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On Properties of a Fibonacci Restricted Lie Algebra

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Abstract. Let $R = K[t_i | i \geq 0]/(t_i^p | i \geq 0)$ be the truncated polynomial ring, where K is a field of characteristic 2. Let $\partial_i = \frac{\partial}{\partial t_i}$, $i \geq 1$, denote the respective derivations. Consider the operators

$$\begin{aligned} v_1 &= \partial_1 + t_0(\partial_2 + t_1(\partial_3 + t_2(\partial_4 + t_3(\partial_5 + t_4(\partial_6 + \cdots))))); \\ v_2 &= \qquad\quad \partial_2 + t_1(\partial_3 + t_2(\partial_4 + t_3(\partial_5 + t_4(\partial_6 + \cdots)))). \end{aligned}$$

Let $\mathcal{L} = \mathrm{Lie}(v_1, v_2)$ and $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2) \subset \operatorname{Der} R$ be the Lie algebra and the restricted Lie algebra generated by these derivations, respectively. These algebras were introduced by the first author and called Fibonacci Lie algebras. It was established that $\mathbf{L}$ has polynomial growth and a nil p-mapping. The latter property is a natural analogue of periodicity of Grigorchuk and Gupta-Sidki groups. We also proved that $\mathbf{L}$, the associative algebra generated by these derivations $\mathbf{A} = \mathrm{Alg}(v_1, v_2) \subset \mathrm{End}(R)$, and the augmentation ideal of the restricted enveloping algebra $u_0(\mathbf{L})$ are direct sums of two locally nilpotent subalgebras.

The goal of the present paper is to study Fibonacci Lie algebras in more details. We give a clear basis for the algebras $\mathbf{L}$ and $\mathcal{L}$. We find functional equations and recurrence formulas for generating functions of $\mathbf{L}$ and $\mathcal{L}$, also we find explicit formulas for these functions. We determine the center, terms of the lower central series, values of regular growth functions, and terms of the derived series of $\mathcal{L}$. We observed before that $\mathbf{L}$ is not just infinite dimensional. Now we introduce one more restricted Lie algebra $\mathbf{G} = \mathrm{Lie}_p(\partial_1, v_2)$ and prove that it is just infinite dimensional. Finally, we formulate open problems.
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1. Introduction

The famous finitely generated periodic group was constructed by Grigorchuk [6]. Similar groups were constructed by Gupta and Sidki [9], their groups act on

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trees. All these groups are self-similar, we refer the reader for further details and developments to [7]. The study of these groups lead to investigation of group rings and other related associative algebras [22]. In particular, there appeared self-similar associative algebras defined by matrices in a recurrent way [4, 5]. The recurrent matrices were also applied to number theoretical problems [1, 2]. Sidki suggested two examples of self-similar associative matrix algebras [23]. We construct more general families of self-similar associative algebras and study their properties in [18]. In particular, our example generalizes the second example of Sidki [23] and we obtain a realization of our examples of self-similar Lie algebras in terms of self-similar matrices [18].

Let X be a set in an associative algebra A, then by $\mathrm{Alg}(X)$ denote the associative subalgebra generated by X, similarly by $\mathrm{Lie}(X)$ (or $\mathrm{Lie}_p(X)$) denote the (restricted) Lie subalgebra generated by X. For necessary definitions and properties of restricted Lie algebras we refer the reader to [10, 24, 3]. Let L be a restricted Lie algebra. By $u(L)$ denote the *restricted enveloping algebra* of L and $u_0(L)$ the *augmentation ideal* of $u(L)$. Suppose that L is a restricted Lie algebra and $H \subset L$ a Lie subalgebra, i.e. H is a vector subspace that is closed under the Lie bracket. Then denote by H_p the *restricted hull* of H, which is the restricted subalgebra of L generated by H.

Recall the notion of growth. Let A be an associative (or Lie) algebra generated by a finite set X. Denote by $A^{(X,n)}$ the subspace of A spanned by all monomials in X of length not exceeding n. If A is a restricted Lie algebra, then we define $A^{(X,n)} = \langle\, [x_1,\dots,x_s]^{p^k} \mid x_i \in X,\ sp^k \le n\rangle_K$ [14]. One considers the regular *growth function*:

$$\gamma_A(n) = \gamma_A(X,n) = \dim_K A^{(X,n)}, \quad n \ge 1.$$

Define one more growth function

$$\lambda_A(X,n) = \gamma_A(X,n) - \gamma_A(X,n-1), \quad n \ge 1,$$

where $\gamma_A(X,0) = 0$. The growth functions clearly depend on the choice of the generating set X. It is easy to see that the exponential growth is the highest possible growth for Lie and associative algebras. The growth function $\gamma_A(n)$ is compared with the polynomial functions n^k, $k \in \mathbb{R}^+$, by computing the *upper and lower Gelfand-Kirillov dimensions* [11]:

$$\overline{\mathrm{GKdim}}\, A = \overline{\lim_{n\to\infty}} \frac{\ln \gamma_A(n)}{\ln n};$$
$$\underline{\mathrm{GKdim}}\, A = \underline{\lim_{n\to\infty}} \frac{\ln \gamma_A(n)}{\ln n}.$$

We shall mainly use a little bit different growth function. Namely, we consider a weight function $\mathrm{wt} : X \to \mathbb{R}^+$, and the *weight growth function* with respect to it

$$\tilde\gamma_A(r) = \dim_K \left\langle x_{i_1}\cdots x_{i_k} \;\middle|\; \sum_{j=1}^{k} \mathrm{wt}\, x_{i_j} \le r, k \ge 1 \right\rangle, \quad r \in \mathbb{R}^+.$$

The standard arguments prove that this growth function yields the same Gelfand-Kirillov dimensions [11]. We get the regular growth function if we assign to all elements X the same weight equal to 1.

2. Fibonacci restricted Lie algebra, its properties

In this paper we continue the study of the self-similar Lie algebra introduced by the first author in [15], see also further developments in [21], [17]. In this section we give main definitions, describe known results, and outline main results of the present paper.

Throughout the paper the ground field K is of characteristic 2. Consider the truncated polynomial ring $R = K[t_i | i \geq 0]/(t_i^p | i \geq 0)$. Let $\partial_i = \frac{\partial}{\partial t_i}$, $i \geq 0$, be the partial derivatives of this ring. Denote by $v(t)$ the action of $v \in \operatorname{Der} R$ on $t \in R$. We define the following two derivations of R:

$$\begin{aligned} v_1 &= \partial_1 + t_0(\partial_2 + t_1(\partial_3 + t_2(\partial_4 + t_3(\partial_5 + t_4(\partial_6 + \cdots))))); \\ v_2 &= \qquad\qquad \partial_2 + t_1(\partial_3 + t_2(\partial_4 + t_3(\partial_5 + t_4(\partial_6 + \cdots)))). \end{aligned}$$

The action on R and products of such operators are well-defined; these operators are so called *special derivations*, see [20, 17, 16]. Denote by $\tau : R \to R$ the endomorphism given by $\tau(t_i) = t_{i+1}$ for $i \geq 0$. Let also $\tau(\partial_i) = \partial_{i+1}$, $i \geq 1$. Then we can write these derivations recursively:

$$\begin{aligned} v_1 &= \partial_1 + t_0 \tau(v_1); \\ v_2 &= \tau(v_1). \end{aligned}$$

Let $\mathcal{L} = \operatorname{Lie}(v_1, v_2)$ be the Lie subalgebra of $\operatorname{Der} R$ generated by v_1 and v_2, called also the *Fibonacci Lie algebra* [15]. Similarly, define

$$v_i = \tau^{i-1}(v_1) = \partial_i + t_{i-1}(\partial_{i+1} + t_i(\partial_{i+2} + t_{i+1}(\partial_{i+3} + \cdots))), \quad i = 1, 2, \ldots.$$

We also can write

$$v_i = \partial_i + t_{i-1} v_{i+1}, \quad i = 1, 2, \ldots. \tag{1}$$

Lemma 2.1 ([15],[17]). *The following commutation relations hold in $\mathcal{L} = \operatorname{Lie}(v_1, v_2)$:*

1. $[v_i, v_{i+1}] = v_{i+2}$ *for* $i = 1, 2, \ldots$ *;*
2. $[v_i, v_{i+2}] = t_{i-1} v_{i+3}$ *for* $i = 1, 2, \ldots$ *;*
3. *in general, for all* $1 \leq i < j$ *we have*
$$[v_i, v_j] = \left(\prod_{i-1 \leq k \leq j-3} t_k \right) v_{j+1};$$
4. *for all* $n \geq 1$, $j \geq 0$ *we have the action*
$$v_n(t_j) = \begin{cases} t_{n-1} t_n \cdots t_{j-2}, & n < j, \\ 1, & n = j, \\ 0, & n > j; \end{cases}$$

5. *for all* $k, n \geq 1$

$$[\partial_n, v_k] = \begin{cases} t_{k-1}t_k \cdots t_{n-1}v_{n+2}, & k < n+1, \\ v_{n+2}, & k = n+1, \\ 0, & k > n+1. \end{cases}$$

Let $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2) \subset \mathrm{Der}\, R$ denote the *restricted subalgebra* generated by $\{v_1, v_2\}$, it will be also referred to as the *Fibonacci restricted Lie algebra*. Remark that $\mathbf{L} = \mathcal{L}_p$, the restricted hull of $\mathcal{L}$. We also consider the associative algebra generated by these derivations $\mathbf{A} = \mathrm{Alg}(v_1, v_2) \subset \mathrm{End}\, R$.

Lemma 2.2 ([15]). *Let* $\mathrm{char}\, K = 2$. *The restricted Lie algebra* $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2)$ *has the following* p*-mapping:*

$$v_i^2 = t_{i-1}v_{i+2}, \qquad i = 1, 2, \ldots. \tag{2}$$

The Fibonacci restricted Lie algebra is also known to have the following properties.

Theorem 2.3 ([15, 17]). *Let* $\mathrm{char}\, K = 2$, $\mathcal{L} = \mathrm{Lie}(v_1, v_2)$, $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2)$ *and* $\mathbf{A} = \mathrm{Alg}(v_1, v_2)$. *Denote* $\lambda = (1 + \sqrt{5})/2$. *Then*

1. $\mathrm{GKdim}\, \mathbf{L} = \underline{\mathrm{GKdim}}\, \mathbf{L} = \mathrm{GKdim}\, \mathcal{L} = \underline{\mathrm{GKdim}}\, \mathcal{L} = \ln 2 / \ln \lambda$;
2. $\mathrm{GKdim}\, \mathbf{A} = \underline{\mathrm{GKdim}}\, \mathbf{A} = 2 \ln 2 / \ln \lambda$;
3. $\mathbf{L}$ *has a nil-*p*-mapping.*
4. $\mathbf{L}$, $\mathbf{A}$, *and the augmentation ideal of the restricted enveloping algebra* $\mathbf{u} = u_0(\mathbf{L})$ *are direct sums of two locally nilpotent subalgebras*

$$\mathbf{L} = \mathbf{L}_+ \oplus \mathbf{L}_-, \quad \mathbf{A} = \mathbf{A}_+ \oplus \mathbf{A}_-, \quad \mathbf{u} = \mathbf{u}_+ \oplus \mathbf{u}_-.$$

In this paper we suggest a slight modification of $\mathbf{L}$, namely, we consider the restricted Lie algebra $\mathbf{G} = \mathrm{Lie}_p(\partial_1, v_2)$, and the respective hull $\mathbf{B} = \mathrm{Alg}(\partial_1, v_2)$. We can realize the algebras $\mathbf{L}$, $\mathbf{A}$ ($\mathbf{G}$ and $\mathbf{B}$ as well) in terms of self-similar matrices [18], thus answering the question of Grigorchuk posed to the authors. For more constructions and detailed definitions see [18].

Theorem 2.4 ([18]). *Let* $\mathrm{char}\, K = 2$. *Consider the ring of self-similar matrices* $\mathbf{C} = \mathrm{Alg}(s_1, r_1)$ *given by recursion:*

$$s_i = \begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix}_{(i)}; \qquad r_i = \begin{pmatrix} s_{i+1} & r_{i+1} \\ 0 & s_{i+1} \end{pmatrix}_{(i)}, \quad i \geq 1.$$

Then $\mathbf{C}$ *contains subalgebras:*

1. $\mathrm{Lie}_p(r_1, r_2)$, *isomorphic to the Fibonacci restricted Lie algebra* $\mathbf{L}$;

2. $\mathbf{D} = \mathrm{Alg}(r_1, r_2)$, *isomorphic to the hull* $\mathbf{A}$ *of* $\mathbf{L}$*;*

3. $\mathrm{Lie}_p(s_1, r_1)$, *isomorphic to* $\mathbf{G}$*;*

4. $\mathbf{C} = \mathrm{Alg}(s_1, r_1)$, *isomorphic to the hull* $\mathbf{B}$ *of* $\mathbf{G}$*.*

Let us also comment on further examples in case of arbitrary positive characteristic $p > 0$. First of all, one can consider the restricted Lie algebra generated by the same operators acting on the truncated polynomial ring R. Then the algebra $\mathbf{L}$ still has a nil p-mapping in case $p = 3$ [17], but for bigger characteristics the situation looks complicated. Another approach is to modify the operators as follows [21]:

$$\begin{aligned} v_1 &= \partial_1 + t_0^{p-1}(\partial_2 + t_1^{p-1}(\partial_3 + t_2^{p-1}(\partial_4 + t_3^{p-1}(\partial_5 + t_4^{p-1}(\partial_6 + \cdots))))), \\ v_2 &= \qquad\qquad \partial_2 + t_1^{p-1}(\partial_3 + t_2^{p-1}(\partial_4 + t_3^{p-1}(\partial_5 + t_4^{p-1}(\partial_6 + \cdots)))), \end{aligned}$$

and consider the restricted Lie algebra $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2)$. Remark that in case $p = 2$ we still have the same example. The virtue of this example is the following.

Theorem 2.5 ([21]). *Let* $\mathrm{char}\, K = p > 0$ *and* v_1, v_2 *as above. Then the restricted Lie algebra* $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2)$ *has a nil* p*-mapping.*

This property is analogous to the *periodicity* of the Grigorchuk and Gupta-Sidki groups [6, 9]. One more example in case of arbitrary characteristic was suggested in [21] and studied in details [19], see also [12].

The goal of the present paper is to study Fibonacci Lie algebras in more details. Let us briefly outline the structure of the paper. In Section 3 we describe weight functions, a $\mathbb{Z} \oplus \mathbb{Z}$-gradation and other known properties of the Fibonacci Lie algebra. In Section 4 we correct erroneous functional equation and recurrence formula of [17] for the generating function of $\mathbf{L}$ (Theorem 4.2). We also find a functional equation for the generating function of $\mathcal{L}$ (Theorem 4.3). We give a clear basis for algebras $\mathbf{L}$ and $\mathcal{L}$ (Section 5), this allows us to find explicit formulas for the generating functions of $\mathcal{L}$ and $\mathbf{L}$ (Theorem 5.7). In Section 6 we study different series for the algebra $\mathcal{L}$. We show that the center is trivial, determine terms of the lower central series (Theorem 6.4), values of regular growth functions (Corollary 6.5), and terms of the derived series (Theorem 6.6). Next, we introduce one more restricted Lie algebra $\mathbf{G} = \mathrm{Lie}_p(\partial_1, v_2)$, study its properties, and prove that it is just infinite dimensional (Theorem 7.8). Finally, we formulate open problems.

3. Gradation by weights

Throughout the whole paper we assume that $\mathrm{char}\, K = p = 2$. In this section we describe a $\mathbb{Z}\oplus\mathbb{Z}$-gradation of $\mathbf{L}$ by weights and describe its applications established before.

Lemma 3.1 ([17]). *Let* $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2)$, *set* $\lambda = (1+\sqrt{5})/2$, *and* $\bar\lambda = (1-\sqrt{5})/2$. *Introduce the weight and superweight functions*

$$\operatorname{wt} v_n = \operatorname{wt} \partial_n = -\operatorname{wt} t_n = \lambda^n,$$
$$\operatorname{swt} v_n = \operatorname{swt} \partial_n = -\operatorname{swt} t_n = \bar\lambda^{n-2}, \quad n = 1, 2, \ldots .$$

1. *We have a* $\mathbb{Z} \oplus \mathbb{Z}$*-gradation* $\mathbf{L} = \oplus_{a,b\ge 0} \mathbf{L}_{a,b}$, *where* $\mathbf{L}_{a,b}$ *is spanned by monomials with* a *factors* v_1 *and* b *factors* v_2.

2. *The functions are additive on products of homogeneous elements of* $\mathbf{L}$.

3. *Let* $v \in \mathbf{L}_{a,b}$, *where* $a, b \ge 0$. *Then*

$$\operatorname{wt} v = a\lambda + b\lambda^2, \qquad \operatorname{swt} v = -a\lambda + b.$$

Recall the idea of the proof. We introduce a grading on $\mathbf{L}$ such that v_i are homogeneous. Suppose that we have a weight function $\operatorname{wt} v_i = -\operatorname{wt} t_i = a_i \in \mathbb{R}$, $i = 1, 2 \ldots$. We want all summands in (1) to have the same weight, so we assume that

$$a_i = \operatorname{wt} v_i = \operatorname{wt} \partial_i = \operatorname{wt} t_{i-1} + \operatorname{wt} v_{i+1} = -a_{i-1} + a_{i+1}.$$

We get the Fibonacci recurrence relation $a_{i+1} = a_i + a_{i-1}$. Its two basic solutions yield the weight and superweight functions. We also consider the *vector weight function* for homogeneous elements:

$$\mathrm{Wt}(v) = (\operatorname{wt}(v), \operatorname{swt}(v)) \in \mathbb{R}^2, \quad v \in \mathbf{L}_{a,b}.$$

The vectors $\mathrm{Wt}(v_1) = (\lambda, -\lambda), \mathrm{Wt}(v_2) = (\lambda^2, 1) \in \mathbb{R}^2$ are linearly independent and form the *weight lattice*:

$$\Gamma = \mathbb{Z}\,\mathrm{Wt}(v_1) \oplus \mathbb{Z}\,\mathrm{Wt}(v_2) \subset \mathbb{R}^2. \tag{3}$$

Consider a homogeneous element $v \in \mathbf{L}_{a,b}$, $a, b \ge 0$, denote $\mathrm{Gr}(v) = (a, b) \in \mathbb{Z}^2 \subset \mathbb{R}^2$ and draw the respective vector on plane, see Figure 1 below. (we warn that in [17] the function $\mathrm{Gr}(*)$ was denoted as $\mathrm{Wt}(*)$).

We introduce a new coordinate system on the plane. Consider a point $(x, y) \in \mathbb{R}^2$, we define its new coordinates (ξ, η) that we also refer to as the *weight* and the *superweight*

$$\begin{array}{lclcl} \xi &=& \operatorname{wt}(x,y) &=& x\lambda + y\lambda^2, \\ \eta &=& \operatorname{swt}(x,y) &=& -x\lambda + y; \end{array} \qquad (x, y) \in \mathbb{R}^2.$$

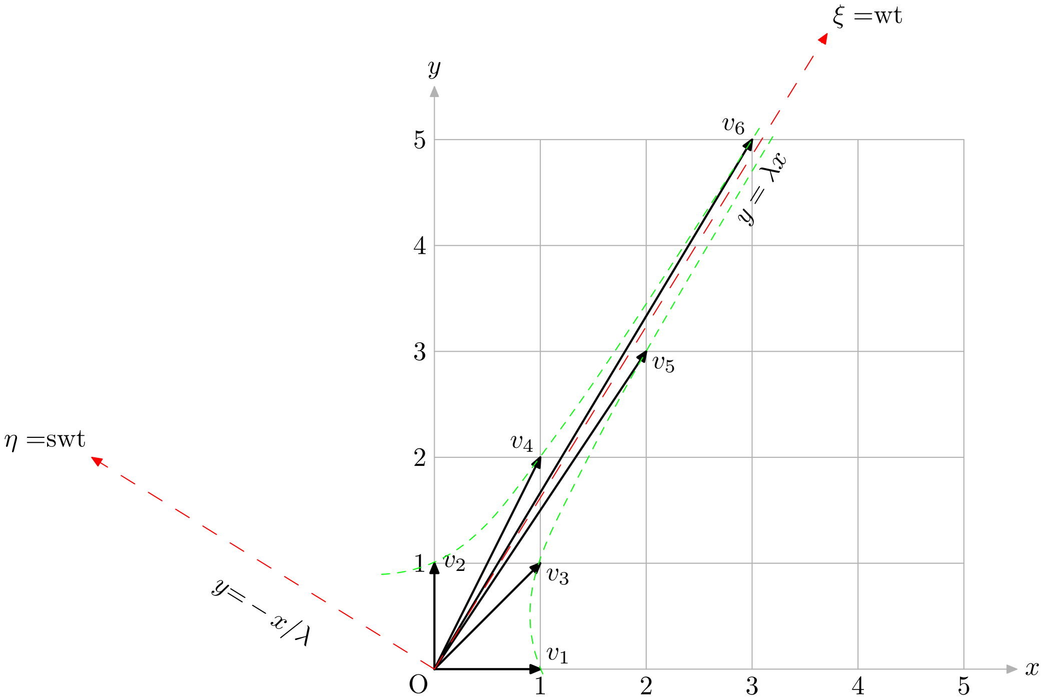
Let $v \in \mathbf{L}_{a,b}$ be a homogeneous element, then by Lemma 3.1, its new coordinates (ξ, η) coincide with the weight and the superweight functions introduced above. Since the superweight is an additive function, we obtain the following *triangular decomposition*.

Corollary 3.2 ([17]). *For* $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2)$, $\mathbf{A} = \mathrm{Alg}(v_1, v_2)$, *and the augmentation ideal of the restricted enveloping algebra* $\mathbf{u} = u_0(\mathbf{L})$ *we have decompositions into direct sums of two subalgebras as follows*

$$\mathbf{L} = \mathbf{L}_+ \oplus \mathbf{L}_-, \quad \mathbf{A} = \mathbf{A}_+ \oplus \mathbf{A}_-, \quad \mathbf{u} = \mathbf{u}_+ \oplus \mathbf{u}_-,$$

where, say, $\mathbf{L}_+ = \langle v \in \mathbf{L} \mid \operatorname{swt} v > 0 \rangle_K$, $\mathbf{L}_- = \langle v \in \mathbf{L} \mid \operatorname{swt} v < 0 \rangle_K$.

Figure 1: the elements v_i, $i = 1, 2, \dots,$ of $\mathcal{L} = \mathrm{Lie}(v_1, v_2)$



Let F_n, $n \geq 0$, be the Fibonacci numbers. We have $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$, $n \in \mathbb{Z}$. The following is known as Binet's formula:

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) = \frac{1}{\sqrt{5}}\left(\lambda^n - \bar{\lambda}^n\right), \quad n \geq 0.$$

Lemma 3.3 ([17]). *Let $v_n \in \mathbf{L}$ be as above. Then $\mathrm{Gr}(v_n) = (F_{n-2}, F_{n-1})$ and $\mathrm{Wt}(v_n) = (\lambda^n, \bar{\lambda}^{n-2})$ for $n \geq 1$.*

The next embedding follows from the multiplication rules, (see Lemma 2.1 and Lemma 2.2), moreover, it was established for an arbitrary characteristic [17].

Lemma 3.4 ([17]). *Consider the vector space*

$$H = \langle v_1, v_2, v_3,\ t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_{n-4}^{\alpha_{n-4}}\, v_n \mid n \geq 4, \alpha_i \in \{0,1\}\rangle_K. \tag{4}$$

Then H is a Lie subalgebra of $\mathrm{Der}\, R$ and $\mathcal{L} \subset H$.

It was shown that homogeneous elements of $\mathbf{L}$, $\mathbf{A}$ and the augmentation ideal of the restricted enveloping algebra $\mathbf{u} = u_0(\mathbf{L})$ lie in specific regions of the plane.

Theorem 3.5 ([17]). *Let $\mathrm{char}\, K = 2$, $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2)$, $\mathbf{A} = \mathrm{Alg}(v_1, v_2)$, and $\mathbf{u} = u_0(\mathbf{L})$. Consider a homogeneous element v of any of these algebras. Denote $\mathrm{Gr}(v) = (x, y)$, $\mathrm{Wt}(v) = (\xi, \eta)$. Then*

1. *the elements $v \in \mathbf{L}$ lie in the strip $\lambda x - \lambda^3 < y < \lambda x + \lambda^2$;*

2. *the elements* $v \in \mathbf{A}$ *lie in the strip* $\lambda x - \lambda^4 < y < \lambda x + \lambda^3$;
3. *set* $\theta = \ln 2/\ln\lambda$, $\kappa = \theta/(1+\theta) \approx 0.59$. *There exists* $C > 0$ *such that the elements* $v \in \mathbf{u}$ *lie in the region* $|\eta| < C\xi^\kappa$;
4. *the elements of all three algebras satisfy* $|\eta| \geq \lambda^2/\xi$.

The first claim can be seen on Figure 2 below. This observation yields the following interesting result.

Theorem 3.6 ([17]). *Let* $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2)$, $\mathbf{A} = \mathrm{Alg}(v_1, v_2)$, *and* $\mathbf{u} = u_0(\mathbf{L})$. *Then the decompositions above*

$$\mathbf{L} = \mathbf{L}_+ \oplus \mathbf{L}_-, \quad \mathbf{A} = \mathbf{A}_+ \oplus \mathbf{A}_-, \quad \mathbf{u} = \mathbf{u}_+ \oplus \mathbf{u}_-,$$

are decompositions into sums of two locally nilpotent subalgebras.

4. Generating functions of L and $\mathcal{L}$, functional equations

In this section we correct erroneous functional equation and recurrence formula [17] for the generating function of $\mathbf{L}$. We also find a functional equation for the generating function of $\mathcal{L}$.

Recall that $\mathcal{L} = \mathrm{Lie}(v_1, v_2) \subset \mathrm{Der}\, R$ is the Lie algebra (i.e. we use brackets only) generated by v_1, v_2 and $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2)$ the restricted Lie algebra generated by these elements. Suppose that H is a Lie subalgebra of a restricted Lie algebra L and $H_p \subset L$ its restricted hull. Let $\{h_i | i \in I\}$ be a basis of H, then it is well-known that it is sufficient to add the p-th powers of the basic elements, namely $H_p = H \oplus \langle h_i^{p^n} \mid i \in I, n \geq 1\rangle_K$ [10]. Remark that in our case $\mathbf{L} = \mathcal{L}_p$, we use (2), and that squares of other basic elements are trivial and $v_i^4 = 0$, $i \geq 1$. We get

$$\mathbf{L} = \mathcal{L} \oplus \langle t_{n-1} v_{n+2} \mid n \geq 1\rangle_K. \tag{5}$$

moreover, the some above is direct (see [17] or Corollary 5.2).

By Lemma 3.1, we have homogeneous decompositions according to multi-degrees with respect to the generating set $X = \{v_1, v_2\}$:

$$\mathbf{L} = \bigoplus_{a+b\geq 1} \mathbf{L}_{a,b}, \quad \mathcal{L} = \bigoplus_{a+b\geq 1} \mathcal{L}_{a,b}.$$

Consider the *Hilbert series in two variables*

$$\mathcal{H}_X(\mathbf{L}, x, y) = \sum_{a+b\geq 1} h_{ab} x^a y^b, \qquad h_{ab} = \dim \mathbf{L}_{a,b};$$
$$\mathcal{H}_X(\mathcal{L}, x, y) = \sum_{a+b\geq 1} h^*_{ab} x^a y^b, \qquad h^*_{ab} = \dim \mathcal{L}_{a,b}.$$

The subscript X will often be omitted. We also can consider the Hilbert series in one variable

$$\mathcal{H}(\mathbf{L}, z) = \mathcal{H}_X(\mathbf{L}, z, z), \quad \mathcal{H}(\mathcal{L}, z) = \mathcal{H}_X(\mathcal{L}, z, z).$$

Lemma 4.1.

$$\mathcal{H}(\mathbf{L}/\mathcal{L},x,y)=\sum_{n=0}^{\infty}x^{2F_{n-1}}y^{2F_n}=x^2+y^2+x^2y^2+x^2y^4+x^4y^6+x^6y^{10}+\cdots. \qquad (6)$$

Proof. Consider the powers in (5), by Lemma 3.3, we get $\mathrm{Gr}(t_{n-1}v_{n+2})=\mathrm{Gr}(v_n^2)=2\,\mathrm{Gr}(v_n)=(2F_{n-2},2F_{n-1})$, $n\geq 1$. The result follows. ∎

Theorem 4.2. *The Hilbert series of* $\mathbf{L}=\mathrm{Lie}_p(v_1,v_2)$ *satisfies the functional equation*

$$\mathcal{H}(\mathbf{L},x,y)=\mathcal{H}(\mathbf{L},y,xy)\left(1+\frac{x}{y}\right)-\sum_{n=0}^{\infty}x^{2F_n+1}y^{2F_{n+1}-1},$$

where $F_0=0,\ F_1=1,\ \dots,$ *are the Fibonacci numbers.*

Proof. Recall that $\mathbf{L}$ is generated by $X=\{v_1,v_2\}$. Let $B\subset\mathbf{L}$ be the restricted ideal generated by v_2 and v_1^2. Then $\mathbf{L}=\langle v_1\rangle_K\oplus B$ and it is well-known [3] that the algebra B is generated by the set

$$Y=\{v_2,[v_1,v_2],v_1^2\}=\{v_2,v_3,t_0v_3\}.$$

Consider homogeneous elements in Y. Remark that we can use t_0v_3 at most once. We have $\mathrm{wt}(v_1)=\lambda$, $\mathrm{wt}(v_2)=\lambda^2=1+\lambda$, $\mathrm{wt}(v_3)=1+2\lambda$, and $\mathrm{wt}(t_0v_3)=2\lambda$. Fix numbers $a,b\geq 0$. Let $P_{a,b,0}$ denote the space of homogeneous elements in $Y\setminus\{t_0v_3\}$ of degrees a, b with respect to v_2 and v_3, respectively, where $a+b\geq 1$. Then $\mathrm{wt}\,P_{a,b,0}=a\,\mathrm{wt}(v_2)+b\,\mathrm{wt}(v_3)=a(1+\lambda)+b(1+2\lambda)=b\lambda+(a+b)(1+\lambda)=b\,\mathrm{wt}(v_1)+(a+b)\,\mathrm{wt}(v_2)$. Hence,

$$P_{a,b,0}\subset\mathbf{L}_{b,a+b}. \qquad (7)$$

Similarly, let $P_{a,b,1}$ denote the space of homogeneous elements in $Y=\{v_2,v_3,t_0v_3\}$ of degrees a, b, and 1, respectively, where $a,b\geq 0$. We get $\mathrm{wt}\,P_{a,b,1}=a(1+\lambda)+b(1+2\lambda)+2\lambda=(b+2)\lambda+(a+b)(1+\lambda)=(b+2)\,\mathrm{wt}(v_1)+(a+b)\,\mathrm{wt}(v_2)$. Thus,

$$P_{a,b,1}\subset\mathbf{L}_{b+2,a+b}. \qquad (8)$$

We get a vector space decomposition

$$\mathbf{L}=\langle v_1\rangle_K\mathop{\oplus}_{a+b\geq 1}P_{a,b,0}\mathop{\oplus}_{a,b\geq 0}P_{a,b,1}. \qquad (9)$$

Recall that we have the embedding $\tau:\mathbf{L}\hookrightarrow\mathbf{L}$ given by $\tau(v_i)=v_{i+1}$ for $i\geq 1$. We have

$$P_{a,b,0}=\tau(\mathbf{L}_{a,b}),\qquad P_{a,b,1}=t_0\tau(\mathcal{L}_{a,b+1}). \qquad (10)$$

The first equality is clear. Let us explain the second one. (In we [17] we erroneously wrote that $P_{a,b,1}=t_0\tau(\mathbf{L}_{a,b+1})$, and this error led to the wrong functional equation.) The space $P_{a,b,1}$ is spanned by restricted monomials w in $\{v_2,v_3,t_0v_3\}$ of degrees $a,b,1$, respectively. Due to the degree one in t_0v_3 this monomial cannot be a pth power. So, w must be a Lie monomial, then $w=t_0w_0$, where w_0 is a

Lie monomial in $\{v_2, v_3\}$ of degrees a and $b+1$. We get $w = t_0\tau(w_1)$, where $w_1 \in \mathcal{L}_{a,b+1}$. Conversely, any element from $t_0\tau(\mathcal{L}_{a,b+1})$ is a Lie monomial and we can rewrite it as an element of $P_{a,b,1}$ by replacing one of elements v_3 by $t_0 v_3$.

From (9), (7), (8), and (10) we get

$$\begin{aligned}\mathcal{H}(\mathbf{L}, x, y) &= x + \sum_{a+b\geq 1} \dim P_{a,b,0}\, x^b y^{a+b} + \sum_{a,b\geq 0} \dim P_{a,b,1}\, x^{b+2} y^{a+b} \\ &= x + \sum_{a+b\geq 1} h_{a,b}\, y^a (xy)^b + \frac{x}{y} \sum_{a,b\geq 0} h^*_{a,b+1}\, y^a (xy)^{b+1} \\ &= x + \mathcal{H}(\mathbf{L}, y, xy) + \frac{x}{y} \sum_{a\geq 0, b\geq 1} h^*_{a,b}\, y^a (xy)^b. \end{aligned} \tag{11}$$

Let us compute $\sum_{a\geq 0, b\geq 1} h^*_{a,b}\, x^a y^b$. We take the generating function for the whole of $\mathbf{L}$ and subtract $\sum_{a\geq 0,\, b=0} h_{a,0} x^a = x + x^2$, which enumerates monomials in v_1 only, these are $\{v_1, v_1^2\}$. Also we subtract the generating function of pure pth powers (6) (except for the term x^2 that was subtracted before). We obtain

$$\sum_{a\geq 0, b\geq 1} h^*_{a,b}\, x^a y^b = \mathcal{H}(\mathbf{L}, x, y) - x - \sum_{n=0}^{\infty} x^{2F_{n-1}} y^{2F_n}. \tag{12}$$

Finally, we use (11), (12) and get

$$\begin{aligned}\mathcal{H}(\mathbf{L}, x, y) &= x + \mathcal{H}(\mathbf{L}, y, xy) + \frac{x}{y}\left(\left(\mathcal{H}(\mathbf{L}, x, y) - x - \sum_{n=0}^{\infty} x^{2F_{n-1}} y^{2F_n}\right)\Bigg|_{\substack{x=y,\\ y=xy}}\right) \\ &= \mathcal{H}(\mathbf{L}, y, xy)\left(1 + \frac{x}{y}\right) - \sum_{n=0}^{\infty} x^{2F_n+1} y^{2F_{n+1}-1}. \end{aligned}$$ ■

Theorem 4.3. *The Hilbert series for $\mathcal{L} = \mathrm{Lie}(v_1, v_2)$ satisfies the functional equation*

$$\mathcal{H}(\mathcal{L}, x, y) = \mathcal{H}(\mathcal{L}, y, xy)(1 + x/y) - x^2.$$

Proof. We use Lemma 4.1 and Theorem 4.2

$$\begin{aligned}\mathcal{H}(\mathcal{L}, x, y) &= \mathcal{H}(\mathbf{L}, x, y) - \sum_{n=0}^{\infty} x^{2F_{n-1}} y^{2F_n} = \\ &= \mathcal{H}(\mathbf{L}, y, xy)\left(1 + \frac{x}{y}\right) - \sum_{n=0}^{\infty} x^{2F_n+1} y^{2F_{n+1}-1} - \sum_{n=0}^{\infty} x^{2F_{n-1}} y^{2F_n} \\ &= \left(\mathcal{H}(\mathcal{L}, y, xy) + \sum_{n=0}^{\infty} y^{2F_{n-1}} (xy)^{2F_n}\right)\left(1 + \frac{x}{y}\right) - \sum_{n=0}^{\infty} x^{2F_n+1} y^{2F_{n+1}-1} - \sum_{n=0}^{\infty} x^{2F_{n-1}} y^{2F_n} \\ &= \mathcal{H}(\mathcal{L}, y, xy)(1 + x/y) - x^2. \end{aligned}$$ ■

Corollary 4.4. *Set $\mathcal{H}_1(x, y) = x + y$ and define recursively*

1. *$\mathcal{H}_{i+1}(x, y) = \mathcal{H}_i(y, xy)(1 + x/y) - \sum_{n=0}^{\infty} x^{2F_n+1} y^{2F_{n+1}-1}$, $i \geq 2$. Then the sequence of series $\mathcal{H}_i(x, y)$, $i \in \mathbb{N}$, converges in degree topology to $\mathcal{H}(\mathbf{L}, x, y)$.*

2. $\mathcal{H}_{i+1}(x,y)=\mathcal{H}_i(y,xy)(1+x/y)-x^2$, $i\geq 2$. *Then the sequence of series* $\mathcal{H}_i(x,y)$, $i\in\mathbb{N}$, *converges in degree topology to* $\mathcal{H}(\mathcal{L},x,y)$.

Proof. Consider the first claim only. Let $\mathcal{H}(\mathbf{L},x,y)=\sum_{a+b\geq 1}h_{a,b}x^a y^b$ and $\mathcal{H}_i(x,y)=\sum_{a+b\geq 1}h_{a,b}^{(i)}x^a y^b$. Let us prove by induction on $j=1,2,\dots$ that $\mathcal{H}(\mathbf{L},x,y)$ and $\mathcal{H}_{2j-1}(x,y)$ have the same coefficients $h_{n,m}=h_{n,m}^{(2j-1)}$ for $n+m\leq j$. Let $j=1$ then by Lemma 3.1 we have $\mathcal{H}(\mathbf{L},x,y)=x+y+\dots$ and the base of induction is true.

Let $j>1$. By Theorem and the recursive relation we have

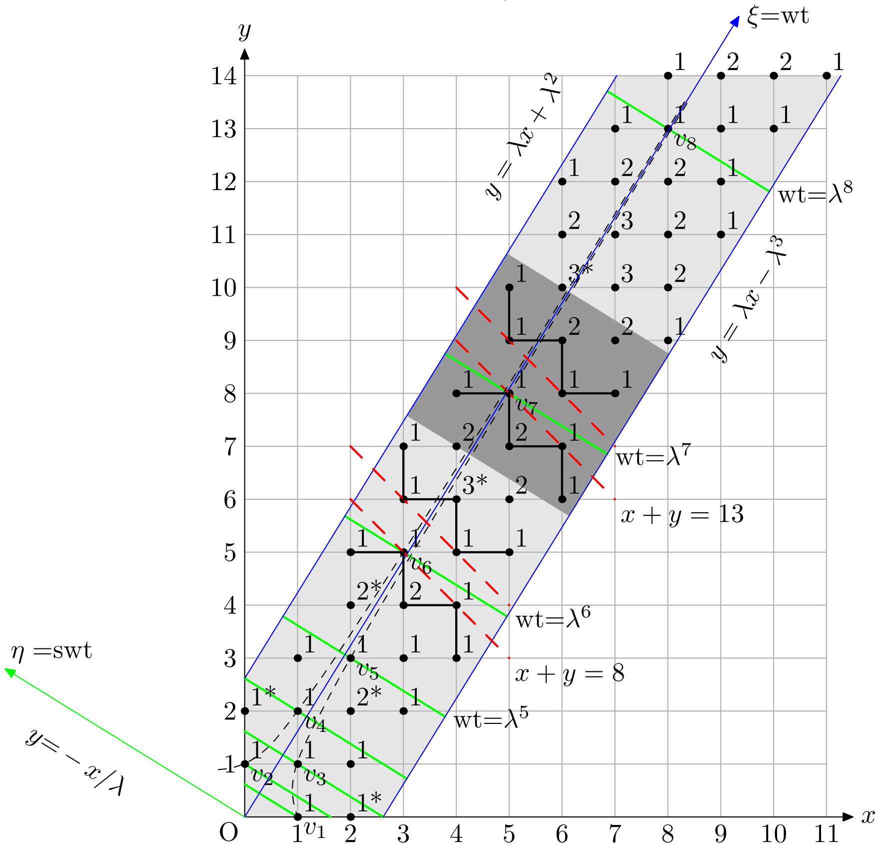
$$h_{n,m}=h_{m-n,n}+h_{m-n+2,n-1}-\delta_{n,2F_q+1}\delta_{m,2F_{q+1}-1},$$
$$h_{n,m}^{(i+1)}=h_{m-n,n}^{(i)}+h_{m-n+2,n-1}^{(i)}-\delta_{n,2F_q+1}\delta_{m,2F_{q+1}-1},\quad i\geq 1,$$

for all $n,m\geq 0$. Here $h_{a,b}$, $h_{a,b}^{(i)}$ are zero if either of indices a,b is negative. One checks that after the iteration the values $h_{0,1}=h_{1,0}=1$ remain the same. We compare sums of indices in the relations above. Assume that $n+m\geq 2$. We have $n+m>(m-n)+n$ if $n>0$ and $n+m>(m-n+2)+(n-1)$ if $n>1$. In case $n=0$ or $n=1$ we apply this relations again and see that sums of indices do decrease. Fix numbers n,m such that $2\leq n+m\leq j$ and consider coefficients $h_{n,m}$ and $h_{n,m}^{(2j-1)}$. We apply two iterations and conclude that they are expressed in the same way via $h_{a,b}$ and $h_{a,b}^{(2j-3)}$, respectively, where $a+b<n+m\leq j$ and the latter coefficients coincide by inductive assumption. Hence $h_{n,m}=h_{n,m}^{(2j-1)}$. The induction step is proved. ■

We use Corollary 4.4 and compute the first terms of the Hilbert series for $\mathbf{L}$ and $\mathcal{L}$. The coefficients below are dimensions of the respective homogeneous components depicted in Figure 2 and written by rows. We analyze these coefficients in the next section in more details.

$$\begin{aligned}\mathcal{H}(\mathbf{L},x,y)=x&+x^2\\&+y+xy+x^2y\\&+y^2+xy^2+2x^2y^2+x^3y^2\\&+xy^3+x^2y^3+x^3y^3+x^4y^3\\&+2x^2y^4+2x^3y^4+x^4y^4\\&+x^2y^5+x^3y^5+x^4y^5+x^5y^5\\&+x^3y^6+3x^4y^6+2x^5y^6+x^6y^6\\&+x^3y^7+2x^4y^7+2x^5y^7+x^6y^7\\&+x^4y^8+x^5y^8+x^6y^8+x^7y^8\\&+x^5y^9+2x^6y^9+2x^7y^9+x^8y^9\\&+x^5y^{10}+3x^6y^{10}+3x^7y^{10}+2x^8y^{10}\\&+2x^6y^{11}+3x^7y^{11}+2x^8y^{11}+x^9y^{11}\\&+x^6y^{12}+2x^7y^{12}+2x^8y^{12}+x^9y^{12}\\&+x^7y^{13}+x^8y^{13}+x^9y^{13}+x^{10}y^{13}+\cdots\end{aligned}$$

Figure 2: Components $\mathbf{L}_{a,b}$, those with v_i^2 are marked by $*$



5. Generating functions of $\mathcal{L}$ and L, explicit formulas

The next description of a basis of $\mathcal{L}$ strengthens Lemma 3.4. Introduce sets

$$
\begin{aligned}
T_n &= \begin{cases} \{v_n\}, & n = 1,2,3; \\ \{t_0^{\alpha_0}\cdots t_{n-4}^{\alpha_{n-4}}\, v_n \mid \alpha_i \in \{0,1\}\}, & n \ge 4; \end{cases}\\
\tilde{T}_n &= \begin{cases} \{v_n\}, & n = 1,2; \\ T_n \cup \{t_{n-3}\, v_n\}, & n \ge 3; \end{cases}\\
T &= \cup_{n=1}^{\infty} T_n;\\
\tilde{T} &= \cup_{n=1}^{\infty} \tilde{T}_n.
\end{aligned}
$$

Lemma 5.1. *The Lie algebra* $\mathcal{L} = \mathrm{Lie}(v_1, v_2)$ *has the basis*

$$T = \{v_1, v_2, v_3,\ t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_{n-4}^{\alpha_{n-4}}\, v_n \mid n \ge 4, \alpha_i \in \{0,1\}\}.$$

Proof. By Lemma 3.4, $\mathcal{L}$ is contained in the linear span of these elements. Let us check by induction on n that $T_n \subset \mathcal{L}$. Clearly $T_1, T_2, T_3 \subset \mathcal{L}$. Let $n \ge 3$. By inductive assumption $T_n = \{t_0^{\alpha_0}\cdots t_{n-4}^{\alpha_{n-4}}\, v_n \mid \alpha_i \in \{0,1\}\} \subset \mathcal{L}$. We multiply these elements by $v_{n-1}, v_{n-2} \in \mathcal{L}$:

$$[v_{n-1}, t_0^{\alpha_0}\cdots t_{n-4}^{\alpha_{n-4}}\, v_n] = t_0^{\alpha_0}\cdots t_{n-4}^{\alpha_{n-4}}\, v_{n+1} \in \mathcal{L}; \qquad (13)$$

$$[v_{n-2}, t_0^{\alpha_0}\cdots t_{n-4}^{\alpha_{n-4}}\, v_n] = t_0^{\alpha_0}\cdots t_{n-4}^{\alpha_{n-4}} t_{n-3}\, v_{n+1} \in \mathcal{L}. \qquad (14)$$

In this way we can obtain all the elements $T_{n+1} \subset \mathcal{L}$.

One can check linear independence of our elements by considering of the action on variables $t_i \in R$, $i \geq 1$. ■

Corollary 5.2. *The restricted Lie algebra* $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2)$ *has the basis*

$$\tilde{T} = \{v_1, v_2, v_3, t_0^{\alpha_0} t_1^{\alpha_1} \cdots t_{n-4}^{\alpha_{n-4}} v_n \mid n \geq 4, \alpha_i \in \{0,1\}\} \cup \{t_{n-3} v_n \mid n \geq 3\}.$$

Proof. We use (5). ■

Lemma 5.3. *The sets* T_n*,* $n \geq 1$*, are separated by weights as follows*

$$\lambda^{n-1} < \mathrm{wt}(T_n) \leq \lambda^n.$$

Proof. Let $v = t_0^{\alpha_0} \cdots t_{n-4}^{\alpha_{n-4}} v_n \in T_n$. Then $\mathrm{wt}(v) \leq \mathrm{wt}(v_n) = \lambda^n$. On the other hand

$$\mathrm{wt}(v) \geq \lambda^n - \sum_{i=0}^{n-4} \lambda^i > \lambda^n - \frac{\lambda^{n-4}}{1 - 1/\lambda} = \lambda^n - \frac{\lambda^{n-2}}{\lambda^2 - \lambda} = \lambda^n - \lambda^{n-2} = \lambda^{n-1}.$$ ■

Corollary 5.4. *Let* $v \in T_m$ *and* $\mathrm{wt}\, v = \lambda^n$ *then* $n = m$ *and* $v = v_n$*.*

Thus, the line $\mathrm{wt}(x, y) = \lambda x + \lambda^2 y = \lambda^n$ passes through v_n, where $n \geq 1$, and all these parallel lines separate the sets T_n, $n \geq 1$, see Figure 2. Next, we observe that the sets T_n can be decomposed into *lower* and *upper parts*.

Lemma 5.5. *For all* $n \geq 4$ *we have*

1. $T_n = [v_{n-3}, T_{n-1}] \cup [v_{n-2}, T_{n-1}]$*, where all elements on the right hand side are different;*

2. $2\lambda^{n-2} < \mathrm{wt}([v_{n-2}, T_{n-1}]) \leq \lambda^n$ *(the* upper part*);*

3. $\lambda^{n-1} < \mathrm{wt}([v_{n-3}, T_{n-1}]) \leq \lambda^{n-3} + \lambda^{n-1}$ *(the* lower part*).*

Proof. The first claim follows from (13), (14). By Lemma 5.3, $\lambda^{n-2} < \mathrm{wt}(T_{n-1}) \leq \lambda^{n-1}$. Since $\mathrm{wt}(v_{n-2}) = \lambda^{n-2}$, $\mathrm{wt}(v_{n-3}) = \lambda^{n-3}$, we get

$$2\lambda^{n-2} < \mathrm{wt}([v_{n-2}, T_{n-1}]) \leq \lambda^{n-2} + \lambda^{n-1} = \lambda^n;$$
$$\lambda^{n-1} = \lambda^{n-3} + \lambda^{n-2} < \mathrm{wt}([v_{n-3}, T_{n-1}]) \leq \lambda^{n-3} + \lambda^{n-1}.$$ ■

Remark 5.6. The terminology *upper, lower* refers to position of the sets relative to the weight function $\mathrm{wt}(*)$ despite the fact that there is an overlap because of $\lambda^{n-3} + \lambda^{n-1} > 2\lambda^{n-2}$.

Theorem 5.7. *Let* $\mathcal{L} = \mathrm{Lie}(v_1, v_2)$. *Then*

$$\mathcal{H}(\mathcal{L}, x, y) = x + y + xy\left(1 + \sum_{n=0}^{\infty}\prod_{m=0}^{n}(x^{F_{m-1}}y^{F_m} + x^{F_m}y^{F_{m+1}})\right);$$

$$\mathcal{H}(\mathcal{L}, z) = 2z + z^2\left(1 + \sum_{n=0}^{\infty}\prod_{m=0}^{n} z^{F_{m+1}}(1 + z^{F_m})\right).$$

Proof. By Lemma 3.3, $\mathrm{Gr}(v_n) = (F_{n-2}, F_{n-1})$ for $n \geq 1$ and $\mathcal{H}(v_n, x, y) = x^{F_{n-2}}y^{F_{n-1}}$. Since $T_i = \{v_i\}$ for $i = 1, 2, 3$, we have

$$\mathcal{H}(T_1, x, y) = x, \quad \mathcal{H}(T_2, x, y) = y, \quad \mathcal{H}(T_3, x, y) = xy. \tag{15}$$

We use Lemma 5.5 and obtain for $n \geq 4$

$$\begin{aligned}
\mathcal{H}(T_n, x, y) &= \mathcal{H}(T_{n-1}, x, y)(\mathcal{H}(v_{n-3}, x, y) + \mathcal{H}(v_{n-2}, x, y)) \\
&= \mathcal{H}(T_{n-1}, x, y)(x^{F_{n-5}}y^{F_{n-4}} + x^{F_{n-4}}y^{F_{n-3}}) \\
&= \mathcal{H}(T_3, x, y)\prod_{m=4}^{n}(x^{F_{m-5}}y^{F_{m-4}} + x^{F_{m-4}}y^{F_{m-3}}) \\
&= xy\prod_{m=0}^{n-4}(x^{F_{m-1}}y^{F_m} + x^{F_m}y^{F_{m+1}}).
\end{aligned} \tag{16}$$

Since $\mathcal{H}(\mathcal{L}, x, y) = \sum_{n=1}^{\infty}\mathcal{H}(T_n, x, y)$, the result follows. ∎

We can rewrite the formula in other way.

Corollary 5.8.

$$\mathcal{H}(\mathcal{L}, x, y) = x + y + xy\Bigg(1 + (x{+}y)\Bigg(1 + (y{+}xy)\Bigg(1 + (xy{+}xy^2)\Bigg(1 + (xy^2{+}x^2y^3)\Bigg(1 + (x^2y^3{+}x^3y^5)\Bigg(1 + (x^3y^5{+}x^5y^8)\Bigg(1 + \cdots\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg)\Bigg).$$

Remark 5.9. An easy check shows that this function indeed satisfies the functional equation of Theorem 4.3.

Corollary 5.10.

$$\mathcal{H}(\mathbf{L}, x, y) = x + y + xy\left(1 + \sum_{n=0}^{\infty}\prod_{m=0}^{n}(x^{F_{m-1}}y^{F_m} + x^{F_m}y^{F_{m+1}})\right) + \sum_{n=0}^{\infty}x^{2F_{n-1}}y^{2F_n}.$$

Proof. Follows by Lemma 4.1. ∎

Corollary 5.11.

$$\mathcal{H}(\mathcal{L}, z) = 2z + z^2 + 2z^3 + 2\sum_{n=5}^{\infty} z^{1+F_{n-1}}\prod_{m=1}^{n-4}(1 + z^{F_m}).$$

Proof. The identity $F_1 + \cdots + F_n = F_{n+2} - 1$, $n \geq 1$ is well-known. We use (16):

$$\mathcal{H}(T_n, z) = z^2 \prod_{m=0}^{n-4} (z^{F_{m+1}} + z^{F_{m+2}}) = z^2 \prod_{m=0}^{n-4} z^{F_{m+1}} (1 + z^{F_m})$$
$$= z^{2+F_1+\cdots+F_{n-3}} \prod_{m=0}^{n-4} (1 + z^{F_m}) = 2z^{1+F_{n-1}} \prod_{m=1}^{n-4} (1 + z^{F_m}), \quad n \geq 4. \qquad \blacksquare$$

The first coefficients are:

$$\begin{aligned}
\mathcal{H}(\mathcal{L}, z) &= 2z + z^2 + 2z^3 + 2z^4 + 2z^5 + 2z^6 + 4z^7 + 2z^8 \\
&+ 2z^9 + 4z^{10} + 4z^{11} + 4z^{12} + 2z^{13} \qquad \text{(the set } T_7) \\
&+ 2z^{14} + 4z^{15} + 4z^{16} + 6z^{17} + 6z^{18} + 4z^{19} + 4z^{20} + 2z^{21} \qquad \text{(the set } T_8) \\
&+ 2z^{22} + 4z^{23} + 4z^{24} + 6z^{25} + 6z^{26} + 6z^{27} + 8z^{28} \\
&\qquad + 6z^{29} + 6z^{30} + 6z^{31} + 4z^{32} + 4z^{33} + 2z^{34} \qquad \text{(the set } T_9) \\
&+ 2z^{35} + 4z^{36} + 4z^{37} + 6z^{38} + 6z^{39} + 6z^{40} + 8z^{41} + 6z^{42} + 8z^{43} + 10z^{44} + 8z^{45} \\
&\qquad + 10z^{46} + 8z^{47} + 6z^{48} + 8z^{49} + 6z^{50} + 6z^{51} + 6z^{52} + 4z^{53} + 4z^{54} + 2z^{55} + \ldots
\end{aligned} \tag{17}$$

We also compute some coefficients of the growth function in Corollary 6.5.

6. Lower central series, derived series and growth for $\mathcal{L}$

By $Z(L)$ denote the *center* of a Lie algebra L.

Lemma 6.1. *Let* $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2)$. *Then* $Z(\mathbf{L}) = \{0\}$.

Proof. Since $\mathbf{L}$ is $\mathbb{Z} \oplus \mathbb{Z}$-graded, we conclude that $Z = Z(\mathbf{L})$ is also $\mathbb{Z} \oplus \mathbb{Z}$-graded. Assume that $0 \neq v \in Z_{a,b}$, where $Z_{a,b} = Z \cap \mathbf{L}_{a,b}$, $a + b \geq 1$. Let $n \in \mathbb{N}$ be such that $\lambda^{n-1} < \mathrm{wt}(v) = a\lambda + b\lambda^2 \leq \lambda^n$. By Lemma 5.1, we have a basis of the homogeneous component:

$$\mathbf{L}_{a,b} = \langle w = t_0^{\alpha_0} \cdots t_{n-3}^{\alpha_{n-3}} v_n \mid w \in \tilde{T}_n,\ \mathrm{Gr}(w) = (a, b) \rangle.$$

The relation similar to (13) implies that $[v_{n-1}, v] \neq 0$. Thus, $Z = \{0\}$. $\blacksquare$

We consider the basic monomials $\cdots v_n$ for $\mathcal{L}$ and $\mathbf{L}$ described in the previous section. Let $m \geq 1$, and consider those with $n \geq m$:

$$\begin{aligned}
\mathcal{L}_m &= \langle t_0^{\alpha_0} \cdots t_{n-4}^{\alpha_{n-4}} v_n \mid n \geq m, \alpha_i \in \{0, 1\} \rangle; \\
\mathbf{L}_m &= \langle t_0^{\alpha_0} \cdots t_{n-4}^{\alpha_{n-4}} v_n;\ t_{k-3} v_k \mid n \geq m, k \geq m, k \geq 3, \alpha_i \in \{0, 1\} \rangle.
\end{aligned}$$

Lemma 6.2. *1. We get the chain of ideals* $\mathcal{L} = \mathcal{L}_1 \triangleright \mathcal{L}_2 \triangleright \cdots \triangleright \mathcal{L}_m \triangleright \cdots$, *and the chain of restricted ideals* $\mathbf{L} = \mathbf{L}_1 \triangleright \mathbf{L}_2 \triangleright \cdots \triangleright \mathbf{L}_m \triangleright \cdots$;

2. $\mathcal{L}_m = \langle \cup_{n \geq m} T_n \rangle_K$ *for* $m \geq 1$;

3. $\mathcal{L}_m = \langle v \in \mathcal{L} \mid \operatorname{wt}(v) > \lambda^{m-1} \rangle_K$ *for* $m \geq 1$*;*

4. $$\dim \mathcal{L}_m/\mathcal{L}_{m+1} = \begin{cases} 2^{m-3}, & m \geq 3; \\ 1, & m = 1, 2. \end{cases}$$

5. $\dim \mathcal{L}/\mathcal{L}_{m+1} = 2^{m-2} + 1$ *for* $m \geq 2$*.*

Proof. The first claim follows by the multiplication rules. The second claim follows from definitions of $\mathcal{L}_m$ and T_m. The second claim and Lemma 5.3 imply the third. To prove the last two claims observe that $\dim \mathcal{L}_m/\mathcal{L}_{m+1} = |T_m| = 2^{m-3}$ for $m \geq 3$. ■

Consider the *lower central series* $\mathcal{L}^1 = \mathcal{L}$, $\mathcal{L}^{n+1} = [\mathcal{L}, \mathcal{L}^n]$, $n \geq 1$.

Lemma 6.3. *The terms of the lower central series are*

$$\mathcal{L}^n = \bigoplus_{a+b \geq n} \mathcal{L}_{a,b}, \quad n \geq 1.$$

Proof. Recall that $\mathcal{L}^n$ is spanned by all products in $\{v_1, v_2\}$ of length at least n [3]. ■

Some terms can be described more clearly.

Theorem 6.4. *Let* $\mathcal{L} = \operatorname{Lie}(v_1, v_2)$*, and* F_n*,* $n \geq 0$*, be the Fibonacci numbers, then some terms of the lower central series are as follows*

1. $\mathcal{L}^2 = \mathcal{L}_3$*;*

2. $\mathcal{L}^{F_n} = \mathcal{L}_{n+1} \oplus \langle v_n, t_0 v_n \rangle_K$*,* $n \geq 4$*;*

3. $\mathcal{L}^{F_n+1} = \mathcal{L}_{n+1}$*,* $n \geq 2$*;*

4. $\dim \mathcal{L}/\mathcal{L}^{F_n} = 2^{n-2} - 1$*,* $n \geq 4$*;*

5. $\dim \mathcal{L}/\mathcal{L}^{F_n+1} = 2^{n-2} + 1$*,* $n \geq 2$*.*

Proof. The first claim is a partial case of the third one ($n = 2$).

Let an integer lattice point (a, b) be marked by c on Figure 2. This means that $\dim \mathbf{L}_{a,b} = c$, or equivalently, choose n such that $\lambda^{n-1} < \operatorname{wt}(a, b) \leq \lambda^n$, then $|\{v \in \tilde{T}_n \mid \operatorname{Gr}(v) = (a, b)\}| = c$. The next arguments are illustrated by Figure 2 for $n = 7$. Namely, the reader is recommended to observe the respective lattice points in the vicinity of v_7.

We apply Lemma 5.5. The upper part of T_n is $[v_{n-2}, T_{n-1}]$, and the lower part of T_{n+1} is $[v_{n-2}, T_n]$. These two sets are around v_n and consist of the elements of the form $v = [v_{n-2}, w]$. We want to consider a part of these two sets. We want for sure a) to cut from T_n its lower part, satisfying: $\operatorname{wt}([v_{n-3}, T_{n-1}]) \leq \lambda^{n-3} + \lambda^{n-1}$

and b) to cut from T_{n+1} its upper part, satisfying: $\mathrm{wt}([v_{n-1},T_n]) > 2\lambda^{n-1}$. Namely, we consider the smaller area around v_n:

$$\lambda^{n-1}+\lambda^{n-3} < \mathrm{wt}(v) \le 2\lambda^{n-1} \tag{18}$$

(this neighbourhood of v_7 is additionally shaded on Figure 2). Then by our construction, all basic elements v around v_n (18) are of the form $v=[v_{n-2},w]$, where the basic elements w's come from the following neighborhood of v_{n-1}:

$$\lambda^{n-1}+\lambda^{n-3}-\lambda^{n-2} < \mathrm{wt}(w) \le 2\lambda^{n-1}-\lambda^{n-2}. \tag{19}$$

Thus, the neighbourhood (18) of v_n is obtained from the neighbourhood (19) of v_{n-1} by translation along v_{n-2}, which is almost parallel to the axis ξ. To illustrate this effect on Figure 2, we draw thick line segments through boundary points of the respective lower and upper parts around v_6, v_7. We clearly see that one pair of line segments is obtained from another one by translation.

Consider two half-planes specified by two inequalities

$$x+y \ge F_n, \tag{20}$$

$$\mathrm{wt}(x,y) = \lambda x+\lambda^2 y > \mathrm{wt}(v_n) = \lambda^n. \tag{21}$$

By Lemma 3.3, $v_n \in \mathcal{L}_{F_{n-2},F_{n-1}}$, since $F_{n-1}+F_{n-2}=F_n$, the point v_n lies on the first line (such lines are dashed on the picture). Thus, both lines pass through v_n. Moreover, by Corollary 5.4, v_n is the only basic element belonging to the second line, also, $T_n \setminus \{v_n\}$ lies below whereas T_{n+1} lies above the second line. The picture shows that one more point of T_n belongs to the first line (20), namely $(a,b)=\mathrm{Gr}(v_n)+(1,-1)$. Since $\mathrm{Gr}(t_0)=(1,-1)$ we see that $t_0v_n \in \mathcal{L}_{a,b}$. Two homogeneous components containing v_n, and t_0v_n are one-dimensional, because for small n the picture around v_n is computed, for bigger n's it is shifted almost along the axis ξ. Therefore, the area specified by the first inequality (20) contains two more points $\{v_n, t_0v_n\}$ than the area specified by the second inequality (21), where $n \ge 4$.

By Lemma 6.3, the first inequality (20) describes the ideal $\mathcal{L}^{F_n}$. By Claim 3 of Lemma 6.2, the second inequality (21) describes the ideal $\mathcal{L}_{n+1}$. Therefore, $\mathcal{L}^{F_n} = \mathcal{L}_{n+1} \oplus \langle v_n, t_0v_n\rangle_K$, the second claim is proved.

Let us prove the third claim. Again, consider the second inequality (21) and the third inequality, defining $\mathcal{L}^{F_n+1}$ (the respective third lines are also drawn dashed on the picture):

$$x+y \ge F_n+1. \tag{22}$$

Using the picture, we see that the second (21) and third (22) inequalities yield the same points, namely $\cup_{m\ge n+1}T_m$, for any $n \ge 2$. Hence, $\mathcal{L}^{F_n+1} = \mathcal{L}_{n+1} = \langle \cup_{m\ge n+1}T_m\rangle$.

The last two claims follow from Claim 5 of Lemma 6.2. ■

Now let us consider the *regular growth functions* $\gamma_{\mathcal{L}}(X,n)$ and $\lambda_{\mathcal{L}}(X,n)$ with respect to the generating set $X=\{v_1,v_2\}$. In particular, the next statement shows that $\lambda_{\mathcal{L}}(X,n)$, $n\ge 1$, is not a monotone sequence, these numbers are the coefficients of the series (17).

Corollary 6.5. *We compute the regular growth functions at the following points:*

1. $\gamma_{\mathcal{L}}(X, F_n) = 2^{n-2} + 1\,,\ n \geq 2\,;$
2. $\lambda_{\mathcal{L}}(X, F_n) = \lambda_{\mathcal{L}}(X, F_n + 1) = 2\,,\ n \geq 4\,.$
3. $\lambda_{\mathcal{L}}(X, F_n - 1) = 4\,,\ n \geq 6\,.$
4. $\lambda_{\mathcal{L}}(X, F_n + 2) = 4\,,\ n \geq 5\,.$

Proof. The grading yields $\gamma_{\mathcal{L}}(X, F_n) = \dim \mathcal{L}/\mathcal{L}^{F_n+1} = 2^{n-2} + 1$ for $n \geq 2$.

As we remarked above, the line $x + y = F_n$ (drawn dashed on Figure 2) contains two basic points, namely, $\{v_n, t_0 v_n\}$, hence $\lambda_{\mathcal{L}}(X, F_n) = 2$, where $n \geq 4$. The picture shows that the parallel line $x+y = F_n+1$ (also drawn dashed), where $n \geq 4$, also contains two points. The picture is computed for $n = 4, 5, 6, 7$ and for bigger ns the picture around v_n is obtained by the shift from the picture around v_{n-1}. So, $\lambda_{\mathcal{L}}(X, F_n + 1) = 2\,,\ n \geq 4$.

To check the last two claims we just consider two more parallel lines. ∎

Next we study the *derived series*: $\mathcal{L}^{(0)} = \mathcal{L}$, $\mathcal{L}^{(n+1)} = [\mathcal{L}^{(n)}, \mathcal{L}^{(n)}]$, for $n \geq 0$.

Theorem 6.6. *Let* $\mathcal{L} = \mathrm{Lie}(v_1, v_2)$. *The terms of the derived series are as follows:*

1. $\mathcal{L}^{(1)} = \mathcal{L}_3\,;$
2. $\mathcal{L}^{(n)} = \langle t_0^{\alpha_0} \cdots t_{2n-4}^{\alpha_{2n-4}} v_{2n+1} \mid \alpha_j \in \{0,1\}\rangle \oplus \mathcal{L}_{2n+2}$ *for* $n \geq 2\,;$
3. $\dim \mathcal{L}/\mathcal{L}^{(n)} = 3 \cdot 2^{2n-3} + 1$ *for* $n \geq 2\,.$

Proof. The first claim follows by Theorem 6.4 $\mathcal{L}^{(1)} = \mathcal{L}^2 = \mathcal{L}_3$.

Let the Claim 2 be true for $n \geq 1$. Denote

$$\bar{T}_{2n+1} = \{t_0^{\alpha_0} \cdots t_{2n-4}^{\alpha_{2n-4}} v_{2n+1} \mid \alpha_j \in \{0,1\}\}.$$

By the inductive assumption, $\mathcal{L}^{(n)} = \langle \bar{T}_{2n+1} \cup_{i \geq 2n+2} T_i \rangle_K$. Then we have

$$\begin{aligned}\mathcal{L}^{(n+1)} &= [\mathcal{L}^{(n)}, \mathcal{L}^{(n)}] \subset \langle [\bar{T}_{2n+1}, T_{2n+2}] \underset{j \geq 2n+3}{\cup} [T_{2n+1}, T_j] \underset{2n+2 \leq i < j}{\cup} [T_i, T_j] \rangle \\ &\subset \langle \bar{T}_{2n+3} \underset{j \geq 2n+4}{\cup} T_j \rangle = \langle \bar{T}_{2n+3} \rangle \oplus \mathcal{L}_{2n+4},\end{aligned}$$

where the equality $[\bar{T}_{2n+1}, T_{2n+2}] = \bar{T}_{2n+3}$ is easily checked. Thus, the elements of $\mathcal{L}^{(n+1)}$ are expressed via the required monomials.

Let us prove the reverse inclusion. By inductive assumption, the sets $\{v_i \mid i \geq 2n+1\}$ and $\cup_{i \geq 2n+2} T_i$ belong to $\mathcal{L}^{(n)}$. Considering $i \geq 2n+1$ and applying Lemma 5.5 we obtain $[v_i, T_{i+2}] \cup [v_{i+1}, T_{i+2}] = T_{i+3} \in \mathcal{L}^{(n+1)}$. Thus, $\mathcal{L}_{2n+4} = \langle \cup_{i \geq 2n+4} T_i \rangle \subset \mathcal{L}^{(n+1)}$. Considering all products

$$[v_{2n+1}, t_0^{\beta_0} \cdots t_{2n-2}^{\beta_{2n-2}} v_{2n+2}] = t_0^{\beta_0} \cdots t_{2n-2}^{\beta_{2n-2}} v_{2n+3} \in \mathcal{L}^{(n+1)}, \quad \beta_i \in \{0,1\}$$

we obtain that $\bar{T}_{2n+3} \subset \mathcal{L}^{(n+1)}$. Thus, $\mathcal{L}^{(n+1)}$ contains all required monomials. The induction step is proved.

To check the last claim we use Lemma 6.2. Let $n \geq 2$, then

$$\dim \mathcal{L}/\mathcal{L}^{(n)} = \dim \mathcal{L}/\mathcal{L}_{2n+1} + |T_{2n+1} \setminus \bar{T}_{2n+1}| = 1 + 2^{2n-1} + 2^{2n-3} = 1 + 3 \cdot 2^{2n-3}. \qquad \blacksquare$$

Remark 6.7. One can define and compute similar series for the restricted Lie algebra $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2)$ as well.

7. Just infinite dimensional restricted Lie algebra G

By analogy with group theory [7], a restricted Lie algebra L is said to be *just infinite dimensional* if it is infinite dimensional and any proper restricted factor algebra L/J is finite dimensional. We observed earlier that the Fibonacci Lie algebra $\mathbf{L}$ in case of an arbitrary positive characteristic is not just infinite dimensional [19]. So, it differs from the Grigorchuk group that is just infinite.

We suggest a slight modification of $\mathbf{L}$ and the respective hull:

$$\begin{aligned} &\mathbf{G} = \mathrm{Lie}_p(\partial_1, v_2), \quad \mathcal{G} = \mathrm{Lie}(\partial_1, v_2), \quad \mathbf{B} = \mathrm{Alg}(\partial_1, v_2);\\ &\partial_1 = \frac{\partial}{\partial t_1}, \quad v_2 = \partial_2 + t_1(\partial_3 + t_2(\partial_4 + t_3(\partial_5 + t_4(\partial_6 + \cdots)))). \end{aligned}$$

Recall that $R = K[t_i | i \geq 0]/(t_i^p | i \geq 0)$. Consider the subring $\tilde{R} = K[t_i | i \geq 1]/(t_i^p | i \geq 1) \hookrightarrow R$. Set $J = t_0 R \triangleleft R$, then $\tilde{R} \cong R/J$. The ideal J is invariant under $v_1, v_2 \in \mathrm{Der}\, R$ because these operators cannot kill the variable t_0. Hence, J is invariant under the algebras $\mathbf{L} = \mathrm{Lie}_p(v_1, v_2)$ and $\mathbf{A} = \mathrm{Alg}(v_1, v_2)$. The natural epimorphism $\pi : R \twoheadrightarrow \tilde{R}$ yields the following epimorphisms denoted by the same letter:

$$\begin{aligned} &\pi(v_1) = \partial_1, \quad \pi(v_2) = v_2,\\ &\pi : \mathcal{L} = \mathrm{Lie}(v_1, v_2) \twoheadrightarrow \mathcal{G} = \mathrm{Lie}(\partial_1, v_2),\\ &\pi : \mathbf{L} = \mathrm{Lie}_p(v_1, v_2) \twoheadrightarrow \mathbf{G} = \mathrm{Lie}_p(\partial_1, v_2),\\ &\pi : \mathbf{A} = \mathrm{Alg}(v_1, v_2) \twoheadrightarrow \mathbf{B} = \mathrm{Alg}(\partial_1, v_2). \end{aligned}$$

The algebras $\mathbf{G}$, $\mathcal{G}$, and $\mathbf{B}$ can be also realized in terms of self-similar matrices, see Theorem 2.4 and [18].

Consider sets

$$\begin{aligned} T'_n &= \begin{cases} \{\partial_1\}, & n = 1;\\ \{v_n\}, & n = 2, 3, 4;\\ \{t_1^{\alpha_1} \cdots t_{n-4}^{\alpha_{n-4}}\, v_n \mid \alpha_i \in \{0, 1\}\}, & n \geq 5; \end{cases}\\ \tilde{T}'_n &= \begin{cases} T'_n, & n = 1, 2, 3;\\ T'_n \cup \{t_{n-3}\, v_n\}, & n \geq 4; \end{cases}\\ T' &= \cup_{n=1}^{\infty} T'_n;\\ \tilde{T}' &= \cup_{n=1}^{\infty} \tilde{T}'_n. \end{aligned}$$

Similar to Lemma 5.5 we have the following.

Lemma 7.1. *We have a disjoint union of sets*

$$T'_n = [v_{n-3}, T'_{n-1}] \cup [v_{n-2}, T'_{n-1}], \quad n \geq 5.$$

We formally repeat arguments of Lemma 5.1 and obtain the following.

Lemma 7.2. *Let* $\mathcal{G} = \mathrm{Lie}(\partial_1, v_2)$, $\mathbf{G} = \mathrm{Lie}_p(\partial_1, v_2)$. *We have bases:*

$$\begin{aligned} \mathcal{G} &= \langle T' \rangle = \langle \partial_1, v_2, v_3, v_4,\ t_1^{\alpha_1} \cdots t_{n-4}^{\alpha_{n-4}}\, v_n \mid n \geq 5, \alpha_i \in \{0,1\} \rangle, \\ \mathbf{G} &= \langle \tilde{T}' \rangle = \langle \partial_1, v_2, v_3, v_4,\ t_1^{\alpha_1} \cdots t_{n-4}^{\alpha_{n-4}}\, v_n;\ t_{k-3} v_k \mid n \geq 5, \alpha_i \in \{0,1\}, k \geq 4 \rangle. \end{aligned} \tag{23}$$

Similar to Section 4, define generating functions $\mathcal{H}_{\tilde{X}}(\mathcal{G}, x, y)$, $\mathcal{H}_{\tilde{X}}(\mathbf{G}, x, y)$ according to the $\mathbb{Z} \oplus \mathbb{Z}$-gradation by multidegree in $\tilde{X} = \{\partial_1, v_2\}$.

Lemma 7.3. *Then*

$$\mathcal{H}_{\tilde{X}}(\mathcal{G}, x, y) = x + \mathcal{H}_X(\mathcal{L}, y, xy); \quad \mathcal{H}_{\tilde{X}}(\mathbf{G}, x, y) = x + \mathcal{H}_X(\mathbf{L}, y, xy).$$

Proof. Recall that by our definitions, $\mathrm{Wt}(\partial_1) = \mathrm{Wt}(v_1)$. Observe that both functions $\mathcal{H}_X(*, x, y)$, $\mathcal{H}_{\tilde{X}}(*, x, y)$ describe decompositions with respect to the same weight lattice (3). We have semidirect sums that yield the required formulas:

$$\begin{aligned} &\mathcal{G} = \langle \partial_1 \rangle \oplus \mathrm{Lie}(v_2, v_3) = \langle \partial_1 \rangle \oplus \tau(\mathrm{Lie}(v_1, v_2)) = \langle \partial_1 \rangle \oplus \tau(\mathcal{L}); \\ &\mathcal{H}_{\tilde{X}}(\mathcal{G}, x, y) = x + \mathcal{H}_{\tilde{X}}(\mathrm{Lie}(v_2, v_3), x, y) = x + \mathcal{H}_X(\mathrm{Lie}(v_1, v_2), y, xy); \\ &\mathbf{G} = \langle \partial_1 \rangle \oplus \mathrm{Lie}_p(v_2, v_3) = \langle \partial_1 \rangle \oplus \tau(\mathrm{Lie}_p(v_1, v_2)); \\ &\mathcal{H}_{\tilde{X}}(\mathbf{G}, x, y) = x + \mathcal{H}_X(\mathbf{L}, y, xy). \end{aligned}$$
■

Corollary 7.4.

$$\mathcal{H}_{\tilde{X}}(\mathcal{G}, x, y) = x + y + xy + xy^2 \Bigg(1 + \sum_{n=1}^{\infty} \prod_{m=1}^{n} (x^{F_{m-1}} y^{F_m} + x^{F_m} y^{F_{m+1}})\Bigg);$$

$$\mathcal{H}_{\tilde{X}}(\mathbf{G}, x, y) = x + y + xy + xy^2 \Bigg(1 + \sum_{n=1}^{\infty} \prod_{m=1}^{n} (x^{F_{m-1}} y^{F_m} + x^{F_m} y^{F_{m+1}})\Bigg) + \sum_{n=1}^{\infty} x^{2F_{n-1}} y^{2F_n}.$$

Proof. We apply Theorem 5.7. Alternatively, we can use Lemma 7.1. ■

Lemma 7.5. *The generating functions and the regular growth functions of* $\mathcal{L}$ *and* $\mathcal{G}$ *are related as follows:*

1. $\mathcal{H}_X(\mathcal{L}, x, y) = \mathcal{H}_{\tilde{X}}(\mathcal{G}, x, y)(1 + x/y) - x - x^2/y - x^2$;

2. $\mathcal{H}_{\tilde{X}}(\mathcal{G}, z) = (\mathcal{H}_X(\mathcal{L}, z) + 2z + z^2)/2$;

3. $\lambda_{\mathcal{L}}(X, n) = 2\lambda_{\mathcal{G}}(\tilde{X}, n)$, $n \geq 3$;

4. $\gamma_{\mathcal{L}}(X,n) = 2\gamma_{\mathcal{G}}(\tilde{X},n) - 3\,,\ n \ge 3\,.$

Proof. The first claim follows from Lemma 7.3 and Theorem 4.3. Also, we can proceed directly, we have a disjoint union:

$$T_n = T'_n \cup t_0 T'_n, \quad n \ge 4;$$
$$\mathcal{H}(T_n, x, y) = \mathcal{H}(T'_n, x, y)(1 + \mathcal{H}(\{t_0\}, x, y)) = \mathcal{H}(T'_n, x, y)(1 + x/y), \quad n \ge 4.$$

It remains to compare both series taking into account (15). All other claims follow from the first. ∎

Let $m \ge 1$. Similarly to the notations above, let $\mathcal{G}_m$, $\mathbf{G}_m$ be spans of the respective basic monomials $\cdots v_n$ such that $n \ge m$. We get the series of (restricted) ideals of finite codimension

$$\mathcal{G} = \mathcal{G}_1 \supset \mathcal{G}_2 \supset \cdots \supset \mathcal{G}_m \supset \mathcal{G}_{m+1} \supset \cdots\,;$$
$$\mathbf{G} = \mathbf{G}_1 \supset \mathbf{G}_2 \supset \cdots \supset \mathbf{G}_m \supset \mathbf{G}_{m+1} \supset \cdots\,.$$

Similarly, we have

Lemma 7.6. *1.* $\mathcal{G}_m = \langle \cup_{j \ge m} T'_j \rangle\,,\ m \ge 1\,;$

2. $\mathbf{G}_m = \langle \cup_{j \ge m} \tilde{T}'_j \rangle\,,\ m \ge 1\,.$

Remark 7.7. One can compute the growth functions, terms of the lower central series and derived series for the restricted Lie algebra $\mathbf{G}$ similar to Section 6.

But the goal of introducing of this algebra is the following observation.

Theorem 7.8. *The restricted Lie algebra* $\mathbf{G} = \mathrm{Lie}_p(\partial_1, v_2)$ *is just infinite dimensional.*

Proof. Let I be a nontrivial restricted ideal in $\mathbf{G}$ and $0 \ne a \in I$. Assume that $a \in \mathbf{G}_m \setminus \mathbf{G}_{m+1}$, where $m \ge 2$, the case $m = 1$ is treated similarly. Then $a \equiv h v_m \mod \mathbf{G}_{m+1}$, where $0 \ne h \in \mathrm{Alg}(t_1, \dots, t_{m-3})$. If h is a scalar, then we skip to the case (24). Otherwise we have

$$a \equiv (f t_s + g) v_m \mod \mathbf{G}_{m+1}; \qquad f, g \in \mathrm{Alg}(t_1, \dots, t_{s-1}),\ s \le m-3,\ f \ne 0.$$

According to our multiplication rules (Lemma 2.1)

$$I \ni a' = [v_s, a] \equiv f v_m \mod \mathbf{G}_{m+1}.$$

We continue, at the end, if we need to kill t_1, we commute with ∂_1. We obtain the element

$$I \ni b = v_m + \sum_{i=m+1}^{M} r_i v_i,\ r_i \in R. \tag{24}$$

We call the sum above the "tail", it has length $M-m$. Next consider

$$I \ni b' = [v_{m+1}, b] = [v_{m+1}, v_m] + \sum_{i=m+1}^{M} (r_i[v_{m+1}, v_i] + v_{m+1}(r_i)v_i)$$
$$= v_{m+2} + \sum_{j=m+3}^{M+1} r'_j t_j.$$

Indeed, the terms in the first line for $i = m+1$ are trivial, because $r_{m+1} \in \mathrm{Alg}(t_1, \dots, t_{m-2})$ and $v_{m+1}(r_{m+1}) = 0$ by Lemma 2.1. Consider the terms for $i = m+2$. We get $[v_{m+1}, v_{m+2}] = v_{m+3}$ and $v_{m+1}(r_{m+2})v_{m+2}$, since $r_{m+2} \in \mathrm{Alg}(t_1, \dots, t_{m-1})$ we get $v_{m+1}(r_{m+2}) = 0$ by Lemma 2.1. On the other hand "the most far" term contains v_{M+1}. Thus, we obtained an element similar to (24) but with the tail of shorter length $M-m-1$. We continue and conclude that $v_k \in I$ for some number k.

We check by induction that $v_i \in I$ for $i \geq k$. Indeed, let $v_i \in I$, then $[v_{i-1}, v_i] = v_{i+1} \in I$.

We have $v_{n-1}, v_{n-2} \in I$, for $n \geq k+2$. We multiply these elements by the basic elements (23) of $\mathbf{G}$, namely, we use analogues of (13), (14) and obtain that

$$\{t_1^{\alpha_1} \cdots t_{j-4}^{\alpha_{j-4}} v_j \mid j \geq k+3,\ \alpha_s \in \{0,1\}\} \subset I.$$

Also, we have $v_i^2 = t_{i-1}v_{i+2} \in I$ for $i \geq k$, thus $t_{j-3}v_j \in I$ for $j \geq k+2$. Hence, $\tilde{T}'_j \subset I$ for $j \geq k+3$. We conclude that $\mathbf{G}_{k+3} \subset I$.

Recall that we have the epimorphism $\pi : \mathbf{L} \to \mathbf{G}$ and $\pi(\mathbf{L}_m) = \mathbf{G}_m$, $m \geq 1$. Finally, by Lemma 6.2, $\dim \mathbf{G}/I \leq \dim \mathbf{G}/\mathbf{G}_{k+3} \leq \dim \mathbf{L}/\mathbf{L}_{k+3} < \infty$. ∎

Lemma 7.9. *Let* $\mathbf{G} = \mathrm{Lie}_p(\partial_1, v_2)$. *Its center is trivial* $Z(\mathbf{G}) = \{0\}$.

Proof. We proceed as in Lemma 6.1. ∎

8. Open problems

1) Is it true that the homomorphic image $\mathbf{G} = \mathrm{Lie}_p(\partial_1, v_2)$ of the Fibonacci restricted Lie algebra is also just infinite dimensional for bigger primes?

2) It is interesting to find an explicit formula for the generating function of $\mathbf{A}$. For this purpose we need to determine a basis of $\mathbf{A}$. We know that $\mathbf{A}$ is contained in a linear span of the following monomials [17]:

$$t_0^{\alpha_0} \cdots t_{n-4}^{\alpha_{n-4}} t_{n-3}^{\alpha_{n-3}} v_n v_{n-1}^{\beta_{n-1}} \cdots v_1^{\beta_1}, \quad \alpha_j, \beta_j \in \{0,1\},\ n \geq 1.$$

We can easily get all such monomials with $\alpha_{n-3} = 0$ using Lemma 5.1. But a description of monomials with $\alpha_{n-3} = 1$ is not clear.

3) Grigorchuk group is infinitely presented, it has a so called *L-presentation* [13, 8]. Now observe that the restricted Lie algebra $\mathbf{L}$ satisfies the relation $[v_2, v_1^3] = [[[v_2, v_1], v_1], v_1] = [[v_3, v_1], v_1] = [t_0 v_4, v_1,] = t_0(t_0 t_1 v_5) = 0$. Similarly, one has also the relation $[[v_2, v_1^2], v_2^2] = 0$. Recall that we have the embedding

$\tau:\mathbf{L}\hookrightarrow\mathbf{L}$, $\tau(v_i)=v_{i+1}$, $i\ge 1$. This embedding is determined by $\tau(v_1)=v_2$ and $\tau(v_2)=v_3=[v_1,v_2]$. We ask whether $\mathbf{L}=\mathrm{Lie}_p(v_1,v_2)$ is infinitely presented and the following is its L-presentation:

$$\tau^m([v_2,v_1^3])=0,\quad \tau^m([[v_2,v_1^2],v_2^2])=0,\quad m=0,1,2,\ldots.$$

4) Another interesting problem is to determine whether $\mathbf{A}$ is a nil-algebra, where the prime p is arbitrary. The answer is negative for Fibonacci Lie algebras in so called rational cases [12], see also further generalizations in [18]. But the case of other primes remains open, one of such cases is our case $p=2$.

5) Remark that $v_i^4=0$, $i\ge 1$. Denote $g_i=1+v_i$, then $g_i^4=1$ for $i\ge 1$. These elements can be written recursively: $g_i=1+\partial_i+t_{i-1}(1+g_{i+1})$, $i\ge 1$. We define a group generated by two periodic elements $G=\mathrm{Group}(g_1,g_2)\subset\mathrm{End}\,R$. Recall that we have the matrix presentation (Theorem 2.4). Denote $S_i=1+s_i$, and $R_i=1+r_i$, $i\ge 1$. We obtain recursively defined invertible matrices:

$$S_i=\begin{pmatrix}1&0\\1&1\end{pmatrix}_{(i)};\qquad R_i=\begin{pmatrix}S_{i+1}&1+R_{i+1}\\0&S_{i+1}\end{pmatrix}_{(i)},\quad i\ge 1.$$

Since $\mathrm{Alg}(r_1,r_2)\cong\mathrm{Alg}(v_1,v_2)$, we get the same group $G=\mathrm{Group}(g_1,g_2)\cong\mathrm{Group}(R_1,R_2)$. If the answer to the previous question is positive, then G is a periodic group.

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