Upper Bound for the Heat Kernel on Higher-Rank $NA$ Groups

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Abstract. Let $S$ be a semi-direct product $S = N \rtimes A$ where $N$ is a connected and simply connected, non-abelian, nilpotent meta-abelian Lie group and $A$ is isomorphic with $\mathbb{R}^k$, $k > 1$. We consider a class of second order left-invariant differential operators $\mathcal{L}_\alpha$, $\alpha \in \mathbb{R}^k$, on $S$. We obtain an upper bound for the heat kernel for $\mathcal{L}_\alpha$.

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1. Introduction

Let $S$ be a semi-direct product $S = N \rtimes A$ where $N$ is a connected and simply connected, non-abelian, nilpotent Lie group and $A = \mathbb{R}^k$ with the ordinary additive structure (abelian group). In particular, the group operation on $A$ is written additively. The dimension $k$ of $A$ is called the rank of $S$. For $g \in S$ we let $x(g) = x$ and $a(g) = a$ denote the components of $g$ in this product so that $g = (x, a)$. Since $A = \mathbb{R}^k$, its Lie algebra $\mathfrak{a}$ is identified with $A$ and the exponential mapping $\exp_A$ is the identity mapping.

We use the exponential map to identify $N$ with its Lie algebra $\mathfrak{n}$. Thus $N$ is $\mathfrak{n}$ endowed with the Campbell-Hausdorff product and $\exp_N$ is also the identity mapping. The action of $A$ on $N$ is then defined by

$$n^a = e^{\text{ad}(a)}(n), \quad a \in A = \mathfrak{a}, \quad n \in N.$$

The multiplication in $S$ then has the form $(x, a)(y, b) = (xy^a, a + b)$.

We assume that there is a basis $X_1, \ldots, X_{\dim_{\mathfrak{n}}}$ of $\mathfrak{n}$ that diagonalizes the $A$-action, i.e., there are corresponding root functionals $\lambda_1, \ldots, \lambda_{\dim_{\mathfrak{n}}} \in \mathfrak{a}^*$ such that

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for every $a \in \mathfrak{a}$, $[a, X_j] = \lambda_j(a)X_j$, $j = 1, \ldots, \dim \mathfrak{n}$. Thus the groups under study are solvable, non-unimodular, and hence have an exponential volume growth.

Let the usual scalar product on $\mathbb{R}^k$ be denoted by $\langle \cdot, \cdot \rangle$. We use this scalar product to identify $\mathfrak{a}$ with $\mathfrak{a}^*$, the space of linear forms on $\mathfrak{a}$. For $v \in \mathbb{R}^k$ we write

$$\|v\|^2 = v^2 = \langle v, v \rangle = \sum_{i=1}^k v_i^2.$$  

In what follows we consider meta-abelian $N$. Specifically we assume that

$$N = M \rtimes V,$$

where $M$ and $V$ are abelian Lie groups with the corresponding Lie algebras $\mathfrak{m}$ and $\mathfrak{v}$. Let $\{Y_1, \ldots, Y_d\}$ and $\{X_1, \ldots, X_n\}$ be bases for $\mathfrak{m}$ and $\mathfrak{v}$ respectively, consisting of left-invariant vector fields, such that

$$\{Y_1, \ldots, Y_d, X_1, \ldots, X_n\}$$

forms an ordered Jordan-Hölder basis for the Lie algebra $\mathfrak{n}$ of $N$, ordered so that the matrix of $\text{ad}_X$ in this basis is strictly lower triangular for all $X \in \mathfrak{n}$. We use these bases to identify $M = \mathfrak{m}$ and $V = \mathfrak{v}$ with $\mathbb{R}^d$ and $\mathbb{R}^n$ respectively. For $x \in N$ we let $m(x) = m$ and $v(x) = v$ denote the components of $x$ in this product so that

$$x = (m, v).$$

By $e$ we denote the neutral element of the group $N$. We assume also that the

$$\{Y_i\}$$

and

$$\{X_j\}$$

are eigenvectors for the $\text{ad}_H$, $H \in \mathfrak{a}$, action, i.e., there are $\xi_1, \ldots, \xi_d, \vartheta_1, \ldots, \vartheta_n \in \mathfrak{a}^*$ such that for every $H \in \mathfrak{a}$,

$$\text{ad}_H Y_i = [H, Y_i] = \xi_i(H)Y_i, \ 1 \leq i \leq d,$$

$$\text{ad}_H X_j = [H, X_j] = \vartheta_j(H)X_j, \ 1 \leq j \leq n.$$  

Let $x = (m, v) \in S$. We define a left-invariant differential operator on $S$,

$$\mathcal{L}_\alpha = \Delta_\alpha + \sum_{j=1}^d e^{2\xi_j(a)}Y_j^2 + \sum_{j=1}^n e^{2\vartheta_j(a)}X_j^2,$$

where

$$\Delta_\alpha = \sum_{i=1}^k (\partial_{a_i}^2 + 2\alpha_i \partial_{a_i}).$$

Let

$$\rho_o = \sum_{j=1}^d \xi_j + \sum_{j=1}^n \vartheta_j$$

and set

$$\chi(g) = \text{det}(\text{Ad}(g)) = e^{\rho_o(a)}, \ g = (x, a) \in S,$$

where

$$\text{Ad}(g)s = gsg^{-1}, \ s \in S.$$  

A left-invariant Haar measure on $S = N \rtimes A$ is then given by

$$dS(y, b) = e^{-\rho_o(b)}dbdy = \chi(b)^{-1}dbdy,$$

where

$$\int_N \int_A T_{(x, a)}f(y, b)dS(y, b) = \int_N \int_A p_t(x, a; y, b)f(y, b)dS(y, b)$$

for every $a \in \mathfrak{a}$, $[a, X_j] = \lambda_j(a)X_j$, $j = 1, \ldots, \dim \mathfrak{n}$. Thus the groups under study are solvable, non-unimodular, and hence have an exponential volume growth.
be the semigroup of operators on $N \rtimes A$ generated by $L_\alpha$. The kernel $p_t$ is called the heat kernel for $L_\alpha$.

The aim of this note is to give off-diagonal upper bounds for $p_t(x,a;y,b)$, particularly as $(x,a)$ and $(y,b)$ vary. There is a vast body of work related to producing on-diagonal and off-diagonal upper bounds for the heat kernel, both on groups and on Riemannian manifolds. See for example [3, 8, 14] and the references contained there in. Much is known about the growth in $t$. For example Varopoulos in [13] gives two sided on-diagonal estimates for the heat kernel $p_t$, $t \geq 1$, on Lie groups. However, in the context of general solvable Lie groups, very little explicit information seems to be available relating to the growth in the other variables. One exception to this statement is the recent work of Melzi [8] in the special case of the affine group of $\mathbb{R}$. Hence, in particular, $A$ and $N$ are assumed to be one dimensional. We remark that the assumption that $A$ have dimension one is particularly restrictive. (The dimension of $A$ is the rank of the corresponding manifold.) Melzi’s result is the only one in the case of solvable Lie groups that we are aware of.

Before we state our estimate we need to introduce some notation. The neutral element of $S$ is $(e,0)$. Let $d^{R}(g_1,g_2)$ be the left-invariant Riemannian distance between two points $g_1, g_2 \in NA$. We write $\tau(g)$ instead of $d^{R}(g,(e,0))$. We note that $\tau(g)$ is a sub-additive function on $S$. Our main result is the following estimate.

**Theorem 1.1.** For every $q \geq 1$ and for every $t > 0$ there is a constant $C_{t,q} > 0$ such that for every $z \in N$ and every $a \in A$,

$$p_t(e,0;z,a) \leq C_{t,q}e^{-a^2/(32t)+\rho_o(a)}e^{-\tau(z)}.$$  

(5)

**Remark 1.2.** The proof of Theorem 1.1 as written is valid only for $q \geq \|\rho_o\|$. However, it is clear that if we have the above upper bound for some $q_0$ then we also have it with all $1 \leq q \leq q_0$.

**Remark 1.3.** Since $L_\alpha$ commutes with left translation, the same is true for $T_1$. Hence, from (4),

$$p_t(x,a;y,b) = p_t(e,0;(x,a)^{-1}(y,b)).$$

Thus Theorem 1.1 immediately yields an estimate for $p_t$ on $S \times S$.

**Remark 1.4.** Notice that since $S$ is non-unimodular the heat kernel $p_t(x,a;y,b)$ is not symmetric with respect to the left-invariant Haar measure $dS$. Specifically,

$$p_t(y,b;x,a) = e^{-\rho_o(a)}e^{\rho_o(b)}p_t(x,a;y,b).$$

**Remark 1.5.** An upper bound for the constant $C_{t,q}$ in (5), for $t > 1$, can be obtained easily by methods of this paper. However, we do not care about the precise value as the result is not optimal – what one can extract from our proof is the bound of the form $t^{-\gamma}e^{\omega t}$ for some $\omega = \omega_q > 1$ and $\gamma = \gamma_q > 0$ (see the proofs of Lemmas 4.3 and 4.4, and Remark 4.5). Hence, as a corollary we get that for
every \( q > 0 \) there are positive constants \( C, \gamma, \omega \) such that for \( t > 1 \),
\[
p_t(e, 0; z, a) \leq Ct^{-\gamma}e^{-\alpha^2/(32t)+\rho_0(a)e^{-\gamma t(z)}}.
\]
\( (6) \)

2. Homogeneous norms and the Riemannian metric

Let \( \xi_i, \vartheta_j \) be the linear forms defined in §1. We set
\[
A^+ = \text{Int}\{a \in \mathbb{R}^k : \xi_i(a) \geq 0 \text{ for } 1 \leq i \leq d \text{ and } \vartheta_j(a) \geq 0 \text{ for } 1 \leq j \leq n\}.
\]
For \( t \in \mathbb{R}^+ \) and \( \rho \in A^+ \), let
\[
\delta^\rho_t = \text{Ad}((\log t)^\rho)|_N.
\]
Then \( t \mapsto \delta^\rho_t \) is a one parameter group of automorphisms (dilations) of \( N \) for which the corresponding eigenvalues on \( n \) are all positive. It is known [2] that \( N \) has \( \delta^\rho_t \)-homogeneous norm: a continuous function \( |\cdot|_\rho \geq 0 \) on \( N \) such that
\[
|\delta^\rho_t x|_\rho = t|x|_\rho.
\]
By \( \|g\| \) we denote \( \ell^2 \)-norm of \( g \in S \) considered as an element of \( \mathbb{R}^{\dim n} \times \mathbb{R}^k \).

**Lemma 2.1.** For every \( \rho \in A^+ \), there are positive constants \( C_1, C_2, p, q \) such that for every \( x \in N \),
\[
C_1\|x\|^p \leq |x|_\rho \leq C_2(\|x\|^q + 1).
\]

**Proof.** Let
\[
p_i = \rho(Y_i), \quad 1 \leq i \leq d,
\]
\[
q_i = \rho(X_i), \quad 1 \leq i \leq n.
\]
For \( x \in N \),
\[
x = \sum_{i=1}^{d} m_i Y_i + \sum_{i=1}^{n} v_i X_i.
\]
we define
\[
\nu(x) = \sum_{i=1}^{d} |m_i|^{1/p_i} + \sum_{i=1}^{n} |v_i|^{1/q_i}.
\]
Then
\[
\delta^\rho_t x = \sum_{i=1}^{d} m_i e^{p_i \log t} Y_i + \sum_{i=1}^{n} v_i e^{q_i \log t} X_i.
\]
Hence
\[
\nu(\delta^\rho_t x) = t \nu(x).
\]
Thus \( \nu \) is a homogeneous norm on \( N \). From homogeniety, there are positive constants \( C_1 \) and \( C_2 \) such that for all \( x \in N \),
\[
C_1 \nu(x) \leq |x|_\rho \leq C_2 \nu(x).
\]

Our lemma follows. \( \blacksquare \)
It is also known that $S$ has a unique left-invariant Riemannian metric $d^R(\cdot, \cdot)$ for which the corresponding Riemannian metric agrees with the obvious scalar product on $\mathfrak{s}$ at the identity. Let

$$\tau(g) = d^R(g, (e, 0)).$$

The following estimate is due to Guivarc'h [4]. It shows how $\tau(g) = \tau(x, a)$ behaves for each component $x \in N$ and $a \in A$. We will use this result in the proofs of Lemmas 4.3 and 4.4.

**Lemma 2.2.** For every $\rho \in A^+$ there is a positive constant $C$ such that for every $x \in N$ and $a \in A$ we have,

$$C^{-1}(\ln(1 + |x|_\rho) + |a|) \leq \tau(x, a) + 1 \leq C(\ln(1 + |x|_\rho) + |a| + 1).$$

3. Skew-product formula

Let $L_\alpha$ be defined by (1). The process $\sigma_t$ in $\mathbb{R}^k$ generated by the operator $\Delta_\alpha$, i.e., the Brownian motion with drift $2\alpha$, is called a *vertical component* of the diffusion generated by $L_\alpha$. The corresponding *horizontal component* is defined as follows.

Let $C^\infty(N)$ be the space of continuous functions $f$ on $N$ for which there exists the limit $\lim_{x \to \infty} f(x)$. For a continuous function $\sigma : [0, \infty) \to \mathbb{R}^k$, we consider the operator

$$L^\sigma = \sum_{j=1}^d e^{2\xi_j(\sigma_t)}Y_j^2 + \sum_{j=1}^n e^{2\theta_j(\sigma_t)}X_j^2.$$

Let $\{U^\sigma(s, t) : 0 \leq s \leq t\}$ be the (unique) family of bounded operators on $C^\infty(N)$ which satisfies

i) $U^\sigma(s, s) = \text{Id}$, for all $s \geq 0$,

ii) $\lim_{h \to 0} U^\sigma(s, s + h)f = f$ in $C^\infty(N)$,

iii) $U^\sigma(s, r)U^\sigma(r, t) = U^\sigma(s, t)$, $0 \leq s \leq r \leq t$. 

In particular $C^{(0,k)}(N)$ is a Banach space with the norm $\|f\|_{0,k}$. For a continuous function $\sigma : [0, \infty) \to \mathbb{R}^k$, we consider the operator

$$C^{(k,\ell)}(N) = \{f : \tilde{X}^l \mathcal{X}^j f \in C^\infty(N) \text{ for every } |I| < k + 1 \text{ and } |J| < \ell + 1\}$$

and

$$\|f\|_{(k,\ell)} = \sup_{|I|=k, |J|=\ell} \|\tilde{X}^l \mathcal{X}^j f\|_{\infty},$$

$$\|f\|_{(k,\ell)} = \sup_{|I|\leq k, |J|\leq \ell} \|\tilde{X}^l \mathcal{X}^j f\|_{\infty}. \quad (7)$$

In particular $C^{(0,k)}(N)$ is a Banach space with the norm $\|f\|_{0,k}$.
iv) \( \partial_s U^\sigma(s,t)f = -L^\sigma U^\sigma(s,t)f \) for every \( f \in C^{(0,2)}(N) \),

v) \( \partial_t U^\sigma(s,t)f = U^\sigma(s,t)L^\sigma f \) for every \( f \in C^{(0,2)}(N) \),

vi) \( U^\sigma(s,t) : C^{(0,2)}(N) \to C^{(0,2)}(N) \).

The operator \( U^\sigma(s,t) \) is a convolution operator with a probability measure with a smooth density, i.e., \( U^\sigma(s,t)f = f * P^\sigma_{t,s} \). In particular, \( U^\sigma(s,t) \) is left-invariant.

By iii), \( P^\sigma_{t,r} * P^\sigma_{r,s} = P^\sigma_{t,s} \) for \( t \geq r \geq s \). Existence of \( U^\sigma(s,t) \) follows from [12].

A stochastic process (evolution) in \( N \) corresponding to transition probabilities \( P^\sigma_{t,s} \) is called a horizontal component of the diffusion generated by \( L^\alpha \).

Let \( U^\sigma(s,t) \) and \( P^\sigma_{t,s} \) be as above. For \( f \in C^\infty_c(N \times \mathbb{R}^k) \) and \( t \geq 0 \), we put

\[
T_t f(x,a) = \mathbb{E}_a U^\sigma(0,t)f(x, \sigma_t) = \mathbb{E}_a (f * P^\sigma_{t,0})(x, \sigma_t),
\]

where the expectation is taken with respect to the distribution of the process \( \sigma_t \) (Brownian motion with drift) in \( \mathbb{R}^k \) with the generator \( \Delta^\alpha \) and starting from \( a \), i.e., \( \sigma_0 = a \). The operator \( U^\sigma(0,t) \) acts on the first variable of the function \( f \) (as a convolution operator).

We have the following theorem which plays a crucial role in the proof of the upper bound for the heat kernel \( p_t(\cdot,\cdot) \).

**Theorem 3.1.** The family \( T_t \) defined in (8) is the semigroup of operators generated by \( L^\alpha \). That is, for \( f \in C^\infty(S) \),

\[
\partial_t T_t f = L^\alpha T_t f
\]

and

\[
\lim_{t \to 0} T_t f = f.
\]

We refer to formula (8) as the skew-product formula. In the case of \( A = \mathbb{R} \) it was proved in [1]. Higher rank case was considered in [9]. Recently a more general skew-product formula was proved in [10].

### 4. Proof of Theorem 1.1

#### 4.1. Upper bound for \( P^\sigma_{t,0} \)

By the skew-product formula (8) we have,

\[
T_t f(x,a) = \mathbb{E}_a U^\sigma(0,t)f(x, \sigma_t),
\]

\[
= \mathbb{E}_a \int_N f(xy^{-1}, \sigma_t) P^\sigma_{t,0}(y)dy.
\]

We recall the result from [11]. In order to state this result let, for a continuous function \( \sigma : [0, \infty) \to A = \mathbb{R}^k \),

\[
A^\sigma_{M,i}(s,t) = \int_s^t e^{2\xi_i(\sigma(u))} du, \quad i = 1, \ldots, d,
\]

\[
A^\sigma_{V,j}(s,t) = \int_s^t e^{2\vartheta_j(\sigma(u))} du, \quad j = 1, \ldots, n,
\]

(10)
and

\[ A_{M,\Sigma}^\sigma(s, t) = \sum_{i=1}^d A_{M,i}^\sigma(s, t), \quad A_{V,\Sigma}^\sigma(s, t) = \sum_{j=1}^n A_{V,j}^\sigma(s, t), \]

\[ A_{M,\Pi}^\sigma(s, t) = \prod_{i=1}^d A_{M,i}^\sigma(s, t), \quad A_{V,\Pi}^\sigma(s, t) = \prod_{j=1}^n A_{V,j}^\sigma(s, t). \]

We also set

\[ A_{N,\Pi}^\sigma(s, t) = A_{M,\Pi}^\sigma(s, t) A_{V,\Pi}^\sigma(s, t), \]

\[ A_{N,\Sigma}^\sigma(s, t) = A_{M,\Sigma}^\sigma(s, t) + A_{V,\Sigma}^\sigma(s, t). \]

Finally, for \( k \in \mathbb{N} \), we let

\[ \phi_k(m) = \left( \frac{\|m\|^{1/k}}{\|m\|^{1/k} + 1} \right)^k, \quad m \in M. \]

We also let \( k_o \) be the smallest non-negative integer such that

\[ (\text{ad}_X)^{k_o+1}m = 0, \quad \forall X \in \mathfrak{v}. \]

Note that if \( k_o = 0 \), then \( \mathfrak{v} \) centralizes \( m \); hence \( N \) is abelian. Thus our hypotheses imply that \( k_o > 0 \).

**Theorem 4.1** ([11, Theorem 1.5]). There are positive constants \( C, D \) and \( k_o \in \mathbb{N} \) such that for all \((m, v) \in N, \)

\[
P^\sigma_{t,0}(m, v) \leq CA_{N,\Sigma}^\sigma(0, t)^{-1/2}(\|m\|^{1/(2k_o)} + 1 + A_{V,\Sigma}^\sigma(0, t)^{1/2})
\times \exp\left(-D\frac{\|v\|^2}{A_{V,\Sigma}^\sigma(0, t)} - D\frac{\|m\|^{1/k_o}}{A_{N,\Sigma}^\sigma(0, t)} A_{V,\Sigma}^\sigma(0, t) \phi_{2k_o}(m)\right). \tag{11}
\]

In case when \( N \) is the \( 2n + 1 \)-dimensional Heisenberg group the above estimate is proved in [10].

To simplify notation we write

\[ m_{k_o} = \|m\|^{\frac{1}{2k_o}} + 1, \]

and

\[ \mathcal{E}_1^\sigma(v) = \exp\left(-\frac{D\|v\|^2}{A_{V,\Sigma}^\sigma(0, t)}\right), \]

\[ \mathcal{E}_2^\sigma(m) = \exp\left(-D\frac{\|m\|^{\frac{1}{k_o}}}{A_{N,\Sigma}^\sigma(0, t)} A_{V,\Sigma}^\sigma(0, t) \phi_{2k_o}(m)\right). \]

**4.2. Apriori \( L^2 \)-weighted estimates.** For \( t \geq 0 \) and \( y = (m, v) \in N \), define

\[
C(t, y) = C(t, m, v) = m_{k_o}^2 E_0 A_{N,\Pi}^\sigma(0, t)^{-1} \mathcal{E}_1^\sigma(v)^2 \mathcal{E}_2^\sigma(m)^2
+ E_0 A_{N,\Pi}^\sigma(0, t)^{-1} A_{V,\Sigma}^\sigma(0, t) \mathcal{E}_1^\sigma(v)^2 \mathcal{E}_2^\sigma(m)^2. \tag{12}
\]

The following result will be applied in §4.5 to \( \xi(y) = e^{\sigma(v)} \) with appropriate \( q \in \mathbb{R} \).
Lemma 4.2. There exists a constant $C > 0$, such that for any measurable function $\xi : N \to \mathbb{R}$,

$$
\int_N \int_{\mathbb{R}^k} \left( p_t(e, 0; y, b) e^{b^2/(8t) - \rho_b(b)/2} \xi(y) \right)^2 e^{-\rho_b(b) db} dy \leq C t^{-k/2} \int_N C(t, y) \xi(y)^2 dy.
$$

(13)

Proof. By (4) and (9) for $f = \delta_y \otimes \phi$, where $y = (m, v) \in N$ and $\phi \in L^2(\mathbb{R}^k, e^{-b^2/4t} db)$ we have,

$$
|T_t f(e, 0)| = \left| \int_{\mathbb{R}^k} p_t(e, 0; y, b) \phi(b) \chi(b)^{-1} db \right| = |E_0 P_{t, 0}^\sigma(y^{-1}) \phi(\sigma_t)| \leq E_0 P_{t, 0}^\sigma(y^{-1}) |\phi(\sigma_t)|.
$$

Since the right hand side of (11) in Theorem 4.1 is symmetric, i.e., has the same value for $y = (m, v)$ and $y^{-1} = (-m, -v)$, we bound $P_{t, 0}^\sigma(y^{-1})$ using (11), and get

$$
\left| \int_{\mathbb{R}^k} p_t(e, 0; y, b) \phi(b) \chi(b)^{-1} db \right| \leq C m_{k_0} E_0 A_{N, \Pi}^\sigma(0, t)^{-\frac{1}{2}} E_1^\sigma(v) E_2^\sigma(m) |\phi(\sigma_t)|
$$

$$
+ C |E_0 A_{N, \Pi}^\sigma(0, t)^{-\frac{1}{2}} A_{N, \Sigma}^\sigma(0, t)^{1/2} E_1^\sigma(v) E_2^\sigma(m) |\phi(\sigma_t)|.
$$

Using the Cauchy-Schwarz inequality to the expectations on the right we bound

the above integral by

$$
C m_{k_0} (E_0 A_{N, \Pi}^\sigma(0, t)^{-1} E_1^\sigma(v) E_2^\sigma(m)^2)^{1/2} (E_0 \phi^2(\sigma_t))^{1/2}
$$

$$
+ C (E_0 A_{N, \Pi}^\sigma(0, t)^{-1} A_{N, \Sigma}^\sigma(0, t) E_1^\sigma(v) E_2^\sigma(m)^2)^{1/2} (E_0 \phi^2(\sigma_t))^{1/2}.
$$

Thus,

$$
\left( \int_{\mathbb{R}^k} p_t(e, 0; y, b) \phi(b) \chi(b)^{-1} db \right)^2 \leq CC(t, m, v) E_0 \phi^2(\sigma_t).
$$

Therefore, for all $\phi \in L^2(\mathbb{R}^k)$, and $y = (m, v) \in N$,

$$
\left| \int_{\mathbb{R}^k} p_t(e, 0; y, b) \phi(b) \chi(b)^{-1} db \right| \leq CC(t, m, v) E_0 \phi^2(\sigma_t)^{1/2}.
$$

Since

$$
E_0 \phi^2(\sigma_t) = C t^{-k/2} \int_{\mathbb{R}^k} \phi^2(b) e^{-b^2/4t} db,
$$

we get that for all $\phi \in L^2(\mathbb{R}^k, e^{-b^2/4t} db),$

$$
\left| \int_{\mathbb{R}^k} p_t(e, 0; y, b) \phi(b) \chi(b)^{-1} db \right| \leq C t^{-k/4} C(t, m, v)^{1/2} \left( \int_{\mathbb{R}^k} \phi^2(b) e^{-b^2/4t} db \right)^{1/2}.
$$

It follows that

$$
\phi \mapsto \int_{\mathbb{R}^k} p_t(e, 0; y, b) \phi(b) e^{b^2/8t} \chi(b)^{-1} db
$$

defines a continuous linear functional on $L^2(\mathbb{R}^k, db)$ with norm bounded by

$$
C t^{-k/4} C(t, y)^{1/2}.
$$
Hence,
\[
\int_{\mathbb{R}^b} \left( p_t(e, 0; y, b) e^{b^2/8t} \right)^2 \chi(b)^{-2} db \leq Ct^{-k/2} C(t, y),
\]
and consequently, (13) follows.

\[\square\]

4.3. A geometric ingredient.

Lemma 4.3. Let $\gamma \in \mathbb{R}$ be given. There exist positive constants $C, D$ such that for every $t > 0$ and $2q \geq |\gamma| \|\rho_o\|$ and every $b \in A$ and $y, z \in N$,
\[
e^{-q^2(y)} e^{-q^2(b^{-1} y^{-1} z b) \gamma} + \gamma \rho_o(b) \leq e^{64C^2q^2 t + 2qD} e^{-q^2(z)}.
\]

Proof. For simplicity, for $g \in S$, we write $|g| = e^\tau(g)$.

Since $\tau(gh) \leq \tau(g) + \tau(h)$ it follows that $|\cdot|$ is sub-multiplicative, i.e., $|gh| \leq |g| |h|$ for $g, h \in S$.

By Lemma 2.2 there are positive constants $C, D$ such that, for $b \in A$,
\[
\tau(b) \leq C \|b\| + D.
\]

Obviously, the above inequality holds also for $b^{-1} = -b$. Thus,
\[
|b|^q |b|^{-q} e^{-\frac{1}{16t} b - \gamma \rho_o(b)} \leq e^{2q(C \|b\| + D) + |\gamma| \|\rho_o\| \|b\|} = e^{(2q+C+|\gamma|\|\rho_o\|)\|b\|+2qD}.
\]

Consequently, with $2q \geq |\gamma| \|\rho_o\|$ and $u = (16t)^{-1/2} b$ we have
\[
|b^{-1} q |b| q e^{-\frac{1}{16t} b - \gamma \rho_o(b)} \leq e^{(2qC+\gamma \|\rho_o\|)\|b\|+\frac{16q^2}{16t}} e^{2qD}
\]
\[
\leq e^{4qC\|b\|-\frac{\|u\|^2}{16t}} e^{2qD}
\]
\[
= e^{4qC(16t)^{1/2}} \|u\|-\|u\|^2 e^{2qD}
\]
\[
= e^{-\|u\|-2q(16t)^{1/2}+64C^2q^2} e^{2qD}
\]
\[
\leq e^{64C^2q^2 t + 2qD}.
\]

Now, using sub-multiplicativity of $|\cdot|$, we get for $b \in A$,
\[
|b y^{-1} z| = |bb^{-1} y^{-1} z b b^{-1}| \leq |b| |b^{-1} y^{-1} z b| |b^{-1}|.
\]

In particular,
\[
|b^{-1} y^{-1} z b|^{-q} \leq |y^{-1} z|^{-q} |b^{-1}|^q |b|^q.
\]

Of course, $|z| = |y(y^{-1} z)| \leq |y| |y^{-1} z|$, and so
\[
|y|^{-q} \leq |z|^{-q} |y^{-1} z|^q.
\]

Hence, by (15) and (16),
\[
|y|^{-q} |b^{-1} y^{-1} z b|^{-q} e^{-\frac{1}{16t} b - \gamma \rho_o(b)} \leq |b^{-1}|^q |b|^q e^{-\frac{1}{16t} b - \gamma \rho_o(b)} |z|^{-q},
\]
and so the lemma follows from (14).

\[\square\]
4.4. Moments estimates of $C(t, y)$. Let $C(t, y)$ be as defined in (12). We have the following result.

**Lemma 4.4.** For every $q \geq 1$ and for every $t > 0$ there is a constant $C = C_{t, q}$ such that

$$
\int_{\mathbb{R}^n} C(t, y)e^{2q\tau(y)}dy \leq C_{t, q} < +\infty.
$$

**Proof.** The function $C(t, y)$ consists of three terms. Thus, we need to estimate three integrals. By the sub-multiplicativity of the Riemannian distance for $y = mv$, $e^{\tau(y)} \leq e^{\tau(m)}e^{\tau(v)}$. Thus, changing the order of integration, the first integral is

$$
\mathbb{E}_0A_{N, \Pi}^\sigma(0, t)^{-1}\int_{\mathbb{R}^n} \mathcal{E}_1^\sigma(v)^2\mathcal{E}_2^\sigma(m)^2e^{2q\tau(m)}m_k^2dy
\leq \mathbb{E}_0A_{N, \Pi}^\sigma(0, t)^{-1}\int_{\mathbb{R}^d} \mathcal{E}_2^\sigma(m)^2e^{2q\tau(m)}m_k^2dm\int_{\mathbb{R}^n} \mathcal{E}_1^\sigma(v)^2e^{2q\tau(v)}dv. \quad (17)
$$

By Lemmas 2.2 and 2.1 it follows that for every $\rho \in A^\dagger$ there are positive constants $C, r_o, C_1, D_1$ such that for all $y \in N$, with $\|y\| \geq 1$,

$$
e^{\tau(y)} \leq C_1|y|^{D_1} \leq C\|y\|^r_o. \quad (18)
$$

By (18), for every $q \geq 1$ and $y \in N$ we have $e^{q\tau(y)} \leq C\max(1, |y|^{r_oq})$. Let $r = r_oq$. We estimate the above integrals as follows

$$
\int_{\mathbb{R}^n} \mathcal{E}_1^\sigma(v)^2e^{2q\tau(v)}dv = \int_{\mathbb{R}^n} \exp\left(-\frac{2D\|v\|^2}{A_{V, \Sigma}(0, t)}\right)e^{2q\tau(v)}dv
\leq C\int_{\|v\| \leq 1} \exp\left(-\frac{D\|v\|^2}{A_{V, \Sigma}(0, t)}\right)dv + C\int_{\|v\| > 1} \exp\left(-\frac{D\|v\|^2}{A_{V, \Sigma}(0, t)}\right)\|v\|^{2r}dv
\leq CA_{V, \Sigma}^\sigma(0, t)^{n/2} + C_rA_{V, \Sigma}^\sigma(0, t)^{(n+2r)/2}.
$$

Similarly we have

$$
\int_{\mathbb{R}^d} \mathcal{E}_2^\sigma(m)^2e^{2q\tau(m)}m_k^2dm
\leq \int_{\mathbb{R}^d} \mathcal{E}_2^\sigma(m)^2e^{2q\tau(m)}\|m\|^{1/k_o}dm + \int_{\mathbb{R}^d} \mathcal{E}_2^\sigma(m)^2e^{2q\tau(m)}dm
$$
and

$$
\int_{\mathbb{R}^d} \mathcal{E}_2^\sigma(m)^2e^{2q\tau(m)}dm = \int_{\mathbb{R}^d} \exp\left(-\frac{D\|m\|^2}{A_{V, \Sigma}(0, t)}\phi_{2k_o}(m)\right)e^{2q\tau(m)}dm
\leq C\int_{\|m\| \leq 1} \exp\left(-\frac{D\|m\|^2}{2k_oA_{N, \Sigma}^\sigma(0, t)}\right)dm
+ C\int_{\|m\| > 1} \exp\left(-\frac{D\|m\|^2}{2k_oA_{N, \Sigma}^\sigma(0, t)}\right)\|m\|^{2r}dm
\leq CA_{N, \Sigma}^\sigma(0, t)^{dk_o/(k_o+1)} + C_rA_{N, \Sigma}^\sigma(0, t)^{(d+2r)k_o}.
$$
Exactly in the same way

$$\int_{\mathbb{R}^d} E_2^\sigma (m)^2 e^{2q r(m)} \|m\|^{1/k_o} dm \leq CA_{N,\Sigma}(0, t)^{(d/k_o + 1)/2} + CR A_{N,\Sigma}(0, t)^{(d+2r+1)/(k_o k_o)}.$$

Thus the expectation in (17) is bounded by a constant $C_r$ depending on $r$ (and therefore on $q$) times a sum of expectations of the form

$$E_0 A_{N,\Pi}(0, t)^{-1} A_{N,\Sigma}(0, t)^{\gamma} A_{N,\Sigma}(0, t)^{\delta}$$

with $\gamma, \delta > 0$. The above expectations are finite. This follows from the fact that the functionals $A_M$ and $A_V$ have finite moments (positive and negative) which follows from the fact that the exponential functionals $A_{\mu}$ defined in the Appendix in (19) have finite moments (see e.g. [6]).

The remaining integrals can be estimated in the same way.

**Remark 4.5.** Using asymptotic given in Theorem A.1 of the Appendix A one can bound, for $t > 1$, the constant $C_{t,q}$ in Lemma 4.4 and so get the estimate (6).

### 4.5. End of the proof of Theorem 1.1.

**Proof.** We now make use of the fact that $p_t = p_{t/2} * p_{t/2}$ to get an estimate for $p_t(e, 0; z, a)$. Specifically, let

$$\psi(y, b) = p_{t/2}(e, 0; y, b),$$

$$\tilde{\psi}(y, b) = p_{t/2}(e, 0; y, b) e^{b^2/8 t - r_0(b)/2} e^{q r(y)}.$$

Thus,

$$p_t(e, 0; z, a) = \int_S p_{t/2}(e, 0; y, b) p_{t/2}(e, 0; (y, b)^{-1} (z, a)) dS(y, b)$$

$$= \psi * \tilde{\psi}(z, a)$$

$$= \int_S \psi(y, b) \psi((y, b)^{-1} (z, a)) dS(y, b)$$

$$= \int_S \psi(y, b) \psi((y, b)^{-1} zb, a - b)) dS(y, b)$$

$$= \int_S \tilde{\psi}(y, b) \tilde{\psi}((y, b)^{-1} zb, a - b)) dS(y, b)$$

$$\times e^{-b^2/8 t - r_0(b)/2} e^{-q r((y, b)^{-1} zb)} dS(y, b),$$

where $dS(y, b)$ is the left-invariant Haar measure on $S$ (see (3)). Since

$$b^2 + (a - b)^2 \geq (\|b\| + \|a - b\|)^2/2 \geq a^2/2,$$
we get from Lemma 4.3 with $\gamma = -1/2$,

\[
p_t(e, 0; z, a) \leq e^{-\frac{a^2}{2} + \frac{\rho_0(a)}{2}} \int_S \tilde{\psi}(y, b)^2 e^{-q\tau(y)} \tilde{\psi}(b^{-1}y^{-1}zb, a - b) e^{-\frac{a^2}{4} e^{-q\tau(b^{-1}y^{-1}zb)}} dS(y, b) \leq c_{t, q} e^{-\frac{a^2}{(32t) + \rho_0(a)/2}} e^{-q\tau(z)} \int_S \tilde{\psi}(y, b)^2 e^{-q\tau(z)} dS(y, b)
\]

\[
= c_{t, q} e^{-\frac{a^2}{(32t) + \rho_0(a)/2}} e^{-q\tau(z)} \int_S \tilde{\psi}(y, b) \tilde{\psi}((y, b)^{-1}(z, a)) e^{-\rho_0(-b)/2} dS(y, b),
\]

where $c_{t, q} = e^{64C^2q^2t + 2qD}$.

For any function $f$ on $S$,

\[
\int_S f(s^{-1}) dS(s) = \int_S f(s) \chi(s) dS(s),
\]

where $\chi(s) = e^{\rho_0(s)}$ is the modular function for $dS(s)$. Thus

\[
\int_S \tilde{\psi}((y, b)^{-1}(z, a))^2 e^{-\rho_0(-b)} dS(y, b) = \int_S \tilde{\psi}((y, b)(z, a))^2 dS(y, b) = e^{\rho_0(a)} \|\tilde{\psi}\|_{L^2(dS)}^2.
\]

Hence from the Cauchy-Schwartz inequality

\[
p_t(e, 0; z, a) = \psi * \psi(z, a) \leq c_{t, q} e^{-a^2/(32t) + \rho_0(a)} e^{-q\tau(z)} \|\tilde{\psi}\|_{L^2(dS)}^2.
\]

By Lemma 4.2 applied to $\xi(y) = e^{q\tau(y)}$ and Lemma 4.4 with $q \geq \|\rho_o\|$,

\[
\|\tilde{\psi}\|_{L^2(dS)}^2 \leq C_{t, q}.
\]

Our theorem follows. \(\blacksquare\)

A. Exponential functionals of Brownian motion

Let $b_s, s \geq 0$, be the Brownian motion on $\mathbb{R}$ normalized so that $\text{Var} b_s = 2s$. For $\mu \in \mathbb{R}$ and $t > 0$, define the following exponential functional

\[
A_t^{(\mu)} = \int_0^t e^{2(b_s + \mu s)} ds. \tag{19}
\]

Exponential functionals $A_t^{(\mu)}$ have been thoroughly studied and their properties are very well known. The reason for such an interest is that these functionals play an important role in financial mathematics (see e.g. [15, 6, 7] and the references therein).

The following theorem is a simplified version of the result proved in [5] (all the constants are explicit in [5]; notice also that we use a different normalization of Brownian motion than in [5]). Let, for $(\mu, m) \in \mathbb{R}^2$,

\[
\Delta_t^{(\mu, m)} = E_0[(A_t^{(\mu)})^{-m}].
\]
Theorem A.1 ([5, Theorem 2.2]). The following limits exists:

\[
\begin{align*}
\lim_{t \to \infty} t^{3/2} e^{\mu^2 t} \Delta_t^{(\mu,m)} &= C_1 \text{ if } 2m > \mu, \mu > 0, \\
\lim_{t \to \infty} t^{1/2} e^{\mu^2 t} \Delta_t^{(\mu,m)} &= C_2 \text{ if } 2m = \mu, \mu > 0, \\
\lim_{t \to \infty} e^{4m(\mu-m)t} \Delta_t^{(\mu,m)} &= C_3 \text{ if } m < \mu, 2m < \mu, \\
\lim_{t \to \infty} t^{-1} \Delta_t^{(\mu,m)} &= C_4 \text{ if } m = \mu, \mu < 0, \\
\lim_{t \to \infty} \Delta_t^{(\mu,m)} &= C_5 \text{ if } m > \mu, \mu < 0, \\
\lim_{t \to \infty} t^{1/2} \Delta_t^{(\mu,m)} &= C_6 \text{ if } m > 0, \mu = 0.
\end{align*}
\]

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