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Unitary Highest Weight Modules over Block Type Lie Algebras $\mathcal{B}(q)$

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Abstract. We classify the unitary quasifinite irreducible highest weight modules over the Block type Lie algebras $\mathcal{B}(q)$ for all non-zero values of the parameter q. The algebra $\mathcal{B}(q)$ contains the Virasoro algebra as a subalgebra and thus is likely to have applications in conformal field theory.
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1. Introduction

In the late 50s, Block [3] introduced a class of infinite dimensional simple Lie algebras, which are usually referred to as *Block type Lie algebras* nowadays. These Lie algebras are analogues of the Zassenhaus algebras in characteristic zero. Their structure theory has been extensively studied in the last twenty years (see, e.g., [5, 16, 25–28]). However, the understanding of their general representations is quite difficult due to intrinsic difficulties of the subject matter. Even the study of the so-called *quasifinite modules*, which have finite dimensional graded subspaces, is a rather nontrivial problem (see, e.g., [18, 19, 21, 22]). The development of representations of W-infinity algebras suffer from the same difficulty before the appearance of the breakthrough work of Kac and Radul [11], where the notion of quasifinite modules over infinite dimensional graded Lie algebras was firstly proposed, and has been widely used in the representation theory of various infinite dimensional Lie (super)algebras (see, e.g., [1, 9, 13, 14, 17, 20]). Such modules are close in spirit to finite dimensional modules of finite dimensional Lie algebras [14], and are also quite natural from the viewpoint of the free field realization [2].

The study of quasifinite modules over Block type Lie algebras was started in [18, 19], partially motivated by Mathieu's classification [15] of Harish-Chandra modules over the Virasoro algebra. The Block type Lie algebra $\mathcal{B}$ studied in [18] is a complex Lie algebra with a basis $\{L_{\alpha,i}, C \mid \alpha, i \in \mathbb{Z}, i \geq -1\}$ and commutation

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relations

$$[L_{\alpha,i}, L_{\beta,j}] = (\beta(i+1) - \alpha(j+1))L_{\alpha+\beta,i+j} + \alpha\delta_{\alpha+\beta,0}\delta_{i+j,-2}C,$$
$$[C, L_{\alpha,i}] = 0, \quad \forall \alpha, \beta, i, j.$$

It was shown [18] that any quasifinite irreducible $\mathcal{B}$-module is either a highest or lowest weight module. In particular, the unitary ones were proved to be trivial.

Recently, a one parameter family of Block type complex Lie algebras $\mathcal{B}(q)$ was extensively studied [21–23], where q is any fixed complex parameter. The algebra has a basis $\{L_{\alpha,i}, C \mid \alpha, i \in \mathbb{Z},\ i \geq 0\}$ and the commutation relations

$$\begin{aligned}&[L_{\alpha,i}, L_{\beta,j}] = (\beta(i+q) - \alpha(j+q))L_{\alpha+\beta,i+j} + \frac{\alpha^3 - \alpha}{12}\delta_{\alpha+\beta,0}\delta_{i+j,0}C,\\ &[C, L_{\alpha,i}] = 0, \quad \forall \alpha, \beta, i, j.\end{aligned} \tag{1.1}$$

These Lie algebras are in fact subalgebras of some very special cases of generalized Block algebras in [5]. It was shown in [21] that any quasifinite irreducible $\mathcal{B}(q)$-module is either a highest or lowest weight module, or else a uniformly bounded module. For the precise definitions of these $\mathcal{B}(q)$-modules, see Section 3 or [21].

Some features of $\mathcal{B}(q)$ (see Section 2, or [21–23] for details) are noteworthy. When $q \neq 0$, $\mathcal{B}(q)$ contains a Virasoro subalgebra

$$\mathrm{Vir} := \mathrm{span}\left\{ L_\alpha = -\frac{1}{q}L_{\alpha,0},\ C_{\mathrm{Vir}} = \frac{1}{q^2}C \mid \alpha \in \mathbb{Z} \right\}.$$

[In contrast, $\mathcal{B}$ does not contain the Virasoro subalgebra.] Therefore, it is likely that $\mathcal{B}(q)$ has applications in conformal field theory, thus is a more interesting object to study.

There exist interesting relationships among $\mathcal{B}(q)$'s at particular values of the parameter q. For example, $\mathcal{B}(1)$ can be embedded into $\mathcal{B}(n)$ with integer $n \geq 1$ via $L_{\alpha,i} \mapsto \frac{1}{n}L'_{\alpha,ni}$. There is also a close relation between $\mathcal{B}(1)$ and the $\mathcal{W}_\infty$ algebra. One can view $\mathcal{W}_\infty$ as a deformation of $\mathcal{B}(1)$ in the sense that $\mathcal{W}_\infty$ is isomorphic to $\mathcal{B}(1)$ as a vector space, and admits a natural filtration for which the associated graded object $\mathrm{gr}(\mathcal{W}_\infty)$ is isomorphic to $\mathcal{B}(1)$ as a Lie algebra. ($\mathcal{B}$ is related to $\mathcal{W}_{1+\infty}$ in a similar way [18]). The W-infinity algebras, especially $\mathcal{W}_\infty$ and $\mathcal{W}_{1+\infty}$, have been extensively studied in the literature (see, e.g., [1, 2, 6, 9, 11, 12]) due to their connections with physics, e.g., conformal field theory and the quantum Hall effect. This provides further motivation for studying the Lie algebra $\mathcal{B}(q)$.

A quasifinite irreducible highest weight $\mathcal{B}(q)$-module with highest weight Λ is specified by a series of numbers (see Section 3 for details):

$$(c, h_0, h_1, \ldots), \quad \text{where} \quad c = \Lambda(C),\ h_i = -\Lambda(L_{0,i}) \quad \text{for} \quad i \in \mathbb{Z}_+.$$

The aim of the present paper is to classify the *unitary* quasifinite irreducible highest weight representations of $\mathcal{B}(q)$. It is the unitary highest weight representations that will be most relevant if the algebra $\mathcal{B}(q)$ turns out to have applications in quantum physics.

Recall that unitarisability of modules needs to be defined with respect to a given conjugate-linear Lie algebra anti-involution of $\mathcal{B}(q)$ (which specifies a real form of the Lie algebra). A $\mathcal{B}(q)$-module is unitary if it admits a positive definite Hermitian form, that is contravariant with respect to the conjugate-linear anti-involution.

When q is not real, we show in Lemma 2.2 that $\mathcal{B}(q)$ admits no sensible conjugate-linear anti-involution, thus one can not discuss unitarisability of a $\mathcal{B}(q)$-module in this case unless it is the trivial 1-dimensional module. When q is real, $\mathcal{B}(q)$ admits a conjugate-linear Lie algebra anti-involution (c.f. (2.3) and Definition 4.1) given by

$$\omega : L_{\alpha,i} \mapsto L_{-\alpha,i}, \quad C \mapsto C.$$

The following theorem is the main result of this paper, which gives a complete classification of the quasifinite irreducible highest weight $\mathcal{B}(q)$-modules which are unitary with respect to ω.

Theorem 1.1. *Let $L(\Lambda)$ be a quasifinite irreducible highest weight $\mathcal{B}(q)$-module.*

(1) *When q is not real, $L(\Lambda)$ is unitary if and only if it is the trivial 1-dimensional module.*

(2) *When q is real, $L(\Lambda)$ is unitary if and only if $h_i = \delta_{2q+i,0}h$ with $h \in \mathbb{R}$ for $i \geq 1$, and one of the following conditions holds:*

 (i) *$c \geq q^2$ and $qh_0 \geq 0$, or*

 (ii) *there exists integer $m \geq 2$, and $r, s \in \mathbb{Z}$ with $1 \leq s \leq r < m$ such that*

$$c = c_{m,q} = q^2 - \frac{6q^2}{m(m+1)}, \quad h_0 = h_{m,q}^{r,s} = \frac{q((m+1)r - ms)^2 - q}{4m(m+1)}.$$

A crucial role is played by the Virasoro subalgebra in the study of the unitary quasifinite irreducible highest weight $\mathcal{B}(q)$-modules. In fact condition (2) in the above theorem arises from the characterisation of the unitary irreducible highest weight modules over the Virasoro algebra [7, 8] (see Theorem 2.3).

This paper is organized as follows. In Section 2, we present some general features of $\mathcal{B}(q)$, together with some related results. Next, in Section 3, we introduce the relevant definitions and terminology, and recall some results obtained in [21] on quasifinite highest weight $\mathcal{B}(q)$-modules. We also define a special class of quasifinite irreducible highest weight $\mathcal{B}(q)$-modules, called *primitive modules*, which include all the unitary ones (Remark 4.4). Section 4 is devoted to the proof of Theorem 1.1. Section 5 gives some general discussions on the results of this paper.

We will work over the field $\mathbb{C}$ of complex numbers. Throughout the paper, we assume that the parameter $q \in \mathbb{C}$ of $\mathcal{B}(q)$ is nonzero.

2. Some structural properties of $\mathcal{B}(q)$

In this section we discuss some general properties of the Block type Lie algebra $\mathcal{B}(q)$. Lemma 2.1 and Lemma 2.2 concern conjugate-linear Lie algebra anti-involutions of $\mathcal{B}(q)$, which will be important for the remainder of the paper. Some other results presented here are not used in a strict sense, but they are interesting in their own right.

2.1. Structural features of $\mathcal{B}(q)$. For any pair of positive integers q_1 and q_2 such that $q_1|q_2$, we have the embeddings of the Block type Lie algebras $\mathcal{B}(q)$'s at different parameters:

$$\mathcal{B}(\varepsilon q_2^{-1}) \underset{\neq}{\hookrightarrow} \mathcal{B}(\varepsilon q_1^{-1}) \underset{\neq}{\hookrightarrow} \mathcal{B}(\varepsilon) \underset{\neq}{\hookrightarrow} \mathcal{B}(\varepsilon q_1) \underset{\neq}{\hookrightarrow} \mathcal{B}(\varepsilon q_2), \text{ where } \varepsilon = \pm 1.$$

Hence, corresponding to each $q \neq 0$ such that $q \in \mathbb{Z}$ or $q^{-1} \in \mathbb{Z}$, there exists a subalgebra chain of $\mathcal{B}(q)$. The fact has been successfully applied to study both structures and representations of $\mathcal{B}(q)$. In the case $q \in \frac{1}{2}\mathbb{Z}_-^*$ ($\mathbb{Z}_-^* = \{-1, -2, \dots\}$), they are used to determine the intermediate series $\mathcal{B}(q)$-modules [21], and the automorphism groups of $\mathcal{B}(q)$ [24].

We now discuss relationships between $\mathcal{B}(1)$ and W-infinity algebras, which were briefly alluded to in the Introduction. Recall that the $\mathcal{W}_{1+\infty}$ algebra is defined as the universal central extension of the Lie algebra of differential operators on the circle (the cocycle appeared for the first time in [10]), which has basis $\{x^\alpha D^i, C \mid \alpha \in \mathbb{Z}, i \in \mathbb{Z}_+\}$ with $D = x\frac{d}{dx}$, and relations

$$\begin{aligned}[x^\alpha D^i, x^\beta D^j] &= x^{\alpha+\beta}((D+\beta)^i D^j - D^i(D+\alpha)^j) \\ &\quad + \delta_{\alpha+\beta,0}(-1)^i i! j! \binom{\alpha+i}{i+j+1} C.\end{aligned}$$

Setting $(\mathcal{W}_{1+\infty})_{[-2]} = \mathbb{C}C$ and

$$(\mathcal{W}_{1+\infty})_{[n]} = \operatorname{span}\{x^\alpha D^i, C \mid \alpha \in \mathbb{Z},\ 0 \le i \le n+1\} \ \text{ for } \ n \ge -1,$$

we obtain a filtration of $\mathcal{W}_{1+\infty}$:

$$\{0\} = (\mathcal{W}_{1+\infty})_{[-3]} \subset (\mathcal{W}_{1+\infty})_{[-2]} \subset (\mathcal{W}_{1+\infty})_{[-1]} \subset (\mathcal{W}_{1+\infty})_{[0]} \subset \cdots. \tag{2.1}$$

The $\mathcal{W}_{1+\infty}$ algebra is the most fundamental W-infinity algebra since all others can be viewed as its subalgebras. The most important subalgebra of $\mathcal{W}_{1+\infty}$ is the $\mathcal{W}_\infty$ algebra, which can be obtained by omitting the unique spin-1 current (expressed as x^α) from $\mathcal{W}_{1+\infty}$. So, the $\mathcal{W}_\infty$ algebra has basis $\{x^\alpha D^i, C \mid \alpha, i \in \mathbb{Z}, i \ge 1\}$. Similarly, if we define

$$(\mathcal{W}_\infty)_{[-1]} = \mathbb{C}C, \ \ (\mathcal{W}_\infty)_{[n]} = \operatorname{span}\{x^\alpha D^i, C \mid \alpha \in \mathbb{Z},\ 1 \le i \le n+1\} \ \text{ for } \ n \ge 0,$$

we obtain a filtration of $\mathcal{W}_\infty$:

$$\{0\} = (\mathcal{W}_\infty)_{[-2]} \subset (\mathcal{W}_\infty)_{[-1]} \subset (\mathcal{W}_\infty)_{[0]} \subset \cdots. \tag{2.2}$$

It is easy to check that under the map

$$x^\alpha D^{i+1} \mapsto L_{\alpha,i},\ C \mapsto -2C \text{ (resp. } x^\alpha D^{i+1} \mapsto L_{\alpha,i},\ C \mapsto C),$$

the associated graded Lie algebra of the filtered Lie algebra $\mathcal{W}_\infty$ (resp. $\mathcal{W}_{1+\infty}$) corresponding to the natural filtration (2.2) (resp. (2.1)) is isomorphic to $\mathcal{B}(1)$ (resp. $\mathcal{B}$).

The quasifinite highest weight modules and all the unitary ones over $\mathcal{W}_\infty$ and $\mathcal{W}_{1+\infty}$ were classified and constructed by Kac, Liberati and Radul in [9, 11]. The authors in [1, 9] observed that the list of unitary quasifinite highest weight modules over $\mathcal{W}_\infty$ is much richer than that over $\mathcal{W}_{1+\infty}$, even though $\mathcal{W}_\infty$ is a subalgebra of $\mathcal{W}_{1+\infty}$.

2.2. Conjugate-linear Lie algebra anti-involution of $\mathcal{B}(q)$. We now discuss conjugate-linear Lie algebra anti-involutions of $\mathcal{B}(q)$. By conjugate-linearity of a map ω, we mean that $\omega(aX+bY)=\bar{a}\omega(X)+\bar{b}\omega(Y)$ for all $a,b\in\mathbb{C}$ and $X,Y\in\mathcal{B}(q)$, where $\bar{a}$ and $\bar{b}$ are the complex conjugates of a and b.

When q is real, we consider the following conjugate-linear map ω of $\mathcal{B}(q)$:

$$\omega(L_{\alpha,i})=L_{-\alpha,i},\quad \omega(C)=C. \tag{2.3}$$

One can easily prove the following result.

Lemma 2.1. *Assume that q is real. Then the map ω defined above is a conjugate-linear Lie algebra anti-involution of $\mathcal{B}(q)$, i.e., it satisfies*

$$\omega^2=\mathrm{id},\quad \omega(aX)=\bar{a}\omega(X),\quad \omega([X,Y])=[\omega(Y),\omega(X)]$$

for $a\in\mathbb{C}$ and $X,Y\in\mathcal{B}(q)$.

However, the case with non-real q is totally different.

Lemma 2.2. *If q is not real, $\mathcal{B}(q)$ does not admit any conjugate-linear Lie algebra anti-involution.*

This will be proven in the next subsection.

As we have already discussed in Section 1, $\mathcal{B}(q)$ contains a Virasoro subalgebra Vir spanned by $C_{\mathrm{Vir}}=\frac{1}{q^2}C$ and $L_\alpha=-\frac{1}{q}L_{\alpha,0}$ with $\alpha\in\mathbb{Z}$. It is entirely straightforward to deduce the following relations from (1.1),

$$\begin{aligned}&[L_\alpha,L_\beta]=(\alpha-\beta)L_{\alpha+\beta}+\frac{\alpha^3-\alpha}{12}\delta_{\alpha+\beta,0}C_{\mathrm{Vir}},\\&[C_{\mathrm{Vir}},L_\alpha]=0,\quad \forall\alpha,\beta,\end{aligned}$$

which are the standard commutation relations of the Virasoro algebra.

The map ω defined by (2.3) restricts to $L_\alpha\mapsto L_{-\alpha}, C_{\mathrm{Vir}}\mapsto C_{\mathrm{Vir}}$ on Vir, yielding a conjugate-linear Lie algebra anti-involution of the Virasoro subalgebra. An irreducible highest weight module $L(\Lambda)$ over Vir is specified by a pair of numbers $(c_{\mathrm{Vir}},h_{\mathrm{Vir}})$, where $c_{\mathrm{Vir}}=\Lambda(C_{\mathrm{Vir}})$ and $h_{\mathrm{Vir}}=\Lambda(L_0)$. The following theorem is the celebrated classification of the unitary highest weight modules over the Virasoro algebra, which had a major impact on the development of conformal field theory in the 80s.

Theorem 2.3 ([7, 8])**.** *The irreducible highest weight* Vir*-module $L(c_{\mathrm{Vir}},h_{\mathrm{Vir}})$ is unitary if and only if one of the following conditions holds:*

(1) $c_{\mathrm{Vir}} \geq 1$ *and* $h_{\mathrm{Vir}} \geq 0$,
(2) *there exists integer* $m \geq 2$, *and* $r, s \in \mathbb{Z}$ *with* $1 \leq s \leq r < m$ *such that*

$$c_{\mathrm{Vir}} = c_m = 1 - \frac{6}{m(m+1)}, \quad h_{\mathrm{Vir}} = h_m^{r,s} = \frac{((m+1)r - ms)^2 - 1}{4m(m+1)}.$$

The classification of unitary quasifinite highest weight $\mathcal{B}(q)$-modules essentially reduces to this theorem, as we shall see later.

2.3. Proof of Lemma 2.2.

Now we give the proof of Lemma 2.2.

Proof of Lemma 2.2. Assume that θ is a conjugate-linear anti-involution of $\mathcal{B}(q)$. Recall the assumption that q is not real. Since $\mathcal{H} = \mathrm{span}\{L_{0,0}, C\}$ is the unique maximal abelian subalgebra of $\mathcal{B}(q)$ which acts semisimply on $\mathcal{B}(q)$ in the adjoint representation, and the fact that $\mathbb{C}C$ is the center of $\mathcal{B}(q)$, we have $\theta(\mathcal{H}) = \mathcal{H}$ and $\theta(\mathbb{C}C) = \mathbb{C}C$. Furthermore, since $\theta^2 = \mathrm{id}$, we see that

$$\theta(C) = \epsilon C, \quad \theta(L_{0,0}) = aL_{0,0} + bC,$$

where $\epsilon\bar{\epsilon} = 1$ and $a\bar{a} = 1$. Set $\theta(L_{\alpha,i}) = \sum_{(\beta,j)\in(\mathbb{Z},\mathbb{Z}_+)} a_{\beta,j}^{\alpha,i} L_{\beta,j} + b_{\alpha,i}C$. Applying θ to $[L_{0,0}, L_{0,i}] = 0$, we obtain $a_{\beta,j}^{0,i} = 0$ for $\beta \neq 0$. Putting $a_j^i := a_{0,j}^{0,i}$. Then $\theta(L_{0,i}) = \sum_{j\in\mathbb{Z}_+} a_j^i L_{0,j} + b_{0,i}C$. Applying θ to $[L_{0,0}, L_{\alpha,0}] = q\alpha L_{\alpha,0}$, we have

$$-qa \sum_{(\beta,j)\in(\mathbb{Z},\mathbb{Z}_+)} \beta a_{\beta,j}^{\alpha,0} L_{\beta,j} = \bar{q}\alpha \sum_{(\beta,j)\in(\mathbb{Z},\mathbb{Z}_+)} a_{\beta,j}^{\alpha,0} L_{\beta,j} + \bar{q}\alpha b_{\alpha,0}C.$$

This implies that $b_{\alpha,0} = 0$ for $\alpha \neq 0$, and $a = \pm\frac{\bar{q}^2}{|q|^2}$.

In the case $a = \frac{\bar{q}^2}{|q|^2}$, we have $a_{\beta,j}^{\alpha,0} = 0$ for $\beta \neq -\alpha$. Putting $a_{\alpha,j} := a_{-\alpha,j}^{\alpha,0}$. Then $\theta(L_{\alpha,0}) = \sum_{j\in\mathbb{Z}_+} a_{\alpha,j} L_{-\alpha,j}$ for $\alpha \neq 0$. Applying θ to

$$[L_{-\alpha,0}, L_{\alpha,0}] = 2q\alpha L_{0,0} - \frac{\alpha^3 - \alpha}{12} C, \tag{2.4}$$

we get $a_{\alpha,j} = 0$ for $j > 0$, $a_{\alpha,0}a_{-\alpha,0} = \frac{\bar{q}^4}{|q|^4}$, $\epsilon = \frac{\bar{q}^4}{|q|^4}$ and $b = 0$. Moreover, from

$$\theta^2(L_{\alpha,0}) = L_{\alpha,0}, \tag{2.5}$$

we obtain $\bar{a}_{\alpha,0}a_{-\alpha,0} = 1$, and from the commutation relation

$$[L_{\alpha,0}, L_{\beta,0}] = q(\beta - \alpha)L_{\alpha+\beta,0} \quad \text{with} \quad \alpha \neq -\beta, \tag{2.6}$$

we obtain $qa_{\alpha,0}a_{\beta,0} = \bar{q}a_{\alpha+\beta,0}$ for $\alpha \neq -\beta$. Hence we have

$$a_{\alpha,0} = \left(\frac{q}{|q|}\right)^{2(\alpha-1)} a_{1,0}^{\alpha} \quad \text{with} \quad a_{1,0} = \frac{\bar{q}^4}{|q|^4}\bar{a}_{1,0} \neq 0.$$

Applying θ to $[L_{-1,0}, [L_{2,0}, L_{0,i}]] = 2(3q + i)[L_{1,0}, L_{0,i}]$, we have

$$\sum_{j\in\mathbb{Z}_+} (|q|^2(3q + j) - q^2(3\bar{q} + i))(q + j)a_j^i L_{-1,j} = 0. \tag{2.7}$$

For any fixed i, by (2.7), there exists only one j such that $a_j^i \neq 0$, and

$$j = \frac{q^2(3\bar{q}+i)}{|q|^2} - 3q \in \mathbb{Z}_+,$$

which is impossible.

In the case $a = -\frac{\bar{q}^2}{|q|^2}$, we have $a_{\beta,j}^{\alpha,0} = 0$ for $\beta \neq \alpha$. Putting $a_{\alpha,j} := a_{\alpha,j}^{\alpha,0}$. Then $\theta(L_{\alpha,0}) = \sum_{j\in\mathbb{Z}_+} a_{\alpha,j} L_{\alpha,j}$ for $\alpha \neq 0$. Similar to the above, by (2.4), we have $a_{\alpha,j} = 0$ for $j > 0$, $a_{\alpha,0}a_{-\alpha,0} = \frac{\bar{q}^4}{|q|^4}$, $\epsilon = -\frac{\bar{q}^4}{|q|^4}$ and $b = 0$. Furthermore, by (2.5), we have $\bar{a}_{\alpha,0}a_{\alpha,0} = 1$, and by (2.6), we have $qa_{\alpha,0}a_{\beta,0} = -\bar{q}a_{\alpha+\beta,0}$ for $\alpha \neq -\beta$. Hence we see that

$$a_{\alpha,0} = (-1)^{\alpha-1}\Big(\frac{q}{|q|}\Big)^{2(\alpha-1)} a_{1,0}^{\alpha} \ \text{ with } \ |a_{1,0}| = 1.$$

Similar as (2.7), applying θ to $[L_{-1,0},[L_{2,0},L_{0,i}]] = 2(3q+i)[L_{1,0},L_{0,i}]$, we have

$$\sum_{j\in\mathbb{Z}_+} (|q|^2(3q+j) - q^2(3\bar{q}+i))(q+j)a_j^i L_{1,j} = 0,$$

which will derive contradiction again as above. ∎

3. Quasifinite highest weight modules

We discuss results on quasifinite highest weight modules (QHWM's) over $\mathcal{B}(q)$. First, we introduce the following notion, which will simplify the presentation to some extent.

Definition 3.1. For any fixed number $q \in \mathbb{C}$, and generalized polynomial $p(t) \in t^q\mathbb{C}[t]$, the *shifted polynomial* $\widetilde{p}(t)$ of $p(t)$ is defined by $\widetilde{p}(t) = t^{-q}p(t) \in \mathbb{C}[t]$.

By denoting $L_{\alpha,i} = x^\alpha t^{q+i}$ (we shall use the two notations interchangeably), we can realize $\mathcal{B}(q)$ defined by (1.1) in the space $\mathbb{C}[x,x^{-1}]\otimes t^q\mathbb{C}[t]\oplus\mathbb{C}C$ with Lie brackets

$$\begin{aligned}[x^\alpha f(t), x^\beta g(t)] &= x^{\alpha+\beta}t^{1-q}(\beta f'(t)g(t) - \alpha f(t)g'(t)) \\ &\quad + \frac{\alpha^3-\alpha}{12}\delta_{\alpha+\beta,0}\widetilde{f}(0)\widetilde{g}(0)C.\end{aligned} \tag{3.1}$$

Here $f'(t) = \frac{df(t)}{dt}$. Note that the central charge term given in (3.1) is consistent with that given by equation (2.1) in [21]. The Lie algebra $\mathcal{B}(q)$ has a natural $\mathbb{Z}$-gradation $\mathcal{B}(q) = \oplus_{\alpha\in\mathbb{Z}}\mathcal{B}(q)_\alpha$ with

$$\mathcal{B}(q)_\alpha = \mathrm{span}\{L_{\alpha,i}\,|\,i\in\mathbb{Z}_+\}\oplus\delta_{\alpha,0}\mathbb{C}C.$$

Putting $\mathcal{B}(q)_\pm = \oplus_{\pm\alpha>0}\mathcal{B}(q)_\alpha$, we have the following triangular decomposition:

$$\mathcal{B}(q) = \mathcal{B}(q)_- \oplus \mathcal{B}(q)_0 \oplus \mathcal{B}(q)_+.$$

Note that $\mathcal{B}(q)_0 = t^q\mathbb{C}[t]\oplus\mathbb{C}C$ is an infinite dimensional commutative subalgebra of $\mathcal{B}(q)$ (but not a Cartan subalgebra). Denote by $\mathcal{B}(q)_0^*$ the dual space of $\mathcal{B}(q)_0$.

Definition 3.2. (1) A module V over $\mathcal{B}(q)$ is called

(a) $\mathbb{Z}$*-graded* if $V = \oplus_{\alpha\in\mathbb{Z}} V_\alpha$ and $\mathcal{B}(q)_\alpha V_\beta \subset V_{\alpha+\beta}$ for all α, β;

(b) *quasifinite* if it is $\mathbb{Z}$-graded and $\dim V_\beta < \infty$ for all β;

(c) a *highest weight module* if there exists some $\Lambda \in \mathcal{B}(q)_0^*$ such that $V = V(\Lambda)$, where $V(\Lambda)$ is a $\mathbb{Z}$-graded module generated by a weight vector $v_\Lambda \in V(\Lambda)_0$ which satisfies $hv_\Lambda = \Lambda(h)v_\Lambda$ for $h \in \mathcal{B}(q)_0$ and $\mathcal{B}(q)_+ v_\Lambda = 0$.

(2) A nonzero vector v in a $\mathbb{Z}$-graded module V is called *singular* if $\mathcal{B}(q)_+ v = 0$.

Suppose $L(\Lambda)$ is an irreducible highest weight $\mathcal{B}(q)$-module with highest weight Λ. By [21, Lemma 3.3], if $L(\Lambda)$ is quasifinite, then there exists a monic polynomial $f(t) \in t^q\mathbb{C}[t]$ such that $(x^{-1}f(t))v_\Lambda = 0$. If such an $f(t)$ is of minimal degree, it is called the *characteristic polynomial* of $L(\Lambda)$, which is uniquely determined by the highest weight Λ. In this case, $\widetilde{f}(t) = t^{-q}f(t)$ is called the *shifted characteristic polynomial* of $L(\Lambda)$.

The highest weight $\Lambda \in \mathcal{B}(q)_0^*$ is determined by the *central charge* $c = \Lambda(C)$, and the *labels* $h_i = -\Lambda(L_{0,i}) = -\Lambda(t^{q+i})$, $i \in \mathbb{Z}_+$. Note here that the definition of labels is slightly different from that given in [21] (in fact, $h_i = \Lambda(L_{0,i})$ defined there), but this is not essential. Define the following *generating series* with variable z:

$$\Delta_\Lambda(z,q) = 2q\sum_{i=0}^{\infty}\frac{h_i}{i!}z^i + \sum_{i=0}^{\infty}\frac{h_{i+1}}{i!}z^{i+1} = -\Lambda((2q+zt)t^q e^{zt}).$$

It was shown in [21, Theorem 1.4] that $L(\Lambda)$ is quasifinite if and only if the generating series $\Delta_\Lambda(z,q)$ is a quasipolynomial, namely $\Delta_\Lambda(z,q)$ can be uniquely written as a finite sum of the form

$$\Delta_\Lambda(z,q) = \sum_{\gamma\in\Gamma} m_\gamma(z)e^{\gamma z}, \tag{3.2}$$

where Γ is a finite subset of $\mathbb{C}$, and $m_\gamma(z) \in \mathbb{C}[z]$. It is well-known [11, Lemma 4.2] that a formal power series is a quasipolynomial if and only if it satisfies a nontrivial linear differential equation with constant coefficients. We call γ the *exponents* of $L(\Lambda)$ with *multiplicities* $m_\gamma(z)$. One can see from the proof of [21, Theorem 1.4] that $\widetilde{f}(\frac{d}{dz})\Delta_\Lambda(z,q) = 0$. Furthermore, the set of roots of $\widetilde{f}(t)$ is exactly Γ by (3.2).

We summarize the above discussion into the following lemma.

Lemma 3.3. *An irreducible highest weight $\mathcal{B}(q)$-module $L(\Lambda)$ associated with $\Delta_\Lambda(z,q)$ is quasifinite if and only if $\widetilde{f}(\frac{d}{dz})\Delta_\Lambda(z,q) = 0$, where $\widetilde{f}(t)$ is the shifted characteristic polynomial, and the set of roots of $\widetilde{f}(t)$ is Γ.*

To emphasize the dependence of $L(\Lambda)$ on the set Γ in this case, we will denote it by $L(\Lambda;\Gamma)$.

Definition 3.4. An irreducible QHWM $L(\Lambda;\Gamma)$ over $\mathcal{B}(q)$ is called *primitive* if

(1) Γ is an empty set, or

(2) the multiplicities of all exponents in Γ are nonzero complex constants.

Note that our definition of primitive modules over $\mathcal{B}(q)$ is similar to those over $\mathcal{W}_{1+\infty}$ and $\hat{D}^{\pm}$ (the central extension of the Lie subalgebras of $\mathcal{W}_{1+\infty}$ fixed by $-\sigma_{\pm}$, where $\sigma_{\pm}$ are anti-involutions of $\mathcal{W}_{1+\infty}$) introduced in [6, 12, 13]. From their elegant work, we can see that the primitive modules over $\mathcal{W}_{1+\infty}$ and $\hat{D}^{\pm}$ are particularly important, especially on free field realization and vertex algebra theory. Due to the following lemma, the primitive $\mathcal{B}(q)$-modules seem to be also interesting. We shall see in the next section that the primitive $\mathcal{B}(q)$-modules include all the unitary quasifinite irreducible highest weight $\mathcal{B}(q)$-modules.

Lemma 3.5. *Let $L(\Lambda;\Gamma)$ be an irreducible QHWM over $\mathcal{B}(q)$. Suppose that $\widetilde{f}(t)$ is the shifted characteristic polynomial of $L(\Lambda;\Gamma)$. We have*

(1) *if $\widetilde{f}(t)=1$, then $(2q+i)h_i=0$ for $i\in\mathbb{Z}_+$;*

(2) *if $\widetilde{f}(t)$ has only simple roots, then there exist nonzero complex constants m_γ such that*

$$(2q+i)h_i=\sum_{\gamma\in\Gamma} m_\gamma\gamma^i \quad \text{for} \quad i\in\mathbb{Z}_+. \tag{3.3}$$

In both cases, $L(\Lambda;\Gamma)$ is primitive.

Proof. (1) Note that $\widetilde{f}(t)=1$ if and only if $\Gamma=\phi$ by Lemma 3.3. Hence $L(\Lambda;\Gamma)$ is primitive. Furthermore, we have $\Delta_\Lambda(z,q)=0$, and so $(2q+i)h_i=0$ for $i\in\mathbb{Z}_+$.

(2) Suppose that $\widetilde{f}(t)=(t-\gamma_1)\cdots(t-\gamma_k)$, where γ_i are different from each other. By Lemma 3.3, the set of exponents is exactly $\Gamma=\{\gamma_1,\ldots,\gamma_k\}$, and there exist k nonzero polynomials $m_i(z)\in\mathbb{C}[z]$ such that

$$\widetilde{f}\Big(\frac{d}{dz}\Big)\sum_{i=1}^{k} m_i(z)e^{\gamma_i z}=0.$$

Direct computation shows that the coefficient of the term $m_i'(z)e^{\gamma_i z}$ on the left hand side of the above equation is $\prod_{1\le j\ne i\le k}(\gamma_i-\gamma_j)\ne 0$, which implies that $m_i(z)$ must be a constant (simply denoted m_i). Hence $L(\Lambda;\Gamma)$ is primitive. Now the formula (3.2) can be written explicitly as follows:

$$2q\sum_{i=0}^{\infty}\frac{h_i}{i!}z^i+\sum_{i=0}^{\infty}\frac{h_{i+1}}{i!}z^{i+1}=\sum_{i=0}^{\infty}\sum_{j=1}^{k} m_j\frac{\gamma_j^i}{i!}z^i.$$

Comparing the coefficients of z^i in the two sides of the above equality, we obtain (3.3). ■

4. Unitary quasifinite highest weight modules

Hereafter we assume that $0\ne q\in\mathbb{R}$. Recall the definition of the conjugate-linear Lie algebra anti-involution ω of $\mathcal{B}(q)$ defined by (2.3).

Definition 4.1. A module V over $\mathcal{B}(q)$ is called *unitary* if there exists a positive definite Hermitian form $\langle\cdot\,,\cdot\rangle$ which is *contravariant* with respect to ω, namely, $\omega(X)$ and X are adjoint operators on V with respect to $\langle\cdot\,,\cdot\rangle$ for $X\in\mathcal{B}(q)$.

Suppose $L(\Lambda)$ is a unitary irreducible QHWM over $\mathcal{B}(q)$ with highest weight vector $v_\Lambda\in L(\Lambda)_0$, and the shifted characteristic polynomial $\widetilde{f}(t)$. Without loss of generality, we assume that $\langle v_\Lambda,v_\Lambda\rangle=1$ through out the paper. The shifted characteristic polynomial $\widetilde{f}(t)$ of $L(\Lambda)$ will be described in Lemma 4.3 and Remark 4.4.

We first prove the following result.

Lemma 4.2. *Let $L(\Lambda)$ be a unitary irreducible QHWM over $\mathcal{B}(q)$. Then c and all h_i are real. Furthermore, if $L(\Lambda)_{-1}\neq 0$, then $c\geq 0$ and $qh_0>0$.*

Proof. We simply denote $L(\Lambda)$ by V. Suppose that $\widetilde{f}(t)$ is the shifted characteristic polynomial of V, and $\deg\widetilde{f}(t)=k$. Since

$$-\bar{h}_i=\langle t^{q+i}v_\Lambda,v_\Lambda\rangle=\langle v_\Lambda,t^{q+i}v_\Lambda\rangle=-h_i,$$

we see that the labels h_i are real. Similarly, the central charge c is also real.

Now assume that $V_{-1}\neq 0$. We claim that $k=\deg\widetilde{f}(t)\geq 1$. Otherwise, $\widetilde{f}(t)=1$, i.e., $L_{-1,0}v_\Lambda=0$. In case $q\neq -1$, applying $L_{-1,i+1}=\frac{1}{q+1}[L_{-1,i},L_{0,1}]$ to v_Λ, by induction we obtain $V_{-1}=0$, a contradiction. In case $q=-1$, applying $[L_{-1,0},L_{0,i}]=(i-1)L_{-1,i}$ to v_Λ, we obtain $L_{-1,i}v_\Lambda=0$ for $i\neq 1$. So, we must have $L_{-1,1}v_\Lambda\neq 0$ by assumption $V_{-1}\neq 0$. Applying $[L_{-1,0},L_{1,i}]=(i-2)L_{0,i}$ to v_Λ, we obtain $L_{0,i}v_\Lambda=0$ for $i\neq 2$. Hence,

$$L_{1,i}L_{-1,1}v_\Lambda=[L_{1,i},L_{-1,1}]v_\Lambda=(1-i)L_{0,i+1}v_\Lambda=0\ \text{ for all }\ i\in\mathbb{Z}_+.$$

On the other hand, it is clear that $L_{\alpha,i}L_{-1,1}v_\Lambda=0$ for $\alpha\geq 2$ and $i\in\mathbb{Z}_+$. Thus $L_{-1,1}v_\Lambda$ is a singular vector in the unitary module V, and we necessarily have $L_{-1,1}v_\Lambda=0$, contradicting the given assumption. Hence the claim holds.

By the definition of the characteristic polynomial, we have

$$(x^{-1}t^q\widetilde{f}(t))v_\Lambda=(x^{-1}f(t))v_\Lambda=0. \tag{4.1}$$

Let $u_i:=(x^{-1}t^{q+i})v_\Lambda$ for $i\geq 0$. Then

$$V_{-1}=\begin{cases}\mathrm{span}\{u_i\,|\,0\leq i<k\} & \text{if } q\neq -1,\\ \mathrm{span}\{u_i,u_{k+1}\,|\,0\leq i<k\} & \text{if } q=-1.\end{cases} \tag{4.2}$$

It is clear from (4.1) that $u_0,\ldots,u_k$ are linearly dependent. However, $u_0,\ldots,u_{k-1}$ are linearly independent, as otherwise there would exist a polynomial $\widetilde{g}(t)$ with $\deg\widetilde{g}(t)<k=\deg\widetilde{f}(t)$ such that $(x^{-1}t^q\widetilde{g}(t))v_\Lambda=0$. This contradicts the fact that f is the characteristic polynomial.

If $q\neq -1$, we have $x^{-1}t^{q+i}\widetilde{f}(t)=\frac{1}{q+1}[x^{-1}t^{q+i-1}\widetilde{f}(t),t^{q+1}]$ for all $i\geq 1$. Applying the $i=1$ case of this relation to v_Λ, we see that $(x^{-1}t^{q+1}\widetilde{f}(t))v_\Lambda=0$, since t^{q+1} acts on v_Λ by multiplication by the scalar $\Lambda(t^{q+1})$. Now applying this relation with arbitrary $i \geq 1$ to v_Λ, we immediately show by induction on i that $(x^{-1}t^{q+i}\widetilde{f}(t))v_\Lambda = 0$ for all $i \geq 1$. This implies that all u_{k+i} with $i \geq 1$ are linear combinations of $\{u_i \,|\, 0 \leq i < k\}$. So, (4.2) holds in case $q \neq -1$. In fact, we have shown that $\{u_i \,|\, 0 \leq i < k\}$ is a basis of V_{-1} in this case.

If $q = -1$, applying $x^{-1}t^{i+2}f(t) = \frac{1}{1+i}[x^{-1}f(t), t^{1+i}]$ with $i \geq 0$ to v_Λ, we obtain $x^{-1}t^{i+2}f(t)v_\Lambda = 0$ for $i \geq 0$. These equalities, together with (4.1), inductively imply that all u_i are linear combinations of $\{u_i, u_{k+1} \,|\, 0 \leq i < k\}$, i.e., (4.2) holds in case $q \neq -1$. This completes the proof of (4.2).

Since $u_0, \ldots, u_{k-1}$ in (4.2) are linearly independent, we have

$$0 < \langle u_i, u_i\rangle = \langle [xt^{q+i}, x^{-1}t^{q+i}]v_\Lambda, v_\Lambda\rangle = \langle -2(q+i)t^{q+2i}v_\Lambda, v_\Lambda\rangle = 2(q+i)h_{2i}. \quad (4.3)$$

Taking $i = 0$ in (4.3) gives $qh_0 > 0$. (We would like to point out here that if $0 > q \in \mathbb{Z}$, then $1 \leq k \leq -q$ by (4.3).) Now, for $\alpha \geq 2$, consider the vector $v_\alpha = (x^{-\alpha}t^q)v_\Lambda \in V_{-\alpha}$. We have

$$(x^{\alpha-1}t^q)v_\alpha = [x^{\alpha-1}t^q, x^{-\alpha}t^q]v_\Lambda = q(1-2\alpha)(x^{-1}t^q)v_\Lambda \neq 0,$$

which implies that $v_\alpha \neq 0$. Then

$$\begin{aligned} 0 < \langle v_\alpha, v_\alpha\rangle &= \langle (x^\alpha t^q)(x^{-\alpha}t^q)v_\Lambda, v_\Lambda\rangle = \langle [x^\alpha t^q, x^{-\alpha}t^q]v_\Lambda, v_\Lambda\rangle \\ &= \langle (-2q\alpha t^q + \tfrac{\alpha^3-\alpha}{12}C)v_\Lambda, v_\Lambda\rangle = 2q\alpha h_0 + \tfrac{\alpha^3-\alpha}{12}c, \end{aligned}$$

which implies that $c > -\frac{24qh_0}{\alpha^2-1}$ for all $\alpha \geq 2$. Note that $qh_0 > 0$. Taking $\alpha \to \infty$, we obtain $c \geq 0$. ∎

Lemma 4.3. *Let $L(\Lambda;\Gamma)$ be a unitary irreducible QHWM over $\mathcal{B}(q)$. Assume that $L(\Lambda;\Gamma)_{-1} \neq 0$, then the shifted characteristic polynomial $\widetilde{f}(t)$ of $L(\Lambda;\Gamma)$ must be of the following form:*

(1) *if $q \neq -1$, then $\widetilde{f}(t) = t$, or $\widetilde{f}(t)$ has only simple nonzero real roots;*
(2) *if $q = -1$, then $\widetilde{f}(t) = t$.*

Moreover, $\deg\widetilde{f}(t) = |\Gamma|$, *where $|\Gamma|$ denotes the number of exponents.*

Proof. We simply denote $L(\Lambda;\Gamma)$ by V. Suppose $\deg\widetilde{f}(t) = k$. Then $k \geq 1$, since $V_{-1} \neq 0$.

(1) First, we show that all the roots of $\widetilde{f}(t)$ are real. Denote by $\mathcal{U}(\mathcal{B}(q))$ the universal enveloping algebra of $\mathcal{B}(q)$. Consider the action of the element $T = -\frac{1}{q+1}(h_1 + t^{q+1}) \in \mathcal{U}(\mathcal{B}(q))$ on $V_{-1} = \mathrm{span}\{u_i \,|\, 0 \leq i < k\}$ (cf. (4.2) with $q \neq -1$). It is easy to show by induction that

$$T^i(u_0) = T^i((x^{-1}t^q)v_\Lambda) = (x^{-1}t^{q+i})v_\Lambda = u_i \quad \text{for} \quad 0 \leq i < k.$$

This in particular implies that $\widetilde{f}(T)(u_0) = 0$ and $\{T^i(u_0) \,|\, 0 \leq i < k\}$ is a basis of V_{-1}. It follows that $\widetilde{f}(t)$ is the characteristic polynomial of the operator T on V_{-1}. Since the operator $T|_{V_{-1}}$ is self-adjoint, all the roots of $\widetilde{f}(t)$ are real.

Next, we claim that all the real roots of $\widetilde{f}(t)$ are simple. Assume that γ is a real root of $\widetilde{f}(t)$ of multiplicity m. Write $\widetilde{f}(t) = (t-\gamma)^m g(t)$ for some polynomial $g(t) \in \mathbb{C}[t]$. For the nonzero vector $u = (T-\gamma)^{m-1} g(T)(u_0) \in V_{-1}$, we have

$$\langle u, u\rangle = \langle g(T)(u_0), (T-\gamma)^{2m-2} g(T)(u_0)\rangle = 0 \quad \text{for} \quad m \geq 2.$$

Hence the unitarity condition implies $m = 1$, and so the claim holds.

Now assume that zero is a root of $\widetilde{f}(t)$. We only need to show that $\widetilde{f}(t) = t$. For $p(t) \in \mathbb{C}[t]$, clearly we can write $p(t) = p(\frac{\partial}{\partial z})e^{zt}|_{z=0}$. More generally, for $p(t) \in t^q\mathbb{C}[t]$, by induction we have $p(t)^i = \widetilde{p}(\frac{\partial}{\partial z})^i (t^{iq} e^{zt})|_{z=0}$ for $i \geq 1$. Let $v_f = (x^{-2} f(t))v_\Lambda \in V_{-2}$. By (3.1)–(3.2) and Lemma 3.3, we have

$$\begin{aligned}
\langle v_f, v_f\rangle &= \langle [x^2 f(t), x^{-2} f(t)] v_\Lambda, v_\Lambda\rangle \\
&= \langle (-4t^{1-q} f(t) f'(t) + \tfrac{1}{2}\widetilde{f}(0)^2 C) v_\Lambda, v_\Lambda\rangle \\
&= -2\langle t^{1-q} (f(t)^2)' v_\Lambda, v_\Lambda\rangle \\
&= -2\big\langle t^{1-q} \widetilde{f}(\tfrac{\partial}{\partial z})^2 (t^{2q} e^{zt})' v_\Lambda, v_\Lambda\big\rangle\big|_{z=0} \\
&= -2\big\langle \widetilde{f}(\tfrac{\partial}{\partial z})^2 (2q + zt) t^q e^{zt} v_\Lambda, v_\Lambda\big\rangle\big|_{z=0} \\
&= 2\widetilde{f}(\tfrac{\partial}{\partial z})^2 \Delta_\Lambda(z, q)\big|_{z=0} = 0,
\end{aligned}$$

which implies that $v_f = 0$. Since $(x^{-1} f(t))v_\Lambda = (x^{-1} t^q \widetilde{f}(t))v_\Lambda = 0$, we have

$$\begin{aligned}
0 &= (xt^q)v_f = [xt^q, x^{-2} f(t)]v_\Lambda \\
&= -(2qx^{-1} f(t) + x^{-1} t f'(t))v_\Lambda = -x^{-1} t f'(t) v_\Lambda \\
&= -(x^{-1} t^{q+1} \widetilde{f}'(t) + q x^{-1} t^q \widetilde{f}(t)) v_\Lambda \\
&= -x^{-1} t^{q+1} \widetilde{f}'(t) v_\Lambda. \qquad (4.4)
\end{aligned}$$

Hereafter $\widetilde{f}'(t)$ denotes the derivative of $\widetilde{f}(t)$. Suppose $k \geq 2$. Then we can write $\widetilde{f}(t) = t(t-\gamma_1)\cdots(t-\gamma_{k-1})$, where γ_i are different nonzero real numbers. Note that $\deg(t\widetilde{f}'(t)) = k$ and that the leading coefficient of $t\widetilde{f}'(t)$ is k. By (4.4) and the uniqueness of the shifted characteristic polynomial, we must have

$$t\widetilde{f}'(t) = k\widetilde{f}(t). \qquad (4.5)$$

Letting $t = \gamma_1$ in (4.5), we obtain two cases: $\gamma_1^2 = 0$ if $k = 2$, and $\gamma_1^2 \prod_{i=2}^{k-1}(\gamma_1 - \gamma_i) = 0$ if $k \geq 3$, each of which is a contradiction. Hence $k = 1$, and so $\widetilde{f}(t) = t$.

(2) Since $q = -1$, we must have $k = 1$ by (4.3). Suppose $\widetilde{f}(t) = t - \gamma$ for some $\gamma \in \mathbb{C}$. By Lemma 3.5(2), there exists a nonzero constant m such that

$$(i-2)h_i = m\gamma^i \quad \text{for} \quad i \in \mathbb{Z}_+.$$

Taking $i = 2$ in the above equation, we obtain $\gamma = 0$. Hence $\widetilde{f}(t) = t$.

The last statement is a direct corollary of (1), (2) and Lemma 3.3. ■

Denote by $e_i(Z)$ the i-th elementary symmetric polynomials on any given set $Z = \{z_1, \ldots, z_n\}$, namely, we define

$$e_i(Z) = \sum_{1 \leq j_1 < \cdots < j_i \leq n} z_{j_1} \cdots z_{j_i} \quad \text{for} \quad 1 \leq i \leq n.$$

Now we prove the main result, Theorem 1.1.

Proof of Theorem 1.1. For notational simplicity we denote $L(\Lambda) = L(\Lambda;\Gamma)$ by V. Let $\widetilde{f}(t)$ be the shifted characteristic polynomial of V, and $\deg\widetilde{f}(t) = k$. From the definition of shifted characteristic polynomial and $\dim V_{-1} < \infty$, it is easy to show that $V_{-1} \neq 0$ if and only if $k \geq 1$. If $q \notin \mathbb{R}$, then there is nothing to prove by Lemma 2.2. Suppose $0 \neq q \in \mathbb{R}$ in what follows.

Claim 1. Cases $k \geq 2$, or $k = 1$ and $\Gamma \neq \{0\}$ can not occur.

Otherwise, $V_{-1} \neq 0$. By Lemma 4.3, we have $q \neq -1$, and we can suppose that $\widetilde{f}(t) = (t-\gamma_1)\cdots(t-\gamma_k)$, where $\gamma_i \in \Gamma$ are different nonzero real numbers. Recall the notation $u_i = (x^{-1}t^{q+i})v_\Lambda$ we introduced in Lemma 4.2. First we claim that $u_k \neq 0$. Otherwise $0 = (x^{-1}t^q\widetilde{f}(t))v_\Lambda = (x^{-1}t^q\widetilde{g}(t))v_\Lambda$, where $\widetilde{g}(t) = \widetilde{f}(t) - t^k$. But, $\deg\widetilde{g}(t) = \deg\widetilde{f}(t) - 1 = k-1 < k$, a contradiction. Applying

$$x^{-1}t^{q+i}\widetilde{f}(t) = \frac{1}{q+1}[x^{-1}t^{q+i-1}\widetilde{f}(t), t^{q+1}]$$

to v_Λ, we have shown, in the arguments in proving (4.2) with $q \neq -1$, that $(x^{-1}t^{q+i}\widetilde{f}(t))v_\Lambda = 0$ for $i \in \mathbb{Z}_+$. Using $(x^{-1}t^q\widetilde{f}(t))v_\Lambda = 0$, $(x^{-1}t^{q+1}\widetilde{f}(t))v_\Lambda = 0$, $u_k \neq 0$, and $\gamma_i \neq 0$, one can easily show that

$$\{L_{-1,j}v_\Lambda \,|\, 1 \leq j < k+1\} \text{ is a basis of } V_{-1}. \tag{4.6}$$

The equalities $(x^{-1}t^{q+i}\widetilde{f}(t))v_\Lambda = 0$, $i \in \mathbb{Z}_+$ also imply that

$$L_{-1,n}v_\Lambda - \sum_{j=0}^{k-1}(-1)^j e_{j+1}(\Gamma)L_{-1,n-j-1}v_\Lambda = 0 \quad \text{for } n \geq k. \tag{4.7}$$

Generalising the left hand side of (4.7), we define

$$X_{\beta,n} = L_{\beta,n} - \sum_{j=0}^{k-1}(-1)^j e_{j+1}(\Gamma)L_{\beta,n-j-1}, \quad \forall \beta \in \mathbb{Z},$$

and let $v_{\alpha,n} = X_{-\alpha,n}v_\Lambda$ for all $\alpha \geq 1$ and $n \geq k+1$. Then clearly, $v_{1,n} = 0$ by (4.7). Using (1.1), we can compute $\langle v_{\alpha,n}, v_{\alpha,n}\rangle$. We have

$$\begin{aligned}
\langle v_{\alpha,n}, v_{\alpha,n}\rangle &= \Big\langle \Big[X_{\alpha,n}, X_{-\alpha,n}\Big]v_\Lambda, v_\Lambda\Big\rangle \\
&= 2\alpha(q+n)h_{2n} - 2\sum_{j=0}^{k-1}(-1)^j\alpha(2q+2n-j-1)e_{j+1}(\Gamma)h_{2n-j-1} \\
&+ \sum_{i=0}^{k-1}\sum_{j=0}^{k-1}(-1)^{i+j}\alpha(2q+2n-i-j-2)e_{i+1}(\Gamma)e_{j+1}(\Gamma)h_{2n-i-j-2} \\
&= \alpha\Big\langle\Big[X_{1,n}, X_{-1,n}\Big]v_\Lambda, v_\Lambda\Big\rangle = \alpha\langle v_{1,n}, v_{1,n}\rangle = 0,
\end{aligned} \tag{4.8}$$

which implies $v_{\alpha,n} = 0$, namely,

$$L_{-\alpha,n}v_\Lambda = \sum_{j=0}^{k-1}(-1)^j e_{j+1}(\Gamma)L_{-\alpha,n-j-1}v_\Lambda \quad \text{for} \quad \alpha \geq 1 \text{ and } n \geq k+1. \tag{4.9}$$

Note that, in the case $\alpha \geq 2$ and $n = k$, the elements of the form $L_{\alpha,0}$, and thus the central charge c, will appear in (4.8). So, the formula (4.8) in generally does not hold for $n = k$, and neither does (4.9) (except $\alpha = 1$, i.e., (4.7)). Applying $[L_{-2,k+1}, L_{1,0}] = (3q+k+1)L_{-1,k+1}$ to v_Λ, by (4.9) with $n = k+1$, we obtain

$$\begin{aligned}
&(3q+k+1)\sum_{j=0}^{k-1}(-1)^j e_{j+1}(\Gamma)L_{-1,k-j}v_\Lambda \\
&= -\sum_{j=0}^{k-1}(-1)^j e_{j+1}(\Gamma)L_{1,0}L_{-2,k-j}v_\Lambda \\
&= -\sum_{j=0}^{k-1}(-1)^j e_{j+1}(\Gamma)[L_{1,0}, L_{-2,k-j}]v_\Lambda \\
&= \sum_{j=0}^{k-1}(-1)^j (3q+k-j)e_{j+1}(\Gamma)L_{-1,k-j}v_\Lambda,
\end{aligned}$$

which gives $\sum_{j=0}^{k-1}(-1)^j(j+1)e_{j+1}(\Gamma)L_{-1,k-j}v_\Lambda = 0$. Hence $L_{-1,1}v_\Lambda, \ldots, L_{-1,k}v_\Lambda$ are linearly dependent vectors in V_{-1}, which contradicts (4.6). So, Claim 1 holds.

Claim 2. If $k = 1$ and $\Gamma = \{0\}$, then $L_{-\alpha,i}v_\Lambda = 0$ for $\alpha \geq 1$ and $i \geq 1$.

In this case, we still have $V_{-1} \neq 0$. By Lemma 4.3, $\widetilde{f}(t) = t$. By Lemma 3.5(2) and Lemma 4.2, there exists $m > 0$ such that

$$(2q+i)h_i = \begin{cases} m & \text{if } i = 0, \\ 0 & \text{if } i \geq 1. \end{cases} \tag{4.10}$$

We first show that Claim 2 holds for case $\alpha = 1$, namely,

$$L_{-1,i}v_\Lambda = 0 \quad \text{for} \quad i \geq 1. \tag{4.11}$$

First, by the definition of shifted characteristic polynomial, $L_{-1,1}v_\Lambda = 0$ since $\widetilde{f}(t) = t$. In case $q \neq -1$, applying $L_{-1,i+1} = \frac{1}{q+1}[L_{-1,i}, L_{0,1}]$ to v_Λ, by induction we immediately obtain (4.11). In case $q = -1$, applying $L_{-1,i+1} = \frac{1}{i-1}[L_{-1,1}, L_{0,i}]$ with $i \geq 2$ to v_Λ, we inductively obtain $L_{-1,i}v_\Lambda = 0$ for $i \geq 3$. Next, we only need to show that $L_{-1,2}v_\Lambda = 0$ in case $q = -1$. Assume that $L_{-1,2}v_\Lambda \neq 0$. By (1.1) and (4.10) with $q = -1$, we have

$$L_{1,i}L_{-1,2}v_\Lambda = [L_{1,i}, L_{-1,2}]v_\Lambda = -iL_{0,i+2}v_\Lambda = ih_{i+2}v_\Lambda = 0 \quad \text{for} \quad i \in \mathbb{Z}_+.$$

On the other hand, it is clear that $L_{\alpha,i}L_{-1,2}v_\Lambda = 0$ for $\alpha \geq 2$ and $i \in \mathbb{Z}_+$. Hence, we have shown that $L_{-1,2}v_\Lambda$ is a singular vector in V_{-1}, a contradiction. So, $L_{-1,2}v_\Lambda = 0$ in case $q = -1$. This proves (4.11).

Furthermore, by (4.11), it is easy to show that Claim 2 also holds for case $\alpha = 2$. Otherwise, one can show that there at least exists a singular vector $L_{-2,i_0}v_\Lambda \in V_{-2}$ with $i_0 \geq 1$ by similar arguments as above, a contradiction. By induction on α, Claim 2 holds.

Claim 3. If $k = 0$, then $L_{-\alpha,i}v_\Lambda = 0$ for $\alpha \geq 1$ and $i \geq 1$.

In this case, $V_{-1} = 0$ and $\widetilde{f}(t) = 1$. As we have shown in Lemma 3.5(1), the labels h_i satisfy

$$(2q + i)h_i = 0 \quad \text{for} \quad i \in \mathbb{Z}_+. \tag{4.12}$$

This can be viewed as a special case of (4.10) with $m = 0$. The case $\alpha = 1$ of Claim 3 clearly holds, since $\widetilde{f}(t) = 1$. By similar arguments in Claim 2 and induction on α, Claim 3 holds.

Now it follows Claims 1–3 that

$$V = \mathcal{U}(\mathcal{B}(q)_- \oplus \mathcal{B}(q)_0)v_\Lambda = \mathcal{U}(\mathrm{Vir}_- \oplus \mathcal{B}(q)_0)v_\Lambda.$$

By (4.10) and (4.12), we have $L_{0,i}v_\Lambda = -\delta_{2q+i,0}h_{-2q}v_\Lambda$ for $i \geq 1$. Regarding V as a Vir-module and using Theorem 2.3, we complete the proof of Theorem 1.1. ∎

Remark 4.4. ¿From the proof of Theorem 1.1, we see that $\widetilde{f}(t) = t$ (Claim 1 and Claim 2) if $V_{-1} \neq 0$, and $\widetilde{f}(t) = 1$ if $V_{-1} = 0$ (Claim 3). Thus the unitary quasifinite irreducible highest weight $\mathcal{B}(q)$-modules are primitive modules by Lemma 3.5.

5. Discussions

Any unitary highest weight Vir-module $L(\Lambda)$ can be *trivially* extended to a unitary $\mathcal{B}(q)$-module by requiring all $L_{a,i}$ with $i > 0$ act trivially. In Claim 2 in the proof of Theorem 1.1, the unitary $\mathcal{B}(q)$-module $L(\Lambda; \Gamma)$ is a primitive $\mathcal{B}(q)$-module with $\Gamma = \{0\}$, which is trivially extended from a unitary highest weight Vir-module $L(c_{\mathrm{Vir}}, h_{\mathrm{Vir}})$ with $h_{\mathrm{Vir}} > 0$.

Given a unitary highest weight Vir-module $L(\Lambda)$, and a real number $h \in \mathbb{R}$, we construct a $\mathcal{B}(q)$-module from $L(\Lambda)$ in the following way. For all $i \geq 1$ and $v \in L(\Lambda)$, $L_{\alpha,i}$ acts on v by $L_{\alpha,i}v = hv$ if $(\alpha, i) = (0, -2q)$, and $L_{\alpha,i}v = 0$ otherwise. Note that if $q \notin \frac{1}{2}\mathbb{Z}^*_-$ or $h = 0$, this degenerates to the trivial extension discussed earlier. If $q \in \frac{1}{2}\mathbb{Z}^*_-$ and $h \neq 0$, we call this $\mathcal{B}(q)$-module an *almost-trivial extension* of Vir-module $L(\Lambda)$. Clearly, this is a unitary quasifinite irreducible highest weight $\mathcal{B}(q)$-module. In Claim 3 in the proof of Theorem 1.1, the unitary $\mathcal{B}(q)$-module $L(\Lambda; \Gamma)$ is a primitive $\mathcal{B}(q)$-module with $\Gamma = \phi$, which is obtained from a unitary highest weight Vir-module $L(c_{\mathrm{Vir}}, 0)$ this way.

It will be interesting to determine the unitary uniformly bounded $\mathcal{B}(q)$-modules. In view of [4], we expect that any unitary irreducible uniformly bounded $\mathcal{B}(q)$-module must be an intermediate series $\mathcal{B}(q)$-module, which gives a unitary version of the conjecture [21, Conjecture 1.6].

We wish to point out that, Kac, Liberati and Radul used natural homomorphisms from the algebra $\mathcal{W}_\infty$ (resp. $\mathcal{W}_{1+\infty}$) to the central extension of the

Lie algebra of infinite matrices with finitely many nonzero diagonals $\widehat{gl}_\infty$ (first to gl_∞, and then lifted to homomorphisms of the corresponding central extensions) in the classification of the unitary quasifinite highest weight modules over $\mathcal{W}_\infty$ (resp. $\mathcal{W}_{1+\infty}$) in [9, 11]. However, to the best of our knowledge, no homomorphisms from $\mathcal{B}(q)$ to gl_∞ or $\widehat{gl}_\infty$ are known, thus it is not clear how to adapt the techniques of [9, 11] to study the unitary quasifinite representations of $\mathcal{B}(q)$.

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