{\beta,\mathbf{0}} = 0. \tag{142}$$

So (141), (142) and Lemma 3.4 imply

$$\begin{aligned} D_{p,q}(t^{1_{[p]}+1_{[q]}}).v_{\beta,\mathbf{0}} &= (D_{p,q}(t^{1_{[p]}+1_{[q]}}) - D_{r,q}(t^{1_{[q]}+1_{[r]}})).v_{\beta,\mathbf{0}} \\ &= [t^{1_{[r]}}\partial_p, t^{1_{[p]}}\partial_r].v_{\beta,\mathbf{0}} \\ &= -t^{1_{[p]}}\partial_r.(\bar{\beta}_p v_{\beta,1_{[r]}}) \\ &= -\bar{\beta}_p v_{\beta,1_{[p]}}, \end{aligned} \tag{143}$$

which coincides with (138).

Subcase 2.2. $\bar{\beta}_p = 0$ and $\bar{\beta}_q \neq 0$.

We go back to Subcase 2.1 by interchanging p and q, namely,

$$D_{p,q}(t^{1_{[p]}+1_{[q]}}).v_{\beta,\mathbf{0}} = -D_{q,p}(t^{1_{[p]}+1_{[q]}}).v_{\beta,\mathbf{0}} = \bar{\beta}_q v_{\beta,1_{[q]}}, \tag{144}$$

which coincides with (138).

Subcase 2.3. $\bar{\beta}_p \neq 0$ and $\bar{\beta}_q \neq 0$.

Pick $r \in \overline{1, l_1+l_2}\backslash\{p,q\}$, then $\bar{\beta}_r = 0$. By Subcase 2.1, we know that

$$D_{r,q}(t^{1_{[q]}+1_{[r]}}).v_{\beta,\mathbf{0}} = \bar{\beta}_q v_{\beta,1_{[p]}}, \tag{145}$$

$$D_{r,p}(t^{1_{[p]}+1_{[r]}}).v_{\beta,\mathbf{0}} = \bar{\beta}_p v_{\beta,1_{[p]}}. \tag{146}$$

So the difference of the above two equations is

$$D_{p,q}(t^{1_{[p]}+1_{[q]}}).v_{\beta,\mathbf{0}} = \bar{\beta}_q v_{\beta,1_{[q]}} - \bar{\beta}_p v_{\beta,1_{[p]}}, \tag{147}$$

which coincides with (138).

This completes the proof of the lemma. ∎

Lemma 3.6. *For any* $p, s \in \overline{1, l_1+l_2}$, $q \in \overline{1,l}\backslash\{p\}$, $\beta \in \Gamma\backslash\{-\mu\}$ *and* $\boldsymbol{i} \in \mathbb{N}^{l_1+l_2}$, *we have*

$$t^{1_{[p]}}\partial_q.v_{\beta,\boldsymbol{i}} = (\beta_q+\mu_q)v_{\beta,\boldsymbol{i}+1_{[p]}} + i_q v_{\beta,\boldsymbol{i}+1_{[p]}-1_{[q]}}, \tag{148}$$

$$(t^{1_{[p]}}\partial_p - t^{1_{[s]}}\partial_s).v_{\beta,\boldsymbol{i}} = (\beta_p+\mu_p)v_{\beta,\boldsymbol{i}+1_{[p]}} - (\beta_s+\mu_s)v_{\beta,\boldsymbol{i}+1_{[s]}} + (i_p - i_s)v_{\beta,\boldsymbol{i}}. \tag{149}$$

Proof. Let $\bar{\alpha} = \alpha+\mu$ for all $\alpha \in \Gamma$. We prove this lemma by induction on $|\mathbf{i}|$. Recall that Lemma 3.4 and Lemma 3.5 have given the proof for the case $|\mathbf{i}| = 0$. Suppose this lemma holds for $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| \leq k$, where $k \geq 0$. Then it suffices to prove (148) and (149) for $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$. We only prove (148) for $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$ in the following. The proof of (149) for $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$ only needs minor changes from Lemma 3.5, which can be derived easily; we omit the details here.

Fix any $\beta \in \Gamma\backslash\{-\mu\}$ and $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$. Define $p_{\mathbf{i}}$, $P(\mathbf{i})$ and $Q(\beta,\mathbf{i})$ as in (121)–(123). Then we prove (148) for $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$ in three cases.

Case 1. $Q(\beta,\mathbf{i}) \neq \emptyset$.

Let $q(\beta,\mathbf{i})=\min Q(\beta,\mathbf{i})$. Recall that (cf. (124))

$$v_{\beta,\mathbf{i}}=\frac{1}{\bar\beta_{q(\beta,\mathbf{i})}}(t^{1_{[p_{\mathbf{i}}]}}\partial_{q(\beta,\mathbf{i})}.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}}-i_{q(\beta,\mathbf{i})}v_{\beta,\mathbf{i}-1_{[q(\beta,\mathbf{i})]}}). \tag{150}$$

Fixing any $p\in\overline{1,l_1+l_2}$, we derive the action of $t^{1_{[p]}}\partial_q$ for $q\in\overline{1,l}\backslash\{p\}$ in two subcases.

Subcase 1.1. $\bar\beta_q=0$ for all $q\in\overline{1,l}\backslash\{p\}$.

Expression (150) and the induction hypothesis give rise to

$$\begin{aligned}t^{1_{[p]}}\partial_q.v_{\beta,\mathbf{i}} &= \frac{1}{\bar\beta_{q(\beta,\mathbf{i})}}\big(t^{1_{[p]}}\partial_q.t^{1_{[p_{\mathbf{i}}]}}\partial_{q(\beta,\mathbf{i})}.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}}-i_{q(\beta,\mathbf{i})}t^{1_{[p]}}\partial_q.v_{\beta,\mathbf{i}-1_{[q(\beta,\mathbf{i})]}}\big)\\ &= \frac{1}{\bar\beta_{q(\beta,\mathbf{i})}}\big(t^{1_{[p_{\mathbf{i}}]}}\partial_{q(\beta,\mathbf{i})}.t^{1_{[p]}}\partial_q.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}}-i_{q(\beta,\mathbf{i})}t^{1_{[p]}}\partial_q.v_{\beta,\mathbf{i}-1_{[q(\beta,\mathbf{i})]}}\\ &\quad+(\delta_{q,p_{\mathbf{i}}}t^{1_{[p]}}\partial_{q(\beta,\mathbf{i})}-\delta_{q(\beta,\mathbf{i}),p}t^{1_{[p_{\mathbf{i}}]}}\partial_q).v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}}\big)\\ &= \frac{1}{\bar\beta_{q(\beta,\mathbf{i})}}\big((i_q-\delta_{q,p_{\mathbf{i}}})t^{1_{[p_{\mathbf{i}}]}}\partial_{q(\beta,\mathbf{i})}.v_{\beta,\mathbf{i}+1_{[p]}-1_{[p_{\mathbf{i}}]}-1_{[q]}}-i_{q(\beta,\mathbf{i})}t^{1_{[p]}}\partial_q.v_{\beta,\mathbf{i}-1_{[q(\beta,\mathbf{i})]}}\\ &\quad+(\delta_{q,p_{\mathbf{i}}}t^{1_{[p]}}\partial_{q(\beta,\mathbf{i})}-\delta_{q(\beta,\mathbf{i}),p}t^{1_{[p_{\mathbf{i}}]}}\partial_q).v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}}\big)\\ &= i_q v_{\beta,\mathbf{i}+1_{[p]}-1_{[q]}}\end{aligned} \tag{151}$$

for $q\in\overline{1,l}\backslash\{p\}$, which coincides with (148).

Subcase 1.2. There exists $q\in\overline{1,l}\backslash\{p\}$ such that $\bar\beta_q\neq 0$.

Take $r=\min\{q\in\overline{1,l}\backslash\{p\}\mid\bar\beta_q\neq 0\}$. First we want to derive the action of $t^{1_{[p]}}\partial_r$.

If $p<p_{\mathbf{i}}$, then the definition of (124) gives

$$t^{1_{[p]}}\partial_r.v_{\beta,\mathbf{i}}=\bar\beta_r v_{\beta,\mathbf{i}+1_{[p]}}+i_r v_{\beta,\mathbf{i}+1_{[p]}-1_{[r]}}. \tag{152}$$

If $p\geq p_{\mathbf{i}}$, then by (150) and the induction hypothesis, we get

$$\begin{aligned}&t^{1_{[p]}}\partial_r.v_{\beta,\mathbf{i}}\\ =&\ \frac{1}{\bar\beta_{q(\beta,\mathbf{i})}}\big(t^{1_{[p]}}\partial_r.t^{1_{[p_{\mathbf{i}}]}}\partial_{q(\beta,\mathbf{i})}.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}}-i_{q(\beta,\mathbf{i})}t^{1_{[p]}}\partial_r.v_{\beta,\mathbf{i}-1_{[q(\beta,\mathbf{i})]}}\big)\\ =&\ \frac{1}{\bar\beta_{q(\beta,\mathbf{i})}}\big(t^{1_{[p_{\mathbf{i}}]}}\partial_{q(\beta,\mathbf{i})}.t^{1_{[p]}}\partial_r.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}}-i_{q(\beta,\mathbf{i})}t^{1_{[p]}}\partial_r.v_{\beta,\mathbf{i}-1_{[q(\beta,\mathbf{i})]}}\\ &+(\delta_{r,p_{\mathbf{i}}}t^{1_{[p]}}\partial_{q(\beta,\mathbf{i})}-\delta_{q(\beta,\mathbf{i}),p}t^{1_{[p_{\mathbf{i}}]}}\partial_r).v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}}\big)\\ =&\ \frac{1}{\bar\beta_{q(\beta,\mathbf{i})}}\big(\bar\beta_r t^{1_{[p_{\mathbf{i}}]}}\partial_{q(\beta,\mathbf{i})}.v_{\beta,\mathbf{i}+1_{[p]}-1_{[p_{\mathbf{i}}]}}+(i_r-\delta_{r,p_{\mathbf{i}}})t^{1_{[p_{\mathbf{i}}]}}\partial_{q(\beta,\mathbf{i})}.v_{\beta,\mathbf{i}+1_{[p]}-1_{[p_{\mathbf{i}}]}-1_{[r]}}\\ &-i_{q(\beta,\mathbf{i})}t^{1_{[p]}}\partial_r.v_{\beta,\mathbf{i}-1_{[q(\beta,\mathbf{i})]}}+(\delta_{r,p_{\mathbf{i}}}t^{1_{[p]}}\partial_{q(\beta,\mathbf{i})}-\delta_{q(\beta,\mathbf{i}),p}t^{1_{[p_{\mathbf{i}}]}}\partial_r).v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}}\big)\\ =&\ \bar\beta_r v_{\beta,\mathbf{i}+1_{[p]}}+i_r v_{\beta,\mathbf{i}+1_{[p]}-1_{[r]}},\end{aligned} \tag{153}$$

where

$$\begin{aligned}&t^{1_{[p_{\mathbf{i}}]}}\partial_{q(\beta,\mathbf{i})}.v_{\beta,\mathbf{i}+1_{[p]}-1_{[p_{\mathbf{i}}]}}\\ =&\ \bar\beta_{q(\beta,\mathbf{i})}v_{\beta,\mathbf{i}+1_{[p]}}+(i_{q(\beta,\mathbf{i})}+\delta_{q(\beta,\mathbf{i}),p})v_{\beta,\mathbf{i}+1_{[p]}-1_{[q(\beta,\mathbf{i})]}}\end{aligned} \tag{154}$$

because of $p \geq p_{\mathbf{i}}$ and (124).

Then we want to derive the action of $t^{1_{[p]}}\partial_q$ for $q \in \overline{1,l}\backslash\{p,r\}$. By (150) and the induction hypothesis, we have

$$\begin{aligned}
& t^{1_{[p]}}(\bar{\beta}_r\partial_q - \bar{\beta}_q\partial_r).v_{\beta,\mathbf{i}} \\
= \; & \frac{1}{\bar{\beta}_{q(\beta,\mathbf{i})}}\big(t^{1_{[p]}}(\bar{\beta}_r\partial_q - \bar{\beta}_q\partial_r).t^{1_{[p_{\mathbf{i}}]}}\partial_{q(\beta,\mathbf{i})}.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}} \\
& -i_{q(\beta,\mathbf{i})}t^{1_{[p]}}(\bar{\beta}_r\partial_q - \bar{\beta}_q\partial_r).v_{\beta,\mathbf{i}-1_{[q(\beta,\mathbf{i})]}}\big) \\
= \; & \frac{1}{\bar{\beta}_{q(\beta,\mathbf{i})}}\big(t^{1_{[p_{\mathbf{i}}]}}\partial_{q(\beta,\mathbf{i})}.t^{1_{[p]}}(\bar{\beta}_r\partial_q - \bar{\beta}_q\partial_r).v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}} \\
& +((\bar{\beta}_r\delta_{q,p_{\mathbf{i}}} - \bar{\beta}_q\delta_{r,p_{\mathbf{i}}})t^{1_{[p]}}\partial_{q(\beta,\mathbf{i})} - \delta_{q(\beta,\mathbf{i}),p}t^{1_{[p_{\mathbf{i}}]}}(\bar{\beta}_r\partial_q - \bar{\beta}_q\partial_r)).v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}} \\
& -i_{q(\beta,\mathbf{i})}t^{1_{[p]}}(\bar{\beta}_r\partial_q - \bar{\beta}_q\partial_r).v_{\beta,\mathbf{i}-1_{[q(\beta,\mathbf{i})]}}\big) \\
= \; & \frac{1}{\bar{\beta}_{q(\beta,\mathbf{i})}}\big(t^{1_{[p_{\mathbf{i}}]}}\partial_{q(\beta,\mathbf{i})}.(\bar{\beta}_r(i_q - \delta_{q,p_{\mathbf{i}}})v_{\beta,\mathbf{i}+1_{[p]}-1_{[p_{\mathbf{i}}]}-1_{[q]}} \\
& -\bar{\beta}_q(i_r - \delta_{r,p_{\mathbf{i}}})v_{\beta,\mathbf{i}+1_{[p]}-1_{[p_{\mathbf{i}}]}-1_{[r]}}) + ((\bar{\beta}_r\delta_{q,p_{\mathbf{i}}} - \bar{\beta}_q\delta_{r,p_{\mathbf{i}}})t^{1_{[p]}}\partial_{q(\beta,\mathbf{i})} \\
& -\delta_{q(\beta,\mathbf{i}),p}t^{1_{[p_{\mathbf{i}}]}}(\bar{\beta}_r\partial_q - \bar{\beta}_q\partial_r)).v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}} - i_{q(\beta,\mathbf{i})}t^{1_{[p]}}(\bar{\beta}_r\partial_q - \bar{\beta}_q\partial_r).v_{\beta,\mathbf{i}-1_{[q(\beta,\mathbf{i})]}}\big) \\
= \; & \bar{\beta}_r i_q v_{\beta,\mathbf{i}+1_{[p]}-1_{[q]}} - \bar{\beta}_q i_r v_{\beta,\mathbf{i}+1_{[p]}-1_{[r]}}
\end{aligned} \tag{155}$$

for any $q \in \overline{1,l}\backslash\{p,r\}$. Since $\bar{\beta}_r \neq 0$, (152), (153) and (155) indicate

$$t^{1_{[p]}}\partial_q.v_{\beta,\mathbf{i}} = \bar{\beta}_q v_{\beta,\mathbf{i}+1_{[p]}} + i_q v_{\beta,\mathbf{i}+1_{[p]}-1_{[q]}} \tag{156}$$

for any $q \in \overline{1,l}\backslash\{p\}$, which coincides with (148).

Case 2. $Q(\beta,\mathbf{i}) = \emptyset$ and $P(\mathbf{i}) \neq \emptyset$.

Let $p(\mathbf{i}) = \min P(\mathbf{i})$. Recall that (cf. (125))

$$v_{\beta,\mathbf{i}} = \frac{1}{\bar{\beta}_{p_{\mathbf{i}}}}\big(t^{1_{[p(\mathbf{i})]}}\partial_{p_{\mathbf{i}}}.v_{\beta,\mathbf{i}-1_{[p(\mathbf{i})]}} - i_{p_{\mathbf{i}}}v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}}\big). \tag{157}$$

The rest proof of this case is analogous to Case 1. We omit the details.

Case 3. $Q(\beta,\mathbf{i}) = \emptyset$ and $P(\mathbf{i}) = \emptyset$.

Observe that $Q(\beta,\mathbf{i}) = \emptyset$ and $P(\mathbf{i}) = \emptyset$ mean

$$i_q = 0 \text{ and } \bar{\beta}_q = 0 \text{ for all } q \in \overline{1,l}\backslash\{p_{\mathbf{i}}\}. \tag{158}$$

So for any $r \in \overline{1,l_1+l_2}\backslash\{p_{\mathbf{i}}\}$, the pair $(\beta, \mathbf{i} - 1_{[p_{\mathbf{i}}]} + 1_{[r]})$ belongs to Case 1 or Case 2, which implies

$$t^{1_{[p]}}\partial_q.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[r]}} = \bar{\beta}_q v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[r]}+1_{[p]}} + (i_q - \delta_{p_{\mathbf{i}},q} + \delta_{r,q})v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[r]}+1_{[p]}-1_{[q]}} \tag{159}$$

for $p \in \overline{1,l_1+l_2}$, $q \in \overline{1,l}\backslash\{p\}$ and $r \in \overline{1,l_1+l_2}\backslash\{p_{\mathbf{i}}\}$. In particular, by (158) we have

$$t^{1_{[p_{\mathbf{i}}]}}\partial_r.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[r]}} = v_{\beta,\mathbf{i}} \quad \text{for } r \in \overline{1,l_1+l_2}\backslash\{p_{\mathbf{i}}\}. \tag{160}$$

So for any $q \in \overline{1,l}\backslash\{p_{\mathbf{i}}\}$, we obtain

$$t^{1_{[p_{\mathbf{i}}]}}\partial_q.v_{\beta,\mathbf{i}} = t^{1_{[p_{\mathbf{i}}]}}\partial_q.t^{1_{[p_{\mathbf{i}}]}}\partial_{r'}.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[r']}} = t^{1_{[p_{\mathbf{i}}]}}\partial_{r'}.t^{1_{[p_{\mathbf{i}}]}}\partial_q.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[r']}} = 0 \quad (161)$$

by (158) and (160), where $r' \in \overline{1,l_1+l_2}\backslash\{p_{\mathbf{i}},q\}$. Fix any $p \in \overline{1,l_1+l_2}\backslash\{p_{\mathbf{i}}\}$. Then (124) or (125) gives

$$t^{1_{[p]}}\partial_{p_{\mathbf{i}}}.v_{\beta,\mathbf{i}} = \bar{\beta}_{p_{\mathbf{i}}}v_{\beta,\mathbf{i}+1_{[p]}} + i_{p_{\mathbf{i}}}v_{\beta,\mathbf{i}+1_{[p]}-1_{[p_{\mathbf{i}}]}}. \quad (162)$$

And for $q \in \overline{1,l}\backslash\{p,p_{\mathbf{i}}\}$,

$$\begin{aligned} t^{1_{[p]}}\partial_q.v_{\beta,\mathbf{i}} &= t^{1_{[p]}}\partial_q.t^{1_{[p_{\mathbf{i}}]}}\partial_{r'}.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[r']}} \\ &= t^{1_{[p_{\mathbf{i}}]}}\partial_{r'}.t^{1_{[p]}}\partial_q.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[r']}} \\ &= \delta_{q,r'}t^{1_{[p_{\mathbf{i}}]}}\partial_{r'}.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[p]}} \\ &= 0 \end{aligned} \quad (163)$$

by (159) and (160), where $r' \in \overline{1,l_1+l_2}\backslash\{p,p_{\mathbf{i}}\}$. So in this case, (148) follows from (161), (162) and (163).

This completes the proof of the lemma. ■

Lemma 3.7. *For any $r \in \overline{1,l}$, $\beta \in \Gamma\backslash\{-\mu\}$ and $\boldsymbol{i} \in \mathbb{N}^{l_1+l_2}$, we have*

$$\partial_r.v_{\beta,\boldsymbol{i}} = (\beta_r + \mu_r)v_{\beta,\boldsymbol{i}} + i_r v_{\beta,\boldsymbol{i}-1_{[r]}}. \quad (164)$$

Proof. Let $\bar{\alpha} = \alpha + \mu$ for all $\alpha \in \Gamma$. We shall prove this lemma by induction on $|\mathbf{i}|$. We have already known that (164) holds when $|\mathbf{i}| = 0$ (cf. (6)). Suppose that it holds for $|\mathbf{i}| \leq k$, where $k \geq 0$. It suffices to prove (164) for $r \in \overline{1,l}$, $\beta \in \Gamma\backslash\{-\mu\}$ and $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$.

For any $\beta \in \Gamma\backslash\{-\mu\}$ and $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$, we define $p_{\mathbf{i}}$, $P(\mathbf{i})$ and $Q(\beta,\mathbf{i})$ as in (121)–(123). Then we give the proof in three cases.

Case 1. $Q(\beta,\mathbf{i}) \neq \emptyset$.

Let $q(\beta,\mathbf{i}) = \min Q(\beta,\mathbf{i})$. Then (124), Lemma 3.6 and the induction hypothesis give

$$\begin{aligned} \partial_r.v_{\beta,\mathbf{i}} &= \frac{1}{\bar{\beta}_{q(\beta,\mathbf{i})}}\big(t^{1_{[p_{\mathbf{i}}]}}\partial_{q(\beta,\mathbf{i})}.\partial_r.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}} + \delta_{r,p_{\mathbf{i}}}\partial_{q(\beta,\mathbf{i})}.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}} - i_{q(\beta,\mathbf{i})}\partial_r.v_{\beta,\mathbf{i}-1_{[q(\beta,\mathbf{i})]}}\big) \\ &= \bar{\beta}_r v_{\beta,\mathbf{i}} + i_r v_{\beta,\mathbf{i}-1_{[r]}} \end{aligned} \quad (165)$$

for $r \in \overline{1,l}$, which coincides with (164).

Case 2. $Q(\beta,\mathbf{i}) = \emptyset$ and $P(\mathbf{i}) \neq \emptyset$.

Let $p(\mathbf{i}) = \min P(\mathbf{i})$. Then (125), Lemma 3.6 and the induction hypothesis imply

$$\begin{aligned} \partial_r.v_{\beta,\mathbf{i}} &= \frac{1}{\bar{\beta}_{p_{\mathbf{i}}}}\big(t^{1_{[p(\mathbf{i})]}}\partial_{p_{\mathbf{i}}}.\partial_r.v_{\beta,\mathbf{i}-1_{[p(\mathbf{i})]}} + \delta_{r,p(\mathbf{i})}\partial_{p_{\mathbf{i}}}.v_{\beta,\mathbf{i}-1_{[p(\mathbf{i})]}} - i_{p_{\mathbf{i}}}\partial_r.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}}\big) \\ &= \bar{\beta}_r v_{\beta,\mathbf{i}} + i_r v_{\beta,\mathbf{i}-1_{[r]}} \end{aligned} \quad (166)$$

for $r \in \overline{1,l}$, which coincides with (164).

Case 3. $Q(\beta, \mathbf{i}) = \emptyset$ and $P(\mathbf{i}) = \emptyset$.

Let $s(\mathbf{i}) = \min\{\overline{1, l_1 + l_2}\backslash\{p_{\mathbf{i}}\}\}$. Note that $(\beta, \mathbf{i} - 1_{[p_{\mathbf{i}}]} + 1_{[s(\mathbf{i})]})$ belongs to Case 1 or Case 2. So we have

$$\begin{aligned} & \partial_r.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[s(\mathbf{i})]}} \\ = \ & \bar{\beta}_r v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[s(\mathbf{i})]}} + (i_r - \delta_{r,p_{\mathbf{i}}} + \delta_{r,s(\mathbf{i})}) v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[s(\mathbf{i})]}-1_{[r]}} \end{aligned} \tag{167}$$

for $r \in \overline{1,l}$. Since $\bar{\beta}_s = 0$ and $i_s = 0$ for $s \in \overline{1,l}\backslash\{p_{\mathbf{i}}\}$, from (126), (167) and Lemma 3.6 we derive

$$\begin{aligned} \partial_r.v_{\beta,\mathbf{i}} &= t^{1_{[p_{\mathbf{i}}]}} \partial_{s(\mathbf{i})}.\partial_r.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[s(\mathbf{i})]}} + \delta_{r,p_{\mathbf{i}}} \partial_{s(\mathbf{i})}.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[s(\mathbf{i})]}} \\ &= t^{1_{[p_{\mathbf{i}}]}} \partial_{s(\mathbf{i})}.(\bar{\beta}_r v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[s(\mathbf{i})]}} + (i_r - \delta_{r,p_{\mathbf{i}}} + \delta_{r,s(\mathbf{i})}) v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}+1_{[s(\mathbf{i})]}-1_{[r]}}) \\ &\quad + \delta_{r,p_{\mathbf{i}}} v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}} \\ &= \bar{\beta}_r v_{\beta,\mathbf{i}} + i_r v_{\beta,\mathbf{i}-1_{[r]}} \end{aligned} \tag{168}$$

for $r \in \overline{1,l}$, which coincides with (164).

So this lemma holds. ■

Lemma 3.8. *For any $\alpha \in \Gamma\backslash\{0\}$, $\beta \in \Gamma\backslash\{-\mu, -\mu - \alpha\}$, $\boldsymbol{i} \in \mathbb{N}^{l_1+l_2}$ and $p, q \in \overline{1,l}$ with $p \neq q$, we have*

$$\begin{aligned} D_{p,q}(x^\alpha).v_{\beta,\boldsymbol{i}} &= (\alpha_p(\beta_q + \mu_q) - \alpha_q(\beta_p + \mu_p)) v_{\beta+\alpha,\boldsymbol{i}} \\ &\quad + \alpha_p i_q v_{\beta+\alpha,\boldsymbol{i}-1_{[q]}} - \alpha_q i_p v_{\beta+\alpha,\boldsymbol{i}-1_{[p]}}. \end{aligned} \tag{169}$$

Proof. Let $\bar{\gamma} = \gamma + \mu$ for all $\gamma \in \Gamma$. We prove this lemma by induction on $|\mathbf{i}|$. Since Lemma 3.2 has given the proof for $|\mathbf{i}| = 0$, supposing that this lemma holds for $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| \leq k$, where $k \geq 0$, we only need to prove (169) for $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$. Fix any $\alpha \in \Gamma\backslash\{0\}$, $\beta \in \Gamma\backslash\{-\mu, -\mu-\alpha\}$, $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$, and $p, q \in \overline{1,l}$ with $p \neq q$. We define $p_{\mathbf{i}}$ as in (121), namely,

$$p_{\mathbf{i}} = \min\{r \in \overline{1, l_1 + l_2} \mid i_r \neq 0\}. \tag{170}$$

Then we proceed the proof in several cases.

Case 1. $\big(\ker\alpha \bigcap \big(\sum_{j\in\overline{1,l}\backslash\{p_{\mathbf{i}}\}} \mathbb{F}\partial_j\big)\big)\backslash \ker\bar{\beta} \neq \emptyset$.

If there exists $s \in \overline{1,l}\backslash\{p_{\mathbf{i}}\}$ such that $\alpha_s \neq 0$, then $\alpha_s\bar{\beta}_r - \alpha_r\bar{\beta}_s \neq 0$ for some $r \in \overline{1,l}\backslash\{s, p_{\mathbf{i}}\}$. Fix such s and r. Since Lemma 3.6 gives

$$t^{1_{[p_{\mathbf{i}}]}}(\alpha_s\partial_r - \alpha_r\partial_s).v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}} = (\alpha_s\bar{\beta}_r - \alpha_r\bar{\beta}_s) v_{\beta,\mathbf{i}} + \alpha_s i_r v_{\beta,\mathbf{i}-1_{[r]}} - \alpha_r i_s v_{\beta,\mathbf{i}-1_{[s]}}, \tag{171}$$

we have

$$
\begin{aligned}
D_{p,q}(x^\alpha).v_{\beta,\mathbf{i}} &= \frac{1}{\alpha_s\bar\beta_r-\alpha_r\bar\beta_s}x^\alpha(\alpha_p\partial_q-\alpha_q\partial_p).\big(t^{1_{[p_\mathbf{i}]}}(\alpha_s\partial_r-\alpha_r\partial_s).v_{\beta,\mathbf{i}-1_{[p_\mathbf{i}]}}\\
&\quad -\alpha_s i_r v_{\beta,\mathbf{i}-1_{[r]}}+\alpha_r i_s v_{\beta,\mathbf{i}-1_{[s]}}\big)\\
&= \frac{1}{\alpha_s\bar\beta_r-\alpha_r\bar\beta_s}\Big(t^{1_{[p_\mathbf{i}]}}(\alpha_s\partial_r-\alpha_r\partial_s).x^\alpha(\alpha_p\partial_q-\alpha_q\partial_p).v_{\beta,\mathbf{i}-1_{[p_\mathbf{i}]}}\\
&\quad +(\alpha_p\delta_{q,p_\mathbf{i}}-\alpha_q\delta_{p,p_\mathbf{i}})x^\alpha(\alpha_s\partial_r-\alpha_r\partial_s).v_{\beta,\mathbf{i}-1_{[p_\mathbf{i}]}}\\
&\quad -\alpha_s i_r x^\alpha(\alpha_p\partial_q-\alpha_q\partial_p).v_{\beta,\mathbf{i}-1_{[r]}}+\alpha_r i_s x^\alpha(\alpha_p\partial_q-\alpha_q\partial_p).v_{\beta,\mathbf{i}-1_{[s]}}\Big)\\
&= (\alpha_p\bar\beta_q-\alpha_q\bar\beta_p)v_{\beta+\alpha,\mathbf{i}}+\alpha_p i_q v_{\beta+\alpha,\mathbf{i}-1_{[q]}}-\alpha_q i_p v_{\beta+\alpha,\mathbf{i}-1_{[p]}} \qquad (172)
\end{aligned}
$$

by (171), Lemma 3.6 and the induction hypothesis.

If $\alpha_s=0$ for all $s\in\overline{1,l}\setminus\{p_\mathbf{i}\}$, then $\bar\beta_r\neq 0$ for some $r\in\overline{1,l}\setminus\{p_\mathbf{i}\}$. Fix such r. Since Lemma 3.6 indicates

$$t^{1_{[p_\mathbf{i}]}}\partial_r.v_{\beta,\mathbf{i}-1_{[p_\mathbf{i}]}}=\bar\beta_r v_{\beta,\mathbf{i}}+i_r v_{\beta,\mathbf{i}-1_{[r]}}, \qquad (173)$$

we obtain

$$
\begin{aligned}
&D_{p,q}(x^\alpha).v_{\beta,\mathbf{i}}\\
=\ & \frac{1}{\bar\beta_r}x^\alpha(\alpha_p\partial_q-\alpha_q\partial_p).(t^{1_{[p_\mathbf{i}]}}\partial_r.v_{\beta,\mathbf{i}-1_{[p_\mathbf{i}]}}-i_r v_{\beta,\mathbf{i}-1_{[r]}})\\
=\ & \frac{1}{\bar\beta_r}\Big(t^{1_{[p_\mathbf{i}]}}\partial_r.x^\alpha(\alpha_p\partial_q-\alpha_q\partial_p).v_{\beta,\mathbf{i}-1_{[p_\mathbf{i}]}}+(\alpha_p\delta_{q,p_\mathbf{i}}-\alpha_q\delta_{p,p_\mathbf{i}})x^\alpha\partial_r.v_{\beta,\mathbf{i}-1_{[p_\mathbf{i}]}}\\
&-i_r x^\alpha(\alpha_p\partial_q-\alpha_q\partial_p).v_{\beta,\mathbf{i}-1_{[r]}}\Big)\\
=\ & (\alpha_p\bar\beta_q-\alpha_q\bar\beta_p)v_{\beta+\alpha,\mathbf{i}}+\alpha_p i_q v_{\beta+\alpha,\mathbf{i}-1_{[q]}}-\alpha_q i_p v_{\beta+\alpha,\mathbf{i}-1_{[p]}} \qquad (174)
\end{aligned}
$$

by (173), Lemma 3.6 and the induction hypothesis. So (169) holds.

Case 2. $\ker\alpha\bigcap\big(\sum_{j\in\overline{1,l}\setminus\{p_\mathbf{i}\}}\mathbb{F}\partial_j\big)\subseteq\ker\bar\beta$ and $\ker\alpha\neq\ker\bar\beta$.

Since $\alpha\neq 0$ and $\bar\beta\neq 0$, we have $\alpha_s\neq 0$ for some $s\in\overline{1,l}\setminus\{p_\mathbf{i}\}$, and $\alpha_s\bar\beta_{p_\mathbf{i}}-\alpha_{p_\mathbf{i}}\bar\beta_s\neq 0$ in this case. Fix such s. Pick $r\in\overline{1,l_1+l_2}\setminus\{p_\mathbf{i},s\}$. Set $\tilde\partial_1=\alpha_s\partial_{p_\mathbf{i}}-\alpha_{p_\mathbf{i}}\partial_s$ and $\tilde\partial_2=\alpha_s\partial_r-\alpha_r\partial_s$. Observe that

$$\tilde\partial_1(t^{1_{[r]}+1_{[p_\mathbf{i}]}})\tilde\partial_2-\tilde\partial_2(t^{1_{[r]}+1_{[p_\mathbf{i}]}})\tilde\partial_1=\alpha_s(t^{1_{[r]}}\tilde\partial_2-t^{1_{[p_\mathbf{i}]}}\tilde\partial_1). \qquad (175)$$

So Lemma 3.6 implies

$$
\begin{aligned}
&(t^{1_{[p_\mathbf{i}]}}\tilde\partial_1-t^{1_{[r]}}\tilde\partial_2).v_{\beta,\mathbf{i}-1_{[p_\mathbf{i}]}}\\
=\ &(\alpha_s\bar\beta_{p_\mathbf{i}}-\alpha_{p_\mathbf{i}}\bar\beta_s)v_{\beta,\mathbf{i}}+\alpha_s(i_{p_\mathbf{i}}-1)v_{\beta,\mathbf{i}-1_{[p_\mathbf{i}]}}-\alpha_{p_\mathbf{i}}i_s v_{\beta,\mathbf{i}-1_{[s]}}\\
&-\alpha_s i_r v_{\beta,\mathbf{i}-1_{[p_\mathbf{i}]}}+\alpha_r i_s v_{\beta,\mathbf{i}+1_{[r]}-1_{[p_\mathbf{i}]}-1_{[s]}}. \qquad (176)
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
& D_{p,q}(x^{\alpha}).v_{\beta,\mathbf{i}} \\
= \;& \frac{1}{\alpha_s\bar{\beta}_{p_{\mathbf{i}}}-\alpha_{p_{\mathbf{i}}}\bar{\beta}_s}x^{\alpha}(\alpha_p\partial_q-\alpha_q\partial_p).\Big((t^{1_{[p_{\mathbf{i}}]}}\tilde{\partial}_1-t^{1_{[r]}}\tilde{\partial}_2).v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}} \\
& -\alpha_s(i_{p_{\mathbf{i}}}-1)v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}}+\alpha_{p_{\mathbf{i}}}i_sv_{\beta,\mathbf{i}-1_{[s]}}+\alpha_si_rv_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}}-\alpha_ri_sv_{\beta,\mathbf{i}+1_{[r]}-1_{[p_{\mathbf{i}}]}-1_{[s]}}\Big) \\
= \;& \frac{1}{\alpha_s\bar{\beta}_{p_{\mathbf{i}}}-\alpha_{p_{\mathbf{i}}}\bar{\beta}_s}\Big((t^{1_{[p_{\mathbf{i}}]}}\tilde{\partial}_1-t^{1_{[r]}}\tilde{\partial}_2).x^{\alpha}(\alpha_p\partial_q-\alpha_q\partial_p).v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}} \\
& +(\alpha_p\delta_{q,p_{\mathbf{i}}}-\alpha_q\delta_{p,p_{\mathbf{i}}})x^{\alpha}\tilde{\partial}_1.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}}-(\alpha_p\delta_{q,r}-\alpha_q\delta_{p,r})x^{\alpha}\tilde{\partial}_2.v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}} \\
& -x^{\alpha}(\alpha_p\partial_q-\alpha_q\partial_p).(\alpha_s(i_{p_{\mathbf{i}}}-1)v_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}}-\alpha_{p_{\mathbf{i}}}i_sv_{\beta,\mathbf{i}-1_{[s]}}-\alpha_si_rv_{\beta,\mathbf{i}-1_{[p_{\mathbf{i}}]}} \\
& +\alpha_ri_sv_{\beta,\mathbf{i}+1_{[r]}-1_{[p_{\mathbf{i}}]}-1_{[s]}})\Big) \\
= \;& (\alpha_p\bar{\beta}_q-\alpha_q\bar{\beta}_p)v_{\beta+\alpha,\mathbf{i}}+\alpha_pi_qv_{\beta+\alpha,\mathbf{i}-1_{[q]}}-\alpha_qi_pv_{\beta+\alpha,\mathbf{i}-1_{[p]}} \qquad (177)
\end{aligned}
$$

by (176), Lemma 3.6 and the induction hypothesis. So (169) holds.

Case 3. $\ker\alpha=\ker\bar{\beta}$.

Since $\alpha\neq 0$, $\bar{\beta}\neq 0\neq\bar{\beta}+\alpha$ and $\ker\alpha=\ker\bar{\beta}$, we have $\alpha_s\neq 0$, $\bar{\beta}_s\neq 0$ and $\bar{\beta}_s+\alpha_s\neq 0$ for some $s\in\overline{1,l}$. Fix such s.

If there exists $r\in\overline{1,l}\backslash\{s\}$ such that $\bar{\beta}_r=0$, then $\alpha_r=0$. Fix one such r. Choose $\rho\in\Gamma\backslash\{0\}$ such that $\rho_r\neq 0$. Then $\ker\rho\neq\ker(\bar{\beta}-\rho)$. So Case 1 or Case 2 gives

$$
x^{\rho}(\rho_r\partial_s-\rho_s\partial_r).v_{\beta-\rho,\mathbf{i}}=\rho_r\bar{\beta}_sv_{\beta,\mathbf{i}}+\rho_ri_sv_{\beta,\mathbf{i}-1_{[s]}}-\rho_si_rv_{\beta,\mathbf{i}-1_{[r]}}. \qquad (178)
$$

Thus we have

$$
\begin{aligned}
& D_{p,q}(x^{\alpha}).v_{\beta,\mathbf{i}} \\
= \;& \frac{1}{\rho_r\bar{\beta}_s}x^{\alpha}(\alpha_p\partial_q-\alpha_q\partial_p).(x^{\rho}(\rho_r\partial_s-\rho_s\partial_r).v_{\beta-\rho,\mathbf{i}}-\rho_ri_sv_{\beta,\mathbf{i}-1_{[s]}}+\rho_si_rv_{\beta,\mathbf{i}-1_{[r]}}) \\
= \;& \frac{1}{\rho_r\bar{\beta}_s}(x^{\rho}(\rho_r\partial_s-\rho_s\partial_r).x^{\alpha}(\alpha_p\partial_q-\alpha_q\partial_p).v_{\beta-\rho,\mathbf{i}} \\
& +x^{\alpha+\rho}((\alpha_p\rho_q-\alpha_q\rho_p)(\rho_r\partial_s-\rho_s\partial_r)-(\rho_r\alpha_s-\rho_s\alpha_r)(\alpha_p\partial_q-\alpha_q\partial_p)).v_{\beta-\rho,\mathbf{i}} \\
& -\rho_ri_sx^{\alpha}(\alpha_p\partial_q-\alpha_q\partial_p).v_{\beta,\mathbf{i}-1_{[s]}}+\rho_si_rx^{\alpha}(\alpha_p\partial_q-\alpha_q\partial_p).v_{\beta,\mathbf{i}-1_{[r]}}) \qquad (179)
\end{aligned}
$$

by (178). Moreover, since $\ker\alpha\neq\ker(\bar{\beta}-\rho)$, $\ker(\alpha+\rho)\neq\ker(\bar{\beta}-\rho)$ and $\ker\rho\neq\ker(\bar{\beta}+\alpha-\rho)$, Case 1 and Case 2 give

$$
\begin{aligned}
& x^{\alpha}(\alpha_p\partial_q-\alpha_q\partial_p).v_{\beta-\rho,\mathbf{i}} \\
= \;& (\alpha_q\rho_p-\alpha_p\rho_q)v_{\beta+\alpha-\rho,\mathbf{i}}+\alpha_pi_qv_{\beta+\alpha-\rho,\mathbf{i}-1_{[q]}}-\alpha_qi_pv_{\beta+\alpha-\rho,\mathbf{i}-1_{[p]}}, \qquad (180)
\end{aligned}
$$

$$
x^{\rho}(\rho_r\partial_s-\rho_s\partial_r).v_{\beta+\alpha-\rho,\mathbf{i}}=\rho_r(\bar{\beta}_s+\alpha_s)v_{\beta+\alpha,\mathbf{i}}+\rho_ri_sv_{\beta+\alpha,\mathbf{i}-1_{[s]}}-\rho_si_rv_{\beta+\alpha,\mathbf{i}-1_{[r]}}, \qquad (181)
$$

and

$$
\begin{aligned}
& x^{\alpha+\rho}((\alpha_p\rho_q-\alpha_q\rho_p)(\rho_r\partial_s-\rho_s\partial_r)-(\rho_r\alpha_s-\rho_s\alpha_r)(\alpha_p\partial_q-\alpha_q\partial_p)).v_{\beta-\rho,\mathbf{i}} \\
= \;& \rho_r(\bar{\beta}_s+\alpha_s)(\alpha_p\rho_q-\alpha_q\rho_p)v_{\beta+\alpha,\mathbf{i}}+\rho_ri_s(\alpha_p\rho_q-\alpha_q\rho_p)v_{\beta+\alpha,\mathbf{i}-1_{[s]}} \\
& -\rho_si_r(\alpha_p\rho_q-\alpha_q\rho_p)v_{\beta+\alpha,\mathbf{i}-1_{[r]}}-\alpha_pi_q(\rho_r\alpha_s-\rho_s\alpha_r)v_{\beta+\alpha,\mathbf{i}-1_{[q]}} \\
& +\alpha_qi_p(\rho_r\alpha_s-\rho_s\alpha_r)v_{\beta+\alpha,\mathbf{i}-1_{[p]}}. \qquad (182)
\end{aligned}
$$

So (179) becomes

$$D_{p,q}(x^{\alpha}).v_{\beta,\mathbf{i}} = \alpha_p i_q v_{\beta+\alpha,\mathbf{i}-1_{[q]}} - \alpha_q i_p v_{\beta+\alpha,\mathbf{i}-1_{[p]}} \tag{183}$$

by (180)–(182) and the induction hypothesis, which coincides with (169).

If $\bar{\beta}_r \neq 0$ for all $r \in \overline{1,l}\backslash\{s\}$, which also means $\alpha_r \neq 0$ for all $r \in \overline{1,l}\backslash\{s\}$, we pick $r_1, r_2 \in \overline{1,l}\backslash\{s\}$. Then replacing ∂_r by $\partial = \alpha_{r_1}\partial_{r_2} - \alpha_{r_2}\partial_{r_1}$ and ρ_r by $\partial(\rho)$ respectively in the above discussion from (178) to (183), we can also get

$$D_{p,q}(x^{\alpha}).v_{\beta,\mathbf{i}} = (\alpha_p\bar{\beta}_q - \alpha_q\bar{\beta}_p)v_{\beta+\alpha,\mathbf{i}} + \alpha_p i_q v_{\beta+\alpha,\mathbf{i}-1_{[q]}} - \alpha_q i_p v_{\beta+\alpha,\mathbf{i}-1_{[p]}}. \tag{184}$$

Thus we complete the proof of this lemma. ■

For convenience, we give a total order on $\mathbb{N}^{l_1+l_2}$:

Definition 3.9. We define a total order on $\mathbb{N}^{l_1+l_2}$ by:

$$\mathbf{i} > \mathbf{j} \iff |\mathbf{i}| > |\mathbf{j}|, \text{ or, } |\mathbf{i}| = |\mathbf{j}| \text{ and } i_s > j_s \text{ with } i_p = j_p \text{ for } p \in \overline{s+1, l_1+l_2}. \tag{185}$$

Next, we shall prove Theorem 1.1 for the case $\mu \in \mathbb{F}^l\backslash\Gamma$.

Lemma 3.10. *If $\mu \notin \Gamma$, then $\{v_{\beta,\mathbf{i}} \mid \beta \in \Gamma, \mathbf{i} \in \mathbb{N}^{l_1+l_2}\}$ is an $\mathbb{F}$-basis of V.*

Proof. It is straightforward to prove that $\{v_{\beta,\mathbf{i}} \mid \beta \in \Gamma, \mathbf{i} \in \mathbb{N}^{l_1+l_2}\}$ is a linearly independent set by Lemma 3.7. We omit the details. We only prove that V is spanned by $\{v_{\beta,\mathbf{i}} \mid \beta \in \Gamma, \mathbf{i} \in \mathbb{N}^{l_1+l_2}\}$ here. For any $\beta \in \Gamma$ and $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$, we set

$$V_{\beta+\mu}^{[\mathbf{i}]} = \{v \in V \mid \prod_{p=1}^{l}(\partial_p - (\beta_p + \mu_p))^{j_p}(v) = 0 \text{ for } \mathbf{j} \in \mathbb{N}^{l_1+l_2} \text{ with } \mathbf{j} > \mathbf{i}\}, \tag{186}$$

$$V_{\beta+\mu}^{<\mathbf{i}>} = \bigcup_{\mathbf{j}\in\mathbb{N}^{l_1+l_2};\mathbf{j}<\mathbf{i}} V_{\beta+\mu}^{[\mathbf{j}]} \tag{187}$$

and

$$V^{[\mathbf{i}]} = \bigoplus_{\beta\in\Gamma} V_{\beta+\mu}^{[\mathbf{i}]}, \quad V^{<\mathbf{i}>} = \bigoplus_{\beta\in\Gamma} V_{\beta+\mu}^{<\mathbf{i}>}. \tag{188}$$

Then Lemma 3.7 implies

$$V_{\beta+\mu} = \bigcup_{\mathbf{j}\in\mathbb{N}^{l_1+l_2}} V_{\beta+\mu}^{[\mathbf{j}]} \text{ and } v_{\beta,\mathbf{i}} \in V_{\beta+\mu}^{[\mathbf{i}]}\backslash V_{\beta+\mu}^{<\mathbf{i}>}, \quad \forall \beta \in \Gamma,\ \mathbf{i} \in \mathbb{N}^{l_1+l_2}. \tag{189}$$

Suppose that V cannot be spanned by $\{v_{\beta,\mathbf{i}} \mid \beta \in \Gamma, \mathbf{i} \in \mathbb{N}^{l_1+l_2}\}$. Then this will lead to a contradiction. Let $\mathbf{j} \in \mathbb{N}^{l_1+l_2}$ be the minimal element such that

$$\text{there exist } \alpha \in \Gamma \text{ and } v \in V_{\alpha+\mu}^{[\mathbf{j}]}\backslash V_{\alpha+\mu}^{<\mathbf{j}>} \text{ such that } v \notin \mathrm{Span}_{\mathbb{F}}\{v_{\beta,\mathbf{i}} \mid \beta \in \Gamma, \mathbf{i} \in \mathbb{N}^{l_1+l_2}\}. \tag{190}$$

Since $V^{[\mathbf{0}]}_{\beta+\mu} = V^{(0)}_{\beta+\mu} = \mathbb{F}v_{\beta,\mathbf{0}}$ for $\beta \in \Gamma$ (cf. (6), Definition 3.3), we have $\mathbf{j} \neq \mathbf{0}$. Let $q = \min\{p \in \overline{1, l_1+l_2} \mid j_p \neq 0\}$. Then (186) and Lemma 3.7 show

$$v' = (\partial_q - (\alpha_q + \mu_q))v \in V^{[\mathbf{j}-1_{[q]}]}_{\alpha+\mu}. \tag{191}$$

By the minimality of $\mathbf{j}$ in (190), we have

$$v' \in \mathrm{Span}_{\mathbb{F}}\{v_{\beta,\mathbf{i}} \mid \beta \in \Gamma, \mathbf{i} \in \mathbb{N}^{l_1+l_2}\}. \tag{192}$$

Thus (186), (189), (191) and (192) give

$$v'' = v' - cv_{\alpha,\mathbf{j}-1_{[q]}} \in V^{<\mathbf{j}-1_{[q]}>}_{\alpha+\mu} \text{ for some } c \in \mathbb{F}. \tag{193}$$

Let

$$w = v - \frac{c}{j_q} v_{\alpha,\mathbf{j}}. \tag{194}$$

Then $w \in V^{[\mathbf{j}]}_{\alpha+\mu}$ by (189) and (190). So for $\mathbf{i} > \mathbf{j}$, we have

$$\prod_{p=1}^{l}(\partial_p - (\alpha_p + \mu_p))^{i_p}(w) = 0 \tag{195}$$

by (186). Moreover, since (191), (193), (194) and Lemma 3.7 give

$$(\partial_q - (\alpha_q + \mu_q))(w) = v' - cv_{\alpha,\mathbf{j}-1_{[q]}} = v'', \tag{196}$$

we have

$$\prod_{p=1}^{l}(\partial_p - (\alpha_p + \mu_p))^{j_p}(w) = \prod_{p=1}^{l}(\partial_p - (\alpha_p + \mu_p))^{j_p-\delta_{p,q}}(v'') = 0 \tag{197}$$

by (193). So (195) and (197) show that

$$w = v - \frac{c}{j_q} v_{\alpha,\mathbf{j}} \in V^{<\mathbf{j}>}_{\alpha+\mu}, \tag{198}$$

which indicates $v \in \mathrm{Span}_{\mathbb{F}}\{v_{\beta,\mathbf{i}} \mid \beta \in \Gamma, \mathbf{i} \in \mathbb{N}^{l_1+l_2}\}$. This contradicts (190). So V is spanned by $\{v_{\beta,\mathbf{i}} \mid \beta \in \Gamma, \mathbf{i} \in \mathbb{N}^{l_1+l_2}\}$. This lemma holds. ■

For the case $\mu \notin \Gamma$, we have determined a basis of $V = V(\mu)$, and derived the action of the set (32) on the basis in Lemmas 3.6, 3.7 and 3.8. On the other hand, from (9) and (17) we see that, the explicit action of $\mathscr{S}(l_1, l_2, l_3; \Gamma)$ on module $\mathscr{A}^\mu$ is

$$\begin{aligned} & D_{p,q}(x^\alpha t^{\mathbf{i}}).x^\beta t^{\mathbf{j}} \\ = \; & (\alpha_p(\beta_q + \mu_q) - \alpha_q(\beta_p + \mu_p))x^{\alpha+\beta}t^{\mathbf{i}+\mathbf{j}} + (j_q\alpha_p - i_q(\beta_p + \mu_p))x^{\alpha+\beta}t^{\mathbf{i}+\mathbf{j}-1_{[q]}} \\ & +(i_p(\beta_q + \mu_q) - j_p\alpha_q)x^{\alpha+\beta}t^{\mathbf{i}+\mathbf{j}-1_{[p]}} + (i_pj_q - i_qj_p)x^{\alpha+\beta}t^{\mathbf{i}+\mathbf{j}-1_{[p]}-1_{[q]}} \end{aligned} \tag{199}$$

for $\alpha, \beta \in \Gamma$, $\mathbf{i}, \mathbf{j} \in \mathbb{N}^{l_1+l_2}$ and $p, q \in \overline{1, l}$ with $p \neq q$. So by (199), Proposition 2.5, Lemmas 3.6, 3.7, 3.8 and 3.10, we have:

Lemma 3.11. *If $\mu \notin \Gamma$, then $V = V(\mu) \simeq \mathscr{A}^\mu$ with the module isomorphism:*

$$\psi \;:\; V(\mu) \longrightarrow \mathscr{A}^\mu, \tag{200}$$

$$v_{\beta,\boldsymbol{i}} \longmapsto x^\beta t^{\boldsymbol{i}}. \tag{201}$$

4. Proof of the main theorem (II)

In this section, we consider the case that $V = V(\mu)$ for some $\mu \in \Gamma$. With a shift of the indices, we can always assume that $\mu = 0$. Notice that the contents from Lemma 3.2 to Lemma 3.8 were discussed for general $\mu \in \mathbb{F}^l$. So they still hold for the case $\mu = 0$. In this section, we need to complement the basis of $V = V(0)$, and to derive the acting relations of the set (32) on the basis which were missed from Lemma 3.2 to Lemma 3.8 under the condition $\mu = 0$.

Lemma 4.1. $V_0 \neq \{0\}$.

Proof. Suppose that $V_0 = \{0\}$. Then this will lead to a contradiction. Pick $p \in \overline{1, l_1 + l_2}$ and $q \in \overline{l_1 + 1, l} \backslash \{p\}$. Choose $\rho \in \Gamma \backslash \{0\}$ such that $\rho_q \neq 0$. Then by Lemmas 3.2 and 3.7, we have

$$\partial_s.(x^\rho(\rho_p\partial_q - \rho_q\partial_p).v_{-\rho,2_{[p]}}) = 0 \quad \text{for } s \in \overline{1,l}\backslash\{p\}, \tag{202}$$

$$\partial_p^2.(x^\rho(\rho_p\partial_q - \rho_q\partial_p).v_{-\rho,2_{[p]}}) = 0. \tag{203}$$

So $x^\rho(\rho_p\partial_q - \rho_q\partial_p).v_{-\rho,2_{[p]}} \in V_0$ by (6), which indicates

$$x^\rho(\rho_p\partial_q - \rho_q\partial_p).v_{-\rho,2_{[p]}} = 0. \tag{204}$$

Pick $r \in \overline{l_1 + 1, l} \backslash \{p, q\}$ and choose $\alpha \in \Gamma \backslash \{0, \rho\}$ such that $\alpha_r \neq 0$. Then (204) and Lemma 3.8 give

$$\begin{aligned} 0 &= x^\alpha(\alpha_r\partial_p - \alpha_p\partial_r).x^\rho(\rho_p\partial_q - \rho_q\partial_p).v_{-\rho,2_{[p]}} \\ &= x^{\alpha+\rho}((\alpha_r\rho_p - \alpha_p\rho_r)(\rho_p\partial_q - \rho_q\partial_p) - (\rho_p\alpha_q - \rho_q\alpha_p)(\alpha_r\partial_p - \alpha_p\partial_r)).v_{-\rho,2_{[p]}} \\ &\quad + x^\rho(\rho_p\partial_q - \rho_q\partial_p).x^\alpha(\alpha_r\partial_p - \alpha_p\partial_r).v_{-\rho,2_{[p]}} \\ &= -2\rho_q\alpha_r v_{\alpha,\mathbf{0}} \neq 0, \end{aligned} \tag{205}$$

which is absurd. So we must have $V_0 \neq \{0\}$. This lemma holds. ∎

Observe that (cf. (6))

$$V_0 \neq \{0\} \Leftrightarrow V_0^{(0)} \neq \{0\}. \tag{206}$$

Since $\dim V_0^{(0)} \leq 1$, we have $\dim V_0^{(0)} = 1$. Moreover, we can obtain:

Lemma 4.2. *$V_0^{(0)}$ is a trivial $\mathscr{S}(l_1, l_2, l_3; \Gamma)$-submodule of V.*

Proof. Pick $0 \neq v \in V_0^{(0)}$. Then (6) indicates

$$\partial_p.v = 0 \quad \text{for } p \in \overline{1,l}. \tag{207}$$

Moreover, we obtain

$$t^{1_{[p]}}\partial_{q'}.v,\ (t^{1_{[p]}}\partial_p - t^{1_{[q]}}\partial_q).v \ \in V_0^{(0)} \tag{208}$$

for any $p,q \in \overline{1,l_1+l_2}$ and $q' \in \overline{1,l}\backslash\{p\}$. Namely, they all act on v as scalars. Thus,

$$t^{1_{[p]}}\partial_{q'}.v = [t^{1_{[p]}}\partial_r, t^{1_{[r]}}\partial_{q'}].v = 0 \quad \text{for } p \in \overline{1,l_1+l_2},\ q' \in \overline{1,l}\backslash\{p\}, \tag{209}$$

$$(t^{1_{[p]}}\partial_p - t^{1_{[q]}}\partial_q).v = [t^{1_{[p]}}\partial_q, t^{1_{[q]}}\partial_p].v = 0 \quad \text{for } p,q \in \overline{1,l_1+l_2} \text{ with } p \neq q, \tag{210}$$

where $r \in \overline{1,l_1+l_2}\backslash\{p,q'\}$. Furthermore, Claims 2, 3 and 6 of Lemma 3.2 give

$$D_{p,q}(x^\alpha).v = 0 \quad \text{for any } \alpha \in \Gamma\backslash\{0\},\ p,q \in \overline{1,l}. \tag{211}$$

So by (207), (209), (210), (211) and Proposition 2.5, we derive that $V_0^{(0)}$ is a trivial $\mathscr{S}(l_1,l_2,l_3;\Gamma)$-submodule of V. ■

Throughout the rest of the section, we fix some

$$r_1, r_2 \in \overline{l_1+1,l}\backslash\{1\} \text{ with } r_1 \neq r_2, \tag{212}$$

and fix some

$$\rho \in \Gamma\backslash\{0\} \text{ satisfying } \rho_{r_1} \neq 0 \text{ and } \rho_{r_2} \neq 0. \tag{213}$$

We then use ρ, r_1, r_2 to determine a basis of the vector space V_0.

Definition 4.3. For $k \geq 0$, we define

$$v_{0,k_{[1]}} = \frac{1}{\rho_{r_1}(k+1)} x^\rho(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).v_{-\rho,(k+1)_{[1]}}, \tag{214}$$

where ρ and r_1 are the fixed elements in (212) and (213). Moreover, we define

$$v_{0,\mathbf{i}} = \frac{i_1!}{k!}(t^{1_{[l_1+l_2]}}\partial_1)^{i_{l_1+l_2}}.(t^{1_{[l_1+l_2-1]}}\partial_1)^{i_{l_1+l_2-1}}.\cdots.(t^{1_{[2]}}\partial_1)^{i_2}.v_{0,k_{[1]}} \tag{215}$$

for $\mathbf{i} = (i_1, i_2, \cdots, i_{l_1+l_2}, 0, \cdots, 0) \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k \geq 1$ and $\mathbf{i} \neq k_{[1]}$.

In fact, $v_{0,k_{[1]}}$ in (214) is independent of the choice of r_1 in (212) and ρ in (213). We shall show this after Lemma 4.8.

Lemma 4.4. $v_{0,\mathbf{0}} \in V_0^{(0)}$ *and* $v_{0,\mathbf{0}} \neq 0$.

Proof. Let ρ and r_1 be the fixed elements in (212) and (213). By (214), Lemma 3.2 and Lemma 3.7, we have

$$v_{0,\mathbf{0}} = \frac{1}{\rho_{r_1}} x^\rho(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).v_{-\rho,1_{[1]}} \in V_0^{(0)} \tag{216}$$

because

$$\begin{aligned} &\partial_q.(x^\rho(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).v_{-\rho,1_{[1]}}) \\ = \;&[\partial_q, x^\rho(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1})].v_{-\rho,1_{[1]}} + x^\rho(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).\partial_q.v_{-\rho,1_{[1]}} = 0 \end{aligned} \tag{217}$$

for $q \in \overline{1,l}$.

Suppose that

$$v_{0,\mathbf{0}} = \frac{1}{\rho_{r_1}} x^{\rho}(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).v_{-\rho,1_{[1]}} = 0, \tag{218}$$

then we have

$$\partial_p.(x^{\rho}(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).v_{-\rho,2_{[1]}}) = 0 \quad \text{for } p \in \overline{1,l} \tag{219}$$

by Lemma 3.7 and (218). In other words,

$$x^{\rho}(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).v_{-\rho,2_{[1]}} \in V_0^{(0)}. \tag{220}$$

Choose $\alpha \in \Gamma\backslash\{0,\rho\}$ such that $\alpha_{r_1} \neq 0$. So (220), Lemma 3.8 and Lemma 4.2 give

$$\begin{aligned} 0 &= x^{\alpha}(\alpha_{r_1}\partial_1 - \alpha_1\partial_{r_1}).(x^{\rho}(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).v_{-\rho,2_{[1]}}) \\ &= x^{\alpha+\rho}\big((\alpha_{r_1}\rho_1 - \alpha_1\rho_{r_1})(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}) \\ &\quad -(\rho_{r_1}\alpha_1 - \rho_1\alpha_{r_1})(\alpha_{r_1}\partial_1 - \alpha_1\partial_{r_1})\big).v_{-\rho,2_{[1]}} \\ &\quad +x^{\rho}(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).x^{\alpha}(\alpha_{r_1}\partial_1 - \alpha_1\partial_{r_1}).v_{-\rho,2_{[1]}} \\ &= 2\rho_{r_1}\alpha_{r_1}v_{\alpha,\mathbf{0}} \neq 0, \end{aligned} \tag{221}$$

which is absurd. So we must have

$$v_{0,\mathbf{0}} \neq 0. \tag{222}$$

Thus the lemma holds. ■

Lemma 4.5. *For any $p \in \overline{1,l}$ and $\boldsymbol{i} \in \mathbb{N}^{l_1+l_2}$, we have*

$$\partial_p.v_{0,\boldsymbol{i}} = i_p v_{0,\boldsymbol{i}-1_{[p]}}. \tag{223}$$

Proof. When $\mathbf{i} = \mathbf{0}$, we have $v_{0,\mathbf{0}} \in V_0^{(0)}$, which indicates

$$\partial_p.v_{0,\mathbf{0}} = 0 \quad \text{for all } p \in \overline{1,l} \tag{224}$$

by Lemma 4.2. When $\mathbf{i} = k_{[1]}$ with $k \geq 1$, we have

$$\begin{aligned} \partial_p.v_{0,k_{[1]}} &= \frac{1}{\rho_{r_1}(k+1)}\partial_p.x^{\rho}(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).v_{-\rho,(k+1)_{[1]}} \\ &= \frac{1}{\rho_{r_1}}\delta_{p,1}x^{\rho}(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).v_{-\rho,k_{[1]}} \\ &= \delta_{p,1}kv_{0,(k-1)_{[1]}}, \qquad \forall\, p \in \overline{1,l} \end{aligned} \tag{225}$$

by (214) and Lemma 3.7. When $\mathbf{i} = (i_1, i_2, \cdots, i_{l_1+l_2}, 0, \cdots, 0)$ with $|\mathbf{i}| = k \geq 1$

and $\mathbf{i} \neq k_{[1]}$, (215), (225) and Lemma 4.2 show

$$\begin{aligned}
\partial_p.v_{0,\mathbf{i}} &= \partial_p.(\frac{i_1!}{k!}(t^{1_{[l_1+l_2]}}\partial_1)^{i_{l_1+l_2}}.\cdots.(t^{1_{[2]}}\partial_1)^{i_2}.v_{0,k_{[1]}}) \\
&= \sum_{s=2}^{l_1+l_2}\delta_{p,s}i_s\frac{i_1!}{k!}(t^{1_{[l_1+l_2]}}\partial_1)^{i_{l_1+l_2}}.\cdots.(t^{1_{[s]}}\partial_1)^{i_s-1}.\cdots.(t^{1_{[2]}}\partial_1)^{i_2}.(\partial_1.v_{0,k_{[1]}}) \\
&\quad +\frac{i_1!}{k!}(t^{1_{[l_1+l_2]}}\partial_1)^{i_{l_1+l_2}}.\cdots.(t^{1_{[2]}}\partial_1)^{i_2}.(\partial_p.v_{0,k_{[1]}}) \\
&= \sum_{s=1}^{l_1+l_2}\delta_{p,s}i_s v_{0,\mathbf{i}-1_{[s]}} \\
&= i_p v_{0,\mathbf{i}-1_{[p]}} \qquad \text{for } p\in\overline{1,l}. \qquad (226)
\end{aligned}$$

Thus this lemma holds. ∎

Lemma 4.6. *For any $\boldsymbol{i}\in\mathbb{N}^{l_1+l_2}$, $k\in\mathbb{N}$, $p,q\in\overline{1,l_1+l_2}$ and $p',q'\in\overline{1,l}$, we have*

$$t^{1_{[p]}}\partial_{q'}.v_{0,\boldsymbol{i}} = i_{q'}v_{0,\boldsymbol{i}+1_{[p]}-1_{[q']}},\quad p\neq q', \qquad (227)$$

$$(t^{1_{[p]}}\partial_p - t^{1_{[q]}}\partial_q).v_{0,\boldsymbol{i}} = (i_p-i_q)v_{0,\boldsymbol{i}},\quad p\neq q, \qquad (228)$$

and

$$D_{p',q'}(x^\rho).v_{-\rho,(k+1)_{[1]}} = (k+1)(\rho_{p'}\delta_{q',1}-\rho_{q'}\delta_{p',1})v_{0,k_{[1]}},\quad p'\neq q', \qquad (229)$$

where ρ is the fixed element in (213).

Proof. We divide the proof into several steps.

Step 1. $t^{1_{[p]}}\partial_1.v_{0,\mathbf{i}} = i_1 v_{0,\mathbf{i}+1_{[p]}-1_{[1]}}$ for $\mathbf{i}\in\mathbb{N}^{l_1+l_2}$ and $p\in\overline{2,l_1+l_2}$.

When $\mathbf{i}=\mathbf{0}$, it holds by Lemma 4.2 and Lemma 4.4. Moreover, (214) and (215) give rise to

$$t^{1_{[p]}}\partial_1.v_{0,\mathbf{i}} = \frac{i_1!}{k!}(t^{1_{[l_1+l_2]}}\partial_1)^{i_{l_1+l_2}}.\cdots.(t^{1_{[p]}}\partial_1)^{i_p+1}.\cdots.(t^{1_{[2]}}\partial_1)^{i_2}.v_{0,k_{[1]}} = i_1 v_{0,\mathbf{i}+1_{[p]}-1_{[1]}} \qquad (230)$$

for $p\in\overline{2,l_1+l_2}$, $\mathbf{i}\in\mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}|=k>0$.

Step 2. For any $\mathbf{i}\in\mathbb{N}^{l_1+l_2}$, $p,q\in\overline{2,l_1+l_2}$ and $q'\in\overline{2,l}\backslash\{p\}$, we have

$$t^{1_{[p]}}\partial_{q'}.v_{0,\mathbf{i}} = i_{q'}v_{0,\mathbf{i}+1_{[p]}-1_{[q']}} \text{ and } (t^{1_{[p]}}\partial_p - t^{1_{[q]}}\partial_q).v_{0,\mathbf{i}} = (i_p-i_q)v_{0,\mathbf{i}}. \qquad (231)$$

Firstly, we want to prove (231) for $\mathbf{i}=k_{[1]}$ with $k\geq 0$, namely,

$$t^{1_{[p]}}\partial_{q'}.v_{0,k_{[1]}} = 0, \qquad (t^{1_{[p]}}\partial_p - t^{1_{[q]}}\partial_q).v_{0,k_{[1]}} = 0 \qquad (232)$$

for $k\geq 0$, $p,q\in\overline{2,l_1+l_2}$ and $q'\in\overline{2,l}\backslash\{p\}$. We shall prove it by induction on k.

When $k=0$, (232) follows from Lemma 4.2 and Lemma 4.4. Assume that, (232) holds for $k=s$ with some $s\geq 0$. Then the induction hypothesis and Lemma 4.5 give

$$t^{1_{[p]}}\partial_{q'}.v_{0,(s+1)_{[1]}},\ (t^{1_{[p]}}\partial_p - t^{1_{[q]}}\partial_q).v_{0,(s+1)_{[1]}} \in V_0^{(0)} \qquad (233)$$

for any $p,q\in\overline{2,l_1+l_2}$ and $q'\in\overline{2,l}\backslash\{p\}$. Thus (233) and Lemma 4.2 imply

$$t^{1_{[p]}}\partial_{q'}.v_{0,(s+1)_{[1]}}=-\frac{1}{2}[t^{1_{[p]}}\partial_{q'},(t^{1_{[p]}}\partial_p-t^{1_{[q']}}\partial_{q'})].v_{0,(s+1)_{[1]}}=0 \tag{234}$$

for $p\in\overline{2,l_1+l_2}$ and $q'\in\overline{2,l_1+l_2}\backslash\{p\}$, and

$$t^{1_{[p]}}\partial_{q'}.v_{0,(s+1)_{[1]}}=[t^{1_{[p]}}\partial_r,t^{1_{[r]}}\partial_{q'}].v_{0,(s+1)_{[1]}}=0 \tag{235}$$

for $p\in\overline{2,l_1+l_2}$ and $q'\in\overline{l_1+l_2+1,l}$, where $r\in\overline{2,l_1+l_2}\backslash\{p\}$. Moreover, by (233) and Lemma 4.2 again, we have

$$(t^{1_{[p]}}\partial_p-t^{1_{[q]}}\partial_q).v_{0,(s+1)_{[1]}}=[t^{1_{[p]}}\partial_q,t^{1_{[q]}}\partial_p].v_{0,(s+1)_{[1]}}=0 \tag{236}$$

for $p,q\in\overline{2,l_1+l_2}$. So (232) holds for $k=s+1$. Thus (232) follows from induction on k.

Secondly, we shall prove (231) for $\mathbf{i}=(i_1,i_2,\cdots,i_{l_1+l_2},0,\cdots,0)\in\mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}|=k\geq 1$ and $\mathbf{i}\neq k_{[1]}$.

For any $p\in\overline{2,l_1+l_2}$ and $q'\in\overline{2,l}\backslash\{p\}$, we have

$$\begin{aligned}
t^{1_{[p]}}\partial_{q'}.v_{0,\mathbf{i}} &= \frac{i_1!}{k!}t^{1_{[p]}}\partial_{q'}.(t^{1_{[l_1+l_2]}}\partial_1)^{i_{l_1+l_2}}.\cdots.(t^{1_{[2]}}\partial_1)^{i_2}.v_{0,k_{[1]}}\\
&= \frac{i_1!}{k!}(t^{1_{[l_1+l_2]}}\partial_1)^{i_{l_1+l_2}}.\cdots.(t^{1_{[2]}}\partial_1)^{i_2}.(t^{1_{[p]}}\partial_{q'}.v_{0,k_{[1]}})\\
&\quad+\sum_{s=2}^{l_1+l_2}\delta_{q',s}i_s\frac{i_1!}{k!}(t^{1_{[l_1+l_2]}}\partial_1)^{i_{l_1+l_2}}.\cdots.(t^{1_{[s]}}\partial_1)^{i_s-1}.\\
&\quad\cdots.(t^{1_{[p]}}\partial_1)^{i_p+1}.\cdots.(t^{1_{[2]}}\partial_1)^{i_2}.v_{0,k_{[1]}}\\
&= i_{q'}v_{0,\mathbf{i}+1_{[p]}-1_{[q']}}
\end{aligned} \tag{237}$$

by (215) and (232), where $k=|\mathbf{i}|$, and $i_{q'}=0$ when $q'\in\overline{l_1+l_2+1,l}$. Moreover, (237) implies

$$(t^{1_{[p]}}\partial_p-t^{1_{[q]}}\partial_q).v_{0,\mathbf{i}}=[t^{1_{[p]}}\partial_q,t^{1_{[q]}}\partial_p].v_{0,\mathbf{i}}=(i_p-i_q)v_{0,\mathbf{i}} \tag{238}$$

for any $p,q\in\overline{2,l_1+l_2}$ with $p\neq q$. So this step follows from (232), (237) and (238).

Step 3. $t^{1_{[1]}}\partial_{q'}.v_{0,k_{[1]}}=0$ for all $q'\in\overline{2,l}$ and $k\geq 0$.

The proof of this step is in analogy with that of (232). We omit the details here.

Step 4. For $k\geq 0$ and $p',q'\in\overline{1,l}$ with $p'\neq q'$, we have

$$D_{p',q'}(x^{\rho}).v_{-\rho,(k+1)_{[1]}}=(k+1)(\rho_{p'}\delta_{q',1}-\rho_{q'}\delta_{p',1})v_{0,k_{[1]}}, \tag{239}$$

where ρ is the fixed element in (213).

For any $k \geq 0$ and $q' \in \overline{2,l}\setminus\{r_1\}$, where r_1 is the fixed element in (212), we obtain

$$\begin{aligned}
& D_{r_1,q'}(x^\rho).v_{-\rho,(k+1)_{[1]}} \\
= \;& x^\rho(\rho_{r_1}\partial_{q'} - \rho_{q'}\partial_{r_1}).v_{-\rho,(k+1)_{[1]}} \\
= \;& \frac{1}{\rho_{r_1}}[x^\rho(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}), t^{1_{[1]}}(\rho_{r_1}\partial_{q'} - \rho_{q'}\partial_{r_1})].v_{-\rho,(k+1)_{[1]}} \\
= \;& -\frac{1}{\rho_{r_1}}t^{1_{[1]}}(\rho_{r_1}\partial_{q'} - \rho_{q'}\partial_{r_1}).(x^\rho(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).v_{-\rho,(k+1)_{[1]}}) \\
= \;& -(k+1)t^{1_{[1]}}(\rho_{r_1}\partial_{q'} - \rho_{q'}\partial_{r_1}).v_{0,k_{[1]}} \\
= \;& 0 \qquad (240)
\end{aligned}$$

by (214), Lemma 3.6 and Step 3. So (214) and (240) give

$$\begin{aligned}
D_{p',q'}(x^\rho).v_{-\rho,(k+1)_{[1]}} &= \frac{1}{\rho_{r_1}}(\rho_{p'}D_{r_1,q'}(x^\rho) - \rho_{q'}D_{r_1,p'}(x^\rho)).v_{-\rho,(k+1)_{[1]}} \\
&= (k+1)(\rho_{p'}\delta_{q',1} - \rho_{q'}\delta_{p',1})v_{0,k_{[1]}} \qquad (241)
\end{aligned}$$

for $p', q' \in \overline{1,l}$ with $p' \neq q'$. This completes the proof of the step.

Step 5. $(t^{1_{[1]}}\partial_1 - t^{1_{[2]}}\partial_2).v_{0,k_{[1]}} = kv_{0,k_{[1]}}$ for $k > 0$.

Let ρ and r_1, r_2 be the fixed elements in (212) and (213). Pick $r \in \{r_1, r_2\}\setminus\{2\}$. Then $r \notin \{1,2\}$. Set

$$\tilde{\partial}_1 = \rho_r\partial_2 - \rho_2\partial_r \;\text{ and }\; \tilde{\partial}_2 = \rho_r\partial_1 - \rho_1\partial_r. \qquad (242)$$

Then (214), Steps 2–4 and Lemma 3.6 give

$$\begin{aligned}
& (t^{1_{[1]}}\partial_1 - t^{1_{[2]}}\partial_2).v_{0,k_{[1]}} \\
= \;& \frac{1}{\rho_r}(t^{1_{[1]}}\tilde{\partial}_2 - t^{1_{[2]}}\tilde{\partial}_1 + \rho_1 t^{1_{[1]}}\partial_r - \rho_2 t^{1_{[2]}}\partial_r).v_{0,k_{[1]}} \\
= \;& \frac{1}{\rho_r\rho_{r_1}(k+1)}(t^{1_{[1]}}\tilde{\partial}_2 - t^{1_{[2]}}\tilde{\partial}_1).x^\rho(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).v_{-\rho,(k+1)_{[1]}} \\
= \;& \frac{1}{\rho_r\rho_{r_1}(k+1)}\Big(x^\rho(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).(t^{1_{[1]}}\tilde{\partial}_2 - t^{1_{[2]}}\tilde{\partial}_1).v_{-\rho,(k+1)_{[1]}} \\
& -x^\rho(\rho_{r_1}\tilde{\partial}_2 + \rho_1\delta_{r_1,2}\tilde{\partial}_1).v_{-\rho,(k+1)_{[1]}}\Big) \\
= \;& \frac{1}{\rho_{r_1}}x^\rho(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).v_{-\rho,(k+1)_{[1]}} - v_{0,k_{[1]}} \\
= \;& kv_{0,k_{[1]}} \qquad (243)
\end{aligned}$$

for $k > 0$.

Step 6. $t^{1_{[1]}}\partial_q.v_{0,\mathbf{i}} = i_q v_{0,\mathbf{i}+1_{[1]}-1_{[q]}}$ for $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ and $q \in \overline{2,l}$.

When $\mathbf{i} = \mathbf{0}$, it follows from Lemma 4.2 and Lemma 4.4. For $\mathbf{0} \neq \mathbf{i} \in \mathbb{N}^{l_1+l_2}$, eq. (215), Steps 2, 3 and 5 give rise to

$$\begin{aligned} t^{1_{[1]}}\partial_2.v_{0,\mathbf{i}} &= \frac{i_1!}{k!}t^{1_{[1]}}\partial_2.(t^{1_{[l_1+l_2]}}\partial_1)^{i_{l_1+l_2}}.\cdots.(t^{1_{[2]}}\partial_1)^{i_2}.v_{0,k_{[1]}} \\ &= \frac{i_1!}{k!}\Big(-\sum_{s=3}^{l_1+l_2} i_s i_2 (t^{1_{[l_1+l_2]}}\partial_1)^{i_{l_1+l_2}}.\cdots.(t^{1_{[2]}}\partial_1)^{i_2-1}.v_{0,k_{[1]}} \\ &\quad -i_2(i_2-1)(t^{1_{[l_1+l_2]}}\partial_1)^{i_{l_1+l_2}}.\cdots.(t^{1_{[2]}}\partial_1)^{i_2-1}.v_{0,k_{[1]}} \\ &\quad +i_2(t^{1_{[l_1+l_2]}}\partial_1)^{i_{l_1+l_2}}.\cdots.(t^{1_{[2]}}\partial_1)^{i_2-1}.(t^{1_{[1]}}\partial_1 - t^{1_{[2]}}\partial_2).v_{0,k_{[1]}}\Big) \\ &= i_2\frac{(i_1+1)!}{k!}(t^{1_{[l_1+l_2]}}\partial_1)^{i_{l_1+l_2}}.\cdots.(t^{1_{[2]}}\partial_1)^{i_2-1}.v_{0,k_{[1]}} \\ &= i_2 v_{0,\mathbf{i}+1_{[1]}-1_{[2]}}, \end{aligned} \tag{244}$$

where $k=|\mathbf{i}|$. Moreover, (244) and Step 2 tell

$$\begin{aligned} t^{1_{[1]}}\partial_q.v_{0,\mathbf{i}} &= [t^{1_{[1]}}\partial_2, t^{1_{[2]}}\partial_q].v_{0,\mathbf{i}} \\ &= t^{1_{[1]}}\partial_2.(i_q v_{0,\mathbf{i}+1_{[2]}-1_{[q]}}) - t^{1_{[2]}}\partial_q.(i_2 v_{0,\mathbf{i}+1_{[1]}-1_{[2]}}) \\ &= i_q(i_2+1)v_{0,\mathbf{i}+1_{[1]}-1_{[q]}} - i_q i_2 v_{0,\mathbf{i}+1_{[1]}-1_{[q]}} \\ &= i_q v_{0,\mathbf{i}+1_{[1]}-1_{[q]}} \end{aligned} \tag{245}$$

for $q\in\overline{3,l}$ and $\mathbf{0}\neq\mathbf{i}\in\mathbb{N}^{l_1+l_2}$. So this step holds.

Step 7. $(t^{1_{[1]}}\partial_1 - t^{1_{[p]}}\partial_p).v_{0,\mathbf{i}} = (i_1-i_p)v_{0,\mathbf{i}}$ for $\mathbf{i}\in\mathbb{N}^{l_1+l_2}$ and $p\in\overline{2,l_1+l_2}$.

When $\mathbf{i}=\mathbf{0}$, it follows from Lemma 4.2 and Lemma 4.4. For $\mathbf{0}\neq\mathbf{i}\in\mathbb{N}^{l_1+l_2}$ and $p\in\overline{2,l_1+l_2}$, Step 1 and Step 6 imply

$$(t^{1_{[1]}}\partial_1 - t^{1_{[p]}}\partial_p).v_{0,\mathbf{i}} = [t^{1_{[1]}}\partial_p, t^{1_{[p]}}\partial_1].v_{0,\mathbf{i}} = (i_1-i_p)v_{0,\mathbf{i}}. \tag{246}$$

In summary, Steps 1, 2 and 6 show (227); Steps 2 and 7 indicate (228); Step 4 gives (229). So we complete the proof of this lemma. ∎

Lemma 4.7. *For $\boldsymbol{i}\in\mathbb{N}^{l_1+l_2}$ and $p,q\in\overline{1,l}$ with $p\neq q$, we have*

$$D_{p,q}(x^\rho).v_{-\rho,\boldsymbol{i}} = \rho_p i_q v_{0,\boldsymbol{i}-1_{[q]}} - \rho_q i_p v_{0,\boldsymbol{i}-1_{[p]}}, \tag{247}$$

where ρ is the fixed element in (213).

Proof. When $\mathbf{i}=\mathbf{0}$, (247) follows from Lemma 3.2. So we only need to consider the case $\mathbf{i}\neq\mathbf{0}$. Recall that Lemma 4.6 has given the proof for $\mathbf{i}=k_{[1]}$ with $k\geq 1$. Therefore, we can prove the lemma this way: For any $k\geq 1$, we verify (247) for all $\mathbf{i}\in\mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}|=k$, by induction on $\mathbf{i}$ with the total order defined in Definition 3.9.

Fix any $k\geq 1$. Assume that, (247) holds for all $\mathbf{i}<\mathbf{j}$ with $|\mathbf{i}|=|\mathbf{j}|=k$, where $j_p\neq 0$ for some $p\in\overline{2,l_1+l_2}$. Then it suffices to verify (247) for $\mathbf{i}=\mathbf{j}$. Set

$$s=\max\{p\in\overline{2,l_1+l_2}\mid j_p\neq 0\}. \tag{248}$$

Pick $r\in\{r_1,r_2\}\backslash\{s\}$, where r_1,r_2 are the fixed element in (212). Then $r\neq 1$ and $\rho_r\neq 0$. So by the total order defined in (185), we have

$$\mathbf{j}+1_{[1]}-1_{[s]}<\mathbf{j},\quad j_r=0 \text{ or } \mathbf{j}+1_{[1]}-1_{[r]}<\mathbf{j}. \tag{249}$$

Thus Lemma 3.6, Lemma 4.6 and the induction hypothesis give

$$\begin{aligned} & D_{p,q}(x^\rho).v_{-\rho,\mathbf{j}} \\ = & \frac{1}{\rho_r(j_1+1)}D_{p,q}(x^\rho).\big(t^{1_{[s]}}(\rho_r\partial_1-\rho_1\partial_r).v_{-\rho,\mathbf{j}+1_{[1]}-1_{[s]}}+\rho_1 j_r v_{-\rho,\mathbf{j}+1_{[1]}-1_{[r]}}\big) \\ = & \frac{1}{\rho_r(j_1+1)}\Big((\rho_p\delta_{q,s}-\rho_q\delta_{p,s})x^\rho(\rho_r\partial_1-\rho_1\partial_r).v_{-\rho,\mathbf{j}+1_{[1]}-1_{[s]}} \\ & +t^{1_{[s]}}(\rho_r\partial_1-\rho_1\partial_r).D_{p,q}(x^\rho).v_{-\rho,\mathbf{j}+1_{[1]}-1_{[s]}}+\rho_1 j_r D_{p,q}(x^\rho).v_{-\rho,\mathbf{j}+1_{[1]}-1_{[r]}}\Big) \\ = & \rho_p j_q v_{0,\mathbf{j}-1_{[q]}}-\rho_q j_p v_{0,\mathbf{j}-1_{[p]}} \end{aligned} \tag{250}$$

for $p,q\in\overline{1,l}$ with $p\neq q$, which coincides with (247). So this lemma holds. ■

Lemma 4.8. *For any $\alpha\in\Gamma\backslash\{0\}$, $\boldsymbol{i}\in\mathbb{N}^{l_1+l_2}$ and $p,q\in\overline{1,l}$ with $p\neq q$, we have*

$$D_{p,q}(x^\alpha).v_{-\alpha,\boldsymbol{i}}=\alpha_p i_q v_{0,\boldsymbol{i}-1_{[q]}}-\alpha_q i_p v_{0,\boldsymbol{i}-1_{[p]}}. \tag{251}$$

Proof. We prove this lemma by induction on $|\mathbf{i}|$. When $|\mathbf{i}|=0$, Lemma 3.2 shows

$$D_{p,q}(x^\alpha).v_{-\alpha,\mathbf{0}}=0 \tag{252}$$

for all $\alpha\in\Gamma\backslash\{0\}$ and $p,q\in\overline{1,l}$ with $p\neq q$, which coincides with (251). Suppose that (251) holds for $\mathbf{i}\in\mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}|\leq k$, where $k\geq 0$. Fix some $\beta\in\Gamma\backslash\{0\}$ such that

$$\ker\beta\neq\ker\rho, \tag{253}$$

where ρ is the fixed element in (213). Then we prove (251) for $\mathbf{i}\in\mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}|=k+1$ in two steps.

Step 1. For $\mathbf{i}\in\mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}|=k+1$, and $p,q\in\overline{1,l}$ with $p\neq q$, we have

$$D_{p,q}(x^\beta).v_{-\beta,\mathbf{i}}=\beta_p i_q v_{0,\mathbf{i}-1_{[q]}}-\beta_q i_p v_{0,\mathbf{i}-1_{[p]}}. \tag{254}$$

Since $\ker\beta\neq\ker\rho$, we have $\beta\neq\pm\rho$, $\beta_{s_1}\neq 0$ for some $s_1\in\overline{l_1+1,l}$, and $\beta_{s_1}\rho_{s_2}-\beta_{s_2}\rho_{s_1}\neq 0$ for some $s_2\in\overline{l_1+1,l}\backslash\{s_1\}$. Fix such s_1 and s_2. Set

$$\tilde{\partial}_q=(\beta_{s_1}\rho_{s_2}-\beta_{s_2}\rho_{s_1})(\beta_{s_1}\partial_q-\beta_q\partial_{s_1})-(\beta_{s_1}\rho_q-\beta_q\rho_{s_1})(\beta_{s_1}\partial_{s_2}-\beta_{s_2}\partial_{s_1}) \tag{255}$$

for $q\in\overline{1,l}\backslash\{s_1,s_2\}$, where ρ is the fixed element in (213). Then for any $q\in\overline{1,l}\backslash\{s_1,s_2\}$, we have

$$\tilde{\partial}_q(\beta)=\tilde{\partial}_q(\rho)=0, \tag{256}$$

$$x^\beta\tilde{\partial}_q=((\beta_{s_1}\rho_{s_2}-\beta_{s_2}\rho_{s_1})D_{s_1,q}(x^\beta)-(\beta_{s_1}\rho_q-\beta_q\rho_{s_1})D_{s_1,s_2}(x^\beta)), \tag{257}$$

$$x^\rho\tilde{\partial}_q=\beta_{s_1}\big(\beta_{s_1}D_{s_2,q}(x^\rho)+\beta_{s_2}D_{q,s_1}(x^\rho)+\beta_q D_{s_1,s_2}(x^\rho)\big). \tag{258}$$

Let

$$\bar{\partial}=\beta_{s_1}\partial_{s_2}-\beta_{s_2}\partial_{s_1}\text{ and }\bar{\partial}'=\rho_{s_1}\partial_{s_2}-\rho_{s_2}\partial_{s_1}. \tag{259}$$

Then $\bar{\partial} \in \ker\beta \backslash \ker\rho$ and $\bar{\partial}' \in \ker\rho\backslash\ker\beta$. So Lemma 3.8, Lemma 4.7 and the induction hypothesis give

$$\begin{aligned} x^\beta\tilde{\partial}_q.v_{-\beta,\mathbf{i}} &= \frac{1}{\bar{\partial}(\rho)\bar{\partial}'(\beta)}[x^{-\rho}\bar{\partial}', [x^\beta\bar{\partial}, x^\rho\tilde{\partial}_q]].v_{-\beta,\mathbf{i}} \\ &= \frac{1}{\bar{\partial}(\rho)\bar{\partial}'(\beta)}\Big(\bar{\partial}(\rho)x^{-\rho}\bar{\partial}'.x^{\beta+\rho}\tilde{\partial}_q.v_{-\beta,\mathbf{i}} - x^\beta\bar{\partial}.x^\rho\tilde{\partial}_q.x^{-\rho}\bar{\partial}'.v_{-\beta,\mathbf{i}} \\ &\quad + x^\rho\tilde{\partial}_q.x^\beta\bar{\partial}.x^{-\rho}\bar{\partial}'.v_{-\beta,\mathbf{i}}\Big) \\ &= \beta_{s_1}\Big((\beta_{s_1}\rho_{s_2} - \beta_{s_2}\rho_{s_1})i_q v_{0,\mathbf{i}-1_{[q]}} - (\beta_{s_1}\rho_q - \beta_q\rho_{s_1})i_{s_2}v_{0,\mathbf{i}-1_{[s_2]}} \\ &\quad + (\beta_{s_2}\rho_q - \beta_q\rho_{s_2})i_{s_1}v_{0,\mathbf{i}-1_{[s_1]}}\Big) \end{aligned} \tag{260}$$

for $q \in \overline{1,l}\backslash\{s_1,s_2\}$, $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$. Thus by Lemma 3.6, Lemma 4.6, (255) and (260), we obtain

$$\begin{aligned} &D_{s_1,s_2}(x^\beta).v_{-\beta,\mathbf{i}} \\ = &\frac{1}{\beta_{s_1}(\beta_{s_1}\rho_{s_2} - \beta_{s_2}\rho_{s_1})}[x^\beta\tilde{\partial}_p, t^{1_{[p]}}(\beta_{s_1}\partial_{s_2} - \beta_{s_2}\partial_{s_1})].v_{-\beta,\mathbf{i}} \\ = &\frac{1}{\beta_{s_1}(\beta_{s_1}\rho_{s_2} - \beta_{s_2}\rho_{s_1})}\Big(\beta_{s_1}i_{s_2}x^\beta\tilde{\partial}_p.v_{-\beta,\mathbf{i}+1_{[p]}-1_{[s_2]}} - \beta_{s_2}i_{s_1}x^\beta\tilde{\partial}_p.v_{-\beta,\mathbf{i}+1_{[p]}-1_{[s_1]}} \\ &- t^{1_{[p]}}(\beta_{s_1}\partial_{s_2} - \beta_{s_2}\partial_{s_1}).(x^\beta\tilde{\partial}_p.v_{-\beta,\mathbf{i}})\Big) \\ = &\beta_{s_1}i_{s_2}v_{0,\mathbf{i}-1_{[s_2]}} - \beta_{s_2}i_{s_1}v_{0,\mathbf{i}-1_{[s_1]}} \end{aligned} \tag{261}$$

for $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$, where $p \in \overline{1, l_1+l_2}\backslash\{s_1,s_2\}$. Moreover, (257), (260) and (261) imply

$$\begin{aligned} D_{s_1,q}(x^\beta).v_{-\beta,\mathbf{i}} &= \frac{1}{\beta_{s_1}\rho_{s_2} - \beta_{s_2}\rho_{s_1}}(x^\beta\tilde{\partial}_q + (\beta_{s_1}\rho_q - \beta_q\rho_{s_1})D_{s_1,s_2}(x^\beta)).v_{-\beta,\mathbf{i}} \\ &= \beta_{s_1}i_q v_{0,\mathbf{i}-1_{[q]}} - \beta_q i_{s_1}v_{0,\mathbf{i}-1_{[s_1]}} \end{aligned} \tag{262}$$

for $q \in \overline{1,l}\backslash\{s_1,s_2\}$, $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$. Therefore, (261) and (262) indicate

$$D_{p,q}(x^\beta).v_{-\beta,\mathbf{i}} = \frac{1}{\beta_{s_1}}(\beta_p D_{s_1,q}(x^\beta) - \beta_q D_{s_1,p}(x^\beta)).v_{-\beta,\mathbf{i}} = \beta_p i_q v_{0,\mathbf{i}-1_{[q]}} - \beta_q i_p v_{0,\mathbf{i}-1_{[p]}} \tag{263}$$

for $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$, $p, q \in \overline{1,l}$ with $p \neq q$. So this step holds.

Step 2. For any $\alpha \in \Gamma\backslash\{0\}$, $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k+1$, and $p, q \in \overline{1,l}$ with $p \neq q$, we have

$$D_{p,q}(x^\alpha).v_{-\alpha,\mathbf{i}} = \alpha_p i_q v_{0,\mathbf{i}-1_{[q]}} - \alpha_q i_p v_{0,\mathbf{i}-1_{[p]}}. \tag{264}$$

Fix an arbitrary $\alpha \in \Gamma\backslash\{0\}$. Then we have $\ker\alpha \neq \ker\rho$ or $\ker\alpha \neq \ker\beta$, where ρ is the fixed element in (213). Choose $\gamma \in \{\rho, \beta\}$ such that $\ker\alpha \neq \ker\gamma$. Then, with β replaced by α and ρ by γ in Step 1, we can get

$$D_{p,q}(x^\alpha).v_{-\alpha,\mathbf{i}} = \alpha_p i_q v_{0,\mathbf{i}-1_{[q]}} - \alpha_q i_p v_{0,\mathbf{i}-1_{[p]}} \tag{265}$$

for $\mathbf{i}\in\mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}|=k+1$, $p,q\in\overline{1,l}$ with $p\neq q$. As $\alpha\in\Gamma\backslash\{0\}$ is arbitrary, this completes the proof of the step.

Thus (251) holds for $\mathbf{i}\in\mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}|=k+1$. So this lemma follows from induction on $|\mathbf{i}|$. ∎

Remark 4.9. The above lemma induces that, for any other possible choice of r_1 in (212) and ρ in (213), which we may denote by $\tilde{r}_1$ and $\tilde{\rho}$ respectively, we have

$$\begin{aligned}&\frac{1}{\tilde{\rho}_{\tilde{r}_1}(k+1)}x^{\tilde{\rho}}(\tilde{\rho}_{\tilde{r}_1}\partial_1-\tilde{\rho}_1\partial_{\tilde{r}_1}).v_{-\tilde{\rho},(k+1)_{[1]}}\\ =\;& v_{0,k_{[1]}}=\frac{1}{\rho_{r_1}(k+1)}x^{\rho}(\rho_{r_1}\partial_1-\rho_1\partial_{r_1}).v_{-\rho,(k+1)_{[1]}}\end{aligned}\tag{266}$$

for $k\geq 0$. In other words, $v_{0,k_{[1]}}$ in (214) is independent of the choice of r_1 and ρ.

Lemma 4.10. *For any $\alpha\in\Gamma\backslash\{0\}$, $\boldsymbol{i}\in\mathbb{N}^{l_1+l_2}$ and $p,q\in\overline{1,l}$ with $p\neq q$, we have*

$$D_{p,q}(x^{\alpha}).v_{0,\boldsymbol{i}}=\alpha_p i_q v_{\alpha,\boldsymbol{i}-1_{[q]}}-\alpha_q i_p v_{\alpha,\boldsymbol{i}-1_{[p]}}.\tag{267}$$

Proof. When $\mathbf{i}=\mathbf{0}$, (267) follows from Lemma 4.2. So we only need to consider the case $\mathbf{i}\neq\mathbf{0}$. Fix an arbitrary $k>0$. Then we prove (267) for all $\alpha\in\Gamma\backslash\{0\}$, $\mathbf{i}\in\mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}|=k$, and $p,q\in\overline{1,l}$ with $p\neq q$ in two steps.

Step 1. For any $\alpha\in\Gamma\backslash\{0,\rho\}$, $\mathbf{i}\in\mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}|=k$, $p,q\in\overline{1,l}$ with $p\neq q$, we have

$$D_{p,q}(x^{\alpha}).v_{0,\mathbf{i}}=\alpha_p i_q v_{\alpha,\mathbf{i}-1_{[q]}}-\alpha_q i_p v_{\alpha,\mathbf{i}-1_{[p]}},\tag{268}$$

where ρ is the fixed element in (213).

We prove this step by induction on $\mathbf{i}$ with the total order defined in Definition 3.9. When $\mathbf{i}=k_{[1]}$, by (214) and Lemma 3.8 we have

$$\begin{aligned}D_{p,q}(x^{\alpha}).v_{0,k_{[1]}} &= \frac{1}{\rho_{r_1}(k+1)}D_{p,q}(x^{\alpha}).(x^{\rho}(\rho_{r_1}\partial_1-\rho_1\partial_{r_1}).v_{-\rho,(k+1)_{[1]}})\\ &= \frac{1}{\rho_{r_1}(k+1)}\Big(x^{\rho}(\rho_{r_1}\partial_1-\rho_1\partial_{r_1}).D_{p,q}(x^{\alpha}).v_{-\rho,(k+1)_{[1]}}\\ &\quad +x^{\alpha+\rho}\big((\alpha_p\rho_q-\alpha_q\rho_p)(\rho_{r_1}\partial_1-\rho_1\partial_{r_1})\\ &\quad -(\rho_{r_1}\alpha_1-\rho_1\alpha_{r_1})(\alpha_p\partial_q-\alpha_q\partial_p)\big).v_{-\rho,(k+1)_{[1]}}\Big)\\ &= k(\alpha_p\delta_{q,1}-\alpha_q\delta_{p,1})v_{\alpha,(k-1)_{[1]}}\end{aligned}\tag{269}$$

for all $\alpha\in\Gamma\backslash\{0,\rho\}$, $p,q\in\overline{1,l}$ with $p\neq q$, where ρ and r_1 are the fixed elements in (212) and (213). Namely, (268) holds when $\mathbf{i}=k_{[1]}$. Suppose that, (268) holds for all $\mathbf{i}<\mathbf{j}$ with $|\mathbf{i}|=|\mathbf{j}|=k$ and $j_p\neq 0$ for some $p\in\overline{2,l_1+l_2}$. Then it suffices to prove (268) for $\mathbf{i}=\mathbf{j}$. Since $r_1>1$ (cf. (212)), we have

$$j_{r_1}=0 \text{ or } \mathbf{j}+1_{[1]}-1_{[r_1]}<\mathbf{j}\tag{270}$$

by the total order defined in (185). Thus Lemma 3.8, Lemma 4.7 and the induction hypothesis give

$$\begin{aligned}
& D_{p,q}(x^\alpha).v_{0,\mathbf{j}} \\
= \;& \frac{1}{\rho_{r_1}(j_1+1)} D_{p,q}(x^\alpha).\big(x^\rho(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).v_{-\rho,\mathbf{j}+1_{[1]}} + \rho_1 j_{r_1} v_{0,\mathbf{j}+1_{[1]}-1_{[r_1]}}\big) \\
= \;& \frac{1}{\rho_{r_1}(j_1+1)} \Big(x^\rho(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}).x^\alpha(\alpha_p\partial_q - \alpha_q\partial_p).v_{-\rho,\mathbf{j}+1_{[1]}} \\
& + x^{\alpha+\rho}\big((\alpha_p\rho_q - \alpha_q\rho_p)(\rho_{r_1}\partial_1 - \rho_1\partial_{r_1}) \\
& - (\rho_{r_1}\alpha_1 - \rho_1\alpha_{r_1})(\alpha_p\partial_q - \alpha_q\partial_p)\big).v_{-\rho,\mathbf{j}+1_{[1]}} \\
& + \rho_1 j_{r_1} x^\alpha(\alpha_p\partial_q - \alpha_q\partial_p).v_{0,\mathbf{j}+1_{[1]}-1_{[r_1]}}\Big) \\
= \;& \alpha_p j_q v_{\alpha,\mathbf{j}-1_{[q]}} - \alpha_q j_p v_{\alpha,\mathbf{j}-1_{[p]}}
\end{aligned} \tag{271}$$

for all $\alpha \in \Gamma\backslash\{0,\rho\}$, $p,q \in \overline{1,l}$ with $p \neq q$. So (268) holds for $\mathbf{i} = \mathbf{j}$, which completes the proof of the step.

Step 2. For all $\mathbf{i} \in \mathbb{N}^{l_1+l_2}$ with $|\mathbf{i}| = k$, $p,q \in \overline{1,l}$ with $p \neq q$, we have

$$D_{p,q}(x^\rho).v_{0,\mathbf{i}} = \rho_p i_q v_{\rho,\mathbf{i}-1_{[q]}} - \rho_q i_p v_{\rho,\mathbf{i}-1_{[p]}}, \tag{272}$$

where ρ is the fixed element in (213).

Replacing ρ by $-\rho$, and α by ρ in Step 1, we can similarly prove (272). Here we omit the details.

Since $k > 0$ is arbitrary, we deduce that this lemma holds. ∎

Under the condition that $\mu = 0$, we get a set $\{v_{\beta,\mathbf{i}} \mid \beta \in \Gamma, \mathbf{i} \in \mathbb{N}^{l_1+l_2}\}$ from Definitions 3.3 and 4.3. In analogy with Lemma 3.10, it can be deduced from Lemma 3.7, Lemma 4.4, Lemma 4.5 and Definition 3.9 that:

Lemma 4.11. *If $\mu = 0$, then $\{v_{\beta,\boldsymbol{i}} \mid \beta \in \Gamma, \boldsymbol{i} \in \mathbb{N}^{l_1+l_2}\}$ is an $\mathbb{F}$-basis of V.*

So (199), Proposition 2.5, Lemma 3.6–3.8 and Lemma 4.4–4.11 give

Lemma 4.12. *If $\mu \in \Gamma$, then $V = V(\mu) \simeq V(0) \simeq \mathscr{A}^0$.*

In summary, Theorem 1.1 follows from Lemmas 3.1, 3.11 and 4.12.

References

[1] Bergen, J., and D. S. Passman, *Simple Lie algebras of special type*, J. Algebra **227** (2000), 45–67.

[2] Chen, L., *Multiplicity-one representations of divergence-free Lie algebras*, J. Algebra **352** (2012), 1–61.

[3] Djokovic, D. Ž., and K. Zhao, *Derivations, isomorphisms, and second cohomology of generalized Witt algebras*, Trans. Amer. Math. Soc. **350** (1998), 643–664.

[4] —, *Generalized Cartan type S Lie algebras in characteristic zero*, J. Algebra **193** (1997), 144–179.

[5] Kac, V. G., *Description of filtered Lie algebras with which graded Lie algebras of Cartan type are associated*, Math. USSR-Izvestiya **8** (1974), 801–835.

[6] —, *Lie superalgebras*, Adv. Math. **26** (1977), 8–96.

[7] —, *Classification of infinite-dimensional simple linearly compact Lie superalgebras*, Adv. Math. **139** (1998), 1–55.

[8] —, *Some problems on infinite dimensional Lie algebras and their representations*, Lect. Notes in Math. **933** (1982), 117–126.

[9] Kac, V. G., and A. K. Raina, "Bombay lectures on highest weight representations of infinite dimensional Lie algebras," Adv. Ser. Math. Phys. **2**, 1987.

[10] Kaplansky, I., *The Virasoro algebra*, Commun. Math. Phys. **86** (1982), 49–54.

[11] Kaplansky, I., and L. J. Santharoubane, *Harish-Chandra modules over the Virasoro algebras*, MSRI Publ. **4** (1987), 217–231.

[12] Mathieu, O., *Classification of Harish-Chandra modules over the Virasoro Lie algebra*, Invent. Math. **107** (1992), 225–234.

[13] Mazorchuk, V., *Verma modules over generalized Witt algebras*, Compositio Math. **115** (1999), 21–35.

[14] Osborn, J. M., *New simple infinite-dimensional Lie algebras of characteristic 0*, J. Algebra **185** (1996), 820–835.

[15] Passman, D. S., *Simple Lie algebras of Witt type*, J. Algebra **206** (1998), 682–692.

[16] Penkov, I., and V. Serganova, *Weight representations of the polynomial Cartan type Lie algebras W_n and $\bar{S}_n$*, Math. Res. Lett. **6** (1999), 397–416.

[17] Rao, S. E., *Representations of Witt algebras*, Publ. Res. Inst. Math. Sci. **30** (1994), 191–201.

[18] —, *Irreducible representations of the Lie algebra of the diffeomorphisms of a d-dimensional torus*, J. Algebra **182** (1996), 401–421.

[19] Shen, G., *Graded modules of graded Lie algebras of Cartan type (I)–mixed products of modules*, Sci. China Ser. A **29** (1986), 570–581.

[20] —, *Graded modules of graded Lie algebras of Cartan type (II)–positive and negative graded modules*, Sci. China Ser. A **29** (1986), 1009–1019.

[21] —, *Graded modules of graded Lie algebras of Cartan type (III)–irreducible modules*, Chinese Ann. Math. Ser. B **9** (1988), 404–417.

[22] Su, Y., *Harish-Chandra modules of the intermediate series over the high rank Virasoro algebras and high rank super-Virasoro algebras*, J. Math. Phys. **35** (1994), 2013–2023.

[23] —, *Classification of Harish-Chandra modules over the super-Virasoro algebras*, Comm. Algebra **23** (1995), 3653–3675.

[24] Su, Y., and X. Xu, *Structure of divergence-free Lie algebras*, J. Algebra **243** (2001), 557–595.

[25] Su, Y., X. Xu, and H. Zhang, *Derivation-simple algebras and the structures of Lie algebras of Witt type*, J. Algebra **233** (2000), 642–662.

[26] Su, Y., and J. Zhou, *Some representations of nongraded Lie algebras of generalized Witt type*, J. Algebra **246** (2001), 721–738.

[27] Su, Y., and K. Zhao, *Generalized Virasoro and super-Virasoro algebras and modules of the intermediate series*, J. Algebra **252** (2002), 1–19.

[28] Xu, X., *New generalized simple Lie algebras of Cartan type over a field with characteristic 0*, J. Algebra **224** (2000), 23–58.

[29] —, *Quadratic conformal superalgebras*, J. Algebra **231** (2000), 1–38.

[30] —, *Equivalence of conformal superalgebras to Hamiltonian superoperators*, Algebra Colloq. **8** (2001), 63–92.

[31] Zhao, K., *Generalized Cartan type S Lie algebras in characteristic 0 (II)*, Pacific J. Math. **192** (2000), 431–454.

[32] —, *Weight modules over generalized Witt algebras with 1-dimensional weight spaces*, Forum Math. **16** (2004), 725–748. Preprint, 2000.

[33] —, *Irreducible representations of nongraded Witt type Lie algebras*, J. Algebra **298** (2006), 540–562.

[34] —, *Representations of nongraded Lie algebras of Block type*, Manuscripta math. **119** (2006), 183–216.

[35] —, *Composition series for a family of modules of nongraded Hamiltonian type Lie algebras*, J. Lie Theory **19** (2009), 1–27.

Ling Chen
Beijing International Center
for Mathematical Research
Peking University
Beijing 100871, P. R. China
chenling@amss.ac.cn

Received April 5, 2012
and in final form October 6, 2012