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Corrigendum to "On the Dimension of the Sheets of a Reductive Lie Algebra"

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Abstract. This note is a corrigendum to [5]. As it has been recently pointed out to me by Alexander Premet, [5, Remark 3.12] is incorrect. We explain in this note the impacts of that error in [5], and amend certain of its statements. In particular, we verify that the statement of [5, Theorem 3.13] remains correct in spite of this error.
[5] A. Moreau, On the dimension of the sheets of a reductive Lie algebra, J. Lie Theory **18** (2008), 671–696.
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1. Introduction

Let $\mathfrak{g}$ be a complex simple Lie algebra and G its adjoint group. We investigate in [5] the dimension of the subsets, for $m \in \mathbb{N}$,

$$\mathfrak{g}^{(m)} := \{x \in \mathfrak{g} \mid \dim(Gx) = 2m\},$$

where Gx denotes the adjoint orbit of $x \in \mathfrak{g}$. The irreducible components of the subsets $\mathfrak{g}^{(m)}$ are called the *sheets* of $\mathfrak{g}$, [2, 1]. Thus, for any $m \in \mathbb{N}$,

$$\dim \mathfrak{g}^{(m)} = \max\{\dim \mathcal{S} \ ; \ \mathcal{S} \subset \mathfrak{g}^{(m)}\}, \tag{1}$$

where $\mathcal{S}$ runs through all sheets contained in $\mathfrak{g}^{(m)}$. The sheets are known to be parameterized by the pairs $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$, up to G-conjugacy class, consisting of a Levi subalgebra $\mathfrak{l}$ of $\mathfrak{g}$ and a rigid nilpotent orbit $\mathcal{O}_{\mathfrak{l}}$ in $\mathfrak{l}$, cf. [1]. This parametrization enables to write the dimension of a sheet $\mathcal{S}$ associated with a pair $(\mathfrak{l}, \mathcal{O}_{\mathfrak{l}})$ as the sum of the dimension of the center of $\mathfrak{l}$ and the dimension of the unique nilpotent orbit contained in $\mathcal{S}$, see e.g. [5, Proposition 2.11].

In the classical case, formulas for $\mathfrak{g}^{(m)}$ are given in [5, Theorems 3.3 and 3.13] in term of partitions associated with nilpotent elements of $\mathfrak{g}$. As it has been recently pointed out by Alexander Premet, Remark 3.12 in [5] which claims that *"in the classical case, the dimension of a sheet containing a given nilpotent*

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orbit does not depend on the choice of a sheet containing it" is incorrect. We give here some counter-examples (cf. Examples 3.1 and 3.2; see also [6, Remark 4]). This is true only for the type **A** where each nilpotent element belongs to only one sheet. The error stems from the proof of [5, Proposition 3.11]; see Section 3 for explanations. As a consequence, the proof of [5, Theorems 3.13], partly based on [5, Proposition 3.11], is incorrect too. However its statement remains true. This can be shown through a recent work of Premet and Topley, [6]. In more details, another formula for $\mathfrak{g}^{(m)}$ in term of partitions can be traced out from [6, Corollary 9] and the equality (1). In this note, we verify (cf. Theorems 2.10) that the Premet-Topley formula for $\mathfrak{g}^{(m)}$ coincides with the one of [5, Theorem 3.13].

The note is organized as follows.

In Section 2, we recall some definitions and results of [6] and show that the statement of [5, Theorem 3.13] is correct in spite of the error in [5, Proposition 3.11], see Theorem 2.10(ii). In Section 3, we precisely pin down the error in the proof [5, Proposition 3.11] and describe the impacts of that error in [5]. As a conclusion, we list in Section 4 all corrections which have to be taken into account in [5].

Since the corrections in [5] only concern the types **B**, **C** and **D**, we assume for the remaining of the note that $\mathfrak{g}$ is either $\mathfrak{so}(N)$ or $\mathfrak{sp}(N)$, with $N \geqslant 2$, and ε is 1 or -1 depending on whether $\mathfrak{g} = \mathfrak{so}(N)$ or $\mathfrak{sp}(N)$. Following the notations of [5] (or [6]), we denote by $\mathcal{P}_\varepsilon(N)$ the set of partitions of N associated with the nilpotent elements of $\mathfrak{g}$. For $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_\varepsilon(N)$, we denote by $e(\lambda)$ the corresponding nilpotent element of $\mathfrak{g}$ whose Jordan block sizes are $\lambda_1, \dots, \lambda_n$. We will always assume that $\lambda_1 \geqslant \cdots \geqslant \lambda_n$.

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2. The main result

For the convenience of the reader, we recall here all the necessary definitions and results of [6]. Given a partition $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_\varepsilon(N)$ we set,

$$\Delta(\lambda) := \{1 \leqslant i < n \,;\, \varepsilon(-1)^{\lambda_1} = \varepsilon(-1)^{\lambda_{i+1}} = -1,\ \lambda_{i-1} \neq \lambda_i \geqslant \lambda_{i+1} \neq \lambda_{i+2}\}.$$

Our convention is that $\lambda_0 = 0$ and $\lambda_i = 0$ for all $i > n$. Recall the following result of Kempken and Spaltenstein (also recalled in [5] and [6]):

Theorem 2.1 ([4, 7]). *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathcal{P}_\varepsilon(N)$. Then $e(\lambda)$ is rigid if and only if*

- $\lambda_i - \lambda_{i+1} \in \{0, 1\}$ *for all* $1 \leqslant i \leqslant n$;
- *the set* $\{i \in \Delta(\lambda) \,;\, \lambda_i = \lambda_{i+1}\}$ *is empty.*

Denote by $\mathcal{P}^*_\varepsilon(N)$ the set of $\lambda \in \mathcal{P}_\varepsilon(N)$ such that $e(\lambda)$ is rigid. We call the elements of $\mathcal{P}^*_\varepsilon(N)$ the *rigid partitions*. We first introduce the notion of *admissible sequences*, see [6, §3.1]. This is an extended version of the algorithm described in [5] which takes $\lambda \in \mathcal{P}_\varepsilon(N)$ and returns an element of $\mathcal{P}^*_\varepsilon(N)$ compatible for the induction process of nilpotent orbits.

Let $\mathbf{i}$ be a finite sequence of integers between 1 and n. The procedure of [6] is as follows: the algorithm commences with input $\lambda = \lambda^{\mathbf{i}} \in \mathcal{P}_\varepsilon(N)$ where $\mathbf{i} = \varnothing$ is the empty sequence. At the l^{th} iteration, the algorithm takes $\lambda^{\mathbf{i}} \in \mathcal{P}_\varepsilon(N - 2\sum_{j=1}^{l-1} i_j)$ where $\mathbf{i} = (i_1, \dots, i_{l-1})$ and returns $\lambda^{\mathbf{i}'} \in \mathcal{P}_\varepsilon(N - 2\sum_{j=1}^{l} i_j)$ where $\mathbf{i}' = (i_1, \dots, i_{l-1}, i_l)$ for some i_l. If the output $\lambda^{\mathbf{i}'}$ is a rigid partition then the algorithm terminates after the l^{th} iteration with output $\lambda^{\mathbf{i}'}$. We shall now explicitly describe the l^{th} iteration of the algorithm. If after the $(l-1)^{\text{th}}$ iteration the input $\lambda^{\mathbf{i}}$ is not rigid then the algorithm behaves as follows. Let i_l denote any index in the range $1 \leqslant i \leqslant n$ such that either of the following case occur:

Case 1 $\quad \lambda^{\mathbf{i}}_{i_l} \geqslant \lambda^{\mathbf{i}}_{i_l+1} + 2$;

Case 2 $\quad i_l \in \Delta(\lambda^{\mathbf{i}})$ and $\lambda^{\mathbf{i}}_{i_l} = \lambda^{\mathbf{i}}_{i_l+1}$.

Note that no integer i_l will fulfill both criteria. If $\mathbf{i} = (i_1, \dots, i_{l-1})$ then define $\mathbf{i}' = (i_1, \dots, i_{l-1}, i_l)$. For Case 1 the algorithm has output

$$\lambda^{\mathbf{i}'} = (\lambda^{\mathbf{i}}_1 - 2, \lambda^{\mathbf{i}}_2 - 2, \dots, \lambda^{\mathbf{i}}_{i_l} - 2, \lambda^{\mathbf{i}}_{i_l+1}, \dots, \lambda^{\mathbf{i}}_n)$$

whilst for Case 2 the algorithm has output

$$\lambda^{\mathbf{i}'} = (\lambda^{\mathbf{i}}_1 - 2, \lambda^{\mathbf{i}}_2 - 2, \dots, \lambda^{\mathbf{i}}_{i_l-1} - 2, \lambda^{\mathbf{i}}_{i_l} - 1, \lambda^{\mathbf{i}}_{i_l+1} - 1, \lambda^{\mathbf{i}}_{i_l+2}, \dots, \lambda^{\mathbf{i}}_n).$$

Due to its definition and the classification of rigid partitions the above algorithm certainly terminates after a finite number of steps.

Definition 2.2 ([6, §3.1])**.** We say that a sequence $\mathbf{i} = (i_1, \dots, i_l)$ is an *admissible sequence* for λ if Case 1 or Case 2 occurs at the point i_k for the partition $\lambda^{(i_1,\dots,i_{k-1})}$ for each $k = 1, \dots, l$. An admissible sequence $\mathbf{i}$ for λ is be called *a maximal admissible sequence for* λ if neither Case 1 nor Case 2 occurs for any index i between 1 and n for the partition $\lambda^{\mathbf{i}}$. By convention the empty sequence is admissible for any $\lambda \in \mathcal{P}_\varepsilon(N)$.

As observed in [6, Lemma 6], if $\mathbf{i}$ is an admissible sequence for λ, then $\mathbf{i}$ is maximal admissible if and only if $\lambda^{\mathbf{i}}$ is a rigid partition. We will denote by $|\mathbf{i}| := l$ the length of an admissible sequence for λ.

Definition 2.3. The algorithm as described in [5] corresponds to the special case where in the above algorithm, we define at each step i_l to be the smallest integer which fulfills one the Case 1 or Case 2 criteria, and $\lambda^{\mathbf{i}}$ is rigid. In the sequel, we will refer to the so obtained maximal admissible sequence for λ as the *canonical maximal admissible sequence for* λ and we denote it by $\mathbf{i}^0$. Then we set

$$z_{\mathrm{M}}(\lambda) := |\mathbf{i}^0|.$$

Remark. The integer $z_{\mathrm{M}}(\lambda)$ corresponds to the integer $z(\lambda)$ of [5].

Definition 2.4 ([6, Definition 1]). If $i \in \Delta(\lambda)$ then the pair $(i, i+1)$ is called a *2-step of* λ. If $i > 1$ and $(i, i+1)$ is a 2-step of λ then λ_{i-1} and λ_{i+2} are referred to as the *boundary of* $(i, i+1)$. If $1 \in \Delta(\lambda)$ then λ_3 is referred to as the boundary of $(1, 2)$ (if $n = 2$ then $\lambda_3 = 0$ by convention).

We observe that $\Delta(\lambda)$ is the set of 2-steps of λ, and by $|\Delta(\lambda)|$ its cardinality.

Definition 2.5 ([6, §3.2]). If $i \in \Delta(\lambda)$ then we say that the 2-step $(i, i+1)$ has a *good boundary* if λ_1 and the boundary of $(i, i+1)$ have the opposite parity. If the boundary of a 2-step $(i, i+1)$ of λ is not good then we say that it is *bad* and we refer to $(i, i+1)$ as a *bad 2-step.* Note that $(i, i+1)$ is a bad 2-step of λ if and only if either $i > 1$ and $\lambda_{i-1} - \lambda_i \in 2\mathbb{N}$, or $\lambda_{i+1} - \lambda_{i+2} \in 2\mathbb{N}$.

We denote by $\Delta_{\mathrm{bad}}(\lambda)$ the set of bad 2-steps of λ, and by $|\Delta_{\mathrm{bad}}(\lambda)|$ its cardinality.

Definition 2.6 ([6, Definition 2]). A sequence $1 \leqslant i_1 < \cdots < i_k < n$ with $k \geqslant 2$ is called a *2-cluster of* λ whenever $i_j \in \Delta(\lambda)$ and $i_{j+1} = i_j + 2$ for all j. We say that a 2-cluster $i_1, \ldots, i_k$ has a *bad boundary* if either of the following conditions holds:

- $\lambda_{i_1-1} - \lambda_{i_1} \in 2\mathbb{N}$;
- $\lambda_{i_k+1} - \lambda_{i_k+2} \in 2\mathbb{N}$.

(if $i_1 = 1$ then the first condition should be omitted). A *bad 2-cluster* is one which has a bad boundary, whilst a *good 2-cluster* is one without a bad boundary.

We denote by $\Sigma(\lambda)$ the set of good 2-clusters of λ, and by $|\Sigma(\lambda)|$ its cardinality.

Lemma 2.7 ([6, Lemma 11]). *A good 2-cluster is maximal in the sense that it is not a proper subsequence of any 2-cluster.*

Definition 2.8 (Premet-Topley). For any $\lambda \in \mathcal{P}_{\varepsilon}(\lambda)$, the integer $z_{\mathrm{PT}}(\lambda)$ is defined by the formula:

$$z_{\mathrm{PT}}(\lambda) := s(\lambda) + |\Delta(\lambda)| - |\Delta_{\mathrm{bad}}(\lambda)| + |\Sigma(\lambda)|$$

where

$$s(\lambda) := \sum_{i=1}^{n} [(\lambda_i - \lambda_{i+1})/2] \, .$$

Remark. The integer $z_{\mathrm{PT}}(\lambda)$ corresponds to the integer $z(\lambda)$ of [6].

By [6, Theorem 8], we have that

$$z_{\mathrm{PT}}(\lambda) := \max |\mathbf{i}| \tag{2}$$

where the maximum is taken over all admissible sequences for λ. Hence, by [6, Corollary 9] and the equality (1) of the introduction, we get:

Theorem 2.9 (Premet-Topley). *For any $m \in \mathbb{N}$, we have*

$$\dim \mathfrak{g}^{(m)} = 2m + \max\{z_{\mathrm{PT}}(\lambda) \; ; \; \lambda \in \mathcal{P}_\varepsilon(N) \; s.t \; \dim Ge(\lambda) = 2m\}.$$

The main result of this note is:

Theorem 2.10. (i) *For any $\lambda \in \mathcal{P}_\varepsilon(N)$, we have $z_{\mathrm{M}}(\lambda) = z_{\mathrm{PT}}(\lambda)$.*
(ii) *For any $m \in \mathbb{N}$, we have*

$$\dim \mathfrak{g}^{(m)} = 2m + \max\{z_{\mathrm{M}}(\lambda) \; ; \; \lambda \in \mathcal{P}_\varepsilon(N) \; s.t \; \dim Ge(\lambda) = 2m\}.$$

In other words, the statement of [5, Theorem 3.13] is correct.

Proof. (ii) is a direct consequence of (i) and Theorem 2.9.
(i) We argue by induction on N (the statement is true for small N). Let $N > 2$ and assume the statement true for any $\lambda \in \mathcal{P}_\varepsilon(N')$, with $1 \leqslant N' \leqslant N$, and let $\lambda \in \mathcal{P}_\varepsilon(N)$.

If $\lambda \in \mathcal{P}^*_\varepsilon(N)$, then $z_{\mathrm{PT}}(\lambda) = z_{\mathrm{M}}(\lambda) = 0$ (see Theorem 2.1, Definition 2.2 and equality (2)). So, we can assume that λ is not a rigid partition. In particular, $z_{\mathrm{PT}}(\lambda) > 0$ and $z_{\mathrm{M}}(\lambda) > 0$. To ease notation, we simply denote here by $\mathbf{i} := \mathbf{i}^0$ the canonical maximal sequence for λ. Then recall that by Definition 2.3, $z_{\mathrm{M}}(\lambda) = |\mathbf{i}|$. Set $\lambda' := \lambda^{(i_1)}$. Clearly, $z_{\mathrm{M}}(\lambda') = z_{\mathrm{M}}(\lambda) - 1$. By the induction hypothesis, we have $z_{\mathrm{PT}}(\lambda') = z_{\mathrm{M}}(\lambda')$. Hence, we have to show that:

$$z_{\mathrm{PT}}(\lambda') = z_{\mathrm{PT}}(\lambda) - 1.$$

Our strategy is to compare the formulas for $z_{\mathrm{PT}}(\lambda')$ and $z_{\mathrm{PT}}(\lambda)$ given by Definition 2.8. Recall that i_1 is the smallest integer which fulfills one of the Case 1 or Case 2 criteria for λ. First of all, we observe that if $i \in \Delta(\lambda)$ (resp. $i \in \Delta(\lambda')$), then $i \geqslant i_1$. Indeed, if $i \in \Delta(\lambda)$ and $i < i_1$ (if $i_1 = 1$, it is clear), then either $\lambda_i = \lambda_{i+1}$ and then i fulfills the Case 2 which contradicts the minimality of i_1, or $\lambda_i - \lambda_{i+1} \in 2\mathbb{N} \smallsetminus \{0\}$ and then i fulfills the Case 1 which contradicts the minimality of i_1 too.

We now consider the two situations Case 1 and Case 2 separately.

<u>**Case 1**</u>: $\lambda_{i_1} \geqslant \lambda_{i_1+1} + 2$.

We have,

$$\lambda' = (\lambda_1 - 2, \ldots, \lambda_{i_1-1} - 2, \lambda_{i_1} - 2, \lambda_{i_1+1}, \ldots, \lambda_n),$$

and

$$\begin{aligned} s(\lambda') &= \sum_{i=1}^{i_1-1} [(\lambda_i - \lambda_{i+1})/2] + [(\lambda_{i_1} - 2 - \lambda_{i_1+1})/2] + \sum_{i=i_1+1}^{n} [(\lambda_i - \lambda_{i+1})/2] \\ &= s(\lambda) - 1. \end{aligned}$$

Compare now the other terms appearing in Definition 2.8. Note that $i_1 \in \Delta(\lambda)$ (resp. $i_1 \in \Delta_{\mathrm{bad}}(\lambda)$) if and only if $i_1 \in \Delta(\lambda')$ (resp. $i_1 \in \Delta_{\mathrm{bad}}(\lambda')$) since the passing from λ to λ' preserves the parities. For the same reason, i_1 belongs to a good 2-cluster of λ if and only i_1 belongs to a good 2-cluster of λ'.

Then we discuss two cases depending on whether $i_1 + 1$ is in $\Delta(\lambda)$ or not:

- $i_1 + 1 \in \Delta(\lambda)$.

Once again, we consider two cases:

  - $\lambda_{i_1} - 2 \neq \lambda_{i_1+1}$.
    Then $i_1 + 1 \in \Delta(\lambda')$ too. Moreover, $i_1 + 1 \in \Delta_{\mathrm{bad}}(\lambda')$ if and only if $i_1 + 1 \in \Delta_{\mathrm{bad}}(\lambda)$. Hence, we conclude that $|\Delta(\lambda')| = |\Delta(\lambda)|$, $|\Delta_{\mathrm{bad}}(\lambda')| = |\Delta_{\mathrm{bad}}(\lambda)|$ and $|\Sigma(\lambda')| = |\Sigma(\lambda)|$.

  - $\lambda_{i_1} - 2 = \lambda_{i_1+1}$.
    Then $i_1 + 1 \in \Delta_{\mathrm{bad}}(\lambda)$ since $\lambda_{i_1} - \lambda_{i_1+1} = 2 \in 2\mathbb{N}$. But $i_1 + 1 \notin \Delta(\lambda')$. Therefore, $|\Delta(\lambda')| = |\Delta(\lambda)| - 1$ and $|\Delta_{\mathrm{bad}}(\lambda')| = |\Delta_{\mathrm{bad}}(\lambda)| - 1$. Moreover, if $i_1 + 1$ belongs to a 2-cluster of λ, then it is bad because $\lambda_{i_1} - \lambda_{i_1+1} \in 2\mathbb{N}$. Hence, we have $|\Sigma(\lambda')| = |\Sigma(\lambda)|$.

- $i_1 + 1 \notin \Delta(\lambda)$.

In this case, note that $i_1 + 1 \notin \Delta(\lambda')$. Hence, we conclude that $|\Delta(\lambda')| = |\Delta(\lambda)|$, $|\Delta_{\mathrm{bad}}(\lambda')| = |\Delta_{\mathrm{bad}}(\lambda)|$ and $|\Sigma(\lambda')| = |\Sigma(\lambda)|$.

Case 2: $i_1 \in \Delta(\lambda)$ and $\lambda_{i_1} = \lambda_{i_1+1}$.

By the minimality condition of i_1, we have $\lambda_{i_1-1} = \lambda_{i_1} + 1$ (except for $i_1 = 1$, in which case $\lambda_{i_1-1} = 0$ by convention), and so $\lambda_{i_1-2} = \lambda_{i_1-1}$ because $\varepsilon(-1)^{\lambda_{i_1-1}} = 1$. We have

$$\lambda' = (\lambda_1 - 2, \ldots, \lambda_{i_1-1} - 2, \lambda_{i_1} - 1, \lambda_{i_1+1} - 1, \lambda_{i_1+2}, \ldots, \lambda_n),$$

and

$$\begin{aligned} s(\lambda') &= \sum_{i=1}^{i_1-2} [(\lambda_i - \lambda_{i+1})/2] + \underbrace{[(\lambda_{i_1-1} - \lambda_{i_1} - 1)/2]}_{=0 \text{ since } \lambda_{i_1-1} = \lambda_{i_1}+1} \\ &\quad + [(\lambda_{i_1} - \lambda_{i_1+1})/2] + [\lambda_{i_1+1} - 1 - \lambda_{i_1+2})/2] + \sum_{i=i_1+1}^{n} [(\lambda_i - \lambda_{i+1})/2] \\ &= \begin{cases} s(\lambda) - 1 & \text{if } \lambda_{i_1+1} - \lambda_{i_1+2} \in 2\mathbb{N}; \\ s(\lambda) & \text{if } \lambda_{i_1+1} - \lambda_{i_1+2} \notin 2\mathbb{N}. \end{cases} \end{aligned}$$

(If $i_1 = 0$, we start at the second line and we get the same conclusion.) Also, observe that in Case 2, we have

$$|\Delta(\lambda')| = |\Delta(\lambda)| - 1.$$

Indeed, $i_1 \in \Delta(\lambda)$ but $i_1 \notin \Delta(\lambda')$ and for the indexes $i \neq i_1$ we have here the equivalence: $i \in \Delta(\lambda) \iff i \in \Delta(\lambda')$.

We discuss two cases depending on the parity of $\lambda_{i_1+1} - \lambda_{i_1+2}$.

- $\lambda_{i_1+1} - \lambda_{i_1+2} \in 2\mathbb{N}$.

Then $i_1 \in \Delta_{\mathrm{bad}}(\lambda)$. There are two sub-cases depending on whether $i_1 + 2$ is in $\Delta(\lambda)$ or not:

  * $i_1 + 2 \in \Delta(\lambda)$.
    Then, $i_1 + 2 \in \Delta_{\mathrm{bad}}(\lambda)$ (since $\lambda_{i_1+1} - \lambda_{i_1+2} \in 2\mathbb{N}$) and $i_1 + 2 \in \Delta(\lambda')$. Once again, there are two sub-cases:

    1) $i_1 + 2 \notin \Delta_{\mathrm{bad}}(\lambda')$.
       Then $|\Delta_{\mathrm{bad}}(\lambda')| = |\Delta_{\mathrm{bad}}(\lambda)| - 2$. Moreover, $(i_1, i_1 + 2)$ is a good 2-cluster of λ. Indeed, $i_1 + 2 \notin \Delta_{\mathrm{bad}}(\lambda')$ implies that $\lambda_{i_1+3} - \lambda_{i_1+4} \notin 2\mathbb{N}$. On the other hand, $\lambda_{i_1-1} - \lambda_{i_1} = 1 \notin 2\mathbb{N}$ (if $i_1 = 1$ the first condition in Definition 2.6 should be omitted). But $(i_1, i_1 + 2)$ is not a 2-cluster of λ' since $i_1 \notin \Delta(\lambda')$. Hence, we have $|\Sigma(\lambda')| = |\Sigma(\lambda)| - 1$ by Lemma 2.7.
    2) $i_1 + 2 \in \Delta_{\mathrm{bad}}(\lambda')$.
       Then $|\Delta_{\mathrm{bad}}(\lambda')| = |\Delta_{\mathrm{bad}}(\lambda)| - 1$. The only 2-clusters of λ which are not 2-clusters of λ' are of the form $(i_1, \dots, i_k)$ with $k \geqslant 2$. Assume that there is a good 2-cluster of the form $(i_1, \dots, i_k)$ for λ, with $k \geqslant 2$. The 2-cluster $(i_1, i_1 + 2)$ of λ is bad. Indeed, $\lambda_{i_1+3} - \lambda_{i_1+4} \in 2\mathbb{N}$ since $i_1 + 2 \in \Delta_{\mathrm{bad}}(\lambda')$ and $\lambda'_{i_1+1} - \lambda'_{i_1+2} \notin 2\mathbb{N}$. Hence, $k > 2$. Since $\lambda_{i_1-1} - \lambda_{i_1} \notin 2\mathbb{N}$ and $\lambda_{i_1+1} - \lambda_{i_1+2} \notin 2\mathbb{N}$, the 2-cluster $(i_1, \dots, i_k)$ is good for λ if and only if the 2-cluster $(i_1 + 2, \dots, i_k)$ is good for λ'. On the other direction, the only possible good 2-clusters of λ' which are not good for λ are of the form $(i_2 = i_1 + 2, \dots, i_k)$ with $k \geqslant 3$. By the above argument, if there is such a good 2-cluster for λ', then $(i_1, \dots, i_k)$ is a good 2-cluster for λ. As a consequence, $|\Sigma(\lambda')| = |\Sigma(\lambda)|$.

  * $i_1 + 2 \notin \Delta(\lambda)$.
    Then $|\Delta_{\mathrm{bad}}(\lambda')| = |\Delta_{\mathrm{bad}}(\lambda)| - 1$. Moreover, since $i_1 + 2 \notin \Delta(\lambda)$, then neither i_1 nor $i_1 + 2$ belongs to a 2-cluster for λ. Hence $|\Sigma(\lambda)| = |\Sigma(\lambda')|$.

- $\lambda_{i_1+1} - \lambda_{i_1+2} \notin 2\mathbb{N}$.

In this case, $i_1 \notin \Delta_{\mathrm{bad}}(\lambda)$, $i_1 + 2 \notin \Delta(\lambda)$ and $i_1 + 2 \notin \Delta(\lambda')$. Hence $|\Delta_{\mathrm{bad}}(\lambda')| = |\Delta_{\mathrm{bad}}(\lambda)|$. Moreover, neither i_1 nor $i_1 + 2$ belongs to any 2-cluster. Hence $|\Sigma(\lambda)| = |\Sigma(\lambda')|$.

In all the cases, we can check with the formula of Definition 2.8 that $z_{\mathrm{PT}}(\lambda') = z_{\mathrm{PT}}(\lambda) - 1$ as desired. This concludes the proof of Theorem 2.10. ■

3. Counter-examples for [5, Proposition 3.11]

From now on, we shall denote by $z(\lambda)$ the integer $z_{\mathrm{M}}(\lambda) = z_{\mathrm{PT}}(\lambda)$ for $\lambda \in \mathcal{P}_\varepsilon(N)$. If $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$ and $\mathcal{O}'$ is a rigid nilpotent orbit of $\mathfrak{l}$, we denote by $\mathrm{Ind}_{\mathfrak{l}}^{\mathfrak{g}}(\mathcal{O}')$ the induced nilpotent orbit of $\mathfrak{g}$ from $\mathcal{O}'$ in $\mathfrak{l}$.

Proposition 3.11 of [5] asserts that if a nilpotent element e associated with the partition $\lambda \in \mathcal{P}_\varepsilon(N)$ is induced form a nilpotent orbit in a Levi subalgebra

$\mathfrak{l}$, then $z(\lambda)$ is equal to the dimension of the center of $\mathfrak{l}$. This result is actually incorrect. If it were true, it would imply that all the sheets containing e share the same dimension (see [5, Remark 3.12]). But this is wrong. Below are some counter-examples (see also [6, Remark 4]):

Example 3.1. Assume that $\mathfrak{g} = \mathfrak{so}(8)$ and consider the nilpotent element e of $\mathfrak{g}$ with partition $\lambda = (3,3,1,1) \in \mathcal{P}_1(8) \smallsetminus \mathcal{P}_1^*(8)$. The algorithm yields $z(\lambda) = 2$.

On the other hand, e is induced from two different ways: from the zero orbit in a Levi subalgebra $\mathfrak{l}_1$ *of type* $(3,1;0)$, that is $\mathfrak{l}_1 \simeq \mathfrak{gl}_3 \times \mathfrak{gl}_1 \times 0$ (see the definition after [5, Lemma 3.2] for the meaning of *type*), and from the zero orbit in a Levi subalgebra $\mathfrak{l}_2$ of type $(2;4)$, that is $\mathfrak{l}_1 \simeq \mathfrak{gl}_2 \times \mathfrak{so}_4$. The first one, $\mathfrak{l}_1$, has a center of dimension 2, while the second one, $\mathfrak{l}_2$, has a center of dimension 1. The nilpotent orbit of e has dimension 18 and e lies in two different sheets: one of dimension $\dim \mathfrak{z}(\mathfrak{l}_1) + \dim \mathrm{Ind}_{\mathfrak{l}_1}^{\mathfrak{g}}(0) = 20$ and one of dimension $\dim \mathfrak{z}(\mathfrak{l}_2) + \dim \mathrm{Ind}_{\mathfrak{l}_2}^{\mathfrak{g}}(0) = 19$ (here $\mathfrak{z}(\mathfrak{l}_i)$ denotes the center of $\mathfrak{l}_i$ for $i = 1, 2$). This contradicts Proposition 3.11 of [5], and also Remark 3.12 of the same paper.

Example 3.2. We give now a counter-example in $\mathfrak{sp}(14)$. Consider the partition $\lambda = (4,4,2,2,1,1)$ of $\mathcal{P}_{-1}(14)$. Here, the algorithm yields $z(\lambda) = 2$.

The corresponding nilpotent element is induced from the zero orbit in $\mathfrak{l}_1 \simeq \mathfrak{gl}_1 \times \mathfrak{gl}_3 \times \mathfrak{sp}(6)$, and from the ridid nilpotent orbit $0 \times \mathcal{O}'$ in $\mathfrak{l}_1 \simeq \mathfrak{gl}_2 \times \mathfrak{sp}(10)$ where $\mathcal{O}'$ corresponds to the partition $(2,2,2,2,1,1) \in \mathcal{P}_{-1}^*(10)$. Again the dimensions of the centers of $\mathfrak{l}_1$ and $\mathfrak{l}_2$ lead to different dimensions, 2 and 1 respectively.

The origin of the error can be pined down in the proof of [5, Proposition 3.11]. Let us briefly explain this. Until the end of the section, we are in the notations of [5].

At the end of this proof, the assertion *"Consequently the smallest integer such that one of the situations **(a)** or **(b)** of **Step 1** happens in* $\mathbf{d}^{(p)}$ *is equal to* i_p*"* is incorrect (here $\mathbf{d}$ is an element of $\mathcal{P}_\varepsilon(N)$). And so, the main induction argument of the proof fails. We can see that is incorrect in general on an explicit example. Consider the partition $\mathbf{d} = (4,4,3,3,1,1)$ of $\mathcal{P}_1(16)$. Then the corresponding nilpotent orbit is induced from the zero orbit in $\mathfrak{l} \simeq \mathfrak{gl}_3 \times \mathfrak{gl}_5 \times 0$ and from the rigid nilpotent orbit with partition $(2,2,1,1,1,1)$ in $\mathfrak{l} \simeq \mathfrak{gl}(4) \times \mathfrak{so}(8)$. Consider the second induction. In the notations of the proof, we have: $S = 1$, $i_1 = 4$, $\mathbf{d}^{(0)} = \mathbf{f} = (2,2,1,1,1,1)$, $\mathbf{d} = \mathbf{d}^{(1)} = \widetilde{\mathbf{d}^{(0)}}$ (see [5, Proposition 3.7] for the tilda notation). Then the smallest integer such that one of the situations **(a)** or **(b)** of **Step 1** happens for $\mathbf{d} = \mathbf{d}^{(1)}$ is $3 \neq i_1$.

4. Conclusion

To summarize, we list below all corrections which have to be taken into account in [5] (the numbering of [5] is used):

- Proposition 3.11 (its proof and its statement) is incorrect.

- As a consequence Remark 3.12, the sentence "The results of this section specify that, in the classical case, the dimension of a sheet containing a given nilpotent orbit does not depend on the choice of a sheet containing it" in §1.2, and the sentence "Surprisingly, in the classical case, we will notice that if $\mathrm{Ind}_{\mathfrak{l}_1}(\mathcal{O}_{\mathfrak{l}_1}) = \mathrm{Ind}_{\mathfrak{l}_2}(\mathcal{O}_{\mathfrak{l}_2})$, then $\dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}_1) = \dim \mathfrak{z}_{\mathfrak{g}}(\mathfrak{l}_2)$" in Remark 2.15, are also incorrect.

- The proof of Theorem 3.13 is incorrect, since it uses Proposition 3.11. Nevertheless, its statement remains valid. In particular, Tables 3, 4 and 5 are still correct.

  Remark. There are some misprints in Table 5: line $2m = 48$, the partitions are $[7, 1^5], [5, 3, 2^2], [4^2, 3, 1]$ and not $[4^3], [4^2, 3, 1]$.

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