Infinite Loop Spaces Associated to Affine Kac-Moody Groups

Xianzu Lin

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Abstract. The main purpose of this paper is to construct infinite loop spaces from affine Kac-Moody groups. It is well known that to each infinite class of classical groups over a commutative ring \( R \), we can associate an infinite loop space \( G(R) \) by Quillen’s plus construction. In this paper we generalize this fact to the cases of affine Kac-Moody groups. Roughly speaking, for each commutative ring \( R \) there are seven infinite classes of affine Kac-Moody groups over \( R \), and to each infinite class we can associate an analogous infinite loop space.

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1. Introduction

The loop space \( \Omega X \) of a pointed space \( X \) is the space of pointed maps from the unit circle \( S^1 \) to \( X \) together with the compact-open topology. We say that a pointed space \( X \) is an infinite loop space if there is a sequence of (pointed) spaces \( X_0, X_1, \cdots \) with \( X_0 = X \) and weak homotopy equivalences \( X_n \simeq \Omega X_{n+1} \) for each \( n \geq 0 \).

Example 1.1. Let \( GL(n) \) be the general linear group over \( \mathbb{C} \) and let \( BGL \) be the limit of classifying space \( \lim \rightarrow BGL(n) \). By the Bott periodicity theorem \([1,2]\) we have a weak homotopy equivalence

\[ \mathbb{Z} \times BGL \simeq \Omega^2 (\mathbb{Z} \times BGL) ; \]

thus \( BGL \) is an infinite loop space. Similar results hold for \( BO \) and \( BSp \), where \( O \) (resp. \( Sp \)) is the infinite orthogonal (resp. symplectic) group over \( \mathbb{C} \).

Now we introduce a theorem about construction of infinite loop spaces. First, we need some preliminaries.

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Let $\Sigma_n$ be the symmetric group on the set $\{1, 2, \cdots, n\}$. For any $\sigma \in \Sigma_m$ and $\tau \in \Sigma_n$, $c(m, n) \in \Sigma_{m+n}$ is defined by

\[ c(m, n)(i) = \begin{cases} 
  n + i, & 1 \leq i \leq m, \\
  i - m, & m < i \leq m + n.
\end{cases} \tag{1a} \]

The definition above implies $c(m, n) = c(n, m) - 1$.

**Theorem 1.2.** Given a sequence of topological groups

\[ G(0), G(1), G(2), \cdots, G(n), \cdots \]

together with homomorphisms $\phi_m : \Sigma_m \to G(m)$, $f_m : G(m) \to G(m+1)$, $m > 0$, satisfying,

0) $\phi_0$ is an isomorphism;
1) $f_m \phi_m(\alpha) = \phi_{m+1}(\alpha)$ for each $m > 0$;
2) set $f_{m,n} := f_{m+1} \cdots f_{m+1} f_m$, then $\phi_n(c(n, m))(f_{n,m}(G(n))) \phi_m(c(m, n))$ and $f_{m,n}(G(m))$ are commutative in $G(m+n)$;
3) let $G = \lim_{n \to \infty} G(n)$ and let $\pi' = [\pi, \pi]$ be the commutator subgroup of $\pi = \pi_0(G)$, we have $\pi' = [\pi', \pi']$.

Then $BG^+$ (where $+$ means the Quillen’s plus construction for $BG$ and $\pi' \subseteq \pi_1(BG)$) is an infinite loop space.

**Proof.** Define a topological category $\Xi$ as follows. The objects of $\Xi$ are nonnegative integers, $\text{hom}_\xi(m, n)$ is empty if $m \neq n$ and $\text{hom}_\xi(m, m) = G(m)$. One checks that $(\xi, \oplus, 0, c)$ has a structure of permutative category, $\prod_{n \geq 0} BG(n)$ is the corresponding classifying space. Then the rest of the proof carries over as in [3] p.62.

**Remark 1.3.** This theorem must be well known, but we can not find suitable reference.

**Corollary 1.4.** Let $R$ be a commutative ring and set

\[ GL(\infty, R) = \lim_{n \to \infty} GL(n, R), \]

then $BGL(\infty, R)^+$ is an infinite loop space.

**Proof.** We can easily find natural homomorphisms $\phi_n : \Sigma_n \to GL(n, R)$, $n > 0$ that satisfy conditions of Theorem 1.2.

Similarly, we can show that $BSL(\infty, R)^+$, $BO(\infty, R)^+$, $BSO(\infty, R)^+$ and $BSp(\infty, R)^+$ are all infinite loop spaces. The main purpose of this paper is to construct infinite loop spaces from affine Kac-Moody groups, which are infinite dimensional generalization of algebraic groups. Roughly speaking, for each commutative ring $R$ there are seven infinite classes of affine Kac-Moody groups over $R$, and to each infinite class we can associate an analogous infinite loop space.
This paper is structured as follows. Section 2 is a short review of Kac-Moody algebras and Kac-Moody groups. In Section 3 we construct the infinite loop spaces corresponding to affine Kac-Moody groups of type $A^{(2)}_{n-1}$. In the final section we consider several variations and the other cases. Throughout this paper $R$ will be an arbitrary commutative ring (not necessarily with unit).

2. Kac-Moody Algebras and Kac-Moody Groups

In this section, we give a brief review of the theory of Kac-Moody algebras and Kac-Moody groups, details can be found in [4, 8, 9].

Definition 2.1. A generalized Cartan matrix is a matrix $A = (a_{ij})_{ij=1}^n$ satisfying, $a_{i,i} = 2$, $a_{i,j}$ are non-positive integers for $i \neq j$, and $a_{i,j} \neq 0$ implies $a_{j,i} \neq 0$.

Definition 2.2. The Kac-Moody algebra $g(A)$ associated to a generalized Cartan matrix $A = (a_{ij})_{ij=1}^n$ is the Lie algebra (over $\mathbb{C}$) generated by $3n$ elements $e_i, f_i, h_i, (i = 1, \ldots, n)$ with the following defining relations:

\[ [h_i, h_j] = 0; \quad [h_i, e_j] = a_{ij}e_j; \quad [h_i, f_j] = -a_{ij}f_j; \quad [e_i, f_j] = \delta_{i,j}h_i; \]

\[ (ad e_i)^{-a_{ij}}e_j = 0, \quad (ad f_i)^{1-a_{ij}}f_j = 0, \quad \text{if } i \neq j. \]

Let $A = (a_{ij})_1^n$ be a generalized Cartan matrix. For a pair of indices $i, j$ such that $ij$, set $m_{i,j} = 2, 3, 4$ or 6 if $a_{i,j}a_{j,i} = 0, 1, 2$ or 3 respectively and set $m_{i,i} = 1$. We associate to $A$ a discrete group $W(A)$ (the Weyl group) on $n$ generators $s_1, \ldots, s_n$ with relations $\{(s_is_j)^{m_{i,j}} = 1\}_0<i,j<n$.

We also need another group $W'(A)$ which is defined by $n$ generators $s'_1, \ldots, s'_n$ and the following relations:

\[ s'_{j} s'_{i} s'_{j}^{-1} = s'_{i} s'_{j} s'_{i}^{-1} = s'_{j} s'_{i} s'_{j} \cdots (m_{i,j} \text{ factors on each side}). \]

By the definitions above the map $s'_i \to s_i$ extends to a group homomorphism $\phi : W'(A) \to W(A)$. As $ad e_i$ and $ad f_i$ are locally nilpotent endomorphisms of $g(A)$ (cf.[4, p.33]), the expressions $exp(e_i) = \sum_{n \geq 0} \frac{(ad e_i)^n}{n!}$ and $exp(f_i) = \sum_{n \geq 0} \frac{(ad f_i)^n}{n!}$ make sense. The map $s'_i \to exp(e_i)exp(-f_i)exp(e_i) \in Aut(g(A))$ can be extended to a homomorphism $\psi : W'(A) \to Aut(g(A))(cf.[5, 188])$; we also denote by $s'_i$ the image of $s'_i$ in $Aut(g(A))$.

Let $V$ be the vector space over $\mathbb{Q}$, with basis $\{a_i\}_{i=1,\ldots,n}$ and let $W(A)$ act on $V$ by $s_i(a_j) = a_j - a_{j,i}a_i$. Real roots of $A = (a_{ij})_1^n$ are defined to be elements of $V$ of the form $w(a_i)$, with $w \in W(A)$ and $0 < i \leq n$. Each real root $a$ is an integral linear combination of $\{a_i\}$, the coefficients of which of all positive or negative; the real root $a$ is said to be positive or negative accordingly. Denote by $\Delta$, $\Delta_+$, $\Delta_-$ the sets of all real roots, positive and negative real roots respectively. We say that a set of real roots $\theta$ is pre-nilpotent if there exist $w, w' \in W(A)$ such that all elements of $w(\theta)$ are positive and all elements of $w'(\theta)$ are negative;
if, moreover, \( a, b \in \theta \) and \( a + b \in \triangle \) imply \( a + b \in \theta \), then we said that \( \theta \) is nilpotent.

For \( 0 < i \leq n \) and \( w' \in W'(A) \), the pair of opposite elements \( w'\{e_i, -e_i\} \subset g(A) \) depends only on the real root \( a = \phi(w')(a_i) \) (cf. [9, p.547]); set \( E_a = w'\{e_i, -e_i\} \) and denote by \( L_a \) the \( \mathbb{C} \)-subalgebra of \( g(A) \) generated by \( E_a \).

For each real root \( a \), we denote by \( \mathcal{U}_a \) the group scheme over \( \mathbb{Z} \) isomorphic to \( \text{Spec} \mathbb{Z} \) and whose Lie algebra is the \( \mathbb{Z} \)-subalgebra of \( g(A) \) generated by \( E_a \).

Let \( \theta \) be a nilpotent set of real roots, then \( L_\theta = \bigoplus_{a \in \theta} L_a \) is a nilpotent Lie algebra. Let \( U_\theta \) be the unipotent complex algebraic group whose Lie algebra is \( L_\theta \). The following proposition was proved in [9].

**Proposition 2.3.** There exist a uniquely defined group scheme \( \mathcal{U}_\theta \) over \( \mathbb{Z} \) containing all \( \mathcal{U}_a \) for \( a \in \theta \), whose fibre over \( \mathbb{C} \) is the group \( U_\theta \) and such that for any order on \( \theta \), the product morphism \( \prod_{a \in \theta} \mathcal{U}_a \to \mathcal{U}_\theta \) is an isomorphism of the underlying schemes.

Now we present Tits’ definition of Kac-Moody group associated to a generalized Cartan matrix \( A = (a_{i,j})_{i,j=1}^n \) and a commutative ring \( R \).

Let \( \wedge \) be a free abelian group with basis \( h_1, \cdots, h_n \), and \( \wedge' \) its dual, then there are \( n \) elements \( \alpha_1, \cdots, \alpha_n \in \wedge' \) satisfying \( \langle h_i, \alpha_j \rangle = a_{i,j} \). Set \( \Sigma(R) = \text{Hom}(\wedge', R^*) \), where \( R^* \) is the multiplicative group of invertible elements of \( R \). The group \( W(A) \) also acts on \( \wedge' \) by \( s_i(\lambda) = \lambda - \langle \lambda, h_i \rangle \alpha_i \). The automorphism of \( \Sigma(R) \) induced by \( s_i \) will also be denoted by \( s_i \).

For a real root \( a \), and a nilpotent set of real roots \( \theta \), set \( \mathcal{U}_a(R), \mathcal{U}_\theta(R) \) to be the groups of \( R \) points of \( \mathcal{U}_a \times \text{Spec} R \) and \( \mathcal{U}_\theta \times \text{Spec} R \) respectively. For each pair of roots \( \{a, b\} \), set \( \vartheta(a, b) = (Na + Nb) \cap \triangle \).

The Steinberg group \( \mathbb{S}(R) \) over \( R \) is defined to be the inductive limit of the groups \( \mathcal{U}_a(R) \) and \( \mathcal{U}_\theta(a, b)(R) \), where \( a \in \triangle \) and \( \{a, b\} \) runs over all pre-nilpotent pairs of real roots, relative to all the canonical injections \( \mathcal{U}_c(R) \to \mathcal{U}_\vartheta(a, b)(R) \) for \( c \in \vartheta(a, b) \). For each \( 0 < i \leq n \), \( s'_i = \exp(e_i)\exp(-f_i)\exp(e_i) \) is an automorphism of \( g(A) \) which permutes the \( L_a \) and the \( E_a \); therefore, it induces an automorphism of \( \mathbb{S}(R) \) which we also denote by \( s'_i \).

**Remark 2.4.** For any \( a, b \) in a nilpotent set \( \theta \) of real roots and any \( r, r' \in R \), the following commutation relation holds inside \( \mathbb{S}(R) \):

\[
[x_a(r), x_b(r')] = \prod_{c=ma+nb} x_c(k(a, b; c)r^m r'^n),
\]

where \( c = ma + nb \) runs over \( \vartheta(a, b) - \{a, b\}, k(a, b; c) \in \mathbb{Z} \) and \( x_a : R \to \mathbb{S}(R) \), \( x_b : R \to \mathbb{S}(R) \) denote respectively the homomorphisms associated to \( a \) and \( b \).

**Definition 2.5.** The Kac-Moody group \( G_A(R) \) associated to \( A \) over \( R \) is defined to be the quotient of the free product of \( \mathbb{S}(R) \) and \( \Sigma(R) \) by the following relations.

\[
tx_i(r) t^{-1} = x_i(t(\alpha_i)r); \quad \bar{s}_i t \bar{s}_i^{-1} = s'_i(t);
\]
where $t$ is an element from $\mathfrak{T}(R)$, $r$ is an invertible element of $R$, $u$ is an element from $\mathfrak{S}(R)$, $x_i : R \to \mathfrak{S}(R)$ and $x_{-i} : R \to \mathfrak{S}(R)$ are the homomorphisms associated to $e_i$ and $f_i$, respectively, $\tilde{s}_i(r)$ is the canonical image of $x_i(r)x_{-i}(r^{-1})x_i(r)$ in $\mathfrak{S}(R)$, $\tilde{s}_i = \tilde{s}_i(1)$, and $r^{h_i} \in \mathfrak{T}(R)$ is defined by $r^{h_i}(\lambda) = r^{(\lambda, h_i)}$ for $\lambda \in \Lambda'$.

It is easy to see $G_A(R)$ is functorial in $R$, we call $G_A$ the *Tits functor* associated to $A = (a_{ij})_{i,j=1}^n$. Set $r = 1$ in $\tilde{s}_i(r^{-1}) = \tilde{s}_i r^{h_i}$, we have $\tilde{s}_i^2 = (-1)^{h_i}$; this formula will be used in the next section.

**Remark 2.6.** The above defining relations were given in [8, 196], and are slightly different from that of [9]; in fact the formula $\tilde{s}_i^2 = (-1)^{h_i}$ cannot be derived from the defining relations in [9].

**Remark 2.7.** From the defining relations we see that $G_A(R)$ (as a group) is generated by the images of $\mathfrak{U}_{e_i}(R)$ ($0 < i \leq n$) in $G_A(R)$.

In Section 3 we need the following lemma.

**Lemma 2.8.** Let $A$ be a Cartan matrix of type

\[
\begin{array}{cccccc}
A_2 & e_1 & e_2 & B_3 & e_1 & e_2 & e_3 \\
\end{array}
\]

or

\[
\begin{array}{cccccc}
C_3 & e_1 & e_2 & e_3 \\
\end{array}
\]

then the corresponding Kac-Moody group satisfies $G_A(R) = [G_A(R), G_A(R)]$.

**Proof.** In the case of $A_2$, we have the commutation relation $[x_{e_1}(1), x_{e_2}(r)] = x_{e_1+e_2}(r)$, hence the image of $\mathfrak{U}_{e_1+e_2}(R)$ is contained in $[G_A(R), G_A(R)]$. But the Weyl group acts transitively on the set of real roots, hence the images of $\mathfrak{U}_{e_1}(R)$ and $\mathfrak{U}_{e_2}(R)$ are contained in $[G_A(R), G_A(R)]$ too. Thus by Remark 2.7, we have $G_A(R) = [G_A(R), G_A(R)]$.

In the case of $C_3$, the above proof shows that the image of $\mathfrak{U}_{e_1}(R)$ and $\mathfrak{U}_{e_2}(R)$ is contained in $[G_A(R), G_A(R)]$. A direct computation shows that in $\mathfrak{U}_{\varnothing_{e_2,e_3}}(R)$ we have $[x_{e_1}(r), x_{e_2}(1)] = x_{e_2+e_3}(-r)x_{e_2+2e_3}(-r)$. As the Weyl group acts transitively on the set of short roots too, hence the image of $\mathfrak{U}_{e_2+e_3}(R)$ is contained in $[G_A(R), G_A(R)]$ and so is $\mathfrak{U}_{e_2+e_3}(R)$. But the Weyl group acts transitively on the set of short roots too, hence the image of $\mathfrak{U}_{e_3}(R)$ is also contained in $[G_A(R), G_A(R)]$. By Remark 2.7 again, we have $G_A(R) = [G_A(R), G_A(R)]$. The proof for the case of $B_3$ is similar.

3. Construction of infinite loop spaces associated to $A_{2l-1}^{(2)}$

It is well known that there are seven infinite classes of generalized Cartan matrices of affine type (cf.[4, p.51]), whose Dynkin diagrams are listed below.
To each infinite class and each commutative ring \( R \) we want to associate a sequence of Kac-Moody groups \( G(n) \) that satisfies the conditions of Theorem 1.2. First consider the case of \( A_{2l-1}^{(2)} \), let \( g_l \) (resp. \( G_l(R) \)) be the corresponding Kac-Moody algebra (resp. group). In the following we use the notations of Section 2 freely, sometimes the subscript \( l \) will be added to indicate that the notations are associated to \( A_{2l-1}^{(2)} \). For example, \( V_l \) will be the vector space over \( \mathbb{Q} \), with basis \( \{a_i\}_{i=0,\ldots,l} \). The group \( W_l(A) \) acts on \( V_l \) and \( \triangle_l \) denotes the set of real roots of \( A_{2l-1}^{(2)} \).

In \( g_{l+1} \) set \( e'_i = s_l'(e_{i+1}) \), \( f'_i = s_l'(f_{i+1}) \), \( h'_i = s_l'(h_{i+1}) = h_{i+1} + h_l \) respectively and for \( i < l \) set \( e'_i = e_i \), \( f'_i = f_i \), \( h'_i = h_i \) respectively.

**Lemma 3.1.** In \( g_{l+1} \) we have, for \( i, j \leq l \),

\[
[h'_i, h'_j] = 0; \quad [h'_i, e'_j] = a_{ij} e'_j; \quad [h'_i, f'_j] = -a_{ij} f'_j; \quad \delta_{i,j} h'_i = \delta_{i,j} h'_i.
\]

\[(ad e_{l-1})^3 e'_l = 0; \quad (ad f_{l-1})^3 f'_l = 0.\]
Lemma 3.2. Define a linear map $\tau_w$ of $g_W$ with adjoint representation. Since $[h_{l-1}, c'_l] = -2e'_l$ and $[f_{l-1}, c'_l] = 0$ (this follows from the fact that every root is either positive or negative), the representation theory of $g_0 \cong sl_2(\mathbb{C})$ implies $(ad \ e_{l-1})^3 e_l' = 0$. The proof for the last relation is exactly the same.

By the defining relations of $g_l$, the map $e_i \to e_i'$, $f_i \to f_i'$ extends to an injective Lie algebra homomorphism $\varphi_l : g_l \to g_{l+1}$.

Lemma 3.3. Define a linear map $\tau_l : V_l \to V_{l+1}$ by $\tau_l(a_i) = a_i$ for $i < l$ and $\tau_l(a_l) = 2a_l + a_{l+1}$, then $\tau_l(\Delta^+_l) \subset \Delta^+_l$ and $\varphi_l(E_a) = E_{\tau_l(a)}$ for any $a \in \Delta_l$.

Proof. It is easy to see that the map $s_i \to s'_i$ for $i < l$ and $s_l \to s_l s_{l+1} s_l$ extends to a group homomorphism $\psi_l : W_l(A) \to W_{l+1}(A)$ and for any $v \in V_l$ and $W \in W_l(A)$ we have $\tau_l \cdot W(v) = w_l(W) \cdot \tau_l(v)$. Thus the first assertion follows readily. Similarly, the map $s'_i \to s'_i$ for $i < l$ and $s'_l \to s'_l s'_{l+1} (s'_l)^{-1}$ extends to a group homomorphism $\psi'_l : W'_l(A) \to W'_{l+1}(A)$. One checks that $w_l$ and $w'_l$ are compatible with the homomorphisms $\phi : W'_l(A) \to W_l(A)$ and $\phi : W'_{l+1}(A) \to W_{l+1}(A)$. We also have for any $\omega \in W'_l(A)$, $\varphi_l \cdot \psi_l(\omega) = (\psi_l w'_l(\omega)) \cdot \varphi_l$; recall the homomorphisms $\psi_l : W'_l(A) \to Aut(g(A)_l)$ and $\psi_{l+1} : W'_{l+1}(A) \to Aut(g(A)_{l+1})$ define in Section 2. Now we are ready to prove the second assertion. First, it is true for $a = a_i$, $i \leq l$ by the definition of $\varphi_l$. Let $a = \phi_l(\omega)(a_i)$ be an element of $\Delta_l$, with $\omega \in W'_l(A)$, then $\varphi_l(E_a) = \varphi_l(\omega)(E_{a_i}) = \varepsilon(E_{a_i}) = (\psi_{l+1} w'_l(\omega))(E_{\tau_l(a_i)}) = E_{\psi_{l+1} w'_l(\omega)(\tau_l(a_i))} = E_{\tau_l(a_i)}$. This finishes the proof.

For any $a \in \Delta_l$, let $\mathfrak{U}_a$ be the corresponding group scheme defined in §2, then we can define a homomorphism $\psi_a : \mathfrak{U}_a \to \mathfrak{U}_{\tau_l(a)}$ that is compatible with the map $E_a \to E_{\tau_l(a)}$.

Lemma 3.3. Let $\theta \subset \Delta_l$ be a nilpotent set of real roots, then $\tau_l(\theta) \subset \Delta_{l+1}$ is pre-nilpotent. Let $\theta'$ be the least nilpotent set containing $\tau_l(\theta)$, let $\mathfrak{U}_\theta$ and $\mathfrak{U}_{\theta'}$ be the group schemes in Proposition 2.3, then the homomorphisms $\psi_a : \mathfrak{U}_a(R) \to \mathfrak{U}_{\tau_l(a)}(R)$, $a \in \theta$ extend uniquely to a homomorphism $\psi : \mathfrak{U}_{\theta}(R) \to \mathfrak{U}_{\theta'}(R)$.

Proof. By lemma 3.2 the homomorphism $L_\theta \to L_{\theta'}$ induced by $\varphi_l$ is injective. Thus for $a, b \in \theta$, the commutation relation of $\mathfrak{U}_a(R)$ and $\mathfrak{U}_b(R)$ in $\mathfrak{U}_{\theta}(R)$ is exactly the same as that of $\mathfrak{U}_{\tau_l(a)}(R)$ and $\mathfrak{U}_{\tau_l(b)}(R)$ in $\mathfrak{U}_{\theta'}(R)$. Now the lemma follows readily.

By Lemma 3.2 and Lemma 3.3 the group homomorphisms $\psi_a : \mathfrak{U}_a(R) \to \mathfrak{U}_{\tau_l(a)}(R)$, $a \in \Delta_l$, extend to a group homomorphism $\psi(R) : \mathfrak{S}_l(R) \to \mathfrak{S}_{l+1}(R)$.

Let $\Lambda_1$ be a free abelian groups with basis $h_0, \ldots, h_l$ and $\Lambda'$ its dual. Define linear map $\omega_l : \Lambda_l \to \Lambda_{l+1}$ by $\omega_l(h_i) = h_i$ for $i < l$ and $\omega_l(h_l) = h_l + 2h_{l+1}$. Denote by $\omega'_l$ the dual map of $\omega_l$, then $\omega'_l$ induces a group homomorphism $\omega_l(R) : \mathfrak{S}_l(R) \to \mathfrak{S}_{l+1}(R)$.
From the defining relations of Kac-Moody groups and the constructions of \( \psi(R) \) and \( \omega_l(R) \) we see that the homomorphism of free products \( \psi^* \omega_l(R) : \mathcal{G}(R) \ast T_l \longrightarrow \mathcal{G}_{-1}(R) \ast T_{l+1}(R) \) reduces to a homomorphism \( g_{l} : G_l(R) \longrightarrow G_{l+1}(R) \).

For each \( 0 < i < l \) in \( G_l(R) \) satisfy the following two relations,

\[
\tilde{s}_i \tilde{s}_j \tilde{s}_i = \tilde{s}_j \tilde{s}_i \tilde{s}_j = (-1)^{h_i - 2a_{i,j}} \tilde{s}_i \tilde{s}_j \tilde{s}_i \tilde{s}_j \tilde{s}_i \cdots (m_{i,j} \text{ factors on each side}),
\]

where \( \tilde{s}_i \) is defined by sending \( \{e_i, e_{i+1}\} \) to \( \{-e_{i+1}, e_i\} \) and leaves the other basis vectors invariant.

**Lemma 3.4.** Let \( \tilde{s}_i \), \( 0 < i < l \) in \( G_l(R) \) satisfy the following two relations,

\[
\tilde{s}_i \tilde{s}_j \tilde{s}_i = \tilde{s}_j \tilde{s}_i \tilde{s}_j = (-1)^{h_i - 2a_{i,j}} \tilde{s}_i \tilde{s}_j \tilde{s}_i \tilde{s}_j \tilde{s}_i \cdots (m_{i,j} \text{ factors on each side}).
\]

Let \( \tilde{W}_l \) be the subgroup of \( G_l(R) \) generated by \( \{\tilde{s}_i\}_{0 < i < l} \), then the maps \( s_i \rightarrow \tilde{s}_i \) extend to a group homomorphism \( h_l : W_l' \rightarrow \tilde{W}_l \).

**Proof.** We prove the first assertion and the second assertion will follow directly. As \( \tilde{s}_i^2 = (-1)^{h_i} \) the first relation is equivalent to

\[
(-1)^{h_i} \tilde{s}_j \tilde{s}_i \tilde{s}_i = (-1)^{h_i - 2a_{i,j}} \tilde{s}_i \tilde{s}_j \tilde{s}_i \tilde{s}_j \tilde{s}_i \cdots \tilde{s}_i \tilde{s}_j \tilde{s}_i \tilde{s}_j \tilde{s}_i \cdots (m_{i,j} \text{ factors on each side}),
\]

which is one of the defining relations of \( G_l(R) \). The second relation was proved in Remark 3.7 of [9].

For each \( 0 < i < n \) set

\[
r_i = s_{2i+1} s_{2i} s_{2i-1} s_{2i+1} s_{2i} s_{2i-1}
\]

in \( G_{2n}(R) \) and set \( w_i = h_{2n}(r_i) \). Let \( \sigma(i) \in \Sigma_n \) be the permutation that swaps the \( i \)-th element with the \((i + 1)\)-th one, then the map \( \sigma(i) \rightarrow r_i \) extends to a group homomorphism \( \zeta_n' : \Sigma_n \rightarrow W_{2n}' \). Set \( \zeta_n = h_{2n} \zeta_n' \).

**Remark 3.5.** In fact we can identify \( W_{2n}' \) with the signed permutation group, i.e., the group of linear transformations of \( \mathbb{R}^{2n} \) leaving invariant the set \( \{\pm e_i\} \) of standard basis vectors and their negatives. Then \( r_i \) is the linear isomorphism of \( \mathbb{R}^{2n} \) that sends \( \{e_{2i-1}, e_{2i}\} \) to \( \{e_{2i+1}, e_{2i+2}\} \) and leaves the other basis vectors invariant.

**Theorem 3.6.** Let \( G(R) = \lim_{n \rightarrow \infty} G_n(R) \), then \( \pi = \pi_0(G) \) satisfies \( \pi = [\pi, \pi] \).

Applying Quillen’s plus construction to \( BG(R) \) and \( \pi_1(BG) \cong \pi \), we get an infinite loop space \( BG^+(R) \).
Proof. Condition 1) of Theorem 1.2 follows directly from Lemma 2.8. Thus we only need to verify condition 2) of Theorem 1.2. Set \( f_{m,n} = f_{m+n-1} \cdots f_{m+1} f_m \); we want to show that \( f_{m,n}(G(m)) \) and \( c(n,m)(f_{m,n}(G(n)))c(m,n) \) are commutative in \( G(m+n) \). Set \( s_{nm} := \phi_{2m+2n} t_{n,m}(c(n,m)) \) in the following, recall that \( \phi_{2m+2n} \) is the natural homomorphism \( W(\Lambda_{4m+4n-1}^{(2)}) \to W(\Lambda_{4m+4n-1}^{(2)}) \).

By remark 2.7, \( f_{m,n}(G(m)) \) is generated by the subgroups \( \{ \Upsilon_0(R) \}_{a \in \Theta} \) and \( c(n,m)(f_{m,n}(G(n)))c(m,n) \) is generated by the subgroups \( \{ \Upsilon_a(R) \}_{a \in \Theta'} \), where

\[
\Theta = \{ \pm a_0, \ldots, \pm a_{2m-1}, (s_{2m-1} \cdot s_{2m} \cdots s_{2m+2n-1})(\pm a_{2m+2n}) \}
\]

\[
= \{ \pm a_0, \ldots, \pm a_{2m-1}, \pm(2a_{2m-1} + \cdots + 2a_{2m+2n-1} + a_{2m+2n}) \}
\]

and

\[
\Theta' = s_{nm} \{ \pm a_0, \ldots, \pm a_{2m-1}, (s_{2m-1} \cdot s_{2m} \cdots s_{2m+2n-1})(\pm a_{2m+2n}) \}.
\]

Thus in order to verify condition 2) it suffices to show that for any \( \alpha \in \Theta \) and \( \beta \in \Theta' \), \( \Upsilon_0(R) \) and \( \Upsilon_\alpha'(R) \) are commutative, but this can be deduced from the fact that the subalgebras \( L_{\pm\alpha} \) and \( L_{\pm\beta} \) of \( g_{2m+2n} \) are commutative. Indeed, when \( L_{\pm\alpha} \) and \( L_{\pm\beta} \) are commutative, one checks that \( \{ \alpha, \beta \} \) is a prenilpotent pair and \( \vartheta(a,b) = \{ \alpha, \beta \} \), hence by Remark 2.4 the group \( \Upsilon_\vartheta(R) \) is commutative. Thus in order to finish the proof it suffices to show that for any \( \alpha \in \Theta \) and \( \beta \in \Theta' \), \( L_{\pm\alpha} \) and \( L_{\pm\beta} \) are commutative.

Direct computation shows that

\[
(s_{2m-1} \cdot s_{2m} \cdots s_{2m+2n-1})(\pm a_{2m+2n}) = s_{nm}(\pm a_{2m+2n});
\]

\[
(s_{2m} \cdot s_{2m} \cdots s_{2m+2n-1})(\pm a_{2m+2n}) = s_{mn}(\pm a_{2m+2n});
\]

\[
s_{mn}(a_0) = \pm(a_0 + a_1 + 2(a_2 + \cdots + a_{2m}) + a_{2m+1});
\]

\[
s_{mn}(\pm a_1, \ldots, \pm a_{2m-1}) = \pm(\pm a_{2m+1}, \ldots, \pm a_{2m+2n-1});
\]

\[
s_{mn}(\pm a_{2m+1}, \ldots, \pm a_{2m+2n-1}) = \pm a_1, \ldots, \pm a_{2m-1}.
\]

Thus we only need to show that \( L_{\pm(a_0+a_1+2(a_2+\cdots+a_{2n})+a_{2m+1})} \) is commutative with \( L_{\pm a_0} \), and \( L_{\pm a_{2m+2n}} \) is commutative with \( L_{\pm(2a_{2m-1}+\cdots+2a_{2m+2n-1}+a_{2m+2n})} \). We prove the first assertion, the proof for the second one is similar.

First, we have \( [L_{-a_0}, L_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}] \in L_{a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}} \); but it is well known that the highest root in \( \mathbb{Z}a_1 + \mathbb{Z}a_2 + \cdots + \mathbb{Z}a_{2m+1} \cap \Delta_{2m+2n} \) is \( a_1 + \cdots + a_{2m+1} \). Hence \( [L_{-a_0}, L_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}] = 0 \). We also have \( [h_0, L_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}] = 0 \). Set \( g_0 = L_{a_0} \oplus L_{-a_0} \oplus \mathbb{C} h_0 \) and consider \( g_{2m+2n} \) as a \( g_0 \)-module by restricting of the adjoint representation. By the representation theory of \( g_0 \cong sl_2(\mathbb{C}) \), it follows that

\[
[L_{a_0}, e_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}] = 0.
\]

Similarly, we have

\[
[L_{a_0}, f_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}] = 0
\]

and

\[
[L_{-a_0}, f_{a_0+a_1+2(a_2+\cdots+a_{2m})+a_{2m+1}}] = 0.
\]
This finishes the proof of the theorem. The following Dynkin diagram would illustrate our proof, where \( a_0' \) (resp. \( a_{2m}' \)) denotes \( 2a_{2m-1} + \cdots + 2a_{2m+2n-1} + a_{2m+2n} \) (resp. \( s_{n,m}(a_0) \)).

\[
\begin{array}{cccccccc}
\bullet & \bullet & \bullet & \cdots & \bullet & \bullet & \bullet & \bullet \\
a_1 & a_2 & a_3 & \cdots & a_{2m-1} & a_{2m} & a_{2m+1} & a_{2m+2} \\
\end{array}
\]

**Remark 3.7.** It is easy to see, from the construction above, that \( BG^+(R) \) as an infinite loop space is functorial in \( R \) (see [6, 7] for a delicate exposition of infinite loop spaces and its relation with \( E_\infty \) spaces). Thus we can, as in the classical cases, define a \( K \)-theory of rings by setting

\[
K_1^G(R) := \pi_1(BG^+(R)).
\]

### 4. The constructions in the other cases

The constructions in the other cases are similar. For example, in the case of \( A_l^{(1)} \), let \( g_l \) be the Kac-Moody algebra associated to \( A_l^{(1)} \), and in \( g_{l+1} \) set \( e'_l = s_l'(e_{l+1}) \), \( f'_l = s_l'(f_{l+1}) \), \( h'_l = s_l'(h_{l+1}) = h_{l+1} + h_l \) respectively and for \( i < l \) set \( e'_i = e_i \), \( f'_i = f_i \), \( h'_i = h_i \) respectively. In the case of \( D_l^{(1)} \), set \( e'_l = s_l \cdot s_{l-1}'(e_{l+1}) \), \( f'_l = s_l \cdot s_{l-1}'(f_{l+1}) \), \( h'_l = s_l \cdot s_{l-1}'(h_{l+1}) = h_{l+1} + h_l + h_{l-1} \) respectively. For the rest constructions we just repeat the arguments of the previous section.

**Remark 4.1.** In Section 3 we require that \( \wedge_l \) is freely generated by \( \{h_0, \ldots, h_i\} \), in fact this assumption is not necessary. For example, in the case of \( A_l^{(1)} \) we can set \( \wedge_l \) to be freely generated by \( \{h_1, \ldots, h_i\} \) and add an \( h_0 := -h_1 - \cdots - h_l \). When \( R \) is a field \( K \), the corresponding Kac-Moody group \( G_l(K) \) is isomorphic to \( SL_{l+1}(K[t, t^{-1}]) \), then \( G(\infty, K)^+ \) is of course an infinite loop space. However, we don’t know any explicit realization of \( G_l(R) \) in the general cases.

We can also treat the (topological) affine Kac-Moody groups over \( \mathbb{C} \) (see [5] for the definition), and applying the method of Section 3 we have the following result.

**Theorem 4.2.** Let \( \{A_l\}_{l>2} \) be one of the seven (infinite) classes of affine generalized Cartan matrices and let \( \{G_l\}_{l>2} \) be the associated simply-connected Kac-Moody groups over \( \mathbb{C} \), then we can define for each \( l > 2 \) a natural homomorphism \( f_l : G_l \to G_{l+1} \) such that \( BG = \lim_{l \to \infty} BG_l \) is an infinite loop space.

**Remark 4.3.** In fact there exists a (infinite) classes of classical Lie groups \( \{G(l)\}_{l>2} \) such that \( G_l \) is isomorphic to a central extension of the group of polynomial loops or twisted polynomial loops on \( G(l) \) (cf. [5] §2.8).
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References


Xianzu Lin
College of Mathematics
and Computer Science
Fujian Normal University
Fuzhou, 350108, China
linxianzu126.com

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