Corrigendum to “On the Dimension of the Sheets of a Reductive Lie Algebra”

Anne Moreau

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Abstract. This note is a corrigendum to [5]. As it has been recently pointed out to me by Alexander Premet, [5, Remark 3.12] is incorrect. We explain in this note the impacts of that error in [5], and amend certain of its statements. In particular, we verify that the statement of [5, Theorem 3.13] remains correct in spite of this error.

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1. Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra and $G$ its adjoint group. We investigate in [5] the dimension of the subsets, for $m \in \mathbb{N}$,

$$\mathfrak{g}^{(m)} := \{ x \in \mathfrak{g} \mid \dim(\text{Ad}(x)) = 2m \},$$

where $\text{Ad}(x)$ denotes the adjoint orbit of $x \in \mathfrak{g}$. The irreducible components of the subsets $\mathfrak{g}^{(m)}$ are called the sheets of $\mathfrak{g}$, [2][1]. Thus, for any $m \in \mathbb{N}$,

$$\dim \mathfrak{g}^{(m)} = \max \{ \dim S \mid S \subset \mathfrak{g}^{(m)} \},$$

(1)

where $S$ runs through all sheets contained in $\mathfrak{g}^{(m)}$. The sheets are known to be parameterized by the pairs $(\mathfrak{l}, \mathcal{O}_l)$, up to $G$-conjugacy class, consisting of a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ and a rigid nilpotent orbit $\mathcal{O}_l$ in $\mathfrak{l}$, cf. [1]. This parametrization enables to write the dimension of a sheet $S$ associated with a pair $(\mathfrak{l}, \mathcal{O}_l)$ as the sum of the dimension of the center of $\mathfrak{l}$ and the dimension of the unique nilpotent orbit contained in $S$, see e.g. [5, Proposition 2.11].

In the classical case, formulas for $\mathfrak{g}^{(m)}$ are given in [5] Theorems 3.3 and 3.13 in term of partitions associated with nilpotent elements of $\mathfrak{g}$. As it has been recently pointed out by Alexander Premet, Remark 3.12 in [5] which claims that "in the classical case, the dimension of a sheet containing a given nilpotent
orbit does not depend on the choice of a sheet containing it” is incorrect. We give here some counter-examples (cf. Examples 3.1 and 3.2; see also [6, Remark 4]). This is true only for the type A where each nilpotent element belongs to only one sheet. The error stems from the proof of [5, Proposition 3.11]; see Section 3 for explanations. As a consequence, the proof of [5, Theorems 3.13], partly based on [5, Proposition 3.11], is incorrect too. However its statement remains true. This can be shown through a recent work of Premet and Topley, [6]. In more details, another formula for $g^{(m)}$ in term of partitions can be traced out from [6, Corollary 9] and the equality [1]. In this note, we verify (cf. Theorems 2.10) that the Premet-Topley formula for $g^{(m)}$ coincides with the one of [5, Theorem 3.13].

The note is organized as follows.

In Section 2, we recall some definitions and results of [6] and show that the statement of [5, Theorem 3.13] is correct in spite of the error in [5, Proposition 3.11], see Theorem 2.10(ii). In Section 3, we precisely pin down the error in the proof [5, Proposition 3.11] and describe the impacts of that error in [5]. As a conclusion, we list in Section 4 all corrections which have to be taken into account in [5].

Since the corrections in [5] only concern the types B, C and D, we assume for the remaining of the note that $g$ is either $so(N)$ or $sp(N)$, with $N \geq 2$, and $\varepsilon$ is $1$ or $-1$ depending on whether $g = so(N)$ or $sp(N)$. Following the notations of [5] (or [6]), we denote by $P_\varepsilon(N)$ the set of partitions of $N$ associated with the nilpotent elements of $g$. For $\lambda = (\lambda_1, \ldots, \lambda_n) \in P_\varepsilon(N)$, we denote by $e(\lambda)$ the corresponding nilpotent element of $g$ whose Jordan block sizes are $\lambda_1, \ldots, \lambda_n$. We will always assume that $\lambda_1 \geq \cdots \geq \lambda_n$.

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2. The main result

For the convenience of the reader, we recall here all the necessary definitions and results of [6]. Given a partition $\lambda = (\lambda_1, \ldots, \lambda_n) \in P_\varepsilon(N)$ we set,

$$\Delta(\lambda) := \{1 \leq i < n ; \varepsilon(-1)^{\lambda_i} = \varepsilon(-1)^{\lambda_{i+1}} = -1, \lambda_{i-1} \neq \lambda_i \geq \lambda_{i+1} \neq \lambda_{i+2}\}.$$ 

Our convention is that $\lambda_0 = 0$ and $\lambda_i = 0$ for all $i > n$. Recall the following result of Kempken and Spaltenstein (also recalled in [5] and [6]):

**Theorem 2.1 ([5, 7]).** Let $\lambda = (\lambda_1, \ldots, \lambda_n) \in P_\varepsilon(N)$. Then $e(\lambda)$ is rigid if and only if

- $\lambda_i - \lambda_{i+1} \in \{0, 1\}$ for all $1 \leq i \leq n$;
- the set $\{i \in \Delta(\lambda) ; \lambda_i = \lambda_{i+1}\}$ is empty.
Denote by \( \mathcal{P}_\varepsilon^*(N) \) the set of \( \lambda \in \mathcal{P}_\varepsilon(N) \) such that \( c(\lambda) \) is rigid. We call the elements of \( \mathcal{P}_\varepsilon^*(N) \) the \textit{rigid partitions}. We first introduce the notion of \textit{admissible sequences}, see [6] §3.1. This is an extended version of the algorithm described in [5] which takes \( \lambda \in \mathcal{P}_\varepsilon(N) \) and returns an element of \( \mathcal{P}_\varepsilon^*(N) \) compatible for the induction process of nilpotent orbits.

Let \( \mathbf{i} \) be a finite sequence of integers between 1 and \( n \). The procedure of [5] is as follows: the algorithm commences with input \( \lambda = \lambda^1 \in \mathcal{P}_\varepsilon(N) \) where \( \mathbf{i} = \emptyset \) is the empty sequence. At the \( l^\text{th} \) iteration, the algorithm takes \( \lambda^l = \mathcal{P}_\varepsilon(N - 2 \sum_{j=1}^{l-1} i_j) \) where \( \mathbf{i} = (i_1, \ldots, i_{l-1}) \) and returns \( \lambda^{l'} \in \mathcal{P}_\varepsilon(N - 2 \sum_{j=1}^{l} i_j) \) where \( i' = (i_1, \ldots, i_{l-1}, i_l) \) for some \( i_l \). If the output \( \lambda^{l'} \) is a rigid partition then the algorithm terminates after the \( l^\text{th} \) iteration with output \( \lambda^{l'} \). We shall now explicitly describe the \( l^\text{th} \) iteration of the algorithm. If after the \((l-1)^\text{th}\) iteration the input \( \lambda^l \) is not rigid then the algorithm behaves as follows. Let \( i_l \) denote any index in the range \( 1 \leq i \leq n \) such that either of the following case occur:

\begin{align*}
\text{Case 1} & \quad \lambda^l_i \geq \lambda^{l+1}_i + 1; \\
\text{Case 2} & \quad i_l \in \Delta(\lambda^l) \text{ and } \lambda^l_i = \lambda^{l+1}_i.
\end{align*}

Note that no integer \( i_l \) will fulfill both criteria. If \( \mathbf{i} = (i_1, \ldots, i_{l-1}) \) then define \( i' = (i_1, \ldots, i_{l-1}, i_l) \). For Case 1 the algorithm has output

\[ \lambda^{l'} = (\lambda^l_1 - 2, \lambda^l_2 - 2, \ldots, \lambda^l_{i_l} - 2, \lambda^{l+1}_{i_l+1}, \ldots, \lambda^l_n) \]

whilst for Case 2 the algorithm has output

\[ \lambda^{l'} = (\lambda^l_1 - 2, \lambda^l_2 - 2, \ldots, \lambda^l_{i_l-1} - 2, \lambda^l_{i_l} - 1, \lambda^{l+1}_{i_l+1} - 1, \lambda^l_{i_l+2}, \ldots, \lambda^l_n). \]

Due to its definition and the classification of rigid partitions the above algorithm certainly terminates after a finite number of steps.

\textbf{Definition 2.2 (}[6] §3.1]). We say that a sequence \( \mathbf{i} = (i_1, \ldots, i_l) \) is an \textit{admissible sequence} for \( \lambda \) if Case 1 or Case 2 occurs at the point \( i_k \) for the partition \( \lambda^{(i_1,\ldots,i_{k-1})} \) for each \( k = 1, \ldots, l \). An admissible sequence \( \mathbf{i} \) for \( \lambda \) be called a \textit{maximal admissible sequence} for \( \lambda \) if neither Case 1 nor Case 2 occurs for any index \( i \) between 1 and \( n \) for the partition \( \lambda^l \). By convention the empty sequence is admissible for any \( \lambda \in \mathcal{P}_\varepsilon(N) \).

As observed in [6] Lemma 6], if \( \mathbf{i} \) is an admissible sequence for \( \lambda \), then \( \mathbf{i} \) is maximal admissible if and only if \( \lambda^l \) is a rigid partition. We will denote by \( |\mathbf{i}| := l \) the length of an admissible sequence for \( \lambda \).

\textbf{Definition 2.3.} The algorithm as described in [5] corresponds to the special case where in the above algorithm, we define at each step \( i_l \) to be the smallest integer which fulfills one the Case 1 or Case 2 criteria, and \( \lambda^l \) is rigid. In the sequel, we will refer to the so obtained maximal admissible sequence for \( \lambda \) as the \textit{canonical maximal admissible sequence} for \( \lambda \) and we denote it by \( \mathbf{i}^0 \). Then we set

\[ z_M(\lambda) := |\mathbf{i}^0|. \]
Remark. The integer $z_M(\lambda)$ corresponds to the integer $z(\lambda)$ of [5].

**Definition 2.4** ([6] Definition 1). If $i \in \Delta(\lambda)$ then the pair $(i, i+1)$ is called a 2-step of $\lambda$. If $i > 1$ and $(i, i+1)$ is a 2-step of $\lambda$ then $\lambda_{i-1}$ and $\lambda_{i+2}$ are referred to as the boundary of $(i, i+1)$. If $1 \in \Delta(\lambda)$ then $\lambda_3$ is referred to as the boundary of $(1, 2)$ (if $n = 2$ then $\lambda_3 = 0$ by convention).

We observe that $\Delta(\lambda)$ is the set of 2-steps of $\lambda$, and by $|\Delta(\lambda)|$ its cardinality.

**Definition 2.5** ([6, §3.2]). If $i \in \Delta(\lambda)$ then we say that the 2-step $(i, i+1)$ has a good boundary if $\lambda_1$ and the boundary of $(i, i+1)$ have the opposite parity. If the boundary of a 2-step $(i, i+1)$ of $\lambda$ is not good then we say that it is bad and we refer to $(i, i+1)$ as a bad 2-step. Note that $(i, i+1)$ is a bad 2-step of $\lambda$ if and only if either $i > 1$ and $\lambda_{i-1} - \lambda_i \in 2\mathbb{N}$, or $\lambda_{i+1} - \lambda_{i+2} \in 2\mathbb{N}$.

We denote by $\Delta_{\text{bad}}(\lambda)$ the set of bad 2-steps of $\lambda$, and by $|\Delta_{\text{bad}}(\lambda)|$ its cardinality.

**Definition 2.6** ([6, Definition 2]). A sequence $1 \leq i_1 < \cdots < i_k < n$ with $k \geq 2$ is called a 2-cluster of $\lambda$ whenever $i_j \in \Delta(\lambda)$ and $i_{j+1} = i_j + 2$ for all $j$. We say that a 2-cluster $i_1, \ldots, i_k$ has a bad boundary if either of the following conditions holds:

- $\lambda_{i_{j-1}} - \lambda_{i_j} \in 2\mathbb{N}$;
- $\lambda_{i_{j+1}} - \lambda_{i_{j+2}} \in 2\mathbb{N}$.

(if $i_1 = 1$ then the first condition should be omitted). A bad 2-cluster is one which has a bad boundary, whilst a good 2-cluster is one without a bad boundary.

We denote by $\Sigma(\lambda)$ the set of good 2-clusters of $\lambda$, and by $|\Sigma(\lambda)|$ its cardinality.

**Lemma 2.7** ([6, Lemma 11]). A good 2-cluster is maximal in the sense that it is not a proper subsequence of any 2-cluster.

**Definition 2.8** (Premet-Topley). For any $\lambda \in P_\varepsilon(\lambda)$, the integer $z_{\text{PT}}(\lambda)$ is defined by the formula:

$$z_{\text{PT}}(\lambda) := s(\lambda) + |\Delta(\lambda)| - |\Delta_{\text{bad}}(\lambda)| + |\Sigma(\lambda)|$$

where

$$s(\lambda) := \sum_{i=1}^{n} \left[ (\lambda_i - \lambda_{i+1})/2 \right].$$

Remark. The integer $z_{\text{PT}}(\lambda)$ corresponds to the integer $z(\lambda)$ of [6]. By [6, Theorem 8], we have that

$$z_{\text{PT}}(\lambda) := \max |i|$$

(2)
where the maximum is taken over all admissible sequences for $\lambda$. Hence, by [6, Corollary 9] and the equality (1) of the introduction, we get:

**Theorem 2.9** (Premet-Topley). For any $m \in \mathbb{N}$, we have

$$\dim g^{(m)} = 2m + \max \{ z_{PT}(\lambda) \mid \lambda \in \mathcal{P}_\varepsilon(N) \text{ s.t. } \dim Ge(\lambda) = 2m \}.$$ 

The main result of this note is:

**Theorem 2.10.** (i) For any $\lambda \in \mathcal{P}_\varepsilon(N)$, we have $z_M(\lambda) = z_{PT}(\lambda)$.

(ii) For any $m \in \mathbb{N}$, we have

$$\dim g^{(m)} = 2m + \max \{ z_M(\lambda) \mid \lambda \in \mathcal{P}_\varepsilon(N) \text{ s.t. } \dim Ge(\lambda) = 2m \}.$$ 

*In other words, the statement of [5, Theorem 3.13] is correct.*

**Proof.** (ii) is a direct consequence of (i) and Theorem 2.9.

(i) We argue by induction on $N$ (the statement is true for small $N$). Let $N > 2$ and assume the statement true for any $\lambda \in \mathcal{P}_\varepsilon(N')$, with $1 \leq N' \leq N$, and let $\lambda \in \mathcal{P}_\varepsilon(N)$.

If $\lambda \in \mathcal{P}^*_\varepsilon(N)$, then $z_{PT}(\lambda) = z_M(\lambda) = 0$ (see Theorem 2.1, Definition 2.2 and equality (2)). So, we can assume that $\lambda$ is not a rigid partition. In particular, $z_{PT}(\lambda) > 0$ and $z_M(\lambda) > 0$. To ease notation, we simply denote here by $i := i^0$ the canonical maximal sequence for $\lambda$. Then recall that by Definition 2.3, $z_M(\lambda) = |i|$. Set $\lambda' := \lambda{(i)}$. Clearly, $z_M(\lambda') = z_M(\lambda) - 1$. By the induction hypothesis, we have $z_{PT}(\lambda') = z_M(\lambda')$. Hence, we have to show that:

$$z_{PT}(\lambda') = z_{PT}(\lambda) - 1.$$ 

Our strategy is to compare the formulas for $z_{PT}(\lambda')$ and $z_{PT}(\lambda)$ given by Definition 2.8. Recall that $i_1$ is the smallest integer which fulfills one of the Case 1 or Case 2 criteria for $\lambda$. First of all, we observe that if $i \in \Delta(\lambda)$ (resp. $i \in \Delta(\lambda')$), then $i \geq i_1$. Indeed, if $i \in \Delta(\lambda)$ and $i < i_1$ (if $i_1 = 1$, it is clear), then either $\lambda_i = \lambda_{i+1}$ and then $i$ fulfills the Case 2 which contradicts the minimality of $i_1$, or $\lambda_i - \lambda_{i+1} \in 2\mathbb{N} \smallsetminus \{0\}$ and then $i$ fulfills the Case 1 which contradicts the minimality of $i_1$ too.

We now consider the two situations Case 1 and Case 2 separately.

**Case 1:** $\lambda_{i_1} \geq \lambda_{i_1 + 1} + 2$.

We have,$$
\lambda' = (\lambda_1 - 2, \ldots, \lambda_{i_1 - 1} - 2, \lambda_{i_1} - 2, \lambda_{i_1 + 1}, \ldots, \lambda_n),$

and

$$s(\lambda') = \sum_{i=1}^{i_1-1} \left[ (\lambda_i - \lambda_{i+1})/2 \right] + \left[ (\lambda_{i_1} - 2 - \lambda_{i_1+1})/2 \right] + \sum_{i=i_1+1}^{n} \left[ (\lambda_i - \lambda_{i+1})/2 \right] = s(\lambda) - 1.$$
Compare now the other terms appearing in Definition 2.8. Note that \( i_1 \in \Delta(\lambda) \) (resp. \( i_1 \in \Delta_{\text{bad}}(\lambda) \)) if and only if \( i_1 \in \Delta(\lambda') \) (resp. \( i_1 \in \Delta_{\text{bad}}(\lambda') \)) since the passing from \( \lambda \) to \( \lambda' \) preserves the parities. For the same reason, \( i_1 \) belongs to a good 2-cluster of \( \lambda \) if and only \( i_1 \) belongs to a good 2-cluster of \( \lambda' \).

Then we discuss two cases depending on whether \( i_1 + 1 \) is in \( \Delta(\lambda) \) or not:

- \( i_1 + 1 \in \Delta(\lambda) \).

Once again, we consider two cases:

- \( \lambda_{i_1} - 2 \neq \lambda_{i_1+1} \).
  Then \( i_1 + 1 \in \Delta(\lambda') \) too. Moreover, \( i_1 + 1 \in \Delta_{\text{bad}}(\lambda') \) if and only if \( i_1 + 1 \in \Delta_{\text{bad}}(\lambda) \). Hence, we conclude that \( |\Delta(\lambda')| = |\Delta(\lambda)| \), \( |\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)| \) and \( |\Sigma(\lambda')| = |\Sigma(\lambda)| \).

- \( \lambda_{i_1} - 2 = \lambda_{i_1+1} \).
  Then \( i_1 + 1 \in \Delta_{\text{bad}}(\lambda) \) since \( \lambda_{i_1} - \lambda_{i_1+1} = 2 \in 2\mathbb{N} \). But \( i_1 + 1 \notin \Delta(\lambda') \). Therefore, \( |\Delta(\lambda')| = |\Delta(\lambda)| - 1 \) and \( |\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)| - 1 \). Moreover, if \( i_1 + 1 \) belongs to a 2-cluster of \( \lambda \), then it is bad because \( \lambda_{i_1} - \lambda_{i_1+1} \in 2\mathbb{N} \).
  Hence, we have \( |\Sigma(\lambda')| = |\Sigma(\lambda)| \).

- \( i_1 + 1 \notin \Delta(\lambda) \).

In this case, note that \( i_1 + 1 \notin \Delta(\lambda') \). Hence, we conclude that \( |\Delta(\lambda')| = |\Delta(\lambda)| \), \( |\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)| \) and \( |\Sigma(\lambda')| = |\Sigma(\lambda)| \).

**Case 2:** \( i_1 \in \Delta(\lambda) \) and \( \lambda_{i_1} = \lambda_{i_1+1} \).

By the minimality condition of \( i_1 \), we have \( \lambda_{i_1-1} = \lambda_{i_1} + 1 \) (except for \( i_1 = 1 \), in which case \( \lambda_{i_1-1} = 0 \) by convention), and so \( \lambda_{i_1-2} = \lambda_{i_1-1} \) because \( \varepsilon(-1)^{\lambda_{i_1-1}} = 1 \).

We have

\[
\lambda' = (\lambda_1 - 2, \ldots, \lambda_{i_1-2} - 2, \lambda_{i_1} - 1, \lambda_{i_1+1} - 1, \lambda_{i_1+2}, \ldots, \lambda_n),
\]

and

\[
s(\lambda') = \sum_{i=1}^{i_1-2} \left[ \frac{(\lambda_i - \lambda_{i+1})}{2} + \frac{(\lambda_{i_1} - \lambda_{i_1+1})}{2} \right] + \begin{cases} s(\lambda) - 1 & \text{if } \lambda_{i_1+1} - \lambda_{i_1+2} \in 2\mathbb{N}; \\ s(\lambda) & \text{if } \lambda_{i_1+1} - \lambda_{i_1+2} \notin 2\mathbb{N}. \end{cases}
\]

(If \( i_1 = 0 \), we start at the second line and we get the same conclusion.) Also, observe that in Case 2, we have

\[
|\Delta(\lambda')| = |\Delta(\lambda)| - 1.
\]

Indeed, \( i_1 \in \Delta(\lambda) \) but \( i_1 \notin \Delta(\lambda') \) and for the indexes \( i \neq i_1 \) we have here the equivalence: \( i \in \Delta(\lambda) \iff i \in \Delta(\lambda') \).
We discuss two cases depending on the parity of $\lambda_{i_1+1} - \lambda_{i_1+2}$.

- $\lambda_{i_1+1} - \lambda_{i_1+2} \in 2\mathbb{N}$.

Then $i_1 \in \Delta_{\text{bad}}(\lambda)$. There are two sub-cases depending on whether $i_1 + 2$ is in $\Delta(\lambda)$ or not:

* $i_1 + 2 \in \Delta(\lambda)$.
  Then, $i_1 + 2 \in \Delta_{\text{bad}}(\lambda)$ (since $\lambda_{i_1+1} - \lambda_{i_1+2} \in 2\mathbb{N}$) and $i_1 + 2 \in \Delta(\lambda')$. Once again, there are two sub-cases:

1) $i_1 + 2 \notin \Delta_{\text{bad}}(\lambda')$.
Then $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)| - 2$. Moreover, $(i_1, i_1 + 2)$ is a good 2-cluster of $\lambda$. Indeed, $i_1 + 2 \notin \Delta_{\text{bad}}(\lambda')$ implies that $\lambda_{i_1+3} - \lambda_{i_1+4} \notin 2\mathbb{N}$.

On the other hand, $\lambda_{i_1-1} - \lambda_i = 1 \notin 2\mathbb{N}$ (if $i_1 = 1$ the first condition in Definition 2.6 should be omitted). But $(i_1, i_1 + 2)$ is not a 2-cluster of $\lambda'$ since $i_1 \notin \Delta(\lambda')$. Hence, we have $|\Sigma(\lambda')| = |\Sigma(\lambda)| - 1$ by Lemma 2.7.

2) $i_1 + 2 \in \Delta_{\text{bad}}(\lambda')$.
Then $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)| - 1$. The only 2-clusters of $\lambda$ which are not 2-clusters of $\lambda'$ are of the form $(i_1, \ldots, i_k)$ with $k \geq 2$. Assume that there is a good 2-cluster of the form $(i_1, \ldots, i_k)$ for $\lambda$, with $k \geq 2$.
The 2-cluster $(i_1, i_1 + 2)$ of $\lambda$ is bad. Indeed, $\lambda_{i_1+3} - \lambda_{i_1+4} \in 2\mathbb{N}$ since $i_1 + 2 \in \Delta_{\text{bad}}(\lambda')$ and $\lambda_{i_1+1} - \lambda_{i_1+2} \notin 2\mathbb{N}$. Hence, $k > 2$.

Since $\lambda_{i_1-1} - \lambda_{i_1} \notin 2\mathbb{N}$ and $\lambda_{i_1+1} - \lambda_{i_1+2} \notin 2\mathbb{N}$, the 2-cluster $(i_1, \ldots, i_k)$ is good for $\lambda$ if and only if the 2-cluster $(i_1 + 2, \ldots, i_k)$ is good for $\lambda'$. On the other direction, the only possible good 2-clusters of $\lambda'$ which are not good for $\lambda$ are of the form $(i_2 = i_1 + 2, \ldots, i_k)$ with $k \geq 3$. By the above argument, if there is such a good 2-cluster for $\lambda'$, then $(i_1, \ldots, i_k)$ is a good 2-cluster for $\lambda$. As a consequence, $|\Sigma(\lambda')| = |\Sigma(\lambda)|$.

* $i_1 + 2 \notin \Delta(\lambda)$.
Then $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)| - 1$. Moreover, since $i_1 + 2 \notin \Delta(\lambda)$, then neither $i_1$ nor $i_1 + 2$ belongs to a 2-cluster for $\lambda$. Hence $|\Sigma(\lambda)| = |\Sigma(\lambda')|$. 

- $\lambda_{i_1+1} - \lambda_{i_1+2} \notin 2\mathbb{N}$.

In this case, $i_1 \notin \Delta_{\text{bad}}(\lambda)$, $i_1 + 2 \notin \Delta(\lambda)$ and $i_1 + 2 \notin \Delta(\lambda')$. Hence $|\Delta_{\text{bad}}(\lambda')| = |\Delta_{\text{bad}}(\lambda)|$. Moreover, neither $i_1$ nor $i_1 + 2$ belongs to any 2-cluster. Hence $|\Sigma(\lambda)| = |\Sigma(\lambda')|$.

In all the cases, we can check with the formula of Definition 2.8 that $z_{\text{PT}}(\lambda') = z_{\text{PT}}(\lambda) - 1$ as desired. This concludes the proof of Theorem 2.10.$\blacksquare$

3. Counter-examples for [5, Proposition 3.11]

From now on, we shall denote by $z(\lambda)$ the integer $z_M(\lambda) = z_{\text{PT}}(\lambda)$ for $\lambda \in \mathcal{P}_c(N)$.

If $I$ is a Levi subalgebra of $\mathfrak{g}$ and $O'$ is a rigid nilpotent orbit of $I$, we denote by $\text{Ind}_I^\mathfrak{g}(O')$ the induced nilpotent orbit of $\mathfrak{g}$ from $O'$ in $I$.

Proposition 3.11 of [5] asserts that if a nilpotent element $e$ associated with the partition $\lambda \in \mathcal{P}_c(N)$ is induced form a nilpotent orbit in a Levi subalgebra
I, then \( z(\lambda) \) is equal to the dimension of the center of \( I \). This result is actually incorrect. If it were true, it would imply that all the sheets containing \( e \) share the same dimension (see [5, Remark 3.12]). But this is wrong. Below are some counter-examples (see also [6, Remark 4]):

**Example 3.1.** Assume that \( \mathfrak{g} = \mathfrak{so}(8) \) and consider the nilpotent element \( e \) of \( \mathfrak{g} \) with partition \( \lambda = (3, 3, 1, 1) \in \mathcal{P}_1(8) \setminus \mathcal{P}_1^*(8) \). The algorithm yields \( z(\lambda) = 2 \).

On the other hand, \( e \) is induced from two different ways: from the zero orbit in a Levi subalgebra \( L_1 \) of type \((3, 1; 0)\), that is \( L_1 \cong \mathfrak{gl}_3 \times \mathfrak{gl}_1 \times 0 \) (see the definition after [5, Lemma 3.2] for the meaning of type), and from the zero orbit in a Levi subalgebra \( L_2 \) of type \((2; 4)\), that is \( L_2 \cong \mathfrak{gl}_2 \times \mathfrak{so}_4 \). The first one, \( L_1 \), has a center of dimension 2, while the second one, \( L_2 \), has a center of dimension 1. The nilpotent orbit of \( e \) has dimension 18 and \( e \) lies in two different sheets: one of dimension \( \dim z(L_1) + \dim \text{Ind}_{L_1}^g(0) = 20 \) and one of dimension \( \dim z(L_2) + \dim \text{Ind}_{L_2}^g(0) = 19 \) (here \( z(L_i) \) denotes the center of \( L_i \) for \( i = 1, 2 \)). This contradicts Proposition 3.11 of [5], and also Remark 3.12 of the same paper.

**Example 3.2.** We give now a counter-example in \( \mathfrak{sp}(14) \). Consider the partition \( \lambda = (4, 4, 2, 1, 1) \) of \( \mathcal{P}_{-1}(14) \). Here, the algorithm yields \( z(\lambda) = 2 \).

The corresponding nilpotent element is induced from the zero orbit in \( L_1 \cong \mathfrak{gl}_4 \times \mathfrak{gl}_3 \times \mathfrak{sp}(6) \), and from the rigid nilpotent orbit \( 0 \times O' \) in \( L_1 \cong \mathfrak{gl}_2 \times \mathfrak{sp}(10) \) where \( O' \) corresponds to the partition \((2, 2, 2, 1, 1) \in \mathcal{P}_{-1}^*(10) \). Again the dimensions of the centers of \( L_1 \) and \( L_2 \) lead to different dimensions, 2 and 1 respectively.

The origin of the error can be pinned down in the proof of [5, Proposition 3.11]. Let us briefly explain this. Until the end of the section, we are in the notations of [5].

At the end of this proof, the assertion “Consequently the smallest integer such that one of the situations (a) or (b) of Step 1 happens in \( d(p) \) is equal to \( i_p \)” is incorrect (here \( d \) is an element of \( \mathcal{P}_1(N) \)). And so, the main induction argument of the proof fails. We can see that is incorrect in general on an explicit example. Consider the partition \( d = (4, 4, 3, 3, 1, 1) \) of \( \mathcal{P}_1(16) \). Then the corresponding nilpotent orbit is induced from the zero orbit in \( L \cong \mathfrak{gl}_4 \times \mathfrak{gl}_3 \times 0 \) and from the rigid nilpotent orbit with partition \((2, 2, 1, 1, 1, 1) \in \mathcal{P}_1(4) \times \mathfrak{so}(8) \). Consider the second induction. In the notations of the proof, we have: \( S = 1, i_1 = 4, d^{(0)} = f = (2, 2, 1, 1, 1, 1), d = d^{(1)} = \tilde{d}^{(0)} \) (see [5, Proposition 3.7] for the tilda notation). Then the smallest integer such that one of the situations (a) or (b) of Step 1 happens for \( d = d^{(1)} \) is \( 3 \neq i_1 \).

4. Conclusion

To summarize, we list below all corrections which have to be taken into account in [5] (the numbering of [5] is used):

- Proposition 3.11 (its proof and its statement) is incorrect.
• As a consequence Remark 3.12, the sentence "The results of this section specify that, in the classical case, the dimension of a sheet containing a given nilpotent orbit does not depend on the choice of a sheet containing it" in §1.2, and the sentence "Surprisingly, in the classical case, we will notice that if \( \text{Ind}_{l_1}(C_{l_1}) = \text{Ind}_{l_2}(C_{l_2}) \), then \( \text{dim}_{\mathfrak{g}}(l_1) = \text{dim}_{\mathfrak{g}}(l_2) \)" in Remark 2.15, are also incorrect.

• The proof of Theorem 3.13 is incorrect, since it uses Proposition 3.11. Nevertheless, its statement remains valid. In particular, Tables 3, 4 and 5 are still correct.

Remark. There are some misprints in Table 5: line 2m = 48, the partitions are \([7, 1^5], [5, 3, 2^2], [4^2, 3, 1]\) and not \([4^3], [4^2, 3, 1]\).

References